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Fractional Integrals Associated with the One-Dimensional Dunkl Operator in Generalized Lizorkin Space

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Abstract: This paper explores the realm of fractional integral calculus in connection with the one-dimensional Dunkl operator on the space of tempered functions and Lizorkin type space. The primary objective is to construct fractional integral operators within this framework. By establishing the analogous counterparts of well-known operators, including the Riesz fractional integral, Feller fractional integral, and Riemann–Liouville fractional integral operators, we demonstrate their applicability in this setting. Moreover, we show that familiar properties of fractional integrals can be derived from the obtained results, further reinforcing their significance. This investigation sheds light on the utilization of Dunkl operators in fractional calculus and provides valuable insights into the connections between different types of fractional integrals. The findings presented in this paper contribute to the broader field of fractional calculus and advance our understanding of the study of Dunkl operators in this context.

Keywords: Dunkl theory; fractional Integral; Bessel functions

MSC: 42B30; 33C52; 33C67; 33D67; 33D80; 35K08; 42B25; 42C05

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1. Introduction

On the real line, for a positive real number κ, the Dunkl operator $\mathscr{D}_\kappa$ provides a one-parameter deformation of the ordinary derivative $\frac{d}{dx}$. It is defined as:

$$\mathscr{D}_\kappa := \frac{d}{dx} + \frac{\kappa}{x}(1-s), \tag{1}$$

where s is the reflection operator acting on a function $f(x)$ of a real variable x as $sf(x) := f(-x)$. The Dunkl operator incorporates the additional term $\frac{\kappa}{x}(1-s)$, which accounts for reflection symmetry and introduces a dependence on the parameter κ. This operator plays a fundamental role in generalizing various classical results in harmonic analysis and approximation theory, as explored in the works of Dunkl [1,2] Trimeche [3], de Jeu [4], Rosler [5–7], and others.

Fractional calculus [8–15] has gained significant importance in recent decades as a powerful tool for developing advanced mathematical models involving fractional differential and integral operators. When applied to the Dunkl operator, fractional calculus offers a fresh perspective by incorporating the effects of reflection and asymmetry within the underlying space.

A notable feature of the Dunkl setting is the existence of a natural Riesz transform, which shares similarities with classical singular integrals. In the multidimensional case, S. Thangavelu and Y. Xu [16,17] established the L^p-boundedness of the associated Riesz transform. This study was further extended by Amri and Sifi [18], who considered the general case for $1 < p < \infty$. Additionally, investigations into singular integrals and multipliers were carried out in [18–22]. These contributions have significantly enriched our understanding of the Dunkl operator and its associated Riesz transform.

In this study, our main focus is on the comprehensive exploration of the one-dimensional fractional Dunkl integral within Lizorkin type spaces [10–12], with a specific emphasis on analytic continuation techniques. The obtained operators go beyond the conventional Riesz fractional integral [9] and Feller fractional integral [8,11], as they are specifically tailored to operate within the Dunkl setting. By extending the applicability of these operators to the Dunkl context, we aim to unlock new possibilities and gain deeper insights into the realm of fractional calculus.

To address the challenges posed by the divergence of fractional Dunkl operators, we adopt a unique approach that incorporates the regularization technique for divergent integrals, inspired by the work described in the book by Samko [11,12]. Our methodology involves utilizing specific segments of the Taylor formula associated with the Dunkl operator, as originally formulated by Mourou [23]. This regularization technique plays a pivotal role in extending the fractional integral operators to the domain of $\Re(\alpha) > 0$. As a result, we introduce an alternative normalization scheme for tempered power functions, offering a fresh and insightful perspective on fractional calculus within the Dunkl setting. It is important to note that while Soltani [24] relies on the conventional Taylor series, our approach, based on the Taylor formula of Mourou [23], better suits the specific requirements of the Dunkl operator.

Our paper is organized as follows: In Section 2, we begin by collecting some essential facts about the Dunkl operator and the Lizorkin space. Section 3 focuses on studying the generalized power function and its analytic continuation. Moving on to Section 4, we dedicate that section to the study of extensions of well-known fractional integrals such as the Riesz fractional integral, the Feller fractional integral, and the Weyl fractional integral.

2. Preliminaries

In this section, we introduce some notations and gather some facts about the one-dimensional Dunkl operator.

2.1. The One-Dimensional Dunkl Operator

Let $\kappa \geq 0$, and f be a differentiable function on $\mathbb{R}$. The Dunkl derivative $\mathscr{D}_\kappa f(x)$ is defined by

$$\mathscr{D}_\kappa f(x) = \begin{cases} f'(x) + \kappa \frac{f(x)-f(-x)}{x}, & \text{if} \quad x \neq 0, \\ (2\kappa+1) f'(0), & \text{if} \quad x = 0. \end{cases} \tag{2}$$

We denote by $L^p_\kappa(\mathbb{R})$ $(1 \leq p)$, the Lebesgue space associated with the measure

$$\sigma_\kappa(dx) = \frac{|x|^{2\kappa}}{2^{\kappa+1/2}\Gamma(\kappa+1/2)}\, dx \tag{3}$$

and by $\|f\|_{\kappa,p}$ the usual norm given by

$$\|f\|_{\kappa,p} = \Big(\int_{\mathbb{R}} |f(x)|^p \, \sigma_\kappa(dx) \Big)^{1/p}. \tag{4}$$

Now, consider the so-called *nonsymmetric Bessel function*, also called *Dunkl type Bessel function*, in the rank one case (see [25]) [§10.22(v)]:

$$\mathscr{E}_\kappa(x) := \mathscr{J}_{\kappa-1/2}(ix) + \frac{x}{2\kappa+1}\, \mathscr{J}_{\kappa+1/2}(ix). \tag{5}$$

where the normalized Bessel functions is defined by

$$\begin{aligned} \mathscr{J}_\kappa(x) &:= \Gamma(\kappa+1)\,(2/x)^\kappa \, J_\kappa(x) \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(\kappa+n+1)} \big(\frac{x}{2}\big)^{2n+\kappa}, \quad x > 0. \end{aligned}$$

