symmetry

MDPI

Article

The Equivalence Conditions of Optimal Feedback Control-Strategy Operators for Zero-Sum Linear Quadratic Stochastic Differential Game with Random Coefficients

Chao Tang [1,*] and Jinxing Liu [2]

[1] College of Computer Science, Chengdu University, Chengdu 610106, China
[2] Faculty of Science, Yibin University, Yibin 644000, China; jxliu@yibinu.edu.cn
* Correspondence: tangchao@cdu.edu.cn

Abstract: From the previous work, when solving the LQ optimal control problem with random coefficients (SLQ, for short), it is remarkably shown that the solution of the backward stochastic Riccati equations is not regular enough to guarantee the robustness of the feedback control. As a generalization of SLQ, interesting questions are, "how about the situation in the differential game?", "will the same phenomenon appear in SLQ?". This paper will provide the answers. In this paper, we consider a closed-loop two-person zero-sum LQ stochastic differential game with random coefficients (SDG, for short) and generalize the results of Lü–Wang–Zhang into the stochastic differential game case. Under some regularity assumptions, we establish the equivalence between the existence of the robust optimal feedback control strategy operators and the solvability of the corresponding backward stochastic Riccati equations, which leads to the existence of the closed-loop saddle points. On the other hand, the problem is not closed-loop solvable if the solution of the corresponding backward stochastic Riccati equations does not have the needed regularity.

Keywords: stochastic differential game; random coefficients; robust optimal feedback; backward stochastic Riccati equations

Citation: Tang, C.; Liu, J. The Equivalence Conditions of Optimal Feedback Control-Strategy Operators for Zero-Sum Linear Quadratic Stochastic Differential Game with Random Coefficients. *Symmetry* **2023**, *15*, 1726. https://doi.org/10.3390/sym15091726

Received: 17 July 2023
Revised: 1 September 2023
Accepted: 4 September 2023
Published: 8 September 2023

1. Introduction

In this paper, we consider the following controlled linear stochastic differential equation (SDE, for short) on a finite horizon $[s, T]$:

$$\begin{cases} dx(t) = [A(t)x(t) + B_1(t)u_1(t) + B_2(t)u_2(t)]dt \\ \qquad\qquad + \displaystyle\sum_{i=1}^{d}[C_i(t)x(t) + D_{1i}(t)u_1(t) + D_{2i}(t)u_2(t)]dW_i(t), \quad t \in [s, T] \\ x(s) = \theta, \end{cases} \tag{1}$$

with the following quadratic objective functional:

$$\begin{aligned} J_\theta(u_1, u_2) = &\frac{1}{2}E\bigg\{\langle Gx(T), x(T)\rangle_{\mathbb{R}^n} + \int_s^T [\langle R(t)x(t), x(t)\rangle_{\mathbb{R}^n} \\ &+ \langle N_1(t)u_1(t), u_1(t)\rangle_{\mathbb{R}^{m_1}} + \langle N_2(t)u_2(t), u_2(t)\rangle_{\mathbb{R}^{m_2}}]dt\bigg\}; \end{aligned} \tag{2}$$

for simplicity of notion, sometimes we denote $m = m_1 + m_2$ and:

$$B = \begin{bmatrix} B_1 & B_2 \end{bmatrix}, \quad D_i = \begin{bmatrix} D_{1i} & D_{2i} \end{bmatrix}, \quad i = 1, 2, \ldots, d,$$
$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad N = \begin{bmatrix} N_1 & O \\ O & N_2 \end{bmatrix}.$$

