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On Ulam Stability with Respect to 2-Norm

Janusz Brzdęk

Faculty of Applied Mathematics, AGH University of Science and Technology, Mickiewicza 30, 30-059 Kraków, Poland; brzdek@agh.edu.pl

Abstract: The Ulam stability of various equations (e.g., differential, difference, integral, and functional) concerns the following issue: *how much does an approximate solution of an equation differ from its exact solutions?* This paper presents methods that allow to easily obtain numerous general Ulam stability results with respect to the 2-norms. In four examples, we show how to deduce them from the already known outcomes obtained for classical normed spaces. We also provide some simple consequences of our results. Thus, we demonstrate that there is a significant symmetry between such results in classical normed spaces and in 2-normed spaces.

Keywords: Ulam stability; equation; 2-norm; 2-Banach space

1. Introduction

The Ulam stability is a quite popular subject of investigations, which mainly concerns various equations (e.g., difference, differential, integral, and functional). Very roughly speaking, it deals with the following issue:

How much an approximate solution to an equation differs from the exact solutions of it?

It is motivated by a problem that was raised by Ulam in 1940 for the equation of group homomorphism and the first solution to it provided by Hyers [1] for Banach spaces. For more details and the historical background, we refer to [2], which is the first monograph on the subject (see also [3]). An ample discussion on various possible definitions of such stability is provided in [4]. Numerous examples of recent Ulam stability results as well as further related information and references are also given in [5–9].

Problems of such types are quite natural for difference, differential, functional and integral equations (see, for example, [10–14] and the references therein), and they are closely related to the issues considered in the theories of approximation, optimization and shadowing (see [15]).

We should mention here that, soon after Hyers' publication, a new wider approach was proposed by T. Aoki [16] (see also, for example, ref. [17]). This approach was later complemented in [8,9,18], and the main result that thus arose can be considered to be very representative of Ulam stability. It can be stated as follows (see, for example, ref. [19] [Theorem 1]).

Theorem 1. *Let X be a Banach space, V be a normed space, and $V_0 := V \setminus \{0\}$. Let $\xi \geq 0$ and $s \neq 1$ be real numbers and $g\colon V \to X$ be such that*

$$\|g(x+y) - g(x) - g(y)\| \leq \xi(\|x\|^s + \|y\|^s), \qquad \forall x, y \in V_0\,. \tag{1}$$

Then there is exactly one additive mapping $h : V \to X$ such that

$$\|h(x) - g(x)\| \leq \frac{\xi\|x\|^s}{|1 - 2^{s-1}|}, \qquad \forall x \in V_0. \tag{2}$$

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Let us remind here that $h : V \to X$ is additive if, for every $x, y \in V$,

$$h(x+y) = h(x) + h(y). \quad (3)$$

In [18], an example was given that for $s = 1$, a result analogous to Theorem 2 is not true. Moreover, estimate (2) is optimal (see, for example, ref. [3]) and, in the situation where $s < 0$, each mapping $h : V \to X$ satisfying inequality (1) must be additive and the completeness of space B is not necessary in this case (see, for example, refs. [6,19] for suitable references and further information on this subject).

The next definition (cf. [19] [Definition 1]) is quite abstract, but makes the notion of Ulam stability more precise (here $\mathbb{R}_+$ stands for the real interval $[0, \infty)$ and C^D denotes the family of all mappings from a nonempty set D into a nonempty set C).

Definition 1. *Let m be a positive integer, (F, d) be a metric space, $U \neq \emptyset$ be a set, $\mathcal{D}_0 \subset \mathcal{D} \subset F^U$ and $\mathcal{T} \subset \mathbb{R}_+^{U^m}$ be nonempty, $\mathcal{S} : \mathcal{T} \to \mathbb{R}_+^U$, and $\mathcal{F}_1, \mathcal{F}_2 : \mathcal{D} \to E^{U^m}$. The equation*

$$(\mathcal{F}_1\alpha)(r_1, \dots, r_m) = (\mathcal{F}_2\alpha)(r_1, \dots, r_m) \quad (4)$$

is said to be $\mathcal{S}$-stable in $\mathcal{D}_0$ if, for any $\phi \in \mathcal{D}_0$ and $\delta \in \mathcal{T}$ with

$$d((\mathcal{F}_1\phi)(r_1, \dots, r_m), (\mathcal{F}_2\phi)(r_1, \dots, r_m)) \leq \delta(r_1, \dots, r_m), \qquad \forall r_1, \dots, r_m \in U,$$

there is a mapping $\alpha \in \mathcal{D}$ such that (4) holds for all $r_1, \dots, r_m \in U$ and

$$d(\alpha(r), \phi(r)) \leq (\mathcal{S}\delta)(r), \qquad \forall r \in U.$$

If $(\mathcal{S}\delta)(r) = 0$ for $\delta \in \mathcal{T}$ and $r \in U$, then we say that the equation is hyperstable in $\mathcal{D}_0$.