It is evident to the reader that the Dunkl kernel $\mathcal{E}_\kappa(i\lambda x)$ coincides with the exponential function when the parameter κ is equal to zero, i.e., $\mathcal{E}_0(i\lambda x) = e^{i\lambda x}$. This function also has a close connection with the Wright function.

$$\mathcal{E}_\kappa(x) = \Gamma(\kappa+1/2)\Big[W_{1,\kappa+1/2}(\frac{x^2}{4}) + \frac{x}{2}\,W_{1,\kappa+3/2}(\frac{x^2}{4})\Big], \tag{6}$$

where the Wright function is defined by the series representation, valid in the whole complex plane [26]

$$W_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!\Gamma(\alpha n+\beta)}, \quad \alpha > -1, \quad \beta \in \mathbb{C}. \tag{7}$$

The Wright function provides a powerful tool for dealing with fractional calculus problems, as it allows for the analysis of fractional differential and integral equations in a unified framework, see [26,27].

The function $\mathscr{E}_k(i\xi x)$ satisfies the following eigenvalue problem

$$\mathscr{D}_\kappa(\mathscr{E}_k(i\xi x)) = i\,\xi\,\mathscr{E}_\kappa(i\xi x), \quad \mathcal{E}_k(0)) = 1 \tag{8}$$

and has the *Laplace* representation

$$\mathscr{E}_\kappa(ix) = \frac{\Gamma(\kappa+1/2)}{\Gamma(1/2)\Gamma(\kappa)} \int_{-1}^{1} e^{tx}(1-t)^{\kappa-1}(1+t)^{\kappa}\,dt. \tag{9}$$

The Dunkl transform is defined by [1,3,4]

$$(\mathscr{F}_\kappa f)(\lambda) := \int_{-\infty}^{\infty} f(x)\,\mathcal{E}_\kappa(-i\lambda x)\,\sigma_\kappa(dx). \tag{10}$$

The *Dunkl* transform can be extended to an isometry of $L^2_\kappa(\mathbb{R})$, that is,

$$\int_{\mathbb{R}} |f(x)|^2\,\sigma_\kappa(dx) = \int_{\mathbb{R}} |\widehat{f}_\kappa(\lambda)|^2\,\sigma_\kappa(d\lambda). \tag{11}$$

For any $f \in L^1_\kappa(\mathbb{R}) \cap L^2_\kappa(\mathbb{R})$, the inverse is given by

$$f(x) = \int_{\mathbb{R}} \widehat{f}_\kappa(\lambda)\,\mathcal{E}_\kappa(i\lambda x)\,\sigma_\kappa(d\lambda). \tag{12}$$

As in the classical case, a generalized translation operator was defined in the Dunkl setting side on $L^2_\kappa(\mathbb{R})$ by Trimèche [3]

$$\mathcal{F}_\kappa\{\tau^y f(x);\xi\} := \mathcal{E}_\kappa(i\xi y)\mathcal{F}_\kappa\{f(x);\xi\}, \quad y,\,\xi \in \mathbb{R}. \tag{13}$$

We also define the Dunkl convolution product for suitable functions f and g by

$$f * g(x) = \int_{\mathbb{R}} \tau^{-x} f(y) g(y) \sigma_\kappa(dy).$$

Explicitly, the generalized translation $\tau^x f(y)$ takes the explicit form (see [28] Theorem 6.3.7):

$$\begin{aligned}\tau^x f(y) :=& \frac{1}{2}\int_{-1}^{1} f(\sqrt{x^2+y^2-2xyt})(1+\frac{x-y}{\sqrt{x^2+y^2-2xyt}})h_k(t)dt \\ &+ \frac{1}{2}\int_{-1}^{1} f(-\sqrt{x^2+y^2-2xyt})(1-\frac{x-y}{\sqrt{x^2+y^2-2xyt}})\,h_k(t)\,dt,\end{aligned} \tag{14}$$

where

$$h_\kappa(t) = \frac{\Gamma(\kappa + 1/2)}{2^{2\kappa}\sqrt{\pi}\Gamma(\kappa)}(1+t)(1-t^2)^{\kappa-1}.$$

2.2. The Generalized Lizorkin Space

For a comprehensive treatment of the standard Lizorkin space, we recommend referring to the book [12] §2, where the authors provide a detailed and in-depth analysis of this topic. Additionally, the study of the generalized Lizorkin space has been carried out by Soltani [24]. While we cannot provide a detailed overview of the entire subject here, we can highlight some important points for clarity.

We denote by $S(\mathbb{R})$ the Schwartz space, which is the space of C^∞-functions on $\mathbb{R}$ which are rapidly decreasing as well as their derivatives, endowed with the topology defined by the seminorms

$$\|f\|_{n,m} = \sup_{x\in\mathbb{R}, j\le m} (1+x^2)^n \mathscr{D}_\kappa^j \varphi(x), \quad n,\, m \in \mathbb{N},$$

It is not difficult to check that

$$\mathscr{D}f(x) = f'(x) + \kappa \int_{-1}^{1} f'(xt)dt.$$

From this representation, we see that the operator $\mathscr{D}$ leaves $S(\mathbb{R})$ invariant.

In the context of distribution theory, the space $S'(\mathbb{R})$ denotes the topological dual of $S(\mathbb{R})$, which consists of generalized functions, also known as tempered distributions. The value of a generalized function f as a functional on a test function $\varphi \in S(\mathbb{R})$ is denoted by (f, φ).