1.1. Literature Review

Thanks to the key role of economics and finance in the real world, much attention has been paid by many researchers to mathematical financial problems, including the differential game theory, not only because of its useful basic theory, but also because of its practical applications to financial markets. The differential game (DG, for short), which involves the decision-making of multiple players being affected by each other, could be traced back to 1965 and the work carried out by Isaacs [1], which inspired later research in this field. For research in the early years, we refer to the literature [2–7] and the references therein. Among them, Ho-Bryson-Baron [7] may be the first paper about deterministic two-person zero-sum differential games (DG, for short) where the stochastic perturbation is absent. In 1989, Fleming-Souganidis [6] generalized the DG in the deterministic framework to the stochastic case, which is considered the major work of the research on the problem (SDG). After that, Hamadè-Lepeltier [8] investigated the SDG using the BSDEs approach. Then, Yong and Mou-Yong [9,10] generalized the SDG into a leader–follower case and investigated it in the open-loop form using a Hilbert space method. In 2008, Buckdahn-Li [11] focused on the existence of players' value through the viscosity solution theory. Yu [12] obtained the existence result of the corresponding Riccati equation by virtue of Hamadè's linear transform, which enriched the related Riccati equation theory, and studied the general case of SDG (i.e., $D_{1i} \neq 0$ and $D_{2i} \neq 0$). Very recently, Moon [13] studied the LQ stochastic leader–follower Stackelberg differential games for jump-diffusion systems with random coefficients. In addition, Wang-Wu [14] considered a kind of time-inconsistent linear-quadratic non-zero sum stochastic differential game problem with random jumps. For more details, we refer to [13–18] and the references therein.

To some extent, the problem of SDG can be treated as the generalization of a stochastic LQ optimal control problem with two controls. From this viewpoint, these two problems have a close relationship with each other and benefit from each other. In 2003, Tang [19] obtained the equivalence between the existence of the solution for the Riccati equation and the homomorphism of the stochastic flows derived from the optimally controlled system, which solved the long-standing open problem proposed by Bismut [20] in 1978. After that, Tang [21] improved the well-posedness results in [19] through dynamic programming. In 2017, Lü-Wang-Zhang [22] showed the nonexistence of feedback controls to a solvable SLQ with random coefficients, which is significantly different from its deterministic counterpart. Very recently, Zhang-Dong-Meng [23] investigated the well-posedness of stochastic Riccati equations associated with stochastic linear quadratic optimal control with jumps and random coefficients using an approach similar to that introduced in [21]. For more essential details on SLQ, one can refer to [23–27] and the rich references therein.

1.2. Main Contributions of This Paper

As is well-known, when solving the closed-loop SLQ or SDG, the key point is the solvability of the corresponding backward stochastic Riccati equations in the form of the following:

$$\begin{cases} dP = -F(A,B,C_i,D_i;R,N;P,Q_i)dt + \sum_{i=1}^{d} Q_i dW_i(t), \quad t \in [s,T] \\ P(T) = G, \quad Q := (Q_1, Q_2, \dots, Q_d), \end{cases}$$

which is a highly nonlinear backward stochastic differential equation; the coefficients will be specified later. Fortunately, Tang [19] obtained a unique solution, $(P,Q) \in L^{\infty}_{\mathcal{F}_t}(0,T;\mathcal{S}(\mathbb{R}^n)) \times L^{p}_{\mathcal{F}_t}(\Omega;L^2(0,T;\mathcal{S}(\mathbb{R}^n)))$ for $p \in [1,\infty)$. After that, Lü-Wang-Zhang [22] remarkably showed that it is not regular enough to establish the feedback control because the second part of the solution $Q \in L^{p}_{\mathcal{F}_t}(\Omega;L^2(0,T;\mathcal{S}(\mathbb{R}^n)))$ will be involved in the feedback control (for details please see Remark 3). This phenomenon is interesting and important. As a generalization of SLQ, a natural question is,"will the same phenomenon appear in SDG?" Obviously, the answer is affirmative and this paper will show that the same remarkable phenomenon appears in the SDG problem. Then, inspired by the idea from [16,22], we establish the equivalence between the existence of the robust optimal feedback control-strategy operators (please see Section 2 for the definition) and the solvability of the corresponding backward stochastic Riccati equations. Furthermore, the existence of a closed-loop saddle point will be obtained by way of some suitable conditions. Finally, from the example, the SDG problem is not closed-loop solvable without the needed regularity of the solution for the corresponding backward stochastic Riccati equations.

Note that, in the literature mentioned above, no works focus on the above phenomenon in SDG. Especially, when considering recent works [13,14,18,23] ([14,18] only consider the deterministic coefficients case [13,23] and do not pay attention to the robustness of the feedback control), to the best of our knowledge, the robustness of the feedback control-strategy pair in SDG with random coefficients has not yet been studied. This paper can be viewed as a nontrivial extension of [22] in the differential game case. The main generalizations and technical challenges of this paper are compared with [22] as follows:

(a) The inclusion of two controls appears in Equations (1) and (2).
(b) The explicit dependence of two controls on both the drift and diffusion terms in Equation (1).
(c) The backward stochastic Riccati equations above with a high dimension.