It is easily seen that Equation (4) becomes Equation (3) when $m = 2$, $(\mathcal{F}_1\alpha)(r_1, r_2) = \alpha(r_1 + r_2)$ and $(\mathcal{F}_2\alpha)(r_1, r_2) = \alpha(r_1) + \alpha(r_2)$ for $\alpha \in \mathcal{D}$ and $r_1, r_2 \in U$. So, Theorem 2 says that, for every $s \neq 1$, Equation (3) is $\mathcal{S}$-stable in $\mathcal{D}_0 = \mathcal{D} = B^U$ with $U = V_0$ and $\mathcal{S} : \mathcal{T} \to \mathbb{R}_+^U$ defined by

$$(\mathcal{S}\delta_\xi)(x) = \frac{\eta\|x\|^s}{|1 - 2^{s-1}|}, \qquad \forall \delta_\xi \in \mathcal{T}, x \in U,$$

where

$$\delta_\xi(x, y) = \xi(\|x\|^s + \|y\|^s), \qquad \forall x, y \in U, \xi \in \mathbb{R}_+,$$

and

$$\mathcal{T} = \{\delta_\xi \in \mathbb{R}_+^{U \times U} : \xi \in \mathbb{R}_+\}.$$

However, if $s < 0$, then (as we have already mentioned) a stronger property is valid. Namely, functional Equation (3) is hyperstable in $\mathcal{D}_0 = B^U$, which actually means that each $h \colon V \to X$ fulfilling inequality (1) must be additive, i.e., (3) holds for every $x, y \in V$.

Clearly, inequality (1) can be replaced by various other conditions of the form

$$\|g(x+y) - g(x) - g(y)\| \leq \Phi(x, y), \qquad \forall x, y \in W \subset V, \quad (5)$$

and, for instance, the condition

$$\|g(x+y) - g(x) - g(y)\| \leq \xi\|x\|^p\|y\|^q, \qquad \forall x, y \in V_0, \quad (6)$$

was investigated in [20,21] for fixed real numbers p, q and $\xi > 0$. Also, the Ulam stability of numerous other equations has been studied (including difference, integral and differential equations), and we refer the readers to [2,3] for further details, references and examples. In particular, several authors (see, for example, refs. [22–30]) investigated the stability of some cases of the following very general functional equation:

$$\sum_{i=1}^{m} A_i f\Big(\sum_{j=1}^{n} a_{ij}x_j\Big) = D(x_1, \ldots, x_n) \tag{7}$$

in the class of functions f mapping a module X over a commutative ring $\mathbb{K}$ into a Banach space Y over the field $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, where $D : X^n \to Y$ is a given function satisfying some additional conditions, $A_1, \ldots, A_m \in \mathbb{F} \setminus \{0\}$, and $a_{ij} \in \mathbb{K}$ for $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$. Note that (3) is a particular case of (7). For comments on some other particular cases of this equation, see [10,24,31–35].

Very roughly, we can say (see Definition 1) that an equation is Ulam stable if for every mapping fulfilling the equation approximately (in some sense), there is an accurate solution of the equation that is close to this mapping (in some way).

In mathematics and its applications, the notions of an approximate solution and of the closeness of two mappings can be understood in various ways (see, for example, refs. [31,36–42]). So, for Ulam stability, we should take into account also the nonstandard ways of measuring distance. One of them can be introduced by the concept of 2-norms, and we refer the readers to the third section for more details on this idea.

The study of Ulam stability with regard to 2-norms has recently been quite popular, and numerous papers have been published on this subject (see [19] for a discussion of such outcomes and suitable references).

The authors of such articles mainly use modifications of quite involved reasonings that were applied in classical normed spaces for analogous issues (see, for example, refs. [39,41,43–57]).

In this paper, we show how to easily obtain numerous general Ulam stability results with respect to the 2-norms, by deriving them from the already known outcomes obtained for normed spaces. Thus, we prove that there is significant symmetry between such results in classical normed spaces and in 2-normed spaces.

The methods that we present are very general, but also quite simple. To show them explicitly, we use four examples of stability results of different natures and generalities.

2. Auxiliary Results

Below, we provide four examples of Ulam stability outcomes, which we use later. First, we recall a part of [10] [Theorem 1.2], concerning the approximate solutions of functional Equation (8) that is connected to the branching measures of information (see [58]) ($\mathbb{R}$ denotes the sets of reals and, as before, $\mathbb{R}_+ := [0, \infty)$).

Theorem 2. *Let W and Y be normed spaces, $W_0 := W \setminus \{0\}$, $\eta \in \mathbb{R}_+$, $r \in \mathbb{R}$, $r \neq 1$, and $d : W^2 \to Y$ be such that the functional equation*

$$\alpha(x + y) = \alpha(x) + \alpha(y) + d(x, y), \qquad \forall x, y \in W, \tag{8}$$

has at least one solution $\alpha : W \to Y$. Assume that $h\colon W \to Y$ fulfills the inequality

$$\|h(x + y) - h(x) - h(y) - d(x, y)\| \leq \eta(\|x\|^r + \|y\|^r), \qquad \forall x, y \in W_0.$$

Then the following two statements are valid:

(i) If $r < 0$, then h is a solution to Equation (8).