A generalized function is said to be κ-regular if there exists a locally integrable function f with respect to the measure $\sigma_\kappa(dx)$, such that the integral $\int_{\mathbb{R}} f(x)\varphi(x)\sigma_\kappa(dx)$ is finite for every $\varphi \in S(\mathbb{R})$. The action of the κ-regular generalized function f on a test function φ is denoted as (f, φ) or equivalently $\langle f, \varphi\rangle_\kappa$. Here, the integral on the right-hand side of the equation is denoted by $\langle f, \varphi\rangle_\kappa$. It is important to note that the measure $\sigma_\kappa(dx)$ depends on the specific context and properties of the Dunkl operators. The notation and definitions provided above establish a general framework for understanding κ-regular generalized functions and their evaluation on test functions.

The Dunkl transform is a powerful mathematical tool that acts as a topological isomorphism between the Schwartz space $S(\mathbb{R})$ and itself. This transform extends naturally to generalized functions by considering the Dunkl transform of a generalized function $f \in S'(\mathbb{R})$. The definition of the Dunkl transform for generalized functions can be expressed using duality as follows: for any $\varphi \in S(\mathbb{R})$, the pairing between the Dunkl transform of f and φ is given by

$$(\mathscr{F}_\kappa f,\, \varphi) = (f,\, \mathscr{F}_\kappa \varphi), \quad \varphi \in S(\mathbb{R}).$$

In terms of integral notation, it can be written as:

$$\int_{\mathbb{R}} (\mathscr{F}_\kappa f)(x)\varphi(x)\sigma\kappa(dx) = \int_{\mathbb{R}} f(x)\mathscr{F}_\kappa\varphi(x)\sigma\kappa(dx), \quad \varphi \in S(\mathbb{R}), \tag{15}$$

provided f and $\mathscr{F}_\kappa f$ are κ-regular.

The space $S(\mathbb{R})$ itself is not invariant under multiplication by power functions. However, we can define an invariant subspace by utilizing the Dunkl transforms. This leads us to the set $\Psi_\kappa(\mathbb{R})$ consisting of functions $\varphi \in S(\mathbb{R})$ that satisfy the conditions:

$$\mathscr{D}_\kappa^n \varphi(0) = 0, \quad \text{for } n = 0, 1, 2, \ldots,$$

where $\mathscr{D}_\kappa^n \varphi$ denotes the nth order Dunkl transform of φ. In other words, φ belongs to $\Psi_\kappa(\mathbb{R})$ if all the Dunkl transforms of φ evaluated at the origin are zero. By imposing these conditions, we construct a space of functions that possess certain transformation properties with respect to the Dunkl operators. The generalized Lizorkin space $\Phi_\kappa(\mathbb{R})$ is introduced as the *Dunkl* transform preimage of the space $\Psi_\kappa(\mathbb{R})$ in the space $S(\mathbb{R})$,

$$\Phi_\kappa(\mathbb{R}) = \Big\{\varphi \in S(\mathbb{R}) \ : \ \varphi = \mathscr{F}_\kappa(\psi), \ \psi \in \Psi_\kappa(\mathbb{R})\Big\}. \tag{16}$$

According to this definition, any function $\varphi \in \Phi_\kappa(\mathbb{R})$ satisfies the orthogonality conditions

$$\int_{\mathbb{R}} x^n \, \varphi(x)\sigma_\kappa(dx) = 0, \quad n = 0, 1, 2, \ldots . \tag{17}$$

3. Regularization of Integrals with Power Singularity

In this section, we examine two types of power functions defined on the entire real line

- Even, $|x|^\alpha$;
- Odd, $\mathrm{sgn}(x)\,|x|^\alpha$; where

$$\mathrm{sgn}(x) := \begin{cases} 1, & \text{if} \quad x > 0 \\ -1, & \text{if} \quad x < 0. \end{cases}$$

Other types of tempered power functions can be defined as follows

$$\begin{aligned} x_\pm^\alpha &= \frac{1}{2}\big[|x|^\alpha \pm |x|^\alpha \mathrm{sgn}(x)\big], \\ (\pm i\,x)^\alpha &= |x|^\alpha\big(\cos(\pi\alpha/2) \pm i\,\mathrm{sgn}(x)\sin(\pi\alpha/2)\big). \end{aligned}$$

These tempered power functions capture different aspects of fractional calculus and are used to generalize the concept of differentiation and integration to noninteger orders.

3.1. Taylor–Dunkl Formula

To facilitate the forthcoming discussion on analytic continuation, we begin by presenting an additional formula that proves to be valuable in the process.

Let $f \in C^\infty(\mathbb{R})$; for every $n \in \mathbb{N}$, we have [19]

$$\tau^y f(x) = \sum_{j=0}^{n-1} b_j(x)\mathscr{D}_\kappa^j f(x) + r_n(x, y; f), \tag{18}$$

where

$$\begin{cases} r_{j+1}(x, y; f) = \int_{-|y|}^{|y|} \big(\frac{\mathrm{sgn}(y)}{2|y|^{2\kappa}} + \frac{\mathrm{sgn}(u)}{2|u|^{2\kappa}}\big) r_j(x, u; \mathscr{D}_\kappa f)\,|u|^{2\kappa} du, \\ \\ r_1(x, y; f) = \tau^y f(x) - f(x) \end{cases}$$

and

$$b_{j+1}(x) = \int_{-|y|}^{|y|} \big(\frac{\mathrm{sgn}(y)}{2|y|^{2\kappa}} + \frac{\mathrm{sgn}(u)}{2|u|^{2\kappa}}\big) b_j(u)\,|u|^{2\kappa} du, \quad b_0(x) = 1. \tag{19}$$

Then,

$$b_{2s}(x) = \frac{\Gamma(\kappa + 1/2)}{\Gamma(\kappa + s + 1/2)}\frac{x^{2s}}{s!}, \quad b_{2s+1}(x) = \frac{\Gamma(\kappa + 1/2)}{\Gamma(\kappa + s + 3/2)}\frac{x^{2s+1}}{s!}, \, s = 0, 1, 2, \ldots .$$

From the work of Mourou [23], we can extract the following proposition, which provides a complete asymptotic expansion for $\tau_\kappa f(x)$ as x approaches a.