1.3. Problem Statement

Throughout the paper, we let $\mathbb{R}^n$ be the n-dimensional Euclidean space with Euclidean norm $|\cdot|_{\mathbb{R}^n}$ and Euclidean inner product $\langle\cdot,\cdot\rangle_{\mathbb{R}^n}$. We also let $\mathcal{S}(\mathbb{R}^n)$ be the set of all $(n\times n)$ symmetric matrices, $\mathcal{S}^+(\mathbb{R}^n)$ be the set of $(n\times n)$ positive definite matrices, and $\mathcal{S}^-(\mathbb{R}^n)$ be the set of $(n\times n)$ negative definite matrices. Furthermore, we suppose that $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq s},P)$ is a complete filtered probability space with the natural filtration $(\mathcal{F}_t)_{t\geq s}$ generated by a d-dimensional standard Brownian motion $\{W_t=(W_1,W_2,\ldots,W_d)';s\leq t\leq T\}$ augmented by all P-null sets, where the superscript $(\cdot)'$ denotes the transpose of vectors or matrices.

In Equations (1) and (2) above, $x(\cdot)$ is the state variable in $\mathbb{R}^n$, the matrix-valued stochastic processes $A(\cdot),B_1(\cdot),B_2(\cdot),C_i(\cdot),D_{1i}(\cdot),D_{2i}(\cdot),R(\cdot),N_1(\cdot),N_2(\cdot)$ are given $\mathcal{F}_t^s$-progressively measurable processes, and G is given $\mathcal{F}_T^s$-measurable random variables satisfying suitable assumptions to be given later such that Equation (1) admits a unique strong solution. In what follows, the time variable t is sometimes suppressed in the form A,B_1,C_i,D_{1i}, etc. For any $s\in[0,T)$ and $i=1,2$, we introduce the following collections of all admissible controls:

$$\mathcal{U}_i[s,T]:=\left\{u_i\in\mathbb{R}^{m_i}|u_i(\cdot)\text{ is }\mathcal{F}_t^s\text{-progressively measurable, }E\int_s^T|u_i(t)|^2dt<\infty\right\}.$$

We also need to introduce the following collections of all admissible non-anticipative strategies for Player $i,i=1,2$ on $[s,T]$, which is adopted from Buckdahn and Li [11] and Yu [12]:

$$\mathcal{A}_i[s,T]:=\left\{\gamma_i:\mathcal{U}_j[s,T]\to\mathcal{U}_i[s,T]|\gamma_i(u_j)\equiv\gamma_i(u_j^*)\text{ on }[s,\tau],\text{ if }u_j\equiv u_j^*\text{ on }[s,\tau]\right\},$$

where $j=3-i$ and $\tau:\Omega\to[s,T]$ is any stopping time. In the above collection, for $i=2$, then $j=1$, $\gamma_2(u_1)$ means the admissible strategy for Player 2 to deal with the control of Player 1 in the game. Further, Player 1 aims to maximize (2) and Player 2 wants to minimize it. So, $J_\theta(u_1,u_2)$ can be regarded as the payoff for Player 1 and the cost for Player 2. Above all, the two-person zero-sum linear quadratic stochastic differential game (SDG, for short) can be stated as follows:

Problem (SDG) For each pair (s,θ), find an admissible control $\bar{u}_1 \in \mathcal{U}_1[s,T]$ and an admissible strategy $\bar{\gamma}_2 \in \mathcal{A}_2[s,T]$ such that:

$$J_\theta(\bar{u}_1, \bar{\gamma}_2(\bar{u}_1)) = \inf_{\gamma_2 \in \mathcal{A}_2[s,T]} \sup_{u_1 \in \mathcal{U}_1[s,T]} J_\theta(u_1, \gamma_2(u_1)), \tag{3}$$

which is called Player 1's value, and $(\bar{u}_1, \bar{\gamma}_2)$ is called an optimal control-strategy pair of Player 1. On the other hand, if we find an admissible control $\bar{u}_2 \in \mathcal{U}_2[s,T]$ and an admissible strategy $\bar{\gamma}_1 \in \mathcal{A}_1[s,T]$ such that:

$$J_\theta(\bar{\gamma}_1(\bar{u}_2), \bar{u}_2) = \sup_{\gamma_1 \in \mathcal{A}_1[s,T]} \inf_{u_2 \in \mathcal{U}_2[s,T]} J_\theta(\gamma_1(u_2), u_2), \tag{4}$$

which is called Player 2's value and $(\bar{\gamma}_1, \bar{u}_2)$ is called an optimal control-strategy pair of Player 2.