(ii) If $r \geq 0$ and Y is complete, then there is a unique $\alpha : W \to Y$ such that (8) holds and

$$\|h(x) - \alpha(x)\| \leq \eta|1 - 2^{r-1}|^{-1}\|x\|^r, \qquad \forall x \in W_0.$$

An analogous result is not possible for $r = 1$ and the constant in the inequality of (ii) is optimal in the general situation (see [10]). Let us remind that the situation depicted by statement (i) is called hyperstability, i.e., an approximate (in the sense that we consider) solution of an equation must be a solution to the equation.

The next three examples of Ulam stability outcomes concern difference, differential and integral equations.

In what follows, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{C}$ denote the sets of positive integers, integers, rational numbers, and complex numbers, respectively. Next, $\mathbb{Q}_+ := \mathbb{Q} \cap (0, \infty)$, $T \in \{\mathbb{N}, \mathbb{Z}\}$, X is a Banach space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $p \in \mathbb{N}$, $a_1, \ldots, a_p \in \mathbb{K}$, and $r_1, \ldots, r_p \in \mathbb{C}$ are all the roots of the equation

$$r^p + a_1 r^{p-1} + \ldots + a_{p-1} r + a_p = 0.$$

Let $(b_n)_{n \in T}$ be a sequence in X. The subsequent Ulam stability result is a simplified version of [12] [Theorem 3] for the difference equation

$$x_{n+p} + a_1 x_{n+p-1} + \ldots + a_p x_n + b_n = 0, \qquad \forall n \in T. \tag{9}$$

Theorem 3. *Let $\delta > 0$ and*

$$C_0 := \prod_{i=1}^{p} \big| 1 - |r_i| \big| \neq 0.$$

Let $(y_n)_{n \in T}$ be a sequence in X with

$$\|y_{n+p} + a_1 y_{n+p-1} + \ldots + a_p y_n + b_n\| \leq \delta, \qquad \forall n \in T. \tag{10}$$

Then, there is a sequence $(x_n)_{n \in T}$ in X such that (9) holds and

$$\|y_n - x_n\| \leq \delta\, C_0^{-1}, \qquad \forall n \in T. \tag{11}$$

Moreover, sequence $(x_n)_{n \in T}$ is unique if $T = \mathbb{Z}$ or

$$|r_i| > 1, \qquad \forall i \in \{1, \ldots, p\}. \tag{12}$$

For results related to Theorem 3, we refer to refs. [12,31,59,60],

The next theorem is a simplified version of [14] [Corollary 4.2] (cf. [61]) and concerns the differential equation

$$g^{(p)}(t) + a_1 g^{p-1}(t) + \ldots + a_{p-1} g'(t) + a_p g(t) + G(t) = 0, \qquad \forall t \in I, \tag{13}$$

for $g \in C^p(I, X)$, where I is an open real interval, $C^p(I, X)$ denotes the family of all mappings from I into X that are p-times continuously differentiable, and $G : I \to X$ is continuous ($\Re(z)$ stands for the real part of a complex number z).

Theorem 4. *Let $\delta \in \mathbb{R}_+$ and*

$$D_0 := \prod_{i=1}^{p} |\Re(r_i)| \neq 0.$$

If $r_1, \ldots, r_p \in \mathbb{K}$ and $h \in C^p(I, X)$ is such that

$$\|h^{(p)}(t) + a_1 h^{(p-1)}(t) + \ldots + a_p h(t) + G(t)\| \leq \delta, \qquad \forall t \in I,$$

then there is a solution $g \in C^p(I, X)$ of (13) with

$$\|h(t) - g(t)\| \leq \delta D_0^{-1}, \qquad \forall t \in I.$$

The last example is a simplified version of [62] [Theorem 1] and concerns the stability of the integral equation

$$\psi(x) + \int_a^x N(x, t, \psi(\alpha(x, t))\, dt + G(x) = 0, \qquad \forall x \in J, \tag{14}$$

for $\psi \in \mathcal{C}(J, X)$, where J is a real interval of the form $[a, b]$ or $[a, b)$ (with some real $a < b$), $\mathcal{C}(J, X)$ means the family of all continuous mappings from J into X, $G \in \mathcal{C}(J, X)$ is fixed, $\int$ denotes the Bochner integral, and $N : J \times J \times X \to X$ and $\alpha : J^2 \to J$ are given continuous functions.

Assume that there is a continuous $L : J \times J \times \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\|N(x, t, u_1) - N(x, t, u_2)\| \leq L(x, t, \|u_1 - u_2\|), \qquad \forall x, t \in J,\ u_1, u_2 \in X,$$

and

$$L(x, t, s_1) \leq L(x, t, s_2), \qquad \forall x, t \in J,\ 0 \leq s_1 \leq s_2.$$

We have the following result ($\mathcal{C}(J, \mathbb{R}_+)$ stands for the family of all mappings from J into $\mathbb{R}_+$ that are continuous).