Lemma 1. *Let $f \in C^\infty(\mathbb{R})$ and $a \in \mathbb{R}$; then, one has*

$$\tau_\kappa^a f(x) \sim \sum_{s=0}^{\infty} b_s(x) \mathscr{D}_\kappa^s f(a), \quad as \quad x \to a, \tag{20}$$

3.2. Generalized Power Functions

By considering $|x|^{-\alpha}$ and $\mathrm{sign}(x)|x|^{-\alpha}$ as elements of $\Psi'_\kappa(\mathbb{R})$, we recognize them as κ-regular generalized functions for all $\alpha \in \mathbb{C}$, that is,

$$\langle |x|^{-\alpha}, \varphi \rangle_\kappa = \int_{\mathbb{R}} \frac{1}{|x|^\alpha} \varphi(x) \sigma_\kappa(dx), \tag{21}$$

$$\langle \mathrm{sgn}(x)|x|^{-\alpha}, \varphi \rangle_\kappa = \int_{\mathbb{R}} \frac{\mathrm{sgn}(x)}{|x|^\alpha} \varphi(x) \sigma_\kappa(dx). \tag{22}$$

When considering the functions $|x|^{-\alpha}$ and $\mathrm{sign}(x)|x|^{-\alpha}$ as elements of $S'(\mathbb{R})$ or $\Phi'_\kappa(\mathbb{R})$, they are not κ-regular if $\Re(\alpha) \geq 2\kappa + 1$. To handle these generalized functions, let $\alpha \in \mathbb{C}$ such that $\alpha \neq 2\kappa + 2s + 1$ for $s = 0, 1, 2, \ldots$. For $\varphi \in S(\mathbb{R})$, we can define the generalized power function $|x|^{-\alpha}$ as follows:

$$\begin{aligned}(|x|^{-\alpha}, \varphi) = &\int_{|x|<1} \frac{1}{|x|^\alpha} \left[\varphi(x) - \sum_{s=0}^{m} b_s(x) \mathscr{D}_\kappa^s \varphi(0) \right] \sigma_\kappa(dx) \\ &+ \sum_{s=0}^{[\frac{m}{2}]} \frac{\mathscr{D}_\kappa^{2s} \varphi(0)}{2^{\kappa-1/2}\Gamma(\kappa+s+1/2), s!} \frac{1}{2\kappa+2s+1-\alpha} \\ &+ \int_{|x|\geq 1} \frac{\varphi(x)}{|x|^\alpha} \sigma_\kappa(dx),\end{aligned} \tag{23}$$

where $m > \mathrm{Re}(\alpha) - 2\kappa - 1$. It is important to note that the right-hand side of Equation (23) does not depend on the choice of m as long as $m > \Re(\alpha) - 2\kappa - 1$. Since $\varphi \in S(\mathbb{R})$, Lemma 1 guarantees that

$$\varphi(x) - \sum_{s=0}^{m} b_s(x) \mathscr{D}_\kappa^s \varphi(0) = \mathcal{O}(x^{m+1}) \quad (\text{as} \quad x \to 0).$$

This property ensures the well-definedness of the expression. The mapping $\alpha \to (|x|^{-\alpha}, \varphi)$ from $\mathbb{C}$ to $S'(\mathbb{R})$ can be extended to a holomorphic function on $\mathbb{C} - \{2\kappa + 2s + 1 : s = 0, 1, 2, \ldots\}$, with simple poles at $\alpha = 2\kappa + 2s + 1$. The residues of the function at these poles are given by

$$\mathrm{Res}\big((|x|^{-\alpha}, \varphi); 2\kappa + 2s + 1\big) = -\frac{2^{-\kappa+1/2} \mathscr{D}_\kappa^{2s} \varphi(0)}{\Gamma(\kappa + s + 1/2)\, s!}. \tag{24}$$

When $\alpha = 2\kappa + 2s + 1$ with $s = 0, 1, 2, \ldots$, we define the even, tempered power function $|x|^{-2\kappa-2s-1}$ as

$$(|x|^{-2\kappa-2s-1}, \varphi) = \lim_{\alpha \to 2\kappa+2s+1} \left\{ (|x|^{-\alpha}, \varphi) + \tfrac{\mathscr{D}_\kappa^{2s} \varphi(0)}{2^{\kappa-1/2}\Gamma(\kappa+s+1/2)\, s!} \tfrac{1}{\alpha - 2\kappa - 2n - 1} \right\}. \tag{25}$$

This provides a definition for the even, tempered power $|x|^{-\alpha}$ for all $\alpha \in \mathbb{C}$.