In the case that the two players' values are equal, we call this common value the value of the game and $(\bar{u}_1, \bar{u}_2)$ is called the saddle point, where $\bar{u}_1 = \bar{\gamma}_1(\bar{u}_2)$ and $\bar{u}_2 = \bar{\gamma}_2(\bar{u}_1)$, these two conditions can be replaced by another two matrices (please see Remark 7 in Section 3).

This paper is organized as follows. Some preliminary results and definitions are provided in Section 2. Section 3 is devoted to presenting the main results of this paper. Finally, two examples are provided to illustrate our results.

2. Preliminaries

The following collections will be used in this paper: here, $p, p_1, p_2 \in [1,\infty)$:

$$L^p_{\mathcal{F}_T}(\Omega;\mathbb{R}^n) = \Big\{\xi : \Omega \to \mathbb{R}^n | \xi \text{ is } \mathcal{F}_T\text{-measurable}, E|\xi|^p_{\mathbb{R}^n} < \infty\Big\},$$

$$L^p_{\mathcal{F}_t}(\Omega;C(s,T;\mathbb{R}^n)) = \Big\{x : \Omega \times [s,T] \to \mathbb{R}^n | x(t) \text{ is } \mathcal{F}_t\text{-adapted, continuous,} \left(E \sup_{t\in[s,T]} |x(t)|^p_{\mathbb{R}^n}\right)^{\frac{1}{p}} < \infty\Big\},$$

$$L^{p_1}_{\mathcal{F}_t}(\Omega;L^{p_2}(s,T;\mathbb{R}^n)) = \Big\{x : \Omega \times [s,T] \to \mathbb{R}^n | x(t) \text{ is } \mathcal{F}_t\text{-adapted}, E\left(\int_s^T |x(t)|^{p_2}_{\mathbb{R}^n} dt\right)^{\frac{p_1}{p_2}} < \infty\Big\},$$

$$L^{p_2}_{\mathcal{F}_t}(s,T;L^{p_1}(\Omega;\mathbb{R}^n)) = \Big\{x : \Omega \times [s,T] \to \mathbb{R}^n | x(t) \text{ is } \mathcal{F}_t\text{-adapted}, \int_s^T \left(E|x(t)|^{p_1}_{\mathbb{R}^n}\right)^{\frac{p_2}{p_1}} dt < \infty\Big\}.$$

All the collections above are Banach spaces with the canonical norms. $L^\infty_{\mathcal{F}_t}(\Omega;L^{p_2}(s,T;\mathbb{R}^n))$, $L^{p_1}_{\mathcal{F}_t}(\Omega;L^\infty(s,T;\mathbb{R}^n))$ and $L^\infty_{\mathcal{F}_t}(\Omega;L^\infty(s,T;\mathbb{R}^n))$ can be defined in the same way. For simplicity of notation, we denote $L^{p_1}_{\mathcal{F}_t}(s,T;L^{p_1}(\Omega;\mathbb{R}^n))$ by $L^{p_1}_{\mathcal{F}_t}(s,T;\mathbb{R}^n)$. In **Problem (SDG)**, we adopt the following assumptions:

Assumption 1. ***(AS1)*** *The coefficients of the state Equation (1) satisfy the following:*

$$\begin{cases} A(\cdot) \in L^\infty_{\mathcal{F}_t}(\Omega;L^1(0,T;\mathbb{R}^{n\times n})), B_1(\cdot) \in L^\infty_{\mathcal{F}_t}(\Omega;L^2(0,T;\mathbb{R}^{n\times m_1})), D_{1i}(\cdot) \in L^\infty_{\mathcal{F}_t}(0,T;\mathbb{R}^{n\times m_1}), \\ C_i(\cdot) \in L^\infty_{\mathcal{F}_t}(\Omega;L^2(0,T;\mathbb{R}^{n\times n})), B_2(\cdot) \in L^\infty_{\mathcal{F}_t}(\Omega;L^2(0,T;\mathbb{R}^{n\times m_2})), D_{2i}(\cdot) \in L^\infty_{\mathcal{F}_t}(0,T;\mathbb{R}^{n\times m_2}). \end{cases}$$

