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Examples of Expansions in Fractional Powers, and Applications

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Abstract: We approximate the solution of a generalized form of the Bagley–Torvik equation using Taylor's expansions in fractional powers. Then, we study the fractional Laguerre-type logistic equation by considering the fractional exponential function and its Laguerre-type form. To verify our findings, we conduct numerical tests using the computer algebra program Mathematica©.

Keywords: fractional differential equations; Bagley–Torvik equation; generalized exponential function; Mittag–Leffler function; fractional Laguerre-type logistic equation

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1. Introduction

In a recent article [1], we used expansions in fractional powers to solve, in an elementary way, several multi-term fractional differential equations, which appeared in the literature (see, e.g., [2–9]). The fractional derivative is a critical concept for innumerable applications in the most diverse fields of applied sciences. Several definitions are examined and compared in classic papers (see e.g., [10–12]), where fractional differential equations [13] are also studied.

Without going into this vast field of investigation, in the article above [1], we limited ourselves to considering the Euler's definition for the fractional derivative that falls within the one given by Caputo [14], and we only considered expansions in fractional power series. The powers considered in our expansions enjoy a symmetrical property, being integer multiples of a given number α, $(0 < \alpha < 1)$.

This method has been analyzed in the work of Groza–Jianu [15], where the main results valid for ordinary power series expansions were extended to the case of fractional power exponents.

In this article, in Section 2, we extend the results obtained in [1] by studying a generalization of the classical Bagley–Torvik equation [16] .

Moreover, in Section 3, we introduce the fractional version of the exponential function, which is related to the Mittag–Leffler function [17], frequently used in the framework of studies concerning fractional derivative theory and applications.

As is well known, the exponential function is the basic tool for constructing special functions and polynomials, often through suitable generating functions, which gave rise to symmetric or antisymmetric functions. Extending this function to the fractional case makes the generalization of many classical polynomial sets and functional operators possible. This is the aim of our investigation involving the study of fractional versions of many mathematical special functions, special polynomials and numbers.

Our goal is to show how these generalized entities, depending on a parameter α (with $0 \leq \alpha < 1$), approach their corresponding classical counterparts as α approaches 1. In this way, one can develop fractional versions of classical differential equations, including those related to population dynamics, and define fractional Laplace transforms, as well as fractional special numbers. Related articles are currently being published on these topics.

In Section 4, we recall the fractional-order logistic equation already examined in a preceding article, with the purpose of generalizing that to the Laguerre-type case.

The Laguerre-type exponentials and derivatives are recalled in Section 5 and are extended to the fractional case.

It is worth noting that the Laguerre derivative and the associated Laguerre-type special functions [18,19] determine a symmetry in the space of analytic functions. In fact, the operator $DxD = D + x\,D^2$ introduces a linear differential isomorphism, acting on the space of analytic functions of the x variable. By using this isomorphism, a parallel structure is created within this space, so that the differentiation properties can be immediately derived.

Furthermore, iterations of the Laguerre derivative can be defined, and this parallelism can be iterated too, in an endless way. Therefore, a cyclic construction is created within the space that repeats the same structure at a higher level of differentiation order. It is one of the great cycles that sometimes occur within mathematical theories.

By using the Laguerre-type derivative, we study the Laguerre-type fractional-order logistic equation, while extending the results in [20] to the Laguerre-type case. The traditional fractional logistic equation has been examined in [21,22].

2. The Bagley–Torvik Equation

In 1984, Torvik and Bagley [16] first proposed a fractional order differential equation to model the viscoelastic behavior of geological strata, as well as metals, and glasses. They showed the effectiveness of their approach in describing structures containing elastic and viscoelastic components. The so called Bagley–Torvik equation became a model to test the solution of fractional differential equations, with suitable initial conditions.