Similarly, for $\alpha \in \mathbb{C}$ such that $\alpha \neq 2\kappa + 2s + 2$ with $s = 0,\ 1,\ 2\ \ldots$, we define the odd tempered power function $|x|^{-\alpha}\mathrm{sgn}(x)$ by

$$\left(\frac{\operatorname{sgn}(x)}{|x|^{\alpha}}, \varphi\right) = \int_{|x|<1} \frac{\operatorname{sgn}(x)}{|x|^{\alpha}} \left[\varphi(x) - \sum_{s=0}^{m} b_s(x)\mathscr{D}_\kappa^s \varphi(0)\right] \sigma_\kappa(dx) \tag{26}$$
$$+ \sum_{s=0}^{[\frac{m-1}{2}]} \frac{\mathscr{D}_\kappa^{2s+1}\varphi(0)}{2^{\kappa-1/2}\Gamma(\kappa+s+3/2)\, s!} \frac{1}{2\kappa+2s+2-\alpha}$$
$$+ \int_{|x|\geq 1} \frac{\operatorname{sgn}(x)}{|x|^{\alpha}} \varphi(x)\, \sigma_\kappa(dx) \quad (m > \Re(\alpha) - 2\kappa - 2).$$

It follows that the mapping $\alpha \to (|x|^{-\alpha}\operatorname{sgn}(x), \varphi)$ is analytic on $\mathbb{C} - \{2\kappa + 2s + 2,$ s = $0, 1, 2, \dots\}$, with simple poles at $\alpha = 2\kappa + 2s + 2$ and

$$\operatorname{Res}\big((|x|^{-\alpha}\operatorname{sgn}(x), \varphi); 2\kappa + 2s + 2\big) = -\frac{2^{-\kappa+1/2}\mathscr{D}_\kappa^{2s+1}\varphi(0)}{\Gamma(\kappa+s+3/2)\, s!}.$$

For $\alpha = 2\kappa + 2s + 2$, with $s = 0, 1, 2, \dots$, we define the odd, tempered powers function $\operatorname{sgn}(x)|x|^{-2\kappa-2s-2}$ as

$$(\operatorname{sgn}(x)|x|^{-2\kappa-2s-2}, \varphi) = \lim_{\alpha\to 2\kappa+2s+2} \left\{(\operatorname{sgn}(x)|x|^{-\alpha}, \varphi) + \frac{\mathscr{D}_\kappa^{2s+1}\varphi(0)}{2^{\kappa-1/2}\Gamma(\kappa+s+3/2)\, s!} \frac{1}{\alpha-2\kappa-2s-2}\right\}. \tag{27}$$

4. Fractional-Type Integral and Derivative for the Dunkl Operator

In this section, we embark on a comprehensive exploration of fractional-type integral operators associated with the Dunkl operator. These operators transcend the conventional Riesz fractional integral, Feller fractional integral, and Liouville fractional integral, as they are specifically designed to operate within the Dunkl setting.

4.1. The Riesz–Dunkl Fractional Integral

In this section, our focus lies on extending the Riesz fractional integral to any arbitrary value of $\Re(\alpha) > 0$. As a reminder, the Riesz fractional integral $I^\alpha f$ is defined by

$$(I^\alpha f)(x) = \frac{1}{\gamma(\alpha)} \int_{\mathbb{R}} k_\alpha(x-y) f(y)dy, \tag{28}$$

where $k_\alpha(x)$ is defined as:

$$k_\alpha(x) = \begin{cases} |x|^{\alpha-1}, & \alpha \neq 1, 3, 5, \dots, \\ -|x|^{\alpha-1}\ln|x|, & \alpha = 1, 3, 5, \dots. \end{cases} \tag{29}$$

The normalization factor $\gamma(\alpha)$ depends on the value of α and is given by:

$$\gamma(\alpha) = \begin{cases} \frac{2^{\alpha-1/2}\pi^{1/2}\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{1-\alpha}{2})}, & \alpha \neq 2s+1,\ s = 0, 2, \dots, \\ (-1)^s s! \pi^{1/2} 2^{2s}\Gamma(s+1/2), & \alpha = 2s+1,\ s = 0, 2, \dots. \end{cases} \tag{30}$$

Lemma 2. *Let $\kappa < \alpha < 2\kappa + 1$. Then, the Dunkl transform of $|x|^{\alpha-2\kappa-1}$ exists in the usual sense, and it is given by*

$$\mathscr{F}_\kappa^{-1}(|x|^{-\alpha}) = \frac{\Gamma(\kappa + \frac{1-\alpha}{2})}{2^{\alpha-\kappa-1/2}\Gamma(\frac{\alpha}{2})} |x|^{\alpha-2\kappa-1}.$$

Proof. By using (5), we obtain

$$\mathscr{F}_\kappa^{-1}\big(|x|^{-\alpha}\big)(x) = \int_{-\infty}^{\infty} |u|^{-\alpha}\mathcal{E}_\kappa(iux)\sigma_\kappa(du)$$
$$= \frac{2}{2^{\kappa+1/2}\Gamma(\kappa+\frac{1}{2})} \int_0^\infty \mathscr{J}_{\kappa-1/2}(|x|u)u^{-\alpha+2\kappa}\, du.$$

Making the substitution $t = |x|u$ yields

$$\mathscr{F}_\kappa^{-1}(|x|^{-\alpha})(x) = |x|^{\alpha-2\kappa-1} \int_0^\infty \frac{J_{\kappa-1/2}(u)}{u^{\alpha-\kappa-1/2}}\, du.$$

The result follows from the following Weber formula [29] §13.24:

$$\int_0^\infty \frac{J_\nu(t)}{t^{\nu-\mu+1}}\, dt = \frac{1}{2^{\nu-\mu+1}} \frac{\Gamma(\frac{\mu}{2})}{\Gamma(\nu - \frac{\mu}{2} + 1)}, \quad 0 < \Re(\mu) < \Re(\nu) + \frac{3}{2}. \tag{31}$$