Theorem 5. *Let* $\varepsilon \in \mathcal{C}(J, \mathbb{R}_+)$*,*

$$\sigma_0(x) := \sum_{n=0}^{\infty} \Lambda^n \varepsilon(x) < \infty, \qquad \forall x \in J,$$

and thus the defined $\sigma_0 : J \to \mathbb{R}_+$ *is continuous, where*

$$\Lambda\eta(x) := \int_a^x L\big(x, t, \eta(\alpha(x, t))\big)\, dt, \qquad \forall \eta \in \mathcal{C}(J, \mathbb{R}_+),\ x \in J.$$

Let $\varphi \in \mathcal{C}(J, X)$ *and*

$$I_\varphi(x) := \varphi(x) + \int_a^x N\big(x, t, \varphi(\alpha(x, t))\big)\, dt + G(x), \qquad \forall x \in J.$$

If

$$\|I_\varphi(x)\| \leq \varepsilon(x), \qquad \forall x \in J,$$

then there is a solution $\psi \in \mathcal{C}(J, X)$ *of* (14) *with*

$$\|\psi(x) - \varphi(x)\| \leq \sigma_0(x), \qquad \forall x \in J.$$

3. Auxiliary Information on 2-Normed Spaces

Let us recall that the notion of 2-norm was introduced by Gähler [63] (cf. refs. [64,65]). To avoid any ambiguities, we need some definitions.

Definition 2. *Let* Y *be a linear space over* $\mathbb{K}$ *with a dimension greater than* 1*. We say (cf. [63,64]) that a mapping* $\|\cdot, \cdot\| : Y^2 \to \mathbb{R}_+$ *is a 2-norm if, for every* $x_1, x_2, x_3 \in Y$ *and* $\beta \in \mathbb{K}$*, the following four conditions hold:*

(a) $\|x_1, x_2\| = 0$ *iff* x_1 *and* x_2 *are linearly dependent;*
(b) $\|x_1, x_2\| = \|x_2, x_1\|$*;*
(c) $\|x_1, x_2 + x_3\| \leq \|x_1, x_2\| + \|x_1, x_3\|$*;*
(d) $\|\beta x_1, x_2\| = |\beta| \|x_1, x_2\|$*.*

Let Y be as above, $\|\cdot, \cdot\| : Y \times Y \to \mathbb{R}_+$ be a 2-norm and $(x_i)_{i \in \mathbb{N}}$ be a sequence in Y. Then $(x_n)_{n \in \mathbb{N}}$ is said to be a 2-*Cauchy sequence* if there are linearly independent $z_1, z_2 \in Y$ with

$$\lim_{m,n \to \infty} \|x_m - x_n, z_i\| = 0, \qquad i = 1, 2.$$

$(x_n)_{n \in \mathbb{N}}$ is 2-convergent if there is $x \in Y$ with $\lim_{n\to\infty} \|x_n - x, z\| = 0$ for every $z \in Y$; x is unique, and it is called a limit of $(x_n)_{n \in \mathbb{N}}$ and denoted by $\lim_{n\to\infty} x_n$. A 2-norm is complete if every 2-Cauchy sequence is 2-convergent.

If $\langle\cdot,\cdot\rangle$ is a real inner product in Y, then a 2-norm in Y can be defined by

$$\|x,y\| := \sqrt{\|x\|^2\|y\|^2 - \langle x,y\rangle^2}, \qquad \forall x,y \in Y. \tag{15}$$

Further, if $(Y,\langle\cdot,\cdot\rangle)$ is a Hilbert space, then the 2-norm given by (15) is complete (see [66] [Proposition 2.3]). In $\mathbb{R}^2$ with the natural inner product (given by $\langle(x_1,x_2),(y_1,y_2)\rangle = x_1y_1 + x_2y_2$), the 2-norm depicted by (15) has the form

$$\|(x_1,x_2),(y_1,y_2)\| := |x_1y_2 - x_2y_1|, \qquad \forall (x_1,x_2),(y_1,y_2) \in \mathbb{R}^2. \tag{16}$$

Finally, let us mention that the following two expressions:

(a) $c\|\cdot,\cdot\|_1 + d\|\cdot,\cdot\|_2$;

(b) $\max\left\{c\|\cdot,\cdot\|_1\,,\,d\|\cdot,\cdot\|_2\right\}$,

define 2-norms for any two 2-norms $\|\cdot,\cdot\|_1$ and $\|\cdot,\cdot\|_2$ in a real linear space Y and every $c,d \in (0,\infty)$.

4. Stability of Difference and Functional Equations

Now we start to present the aforementioned new methods of proving Ulam stability results with respect to the 2-norms. In this section, Y is a linear space over $\mathbb{K}$, with dimension greater than 1, $\|\cdot,\cdot\|$ is a 2-norm in Y, and $Z \subset Y$ contains two linearly independent vectors. We begin with an analogue of Theorem 2.

Theorem 6. *Let W, W_0, $r \neq 1$ and d be as in Theorem 2, and $\mu : Z \to \mathbb{R}$. Let $h : W \to Y$ satisfy*

$$\|h(x+y) - h(x) - h(y) - d(x,y), z\| \le (\|x\|^r + \|y\|^r)\mu(z), \qquad \forall x,y \in W_0,\ z \in Z.$$

Then the following two statements are valid.

(i) If $r < 0$, then h is a solution to Equation (8).