We consider the following inhomogeneous Bagley–Torvik-type fractional differential equation (see [15]), with special initial conditions

$$\begin{aligned} &D_t^2 y(t) + A\,D_t^{3/2} y(t) + B\,y(t) = \\ &= c_0 + \frac{c_1}{\Gamma(3/2)} t^{1/2} + \frac{c_2}{\Gamma(2)} t + \frac{c_3}{\Gamma(5/2)} t^{3/2} + \cdots + \frac{c_n}{\Gamma(n/2+1)} t^{n/2} + \ldots, \end{aligned} \tag{1}$$

$$y(0) = 1,\ \ y^{(1/2)}(0) = 0,\ \ y'(0) = 0,\ \ y^{(3/2)}(0) = 0\,. \tag{2}$$

Put

$$y(t) = \sum_{n=0}^{\infty} a_n t^{n/2}\,. \tag{3}$$

Since

$$D^{3/2} t^{n/2} = \frac{\Gamma(n/2+1)}{\Gamma(n/2-1/2)}\, t^{(n-3)/2}, \quad (n \geq 3)\,, \tag{4}$$

and in Equation (1) the derivatives of order 3/2 and 2 appear, we put $a_1 = a_2 = a_3 = 0$ in the expansion (3). As $a_0 = y(0) = 1$, we have

$$y(t) = 1 + a_4\,t^2 + a_5\,t^{5/2} + a_6\,t^3 + a_7\,t^{7/2} + a_8\,t^4 + \ldots, \tag{5}$$

and

$$y'(t) = 2\,a_4\,t + \frac{5}{2}\,a_5\,t^{3/2} + 3\,a_6\,t^2 + \frac{7}{2}\,a_7\,t^{5/2} + 4a_8\,t^3 + \dots .$$

$$D_t^{3/2}y(t) = \frac{\Gamma(3)}{\Gamma(3/2)}\,a_4\,t^{1/2} + \frac{\Gamma(7/2)}{\Gamma(2)}\,a_5\,t + \frac{\Gamma(4)}{\Gamma(5/2)}\,a_6\,t^{3/2} + \frac{\Gamma(9/2)}{\Gamma(3)}\,a_7\,t^2 + \dots ,$$

$$y''(t) = 2\,a_4 + \frac{15}{4}\,a_5\,t^{1/2} + 6\,a_6\,t + \frac{35}{4}\,a_7\,t^{3/2} + 12\,a_8\,t^2 + \dots .$$

Substituting into Equation (1), we find

$$\begin{aligned} &2\,a_4 + \frac{15}{4}\,a_5\,t^{1/2} + 6\,a_6\,t + \frac{35}{4}\,a_7\,t^{3/2} + 12\,a_8\,t^2 + \dots + \\ &+ A\left(\frac{\Gamma(3)}{\Gamma(3/2)}\,a_4\,t^{1/2} + \frac{\Gamma(7/2)}{\Gamma(2)}\,a_5\,t + \frac{\Gamma(4)}{\Gamma(5/2)}\,a_6\,t^{3/2} + \frac{\Gamma(9/2)}{\Gamma(3)}\,a_7\,t^2 + \dots\right) + \\ &+ B\left(1 + a_4\,t^2 + a_5\,t^{5/2} + a_6\,t^3 + a_7\,t^{7/2} + a_8\,t^4 + \dots\right) = \\ &= c_0 + \frac{c_1}{\Gamma(3/2)}t^{1/2} + \frac{c_2}{\Gamma(2)}t + \frac{c_3}{\Gamma(5/2)}t^{3/2} + \dots + \frac{c_n}{\Gamma(n/2+1)}t^{n/2} + \dots . \end{aligned}$$

Equating the coefficients of equal t-powers, we find a triangular system, which recursively gives the a_n coefficients $n \geq 4$ of the solution (3).