□

Proposition 1. *The Dunkl transform of $|x|^{-\alpha} \in \Psi'_\kappa(\mathbb{R})$ is given by*

$$\mathscr{F}_\kappa^{-1}\big(|x|^{-\alpha}\big) = \frac{1}{\gamma_\kappa(\alpha)} \begin{cases} |x|^{\alpha-2\kappa-1}, & \alpha \neq -2s,\ \alpha \neq 2\kappa+2s+1,\ s \in \mathbb{N}_0, \\ |x|^{\alpha-2\kappa-1} \ln \frac{1}{|x|}, & \alpha = 2\kappa+2s+1,\ s \in \mathbb{N}_0, \\ (-1)^s \mathscr{D}_\kappa^{2s}\delta, & \alpha = -2s,\ s \in \mathbb{N}_0, \end{cases}$$

where

$$\gamma_\kappa(\alpha) = \begin{cases} \frac{2^{\alpha-\kappa-1/2}\Gamma(\frac{\alpha}{2})}{\Gamma(\kappa+\frac{1-\alpha}{2})} & \alpha \neq -2s,\ \alpha \neq 2\kappa+2s+1, \\ (-1)^s s! 2^{\kappa+2s+1/2}\Gamma(\kappa+s+1/2), & \alpha = 2\kappa+2s+1, \\ 1, & \alpha = -2s \end{cases}$$

and δ is the Dirac delta distribution.

Proof. From Lemma 2, it is evident that by analytic continuation, for $\alpha \in \mathbb{C}$ such that $\alpha \neq 2\kappa + 2s + 1$ and $\alpha \neq -2s$ for $s = 0, 1, 2, \ldots$, we have:

$$\frac{1}{|x|^\alpha} = \frac{\Gamma(\kappa + \frac{1-\alpha}{2})}{2^{\alpha-\kappa-1/2}\Gamma(\frac{\alpha}{2})} \mathscr{F}_\kappa(|x|^{\alpha-2\kappa-1}). \tag{32}$$

The case $\alpha = -2s$ for $s = 0, 1, 2, \ldots$ follows from the fact that

$$(\mathscr{F}_\kappa \mathscr{D}_\kappa^{2s}\varphi)(x) = (-1)^s |x|^{2s} (\mathscr{F}_\kappa \varphi)(x), \quad \varphi \in S(\mathbb{R}).$$

It remains to consider the case $\alpha = \alpha_s = 2\kappa + 2s + 1$ for $s \in \mathbb{N}_0$. From Equation (32), we have

$$\frac{\partial}{\partial\alpha}\big((\alpha - \alpha_s)(|x|^{-\alpha}, \mathscr{F}_\kappa\varphi)\big) = \frac{\partial}{\partial\alpha}\Big(\eta(\alpha)(|x|^{\alpha-2\kappa-1}, \varphi)\Big), \quad \eta(\alpha) = \frac{\alpha - \alpha_s}{\gamma_\kappa(\alpha)}. \tag{33}$$

By considering (23) and (25), the limit as $\alpha \to \alpha_s$ of the left-hand side of (33) can be evaluated as follows:

$$\lim_{\alpha\to\alpha_s} \frac{\partial}{\partial\alpha}\Big((\alpha - \alpha_k)(|x|^{-\alpha}, \mathscr{F}_\kappa\varphi)\Big) = (|x|^{-2\kappa-2s-1}, \mathscr{F}_\kappa\varphi).$$

The limit of the right-hand side of Equation (33) as $\alpha \to \alpha_s$ can be evaluated as follows:

$$\lim_{\alpha\to\alpha_s} \frac{\partial}{\partial\alpha}\Big(\eta(\alpha)\,(|x|^{\alpha-2\kappa-2}, \varphi)\Big) = \lim_{\alpha\to\alpha_s}((\eta'(\alpha) + \eta(\alpha)\ln|x|)|x|^{\alpha-2\kappa-1}, \varphi).$$

A straightforward computation shows that

$$\lim_{\alpha \to \alpha_s} \eta(\alpha) = \frac{(-1)^{s+1}}{s! 2^{\kappa+2s-1/2} \Gamma(\kappa + s + 1/2)}. \tag{34}$$

Taking into account Equation (17), in the limit as α approaches α_s, we obtain the following expression:

$$(|x|^{-2\kappa-2s-1}, \mathscr{F}_\kappa \varphi) = \frac{(-1)^s}{s! 2^{\kappa+2s-1/2} \Gamma(\kappa + s + 1/2)} (|x|^{2s} \ln \frac{1}{|x|}, \varphi). \tag{35}$$

□

Definition 1. *For $\Re(\alpha) > 0$, we define the Riesz–Dunkl fractional integral $\mathscr{I}_\kappa^\alpha f$ of $f \in \Phi_\kappa(\mathbb{R})$ as:*

$$(\mathscr{I}_\kappa^\alpha f)(x) = \int_{\mathbb{R}} \tau^{-y} \mathscr{K}_{\kappa,\alpha}(x) f(y) \sigma_\kappa(dy) \tag{36}$$

where

$$\mathscr{K}_{\kappa,\alpha}(x) = \frac{1}{\gamma_\kappa(\alpha)} \begin{cases} |x|^{\alpha-2\kappa-1}, & \alpha \neq -2s, \ \alpha \neq 2\kappa + 2s + 1 \\ \ln(\frac{1}{|x|}) \, |x|^{\alpha-2\kappa-1}, & \alpha = 2\kappa + 2s + 1. \end{cases} \tag{37}$$

The following theorem states that the space $\Phi_\kappa(\mathbb{R})$ is closed under the action of the operator $\mathscr{I}_\kappa^\alpha$. This result ensures the consistency and coherence of the space $\Phi_\kappa(\mathbb{R})$ under the Riesz–Dunkl fractional integral. Moreover, the proposition establishes the relationship between the Dunkl transform $\mathscr{F}_\kappa$ and the fractional integral operator $\mathscr{I}_\kappa^\alpha$ and shows the compatibility of the fractional integral operators $\mathscr{I}_\kappa^\alpha$ under composition.