Assumption 2. ***(AS2)*** *The weighting coefficients in Equation (2) satisfy the following:*

$$\begin{cases} R(\cdot) \in L^\infty_{\mathcal{F}_t}(\Omega;L^1(0,T;\mathcal{S}(\mathbb{R}^n))), \quad G \in L^\infty_{\mathcal{F}_T}(\Omega;\mathcal{S}(\mathbb{R}^n)) \\ N_1(\cdot) \in L^\infty_{\mathcal{F}_t}(0,T;\mathcal{S}(\mathbb{R}^{m_1})), \quad N_2(\cdot) \in L^\infty_{\mathcal{F}_t}(0,T;\mathcal{S}(\mathbb{R}^{m_2})). \end{cases}$$

Lemma 1 ([16]). *Suppose the assumption* ***(AS1)*** *holds. For any* $(s,\theta) \in [0,T) \times L^2_{\mathcal{F}_s}(\Omega;\mathbb{R}^n)$, *and* $u_1 \in \mathcal{U}_1 \equiv L^2_{\mathcal{F}_t}(s,T;\mathbb{R}^{m_1}))$, $u_2 \in \mathcal{U}_2 \equiv L^2_{\mathcal{F}_t}(s,T;\mathbb{R}^{m_2}))$, *the state Equation (1) admits a unique solution* $x(\cdot;s,\theta,u_1(\cdot),u_2(\cdot)) \in L^2_{\mathcal{F}_t}(\Omega;C(s,T;\mathbb{R}^n))$.

Remark 1. *Further, under the assumption* ***(AS2)***, *Equation (2) is well-defined for any* $(s,\theta) \in [0,T) \times L^\infty_{\mathcal{F}_s}(\Omega;\mathbb{R}^n)$ *and* $u_1 \in \mathcal{U}_1[s,T]$, $u_2 \in \mathcal{U}_2[s,T]$. *So the* ***Problem (SDG)*** *provided before is meaningful.*

Next, we present some well-known results, which will be used later.

Lemma 2. *For any* $(s,\theta) \in [0,T) \times L^2_{\mathcal{F}_s}(\Omega;\mathbb{R}^n)$, *the stochastic differential equation:*

$$\begin{cases} dx(t) =(\mathbf{A}(t)x(t) + f(t))dt + \sum_{i=1}^{d}(\mathbf{B}_i(t)x(t) + g_i(t))dW_i(t), \\ x(s) =\theta \end{cases} \tag{5}$$

admits a unique $\mathcal{F}_t$*-adapted solution* $x(\cdot) \in L^2_{\mathcal{F}_t}(\Omega;C(s,T;\mathbb{R}^n))$, *if the coefficients* $\mathbf{A}(\cdot) \in L^\infty_{\mathcal{F}_t}(\Omega;L^2(s,T;\mathbb{R}^{n\times n}))$, $\mathbf{B}(\cdot) \in L^\infty_{\mathcal{F}_t}(\Omega;L^2(s,T;(\mathbb{R}^{n\times n})^d))$, $f(\cdot) \in L^2_{\mathcal{F}_t}(s,T;\mathbb{R}^{n\times n})$ *and* $g(\cdot) \in L^2_{\mathcal{F}_t}(s,T;(\mathbb{R}^{n\times n})^d)$.

For details of the existence and uniqueness results of Equation (5), we refer to [27,28].

Lemma 3 ([29]). *Suppose that the coefficient* f *satisfies that:*

(i) $f(\cdot,0,0) \in L^\infty_{\mathcal{F}_t}(\Omega;L^1(s,T;\mathbb{R}^n))$,

(ii) $|f(\cdot,\alpha_1,\beta_1) - f(\cdot,\alpha_2,\beta_2)| \le k_1(\cdot)|\alpha_1 - \alpha_2| + k_2(\cdot)|\beta_1 - \beta_2|$, $\forall \alpha_1,\alpha_2,\beta_1,\beta_2 \in \mathbb{R}^n$, *where* $k_1(\cdot) \in L^\infty_{\mathcal{F}_t}(\Omega;L^1(s,T;\mathbb{R}))$ *and* $k_2(\cdot) \in L^\infty_{\mathcal{F}_t}(\Omega;L^2(s,T;\mathbb{R}))$. *Then, for any* $\xi \in L^\infty_{\mathcal{F}_T}(\Omega;\mathbb{R}^n)$, *the backward stochastic differential equation:*