$$\begin{aligned} &2a_4 + B = c_0 \\ &A\,\frac{\Gamma(3)}{\Gamma(3/2)}\,a_4 + \frac{15}{4}\,a_5 = \frac{c_1}{\Gamma(3/2)} \\ &A\,\frac{\Gamma(7/2)}{\Gamma(2)}\,a_5 + 6\,a_6 = \frac{c_2}{\Gamma(2)} \\ &A\,\frac{\Gamma(4)}{\Gamma(5/2)}\,a_6 + \frac{35}{4}\,a_7 = \frac{c_3}{\Gamma(5/2)} \\ &A\,\frac{\Gamma(9/2)}{\Gamma(3)}\,a_7 + B\,a_4 + 12a_8 = \frac{c_4}{\Gamma(3)} \\ &A\,\frac{\Gamma(5)}{\Gamma(7/2)}\,a_8 + B\,a_5 + \frac{63}{4}\,a_9 = \frac{c_5}{\Gamma(7/2)} \\ &\vdots \\ &A\,\frac{\Gamma(\frac{n+5}{2})}{\Gamma(\frac{n+2}{2})}\,a_{n+3} + B\,a_n + \frac{(n+2)(n+4)}{4}\,a_{n+4} = \frac{c_n}{\Gamma(\frac{n+2}{2})} . \end{aligned}$$

2.1. Convergence Results

Let A and B be positive numbers, and suppose the sequence $\{c_n\}$ is bounded, i.e., $|c_n| \leq C, \forall n$.

We have put, for example, $A = 2$, $B = 1$, and we have found that the coefficients a_n alternate in sign and tend to zero as $n \to +\infty$.

We first prove that the coefficients of the series (3) are in the order of the reciprocal of the Gamma function of $(n-1)/2$, so that the series is absolutely convergent in the whole complex plane. Then, we show that, on the real axis, the Nth remainder term tends to zero when $N \to +\infty$.

Consequently,

$$\left| A\,\Gamma\left(\frac{n+5}{2}\right) a_{n+3} + B\,\Gamma\left(\frac{n+2}{2}\right) a_n + \Gamma\left(\frac{n+2}{2}\right) \frac{(n+2)(n+4)}{4}\, a_{n+4} \right| \leq C\,,$$

assuming $a_{n+3} < 0$, and consequently, $a_n > 0,\ a_{n+4} > 0$, with $a_{n+4} < |a_{n+3}| < a_n$, we have

$$\begin{aligned} &\left| A\,\Gamma\left(\tfrac{n+5}{2}\right) a_{n+3} + B\,\Gamma\left(\tfrac{n+2}{2}\right) a_n + \Gamma\left(\tfrac{n+2}{2}\right) \frac{(n+2)(n+4)}{4}\, a_{n+4} \right| < \\ &\quad < \left| -A\,\Gamma\left(\tfrac{n+5}{2}\right) |a_{n+3}| + B\,\Gamma\left(\tfrac{n+2}{2}\right) a_n \right| \leq C\,, \end{aligned}$$

so that

$$\left| -A\,\Gamma\left(\frac{n+2}{2}\right) |a_{n+3}| + B\,\Gamma\left(\frac{n+2}{2}\right) |a_{n+3}| \right| \leq C\,,$$

$$|a_{n+3}| \leq \frac{C}{|B-A|\,\Gamma\left(\frac{n+2}{2}\right)}\,,$$

and

$$a_n = \mathcal{O}\left(\frac{1}{\Gamma\left(\frac{n-1}{2}\right)} \right).$$

Then, we find

$$\sum_{n=0}^{\infty} |a_n| t^{n/2} \leq \frac{C}{|B-A|} \sum_{n=0}^{\infty} \frac{t^{n/2}}{\Gamma\left(\frac{n-1}{2}\right)}\,.$$

so that the series $\sum_{n=0}^{\infty} a_n t^{n/2}$ is absolutely convergent in the whole complex plane, as its convergence radius is $+\infty$.