(ii) If $r \ge 0$ and $\|\cdot,\cdot\|$ is complete, then there is a unique solution $\alpha : W \to Y$ of (8) with

$$\|h(x) - \alpha(x), z\| \le |1 - 2^{r-1}|^{-1}\|x\|^r\mu(z), \qquad \forall x \in W_0,\ z \in Z. \tag{17}$$

Proof. Fix $z \in Z$ and $k \in \mathbb{N}$. Take $w \in Z$ such that z and w are linearly independent and write

$$\|x\|_k := \|x,z\| + \frac{1}{k}\|x,w\|, \qquad \forall x \in Y.$$

Then, it is easily seen that $\|\cdot\|_k$ is a norm in Y. Next,

$$\|h(x+y) - h(x) - h(y) - d(x,y)\|_k \le \left(\mu(z) + \frac{1}{k}\mu(w)\right)(\|x\|^r + \|y\|^r), \qquad \forall x,y \in W_0.$$

So, in the case $r < 0$, by Theorem 2 with $\eta := \mu(z) + \frac{1}{k}\mu(w)$, h is a solution to (8).

Now, assume that $r \ge 0$ and $\|\cdot,\cdot\|$ is complete. Let $(y_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in Y with respect to $\|\cdot\|_k$. Then

$$\lim_{m,n\to\infty} \|y_n - y_m, v\| = 0$$

for $v \in \{z,w\}$, whence $(y_n)_{n\in\mathbb{N}}$ is a 2-Cauchy sequence. So, there is $y_0 \in Y$ with

$$\lim_{n\to\infty} \|y_n - y_0, v\| = 0, \qquad \forall v \in Y.$$

Hence,

$$\lim_{n\to\infty} \|y_n - y_0\|_k = 0.$$

In this way we have shown that the norm $\|\cdot\|_k$ is complete and consequently, by Theorem 2, there is a unique additive $\alpha_k : W \to Y$ with

$$\|h(x) - \alpha_k(x)\|_k \le \mu_k \|x\|^r, \qquad \forall x \in W_0,$$

where

$$\mu_k := |1 - 2^{r-1}|^{-1}\Big(\mu(z) + \frac{1}{k}\mu(w)\Big).$$

Since $\mu_k \le \mu_1$, the uniqueness of α_k means that $\alpha_k = \alpha_1$ for $k \in \mathbb{N}$, and consequently,

$$\|h(x) - \alpha_1(x)\|_k \le \mu_k \|x\|^r, \qquad \forall x \in W_0.$$

Hence, letting $k \to \infty$, we obtain the inequality in (ii) with $\alpha = \alpha_1$.

The uniqueness of α follows from the fact that each solution $\alpha_0 : W \to Y$ of Equation (8), satisfying the inequality in (ii), also fulfills

$$\|h(x) - \alpha_0(x)\|_1 \le (\mu(z) + \mu(w))|1 - 2^{r-1}|^{-1}\|x\|^r, \qquad \forall x \in W_0.$$

Consequently, by Theorem 2 (with $\eta = \mu(z) + \mu(w)$), α_0 is unique. □

In the next theorem, concerning the stability of difference Equation (9), we only consider the case where $T = \mathbb{Z}$ or (12) holds. The remaining situation is more involved due to the lack of uniqueness of $(x_n)_{n\in T}$ in Theorem 3, and we omit it here to maintain the simplicity of the presentation of the method.

Theorem 7. *Let $\mu : Z \to \mathbb{R}$,*

$$C_0 := \prod_{i=1}^{p} \big|1 - |r_i|\big| \ne 0,$$

and $\|\cdot,\cdot\|$ be complete. Let $(b_n)_{n\in T}$, $(y_n)_{n\in T}$ be sequences in Y with

$$\|y_{n+p} + a_1 y_{n+p-1} + \ldots + a_p y_n + b_n, z\| \le \mu(z), \qquad \forall n \in T,\ z \in Z.$$

Assume that $T = \mathbb{Z}$ or (12) is fulfilled. Then, there exists a unique sequence $(x_n)_{n\in T}$ in Y such that (9) holds and

$$\|y_n - x_n, z\| \le C_0^{-1}\mu(z), \qquad \forall n \in T,\ z \in Z.$$

Proof. It is enough to argue analogously as in the proof of Theorem 6 (ii), with Theorem 2 replaced by Theorem 3. □

5. Stability of Differential and Integral Equations

In the case of Equations (13) and (14), the situation is more involved because the Bochner integral and the derivative depend on the norm. So, in this section, we are only confined to the case where Y is a real Hilbert space with a real inner product $\langle\cdot,\cdot\rangle$, and the 2-norm in Y is given by (15). We start with the stability of (14), where J, G, N, α, L and $\mathcal{C}(J, X)$ are as depicted before in Theorem 5.