$$\begin{cases} dy(t) =f(t,y(t),z_i(t))dt + \sum_{i=1}^{d} z_i(t)dW_i(t), \\ y(T) =\xi \end{cases} \tag{6}$$

admits a unique solution $(y(\cdot),z(\cdot)) \in L^\infty_{\mathcal{F}_t}(\Omega;C(s,T;\mathbb{R}^n)) \times L^p_{\mathcal{F}_t}(\Omega;L^2(s,T;\mathbb{R}^{n\times d}))$, *which is* $\mathcal{F}_t$*-adapted.*

Further, we would like to recall the Pontryagin-type maximum principle for differential games; for further details, we refer to [30] and the references therein.

For any $(\omega,t,x,y,z,u_1,u_2) \in \Omega \times [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n\times d} \times \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$, we define a Hamiltonian function associated with **Problem (SDG)** by (the variable t is suppressed),

$$\begin{aligned} H(t,x,y,z,u_1,u_2) =& \langle Ax + B_1u_1 + B_2u_2, y\rangle + \sum_{i=1}^{d}\langle C_ix + D_{1i}u_1 + D_{2i}u_2, z_i\rangle \\ &+ \frac{1}{2}\Big[\langle Rx,x\rangle + \langle N_1u_1,u_1\rangle + \langle N_2u_2,u_2\rangle\Big]. \end{aligned} \tag{7}$$

Lemma 4. *Let* $(\bar{x}(\cdot), \bar{u}_1(\cdot), \bar{u}_2(\cdot)) \in L^2_{\mathcal{F}_t}(\Omega; C(s,T;\mathbb{R}^n)) \times L^2_{\mathcal{F}_t}(s,T;\mathbb{R}^{m_1}) \times L^2_{\mathcal{F}_t}(s,T;\mathbb{R}^{m_2})$ *be an optimal pair of* ***Problem (SDG)****. Then there exists a pair* $(\bar{y}(\cdot), \bar{z}(\cdot)) \in L^2_{\mathcal{F}_t}(\Omega; C(s,T;\mathbb{R}^n)) \times L^2_{\mathcal{F}_t}(s,T;\mathbb{R}^{n\times d})$ *satisfying the following BSDE:*

$$\begin{cases} d\bar{y}(t) = -\left[A'\bar{y}(t) + \sum_{i=1}^{d}(C_i)'\bar{z}_i(t) + R\bar{x}(t)\right]dt + \sum_{i=1}^{d}\bar{z}_i(t)dW_i(t), \\ \bar{y}(T) = G\bar{x}(T), \end{cases} \tag{8}$$

such that

$$\begin{cases} (B_1)'\bar{y}(\cdot) + \sum_{i=1}^{d}(D_{1i})'\bar{z}_i(\cdot) + N_1\bar{u}_1(\cdot) = 0, \\ (B_2)'\bar{y}(\cdot) + \sum_{i=1}^{d}(D_{2i})'\bar{z}_i(\cdot) + N_2\bar{u}_2(\cdot) = 0. \end{cases}$$

As is well-known, stochastic Hamiltonian systems are very useful for deriving the open-loop form control-strategy for **Problem (SDG)**, which is closely linked with the stochastic Riccati equation. Next, in order to investigate the closed-loop form (SDG), the associated backward stochastic Riccati equation takes the following form:

$$\begin{cases} dP = -F(A,B,C_i,D_i;R,N;P,Q_i)dt + \sum_{i=1}^{d} Q_i dW_i(t), \quad t \in [s,T] \\ P(T) = G, \quad Q := (Q_1, Q_2, \dots, Q_d), \end{cases} \tag{9}$$

where the variable t is suppressed, and for any $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C = (C_1, \dots, C_d) \in (\mathbb{R}^{n\times n})^d$, $D = (D_1, \dots, D_d) \in (\mathbb{R}^{n\times m})^d$; $R \in \mathbb{R}^{n\times n}$, $N \in \mathbb{R}^{m\times m}$, $P \in \mathbb{R}^{n\times n}$, $Q = (Q_1, Q_2, \dots, Q_d) \in (\mathbb{R}^{n\times n})^d$ with P and $Q_i (i = 1, \dots, d)$ being symmetric, the generator F is defined by:

$$\begin{aligned} &F(A,B,C_i,D_i;R,N;P,Q_i) \\ &:= A'P + PA + R + \sum_{i=1}^{d}(C_i)'PC_i + \sum_{i=1}^{d}[(C_i)'Q_i + Q_iC_i] \\ &- \left[PB + \sum_{i=1}^{d}(C_i)'PD_i + \sum_{i=1}^{d}Q_iD_i\right]\left[N + \sum_{i=1}^{d}(D_i)'PD_i\right]^{-1} \\ &\times \left[PB + \sum_{i=1}^{d}(C_i)'PD_i + \sum_{i=1}^{d}Q_iD_i\right]'. \end{aligned} \tag{10}$$

It is well-known that Equation (9) is a backward stochastic differential equation with the generator $F(A,B,C_i,D_i;R,N;P,Q_i)$ being nonlinear in P and Q. For the simplicity of notation, we denote:

$$\begin{aligned} &\widetilde{N}_{11} := N_1 + \sum_{i=1}^{d}(D_{1i})'PD_{1i}, \ \ \widetilde{N}_{22} := N_2 + \sum_{i=1}^{d}(D_{2i})'PD_{2i}, \ \ \widetilde{N}_{12} := \sum_{i=1}^{d}(D_{1i})'PD_{2i}, \\ &\widetilde{N} := N + \sum_{i=1}^{d}(D_i)'PD_i = \begin{bmatrix} \widetilde{N}_{11} & \widetilde{N}_{12} \\ (\widetilde{N}_{12})' & \widetilde{N}_{22} \end{bmatrix}, \\ &\widetilde{B}_1 := PB_1 + \sum_{i=1}^{d}(C_i)'PD_{1i} + \sum_{i=1}^{d}Q_iD_{1i}, \ \ \widetilde{B}_2 := PB_2 + \sum_{i=1}^{d}(C_i)'PD_{2i} + \sum_{i=1}^{d}Q_iD_{2i}, \\ &\widetilde{B} := PB + \sum_{i=1}^{d}(C_i)'PD_i + \sum_{i=1}^{d}Q_iD_i = [\widetilde{B}_1 \quad \widetilde{B}_2]. \end{aligned} \tag{11}$$

Remark 2. *The backward stochastic Riccati equation (BSRE, for short) above becomes the Riccati equation for SLQ when SDG involves only one player (i.e., $B_2, D_{2i}, N_2 \equiv 0$) in the suitable dimensions, for the formal derivation about the Riccati equation for SLQ, we refer to Bismut [20,31]. Yu [12] solved Equation (9) without a stochastic part by virtue of Hamadѐ's linear transform, which needed to restrict the matrices B and $D_i(i = 1, 2 \ldots, d)$ to some special case,*

$$B_j = k_0 K_j, \;\; D_{ij} = k_i K_j, \;\; i = 1, 2, \ldots, d; \; j = 1, 2,$$

where $K_1 \in L^{\infty}_{\mathcal{F}_t}(0, T; \mathcal{S}(\mathbb{R}^{n\times m_1}))$ and $K_2 \in L^{\infty}_{\mathcal{F}_t}(0, T; \mathcal{S}(\mathbb{R}^{n\times m_2}))$. In this paper, we release this restriction under some sharp regularity on feedback control-strategy operators (the definition will be given later).

With the notation above, Equation (10) can be rewritten as:

$$\begin{aligned} F(A, B, C_i, D_i; R, N; P, Q_i) =& A'P + PA + R + \sum_{i=1}^{d}(C_i)'PC_i + \sum_{i=1}^{d}[(C_i)'Q_i + Q_iC_i] \\ & - \widetilde{B}\widetilde{N}^{-1}(\widetilde{B})'. \end{aligned} \tag{12}$$