Furthermore, according to the Leibniz theorem, $\forall t > 0$ on the positive real axis, the alternating series $\sum_{n=0}^{\infty} a_n\, t^{n/2}$ is convergent and the remainder term $R_N(t)$ resulting from the truncation of the series at the index N is bounded by the first neglected term, that is

$$|R_N(t)| = \Big| \sum_{n=N+1}^{\infty} a_n\, t^{n/2} \Big| \leq |a_{N+1}|\, t^{(N+1)/2}\,.$$

2.2. Numerical Results

Assuming $c_n = 1, \forall n \geq 0$, $A = 2$, $B = 1$ and using the above recursion, we find the following the Table of the a_n coefficients $0 \leq n \leq 40$, reported in Figure 1.

$a_0 = 1$

$a_1 = 0$

$a_2 = 0$

$a_3 = 0$

$a_4 = 0$

$a_5 = 0.300901$

$a_6 = -0.166667$

$a_7 = 0.257915$

$a_8 = -0.208333$

$a_9 = 0.191048$

$a_{10} = -0.15$

$a_{11} = 0.118103$

$a_{12} = -0.0861111$

$a_{13} = 0.0614561$

$a_{14} = -0.0418651$

$a_{15} = 0.0277176$

$a_{16} = -0.0177331$

$a_{17} = 0.0110317$

$a_{18} = -0.00666887$

$a_{19} = 0.00392843$

$a_{20} = -0.00225639$

$a_{21} = 0.00126569$

$a_{22} = -0.000693968$

$a_{23} = 0.000372331$

$a_{24} = -0.000195642$

$a_{25} = 0.000100767$

$a_{26} = -0.000050912$

$a_{27} = 0.0000252513$

$a_{28} = -0.0000123024$

$a_{29} = 5.89134 \times 10^{-6}$

$a_{30} = -2.77459 \times 10^{-6}$

$a_{31} = 1.28582 \times 10^{-6}$

$a_{32} = -5.86648 \times 10^{-7}$

$a_{33} = 2.6363 \times 10^{-7}$

$a_{34} = -1.1674 \times 10^{-7}$

$a_{35} = 5.09595 \times 10^{-8}$

$a_{36} = -2.19334 \times 10^{-8}$

$a_{37} = 9.30383 \times 10^{-9}$

$a_{38} = -3.87438 \times 10^{-9}$

$a_{39} = 1.55206 \times 10^{-9}$

$a_{40} = -5.30254 \times 10^{-10}$

Figure 1. The a_n coefficients for $0 \leq n \leq 40$.

The graph of the approximate solution is depicted in Figure 2.

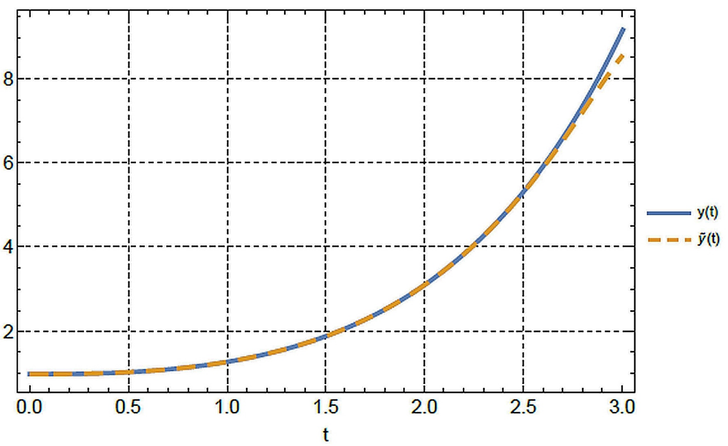

Figure 2. Graph of the solution $\tilde{y}(t)$ using the a_n coefficients vs $y(t)$, obtained using the predictor–corrector method.

3. The Fractional Exponentials

Note that in the particular case $c = \{1, 1, \ldots, 1, \ldots\}$, the second member of the classical Bagley–Torvik equation is an extension of the exponential function:

$$\mathrm{Exp}_{1/2}(t) = 1 + \frac{1}{\Gamma(3/2)} t^{1/2} + \frac{1}{\Gamma(2)} t + \cdots + \frac{1}{\Gamma(n/2+1)} t^{n/2} + \ldots . \tag{6}$$

It is convergent in the whole complex plane, as the same holds for the classical exponential $\exp(t)$.