Theorem 8. *Let $s \in \mathbb{R}$, $Z \subset Y \setminus \{0\}$, $\varepsilon \in \mathcal{C}(J, \mathbb{R}_+)$, and $\varphi \in \mathcal{C}(J, X)$. Let σ_0 and I_φ be as in Theorem 5. Assume that*

$$\|I_\varphi(x), z\| \le \|z\|^s \varepsilon(x), \qquad \forall x \in J,\ z \in Z.$$

Then, the following two statements are valid:

(A) *If $s \ne 1$ and $Z = Y$, then φ is a solution of Equation (14).*

(B) *If $S_Y := \{z \in Y : \|z\| = 1\} \subset Z$, then there is a solution $\psi \in \mathcal{C}(J, X)$ of (14) with*

$$\|\varphi(x) - \psi(x), z\| \le \sigma_0(x)\|z\|, \qquad \forall x \in Y,\ z \in Y.$$

Proof. Let $s \neq 1$ and $Z = Y$. Then,

$$\begin{aligned}\|I_\varphi(x), z\| &= \inf_{q \in \mathbb{Q}_+} q^{-1}\|I_\varphi(x), qz\| \\ &\le \inf_{q \in \mathbb{Q}_+} q^{-1}\varepsilon(x)\|qz\|^s \\ &= \varepsilon(x)\|z\|^s \inf_{q \in \mathbb{Q}_+} q^{s-1} = 0, \qquad \forall x \in J,\ z \in Y,\end{aligned}$$

whence φ fulfills (14) (see condition (a) of Definition 2).

If $S_Y \subset Z$, then for each $x \in J$ there is $z_x \in S_Y$ with $z_x \perp I_\varphi(x)$, whence (15) implies

$$\|I_\varphi(x)\| = \|I_\varphi(x), z_x\| \le \varepsilon(x).$$

So, by Theorem 5, there is a solution $\psi \in \mathcal{C}(J, X)$ of (14) with

$$\|\psi(x) - \varphi(x)\| \le \sigma_0(x), \qquad \forall x \in J.$$

Since, by (15), $\|y, z\| \le \|y\|\|z\|$ for $y, z \in Y$, this yields the inequality in (B). □

Theorem 9. *Let $S_Y \subset Z \subset Y$, $G : I \to X$ be continuous,*

$$D_0 := \prod_{i=1}^{p} |\Re(r_i)| \neq 0,$$

and $\chi : \mathbb{R}_+ \to \mathbb{R}_+$. If $r_1, \ldots, r_p \in \mathbb{K}$ (this simply means that in the case $\mathbb{K} = \mathbb{R}$, $r_1, \ldots, r_p$ are real numbers) and $h \in C^p(I, X)$ satisfies the inequality

$$\|h^{(p)}(t) + a_1 h^{p-1}(t) + \ldots + a_p h(t) + G(t), z\| \le \chi(\|z\|), \qquad \forall t \in I,\ z \in Z,$$

then there exists a solution $g \in C^p(I, X)$ of Equation (13) such that

$$\|h(t) - g(t), z\| \le D_0^{-1}\chi(1)\|z\|, \qquad \forall t \in I,\ z \in Y.$$

Proof. It is enough to argue analogously as in the proof of Theorem 8 (ii), with Theorem 5 replaced by Theorem 4 (with $\delta = \chi(1)$). □

6. Some Consequences

In this section, we show three simple consequences of Theorems 6 and 7. As before, Y is a linear space over $\mathbb{K}$, with dimension greater than 1, and $\|\cdot,\cdot\|$ is a 2-norm in Y. We begin with a corollary resulting from Theorem 6.

Corollary 1. *Let W, W_0, and d be as in Theorem 2, and $h : W \to Y$. Assume that there exist a real number $r \neq 1$ and two linearly independent vectors $z_1, z_2 \in Y$ such that*

$$\mu_i := \sup_{x, y \in W_0} \frac{\|h(x+y) - h(x) - h(y) - d(x, y), z_i\|}{\|x\|^r + \|y\|^r} < \infty, \qquad i = 1, 2.$$

Then, the following two statements are valid:

(i) *If $r < 0$, then h is a solution to Equation (8).*

(ii) *If $r \ge 0$ and $\|\cdot,\cdot\|$ is complete, then there is a unique solution $\alpha : W \to Y$ of (8) with*

$$\|h(x) - \alpha(x), z\| \le |1 - 2^{r-1}|^{-1}\|x\|^r \mu_i, \qquad \forall x \in W_0,\ i = 1, 2.$$

Proof. It is enough to use Theorem 6 with $Z = \{z_1, z_2\}$ and $\mu(z_i) = \mu_i$ for $i = 1, 2$. $\square$

Corollary 2. *Let $\mathbb{R}_0 := \mathbb{R} \setminus \{0\}$ and $h_1, h_2 : \mathbb{R} \to \mathbb{R}$. Assume that there exist $r \neq 1$ and $u_1, u_2 \in \mathbb{R}$ such that $u_1 \neq u_2$ and*

$$\mu_i := \sup_{x,y \in \mathbb{R}_0} \frac{\left|h_1(x+y) - h_1(x) - h_1(y) - (h_2(x+y) - h_2(x) - h_2(y))u_i\right|}{|x|^r + |y|^r} < \infty, \qquad i = 1, 2.$$

Then, the following two statements are valid.

(i) If $r < 0$, then h_1 and h_2 are additive functions.