Theorem 1. *The space $\Phi_\kappa(\mathbb{R})$ is invariant under the operator $\mathscr{I}_\kappa^\alpha$, i.e.,*

$$f \in \Phi_\kappa(\mathbb{R}) \quad \Rightarrow \quad \mathscr{I}_\kappa^\alpha f \in \Phi_\kappa(\mathbb{R}).$$

Furthermore,

$$(\mathscr{F}_\kappa \mathscr{I}_\kappa^\alpha f) = \frac{1}{|x|^\alpha} \mathscr{F}_\kappa f,$$

and

$$\mathscr{I}_\kappa^\alpha \mathscr{I}_\kappa^\beta = \mathscr{I}_\kappa^{\alpha+\beta}, \quad \Re(\alpha), \Re(\delta) > 0.$$

The proof of this theorem is omitted, but it can be established by utilizing Lemma 2 and Proposition 1 mentioned earlier, which provide the necessary tools and results to derive these conclusions.

Utilizing the reflection formula for the gamma function, we have:

$$\Gamma(z)\Gamma(1-z) = \frac{\pi z}{\sin \pi z}, \quad z \notin \mathbb{Z}.$$

In the limit when $\kappa \downarrow 0$, we retrieve the classical *Riesz* and *Feller fractional integral* (see, [11]) §12.1

$$\lim_{\kappa \downarrow 0} \mathscr{I}_\kappa^\alpha f(x) = \frac{1}{2\Gamma(\alpha) \cos(\pi\alpha/2)} \int_{-\infty}^{\infty} \frac{1}{|x-y|^{1-\alpha}} f(y) dy. \tag{38}$$

4.2. Feller–Dunkl Fractional Integral

In this section, we aim to establish an analogous version of the classical Feller fractional integral within the framework of Dunkl operators. The Feller fractional integral, denoted as $J_\kappa^\alpha f(x)$, is defined as follows:

$$J_\kappa^\alpha f(x) = \frac{1}{2\Gamma(\alpha)\sin(\pi\alpha/2)}\int_{-\infty}^{\infty}\frac{\mathrm{sgn}(x-y)}{|x-y|^{1-\alpha}}f(y)dy. \tag{39}$$

The following lemmas play a crucial role in establishing an extension of the Feller integral within the framework of the Dunkl operator.

Lemma 3. *Let $\kappa < \alpha < 2\kappa + 2$. Then, the Dunkl transform of $sgn(x)\,|x|^{-\alpha}$ exists in the usual sense, and it is given by*

$$\mathscr{F}_\kappa^{-1}(sgn(x)|x|^{-\alpha}) = i\,\frac{\Gamma(\kappa+\frac{2-\alpha}{2})}{2^{\alpha-\kappa-1/2}\Gamma(\frac{1+\alpha}{2})}\,sgn(x)\,|x|^{\alpha-2\kappa-1}.$$

Proof. Using (5), we have

$$\begin{aligned}
\mathscr{F}_\kappa^{-1}(\mathrm{sgn}(x)|x|^{-\alpha})(x) &= \int_{-\infty}^{\infty}\mathrm{sgn}(u)|u|^{-\alpha}\mathcal{E}_\kappa(iux)\sigma_\kappa(du)\\
&= \frac{2\,i\,x}{(2\kappa+1)2^{\kappa+1/2}\Gamma(\kappa+\frac{1}{2})}\int_0^\infty \mathscr{J}_{\kappa+1/2}(xu)\,u^{-\alpha+2\kappa+1}\,du\\
&= i\,\mathrm{sgn}(x)\,|x|^{\alpha-2\kappa-1}\int_0^\infty\frac{J_{\kappa+1/2}(t)}{t^{\alpha-\kappa-1/2}}\,dt.
\end{aligned}$$

The Weber Formula (31) achieves the result. □

Lemma 4. *The following holds: for $\alpha \neq 2\kappa + s + 1$ with $s \in \mathbb{Z}_-$, we have*

$$\mathscr{D}_\kappa|x|^{-\alpha} = -\alpha\,|x|^{-\alpha-1}sgn(x).$$

Proof. Let $\kappa < \Re(\alpha) < 2\kappa + 1$ and $\varphi \in S(\mathbb{R})$, we have

$$\begin{aligned}
< \mathscr{D}_\kappa|x|^{-\alpha},\,\varphi >_\kappa &= -< |x|^{-\alpha},\,\mathscr{D}_\kappa\varphi >_\kappa\\
&= -\int_{\mathbb{R}}|x|^{-\alpha}\mathscr{D}_\kappa\varphi(x)\sigma_\kappa(dx)\\
&= -\alpha\int_{\mathbb{R}}|x|^{-\alpha-1}\mathrm{sgn}(x)\varphi(x)\sigma_\kappa(dx)\\
&= -\alpha < |x|^{-\alpha-1}\mathrm{sgn}(x),\,\varphi >_\kappa .
\end{aligned}$$

By analytic continuation for $\alpha \in \mathbb{C}$ such that $\alpha \neq 2\kappa + s + 1,\ s \in \mathbb{N}$, we have

$$\mathscr{D}_\kappa|x|^{-\alpha} = -\alpha\,|x|^{-\alpha-1}\mathrm{sgn}(x),$$

which is the required result.
□

Proposition 2. *The Dunkl transform of $sgn(x)\,|x|^{-\alpha} \in \Psi'_\kappa(\mathbb{R})$ is given by*

$$\mathscr{F}_\kappa^{-1}\big(-i\,|x|^{-\alpha}sgn(x)\big) = \frac{1}{\delta_\kappa(\alpha)}\begin{cases} sgn(x)\,|x|^{\alpha-2\kappa-1}, & \alpha \neq -2s-1,\ \alpha \neq 2\kappa+2s+2,\ s \in \mathbb{N}_0,\\ -|x|^{2s+1}\ln|x|, & \alpha = 2\kappa+2s+2,\ s \in \mathbb{N}_0,\\ (-1)^s\mathscr{D}_\kappa^{2s+1}\delta, & \alpha = -2s-1,\ s \in \mathbb{N}_0. \end{cases}$$