Furthermore, according to the fractional differentiation rule of powers, the following results

$$D^{1/2}\mathrm{Exp}_{1/2}(t) = \mathrm{Exp}_{1/2}(t) . \tag{7}$$

In general, putting

$$\mathrm{Exp}_{\alpha}(t) = 1 + \frac{1}{\Gamma(\alpha+1)} t^{\alpha} + \frac{1}{\Gamma(2\alpha+1)} t^{2\alpha} + \cdots + \frac{1}{\Gamma(n\alpha+1)} t^{n\alpha} + \ldots .$$

we find

$$D^{\alpha}\mathrm{Exp}_{\alpha}(t) = \mathrm{Exp}_{\alpha}(t) .$$

Remark 1. *Recalling the Mittag–Leffler function [17]*

$$E_{\alpha,\beta}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\alpha n + \beta)} , \quad \forall\, x \in \mathbf{C}, \forall\, \alpha, \beta \in \mathbf{R}^+ ,$$

assuming $\beta = 1$, and substituting x with x^{α} results in

$$E_{\alpha,1}(x^{\alpha}) = \mathrm{Exp}_{\alpha}(x) , \quad \text{and} \quad D_x^{\alpha} E_{\alpha,1}(x^{\alpha}) = E_{\alpha,1}(x^{\alpha}) ,$$

so that the fractional exponentials can be reduced to the Mittag–Leffler function.
In particular, we have:

$$E_{\frac{1}{2},1}(x^{1/2}) = \sum_{n=0}^{\infty} \frac{x^{n/2}}{\Gamma(n/2+1)} = \mathrm{Exp}_{1/2}(x) .$$

4. The Fractional-Order Logistic Equation

We consider the fractional-order logistic initial value problem [21]

$$\begin{cases} D_t^{\alpha} P(t) = r\, P(t) \left[1 - \frac{1}{K}\, P(t)\right] , \quad (0 < \alpha < 1) , \\ P(0) = p_0 . \end{cases} \tag{8}$$

In a recent paper [20], we proved the result

Theorem 1. *Setting*

$$P(t) = \sum_{n=0}^{\infty} a_n \frac{t^{\alpha n}}{\Gamma(\alpha\, n + 1)} , \tag{9}$$

the solution of the fractional-order logistic initial value problem in Equation (8) is obtained by computing the a_n coefficients through the following recursion

$$\begin{cases} a_0 = p_0 \\ a_{n+1} = r\left[a_n - \dfrac{1}{K}\displaystyle\sum_{k=0}^{n} \dfrac{a_k\, a_{n-k}\,\Gamma(n\,\alpha+1)}{\Gamma(\alpha(n-k)+1)\,\Gamma(\alpha\, k+1)}\right]. \end{cases} \tag{10}$$

Example

Assuming $\alpha = 1/2, r = 1, K = 1.5$, and putting $y_0 = 0.5$, we find the following Table for the a_n coefficients, $(0 \leq n \leq 36)$, reported in Figure 3.

5. The Laguerre-Type Exponentials

In preceding articles [18,19], the Laguerre-type exponential

$$e_L(x) := \sum_{k=0^\infty} \frac{x^k}{(k!)^2}. \tag{11}$$

has been introduced in connection with the Laguerre-type derivative

$$D_L := DxD = D + x\,D^2. \tag{12}$$

Namely, it is an eigenfunction of this operator, since $\forall a$ (complex constant), resulting in the following equation:

$$D_L e_L(a\,x) = a\,e_L(a\,x).$$

In general, for any integer $n \geq 1$, the higher-order Laguerre-type exponentials