(ii) If $r \geq 0$, then there are unique additive $\alpha_1, \alpha_2 : W \to Y$ with

$$\left|h_1(x) - \alpha_1(x) - \big(h_2(x) - \alpha_2(x)\big)u_i\right| \leq |1 - 2^{r-1}|^{-1}|x|^r \mu_i, \qquad \forall x \in \mathbb{R}_0,\ i = 1, 2. \tag{18}$$

Proof. It is enough to use Corollary 1 with $W = \mathbb{R}$, $Y = \mathbb{R}^2$ and the 2-norm in $\mathbb{R}^2$ defined by (16), $h(x) = (h_1(x), h_2(x))$ for $x \in \mathbb{R}$, $d(x, y) = 0$ for $x, y \in \mathbb{R}$, and $z_i = (u_i, 1)$ for $i = 1, 2$. $\square$

Remark 1. Note that from inequality (18), we also obtain

$$\begin{aligned}
&\left|(u_1 - u_2)(h_2(x) - \alpha_2(x))\right| \\
&\qquad \leq \left|h_1(x) - \alpha_1(x) - \big(h_2(x) - \alpha_2(x)\big)u_1\right| + \left|h_1(x) - \alpha_1(x) - \big(h_2(x) - \alpha_2(x)\big)u_2\right| \\
&\qquad \leq |1 - 2^{r-1}|^{-1}|x|^r(\mu_1 + \mu_2), \qquad \forall x \in \mathbb{R}_0,\ i = 1, 2,
\end{aligned}$$

whence

$$\left|h_2(x) - \alpha_2(x)\right| \leq \frac{|1 - 2^{r-1}|^{-1}|x|^r(\mu_1 + \mu_2)}{|u_1 - u_2|}, \qquad \forall x \in \mathbb{R}_0,\ i = 1, 2.$$

Next,

$$\begin{aligned}
\left|h_1(x) - \alpha_1(x)\right| &\leq \left|h_1(x) - \alpha_1(x) - \big(h_2(x) - \alpha_2(x)\big)u_i\right| + \left|u_i(h_2(x) - \alpha_2(x))\right| \\
&\leq |1 - 2^{r-1}|^{-1}|x|^r\mu_i + \frac{|u_i||1 - 2^{r-1}|^{-1}|x|^r(\mu_1 + \mu_2)}{|u_1 - u_2|}, \qquad \forall x \in \mathbb{R}_0,\ i = 1, 2.
\end{aligned}$$

If we assume additionally that h_1 is bounded on a real nontrivial interval, then the last inequality implies that so must be the additive function α_1 and, consequently, there exists a real constant c_1 such that $\alpha_1(x) = c_i x$ for $x \in \mathbb{R}$ (see refs. [67–69]; for some other examples of related regularity assumptions on h_1, we refer to [69,70]). Clearly, the same concerns h_2.

So, if h_1 and h_2 are bounded on some nontrivial intervals, then there exist real constants c_1, c_2 such that inequality (18) takes the form

$$\left|h_1(x) - c_1 x - \big(h_2(x) - c_2 x\big)u_i\right| \leq |1 - 2^{r-1}|^{-1}|x|^r\mu_i, \qquad \forall x \in \mathbb{R}_0,\ i = 1, 2.$$

For applications in the theory of information of functional equations considered above, we refer to [58]. For other possible applications of functional equations, see [67–69,71].

The next corollary is a consequence of Theorem 7.

Corollary 3. *Assume that*

$$C_0 := \prod_{i=1}^{p} \big|1 - |r_i|\big| \neq 0.$$

Let $u_1, u_2 \in \mathbb{R}$ be such that $u_1 \neq u_2$, and $(\phi_n)_{n\in T}$, $(\gamma_n)_{n\in T}$, $(\psi_n)_{n\in T}$, $(\rho_n)_{n\in T}$ be sequences in $\mathbb{R}$ with

$$\mu_i := \sup_{n\in T} \big|\gamma_{n+p} + a_1\gamma_{n+p-1} + \ldots + a_p\gamma_n + \phi_n - (\rho_{n+p} + a_1\rho_{n+p-1} + \ldots + a_p\rho_n + \psi_n)u_i\big| < \infty, \qquad i = 1,2. \tag{19}$$

Assume that $T = \mathbb{Z}$ or (12) holds. Then, there exist two unique sequences $(\eta_n)_{n\in T}$ and $(\xi_n)_{n\in T}$ in $\mathbb{R}$ such that

$$\eta_{n+p} + a_1\eta_{n+p-1} + \ldots + a_p\eta_n + \phi_n = 0, \qquad \forall n \in T,$$

$$\xi_{n+p} + a_1\xi_{n+p-1} + \ldots + a_p\xi_n + \psi_n = 0, \qquad \forall n \in T,$$

and

$$\big|\gamma_n - \eta_n - (\rho_n - \xi_n)u_i\big| \leq C_0^{-1}\mu_i, \qquad \forall n \in T,\ i = 1,2. \tag{20}$$

Proof. It is enough to argue analogously as in the proofs of Corollaries 1 and 2, but using Theorem 7 instead of Theorem 6. □

Remark 2. Arguing analogously as in Remark 1, we can easily derive from (20) estimations of the differences $|\gamma_n - \eta_n|$ and $|\rho_n - \xi_n|$.