where

$$\delta_\kappa(\alpha) = \begin{cases} \frac{2^{\alpha-\kappa-1/2}\Gamma(\frac{\alpha+1}{2})}{\Gamma(\kappa+\frac{2-\alpha}{2})} & \alpha \neq -2s-1,\ \alpha \neq 2\kappa+2s+2, \\ (-1)^s s! 2^{\kappa+2s+3/2}\Gamma(\kappa+s+3/2), & \alpha = 2\kappa+2s+2, \\ 1, \quad \alpha = -2s-1. \end{cases}$$

Proof. The proof of the proposition can be achieved by utilizing the above lemmas. □

Definition 2. *For $\Re(\alpha) > 0$, we define the Riesz–Dunkl fractional integral $\mathscr{J}_\kappa^\alpha f$ of $f \in \Phi_\kappa(\mathbb{R})$ as:*

$$(\mathscr{J}_\kappa^\alpha f)(x) = \int_{\mathbb{R}} \tau^{-y}\mathscr{G}_{\kappa,\alpha}(x) f(y)\sigma_\kappa(dy) \tag{40}$$

where

$$\mathscr{G}_{\kappa,\alpha}(x) = \frac{1}{\delta_\kappa(\alpha)} \begin{cases} sgn(x)\,|x|^{\alpha-2\kappa-1}, & \alpha \neq 2\kappa+2s+2 \\ sgn(x)\,\ln(\frac{1}{|x|})|x|^{\alpha-2\kappa-1}, & \alpha = 2\kappa+2s+2. \end{cases} \tag{41}$$

In the limit when $\alpha \downarrow 0$, we obtain

$$\lim_{\alpha\downarrow 0}(\mathscr{J}_\kappa^\alpha f) := \mathscr{H}_\kappa f(x) := \frac{\Gamma(\kappa+1)}{\sqrt{\pi}\Gamma(\kappa+1/2)} \lim_{\varepsilon\downarrow 0} \int_{|y|\geq\varepsilon} \tau_\kappa^{-y} f(x)\,\frac{dy}{y}, \tag{42}$$

and

$$(\mathscr{F}_\kappa \mathscr{H}_\kappa f)(x) = -i\,\mathrm{sgn}(x)(\mathscr{F}_\kappa f)(x), \quad f \in \Phi_\kappa(\mathbb{R}).$$

For the special case of $\kappa = 0$ and $\alpha = 0$, the Feller–Dunkl fractional integral coincides with the Hilbert transform. The Hilbert transform is a well-known operator in harmonic analysis and signal processing. It acts as a multiplier with the symbol $-isign(x)$.

It can be easily seen from Propositions 1 and 2 that the operators $\mathscr{I}_\kappa^\alpha$ and $\mathscr{J}_\kappa^\alpha$ are connected by

$$\mathscr{I}_\kappa^\alpha = \mathscr{H}_\kappa \mathscr{J}_\kappa^\alpha.$$

4.3. Riemann–Liouville–Dunkl fractional integrals

The Riemann–Liouville fractional integrals are given by [12] formulas (5.1) and (5.2)

$$I_+^\alpha f(x) := \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{x} (x-y)^{\alpha-1} f(y)dy \tag{43}$$

and

$$I_-^\alpha f(x) := \frac{1}{\Gamma(\alpha)} \int_{x}^{\infty} (y-x)^{\alpha-1} f(y)dy \tag{44}$$

They are related to the Riesz fractional integral I^α and its conjugate J^α by

$$I^\alpha f(x) = \frac{I_+^\alpha f(x) + I_-^\alpha f(x)}{2\cos(\frac{\pi\alpha}{2})},$$
$$J^\alpha f(x) = \frac{I_+^\alpha f(x) - I_-^\alpha f(x)}{2\sin(\frac{\pi\alpha}{2})}.$$

Similarly, the correspondent definition of the Riemann–Liouville–Dunkl fractional integral can be given as follows:

$$\mathscr{I}_{\kappa,+}^\alpha f(x) := \cos(\alpha\pi/2)\mathscr{I}_\kappa^\alpha f(x) + \sin(\alpha\pi/2\mathscr{J}_\kappa^\alpha f(x),$$
$$\mathscr{I}_{\kappa,-}^\alpha f(x) := \cos(\alpha\pi/2)\mathscr{I}_\kappa^\alpha f(x) - \sin(\alpha\pi/2\mathscr{J}_\kappa^\alpha f(x).$$

Proposition 3. *The following holds:*

(1) *For $f \in \Phi$, we have*

$$(\mathscr{F}_\kappa \mathscr{I}^\alpha_{\kappa,\pm} f) = (\mp ix)^{-\alpha} (\mathscr{F}_\kappa f)(x).$$

(2) *For $f \in \Phi$ and $\Re(\alpha), \Re(\beta) > 0$, we have*

$$\mathscr{I}^\alpha_{\kappa,\pm} \mathscr{I}^\beta_{\kappa,\pm} = \mathscr{I}^{\alpha+\beta}_{\kappa,\pm}.$$

(3) *Integration by parts:*

$$\int_{\mathbb{R}} \mathscr{I}^\alpha_{\kappa,+} f(x)\, g(x)\sigma_\kappa(dx) = \int_{\mathbb{R}} f(x)\, \mathscr{I}^\alpha_{\kappa,-} g(x)\sigma_\kappa(dx), \quad f, g \in \Phi_\kappa(\mathbb{R}).$$

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