$$e_{nL}(x) := \sum_{k=0^\infty} \frac{x^k}{(k!)^{n+1}}, \tag{13}$$

satisfy the eigenvalue property concerning the nth order Laguerre derivative

$$\begin{aligned} D_{nL} := Dx\cdots DxDxD = S(n+1,1)\,D + S(n+1,2)\,xD^2 + \cdots + \\ S(n+1,n+1)\,x^n D^{n+1} \end{aligned} \tag{14}$$

where $S(n+1,1)$, $S(n+1,2)$, $\ldots$, $S(n+1,n+1)$ denote the Stirling numbers of the second kind, since

$$D_{nL}e_{nL}(a\,x) = a\,e_{nL}(a\,x).$$

The Laguerre-type special functions have been considered in preceding papers, and the relevant properties have been examined. It turned out that the properties of Laguerre-type special functions exhibit symmetric properties with respect to those of the corresponding ordinary ones. This is a consequence of a differential isomorphism in the space of analytic functions that connects ordinary and Laguerre-type special functions. Such isomorphism is described in [19].

5.1. The Fractional Laguerre-Exponentials

Introducing the fractional Laguerre-type exponential of order $\alpha = 1/2$,

$$\begin{aligned} {}_{L_1}\mathrm{Exp}_{1/2}(t) = 1 + \frac{1}{[\Gamma(3/2)]^2}t^{1/2} + \frac{1}{[\Gamma(2)]^2}t + \\ \frac{1}{[\Gamma(5/2)]^2}t^{3/2} + \cdots + \frac{1}{[\Gamma(n/2+1)]^2}t^{n/2} + \ldots, \end{aligned}$$

we found that

$$D^{1/2}\,x^{1/2}\,D^{1/2}\,{}_{L_1}\mathrm{Exp}_{1/2}(t) = {}_{L_1}\mathrm{Exp}_{1/2}(t)\,.$$

More generally, putting

$$\begin{aligned} {}_{L_n}\mathrm{Exp}_{1/2}(t) = 1 + \frac{1}{[\Gamma(3/2)]^{n+1}}\,t^{1/2} + \frac{1}{[\Gamma(2)]^{n+1}}\,t + \\ \frac{1}{[\Gamma(5/2)]^{n+1}}\,t^{3/2} + \cdots + \frac{1}{[\Gamma(n/2+1)]^{n+1}}\,t^{n/2} + \ldots\,, \end{aligned}$$

and considering the iterated Laguerre-type operator, which embeds $n+1$ fractional derivatives, results in the following equation:

$$D^{1/2}\,x^{1/2}\,D^{1/2}\,x^{1/2}\,D^{1/2}\,\cdots\,x^{1/2}\,D^{1/2}\,{}_{L_n}\mathrm{Exp}_{1/2}(t) = {}_{L_n}\mathrm{Exp}_{1/2}(t)\,.$$

Of course, the results of this section, and the relevant application, could be generalized to any value of α, $(0 < \alpha \leq 1)$; but, for the sake of conciseness, we limit ourselves to the particular case $\alpha = 1/2$ because the technique used is always the same.

5.2. The Laguerre-Type Fractional-Order Logistic Equation

We consider the Laguerre-type fractional-order logistic initial value problem

$$\begin{cases} D_t^{1/2} t^{1/2} D_t^{1/2} P(t) = r\,P(t)\left[1 - \frac{1}{K}\,P(t)\right], \\ \\ P(0) = p_0\,. \end{cases} \tag{15}$$

We prove the following result:

Theorem 2. *Setting*

$$P(t) = \sum_{n=0}^{\infty} a_n\,\frac{t^{n/2}}{\Gamma(n/2+1)}\,, \tag{16}$$

the solution of the considered Laguerre-type fractional-order logistic initial value problem is obtained computing the a_n coefficients using the recursion

$$\begin{cases} a_0 = p_0 \\ a_{n+1} = r\left[a_n\,\dfrac{\Gamma(\frac{n+2}{2})}{\Gamma(\frac{n+3}{2})} - \dfrac{1}{K}\displaystyle\sum_{k=0}^{n} \dfrac{a_k\,a_{n-k}\,[\Gamma(\frac{n+2}{2})]^2}{\Gamma(\frac{n+3}{2})\,\Gamma(\frac{n-k}{2}+1)\,\Gamma(\frac{k}{2}+1)}\right]. \end{cases} \tag{17}$$