7. Conclusions

Roughly speaking, an equation (e.g., difference, differential, functional, and integral) is Ulam stable if every mapping satisfying the equation approximately (in some sense), is close (in some way) to an accurate solution of the equation.

The notions of an approximate solution and the closeness of two mappings always depend on a situation that we consider and therefore may have various meanings. Therefore, it makes sense to consider such notions also with regard to the 2-norms.

In this paper, with four examples (with difference, differential, functional and integral equations), we showed that it is possible to easily derive various general Ulam stability outcomes, with respect to the 2-norms, from the already known results obtained for classical normed spaces.

Our considerations show that there is a significant symmetry between the Ulam stability results in classical normed spaces and in 2-normed spaces.

However, there are several issues that could be investigated further.

For instance, as we already mentioned in the introduction, it is shown in [10] (cf. [18]) that a result analogous to Theorem 2 is not possible for $r = 1$, which means that, then, non-stability (lack of stability) occurs. Moreover, in the case $r \geq 0$ and $r \neq 1$, the constant in the inequality of (ii) (in Theorem 2) is optimal in the general situation (see ref. [3]). We summarize this information and the statements of Theorem 2 in the Table 1.

Table 1. Stability of the Cauchy inhomogeneous Equation (8).

Value of the Power r	Type of Stability
$r < 0$	hyperstability
$r \geq 0$ and $r \neq 1$	stability with uniqueness and the best constant is known
$r = 1$	non-stability (the lack of stability)

With regard to our Theorem 6, it would be interesting to know if the situation is fully symmetric in 2-normed spaces (i.e., as in Table 1, there is lack of stability if $r = 1$ and the constant $|1 - 2^{r-1}|^{-1}$ in (17) is optimum when $r \geq 0$ and $r \neq 1$).

Next, in Theorems 3 and 7, the assumption that $T = \mathbb{Z}$ or (12) holds is important. If it is not fulfilled (i.e., $T = \mathbb{N}$ and (12) does not hold), then the situation in the normed spaces is quite different (see [12] [Theorems 3 and 4]). Namely, if $|r_i| < 1$ for some $i \in \{1, \ldots, n\}$ (and $|r_i| \neq 1$ for $i = 1, \ldots, n$), then there is no uniqueness of $(x_n)_{n\in T}$; actually, the set of all such sequences $(x_n)_{n\in T}$ satisfying (9) and (11) has the same cardinality as the space X, which means that this case can be regarded as somewhat chaotic. If $|r_i| = 1$ for some $i \in \{1, \ldots, n\}$, then there is lack of stability (non-stability occurs) in the following sense: there exist sequences $(y_n)_{n\in T}$ in X such that (10) holds and

$$\sup_{n\in T} \|y_n - x_n\| = \infty$$

for every sequence $(x_n)_{n\in T}$ in X, fulfilling (9). As before, we summarize this information in the next table (Table 2).

Table 2. Stability of difference Equation (9).

Possible Case	Type of Stability
$\lvert r_i\rvert > 1$ for $i = 1, \ldots, n$	stability with uniqueness
$\lvert r_i\rvert \neq 1$ for $i = 1, \ldots, n$ and $T = \mathbb{Z}$	stability with uniqueness
$\lvert r_i\rvert = 1$ for some $i \in \{1, \ldots, n\}$	non-stability
$\lvert r_i\rvert \neq 1$ for $i = 1, \ldots, n$, $\lvert r_j\rvert < 1$ for some $j \in \{1, \ldots, n\}$ and $T = \mathbb{N}$	stability without uniqueness

The cases in Table 2 are a bit complicated, so below, we also present the information contained there in the simple case $n = 2$ (with only two roots r_1 and r_2) in two tables (Tables 3 and 4) (for $T = \mathbb{N}$ and $T = \mathbb{Z}$) of somewhat modified forms.

Table 3. Stability of difference Equation (9) for $n = 2$ and $T = \mathbb{N}$.

	$\lvert r_1\rvert > 1$	$\lvert r_1\rvert = 1$	$\lvert r_1\rvert < 1$
$\lvert r_2\rvert > 1$	stability with uniqueness	non-stability	stability without uniqueness
$\lvert r_2\rvert = 1$	non-stability	non-stability	non-stability
$\lvert r_2\rvert < 1$	stability without uniqueness	non-stability	stability without uniqueness

Table 4. Stability of difference Equation (9) for $n = 2$ and $T = \mathbb{Z}$.

	$\lvert r_1\rvert \neq 1$	$\lvert r_1\rvert = 1$
$\lvert r_2\rvert \neq 1$	stability with uniqueness	non-stability
$\lvert r_2\rvert = 1$	non-stability	non-stability

It would be interesting to know if the situation is analogous (symmetric) as in Tables 3 and 4 also in the 2-normed spaces.

Moreover, it would be nice to obtain results similar to our theorems for other equations. In this regard, we would like to draw the attention of interested readers in particular to the outcomes in publications [6,11,22–30] that were obtained mainly for the normed spaces.

Finally, it would be very desirable to extend the methods presented in this paper to the case of n-norms and thus obtain analogous results, e.g., as in [72,73]. For the necessary definitions of n-normed spaces, stability issues considered in them and related information, we refer to [72–79].

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