Proof. Using the fractional differentiation, we find

$$D_t^{1/2} t^{1/2} D_t^{1/2} P(t) = \sum_{n=0}^{\infty} a_{n+1}\,\frac{\Gamma(\frac{n+3}{2})\,t^{n/2}}{[\Gamma(\frac{n+2}{2})]^2}\,,$$

$$P(t)\cdot\frac{1}{K}\,P(t) = \frac{1}{K}\sum_{n=0}^{\infty}\sum_{k=0}^{n} a_k\,a_{n-k}\,\frac{t^{n/2}}{\Gamma(\frac{n-k}{2}+1)\,\Gamma(\frac{k}{2}+1)}\,.$$

Substituting into the equation, we find

$$\sum_{n=0}^{\infty} a_{n+1} \frac{\Gamma(\frac{n+3}{2})\, t^{n/2}}{[\Gamma(\frac{n+2}{2})]^2} =$$

$$= r \left[\sum_{n=0}^{\infty} a_n \frac{t^{n/2}}{\Gamma(\frac{n+2}{2})} - \frac{1}{K} \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_k\, a_{n-k} \frac{t^{n/2}}{\Gamma(\frac{n-k}{2}+1)\, \Gamma(\frac{k}{2}+1)} \right] =$$

$$= r \sum_{n=0}^{\infty} \left[\frac{a_n}{\Gamma(\frac{n+2}{2})} - \frac{1}{K} \sum_{k=0}^{n} \frac{a_k\, a_{n-k}}{\Gamma(\frac{n-k}{2}+1)\, \Gamma(\frac{k}{2}+1)} \right] t^{n/2},$$

so that the recursion for the a_n coefficients follows. □

5.3. Numerical Results

Assuming $r = 1$, $K = 2$, $y_0 = 1$, and using the above recursion, we find the following Table of the a_n coefficients $0 \le n \le 36$, reported in Figure 4.

Remark 2. *Note that in the Table contained in Figure 3, as well as in that in Figure 4, the values of the coefficients a_n, with even index greater than 0, that is for $n = 2m$, con, m a strictly positive integer, vanish or are so small that they cannot have any influence on the solution. The graph of the solution of problem 15, with a = 1/2, is shown in Figure 5.*

$a_0 = 1$	$a_{19} = -6.43722$
$a_1 = 0.5$	$a_{20} = 8.88178 \times 10^{-17}$
$a_2 = 0$	$a_{21} = 37.9541$
$a_3 = -0.0833333$	$a_{22} = -3.22974 \times 10^{-16}$
$a_4 = 0$	$a_{23} = -267.289$
$a_5 = 0.0333333$	$a_{24} = 2.36848 \times 10^{-15}$
$a_6 = 0$	$a_{25} = 2216.02$
$a_7 = -0.0242063$	$a_{26} = -3.49806 \times 10^{-14}$
$a_8 = 0$	$a_{27} = -21\,364.6$
$a_9 = 0.0280423$	$a_{28} = 3.89783 \times 10^{-13}$
$a_{10} = -3.46945 \times 10^{-19}$	$a_{29} = 236\,999$
$a_{11} = -0.0474796$	$a_{30} = -3.88051 \times 10^{-12}$
$a_{12} = 5.78241 \times 10^{-19}$	$a_{31} = -2.99736 \times 10^{6}$
$a_{13} = 0.110618$	$a_{32} = 7.27596 \times 10^{-11}$
$a_{14} = 9.91271 \times 10^{-19}$	$a_{33} = 4.28714 \times 10^{7}$
$a_{15} = -0.339426$	$a_{34} = -1.09567 \times 10^{-9}$
$a_{16} = 6.93889 \times 10^{-18}$	$a_{35} = -6.88556 \times 10^{8}$
$a_{17} = 1.32683$	$a_{36} = 1.98682 \times 10^{-8}$
$a_{18} = 0$	

Figure 3. The a_n coefficients of the solution (9), for the considered parameters, and $0 \le n \le 36$.

$a_0=1$	$a_{19}=-1.22686$
$a_1=0.56419$	$a_{20}=-4.71545\times 10^{-15}$
$a_2=9.8391\times 10^{-17}$	$a_{21}=3.04327$
$a_3=-0.152438$	$a_{22}=1.37681\times 10^{-14}$
$a_4=-7.37932\times 10^{-17}$	$a_{23}=-8.26784$
$a_5=0.0878662$	$a_{24}=-4.26281\times 10^{-14}$
$a_6=6.91811\times 10^{-17}$	$a_{25}=24.4149$
$a_7=-0.0724426$	$a_{26}=1.41508\times 10^{-13}$
$a_8=-7.39854\times 10^{-17}$	$a_{27}=-77.8646$
$a_9=0.0769943$	$a_{28}=-5.00646\times 10^{-13}$
$a_{10}=1.0896\times 10^{-16}$	$a_{29}=266.722$
$a_{11}=-0.100047$	$a_{30}=1.87761\times 10^{-12}$
$a_{12}=-1.94212\times 10^{-16}$	$a_{31}=-976.654$
$a_{13}=0.153637$	$a_{32}=-7.50135\times 10^{-12}$
$a_{14}=3.70984\times 10^{-16}$	$a_{33}=3806.92$
$a_{15}=-0.272226$	$a_{34}=3.22979\times 10^{-11}$
$a_{16}=-7.72883\times 10^{-16}$	$a_{35}=-15\,738.4$
$a_{17}=0.546654$	$a_{36}=-1.45614\times 10^{-10}$
$a_{18}=1.82486\times 10^{-15}$	

Figure 4. The a_n coefficients for the Laguerre-type fractional logistic equation, with $0 \leq n \leq 36$.

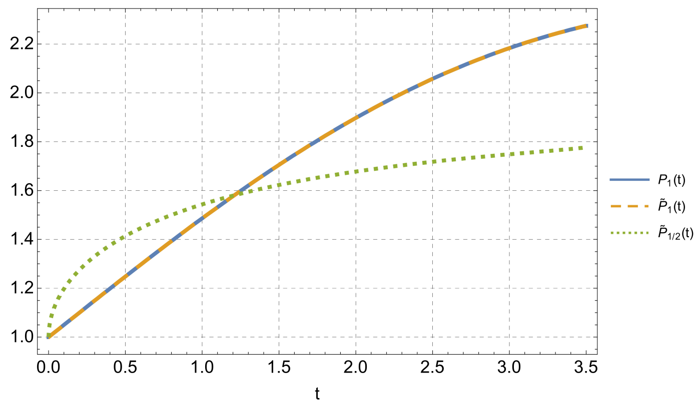

Figure 5. Graph of the solution of the problem (15) ($\alpha = 1/2$), using the a_n coefficients of the Table in Figure 4 (dotted line), compared with the solutions of the Laguerre-type logistic equation ($\alpha = 1$) using a predictor-corrector method (blue line) and a recursion method for approximating the coefficients (orange dashed line).

6. Conclusions

We have presented various findings within the context of fractional derivatives. Additionally, we have introduced a fractional version of the exponential function, which is connected to the Mittag–Leffler function that is commonly found in papers on fractional derivatives. In terms of the fractional derivative, this function shares the same eigenvalue characteristic as the traditional exponential has with respect to the ordinary derivative. As a result, many of the properties associated with analytic functions involving the exponential can be extended to fractional power series.

We are currently working on further articles that will delve deeper into this subject.

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