From the above notations, along with Yu's [12] observation on the following, algebra equations,

$$\begin{cases} \widetilde{N}_{11}u_1(\cdot) + \widetilde{N}_{12}u_2(\cdot) + (\widetilde{B}_1)'x(\cdot) = 0, \\ (\widetilde{N}_{12})'u_1(\cdot) + \widetilde{N}_{22}u_2(\cdot) + (\widetilde{B}_2)'x(\cdot) = 0, \end{cases} \tag{13}$$

solve them in a useful expression for studying the game problem with strategy, then derive the following two control-strategy forms for Player 1 and Player 2, respectively:

$$\begin{cases} \bar{u}_1(\cdot) = -\Big[\widetilde{N}_{11} - \widetilde{N}_{12}\widetilde{N}_{22}^{-1}(\widetilde{N}_{12})'\Big]^{-1}\Big[\widetilde{B}_1 - \widetilde{B}_2\widetilde{N}_{22}^{-1}(\widetilde{N}_{12})'\Big]'x(\cdot), \\ \bar{\gamma}_2(u_1(\cdot)) = -\widetilde{N}_{22}^{-1}\Big[(\widetilde{B}_2)'x(\cdot) + (\widetilde{N}_{12})'u_1(\cdot)\Big] \end{cases} \tag{14}$$

and

$$\begin{cases} \bar{\gamma}_1(u_2(\cdot)) = -\widetilde{N}_{11}^{-1}\Big[(\widetilde{B}_1)'x(\cdot) + \widetilde{N}_{12}u_2(\cdot)\Big], \\ \bar{u}_2(\cdot) = -\Big[\widetilde{N}_{22} - (\widetilde{N}_{12})'\widetilde{N}_{11}^{-1}\widetilde{N}_{12}\Big]^{-1}\Big[\widetilde{B}_2 - \widetilde{B}_1\widetilde{N}_{11}^{-1}\widetilde{N}_{12}\Big]'x(\cdot). \end{cases} \tag{15}$$

Taking advantage of these two expressions, Equation (10) can be rewritten in the following two forms:

(i) From Equation (14), F can be rewritten in the following Form (I):

$$\begin{aligned} & F(A, B, C_i, D_i; R, N; P, Q_i) \\ =& A'P + PA + R + \sum_{i=1}^{d}(C_i)'PC_i + \sum_{i=1}^{d}[(C_i)'Q_i + Q_iC_i] \\ & - \widetilde{B}_2\widetilde{N}_{22}^{-1}(\widetilde{B}_2)' - \Big[\widetilde{B}_1 - \widetilde{B}_2\widetilde{N}_{22}^{-1}(\widetilde{N}_{12})'\Big]\Big[\widetilde{N}_{11} - \widetilde{N}_{12}\widetilde{N}_{22}^{-1}(\widetilde{N}_{12})'\Big]^{-1}\Big[\widetilde{B}_1 - \widetilde{B}_2\widetilde{N}_{22}^{-1}(\widetilde{N}_{12})'\Big]'. \end{aligned} \tag{16}$$

(ii) From Equation (15), F can be rewritten in the following Form (II):

$$\begin{aligned} & F(A, B, C_i, D_i; R, N; P, Q_i) \\ =& A'P + PA + R + \sum_{i=1}^{d}(C_i)'PC_i + \sum_{i=1}^{d}[(C_i)'Q_i + Q_iC_i] \\ & - \widetilde{B}_1\widetilde{N}_{11}^{-1}(\widetilde{B}_1)' - \Big[\widetilde{B}_2 - \widetilde{B}_1\widetilde{N}_{11}^{-1}\widetilde{N}_{12}\Big]\Big[\widetilde{N}_{22} - (\widetilde{N}_{12})'\widetilde{N}_{11}^{-1}\widetilde{N}_{12}\Big]^{-1}\Big[\widetilde{B}_2 - \widetilde{B}_1\widetilde{N}_{11}^{-1}\widetilde{N}_{12}\Big]'. \end{aligned} \tag{17}$$

These two forms are convenient for completing the square of Players' control strategies and also investigating the Players' values.

Remark 3. *In 2003, Tang [19], connected the existence of a solution of the Riccati equation for Problem (SLQ) to the homomorphism of the stochastic flows derived from the optimally controlled system, and obtained a unique solution* $(P, Q) \in L^{\infty}_{\mathcal{F}_t}(0, T; \mathcal{S}(\mathbb{R}^n)) \times L^{p}_{\mathcal{F}_t}(\Omega; L^2(0, T; \mathcal{S}(\mathbb{R}^n)))$ *for a given* $p \in [1, \infty)$*, from which we know that the optimal feedback operator for SLQ (denoted by* Λ*) only with the regularity* $\Lambda \in L^{p}_{\mathcal{F}_t}(\Omega; L^2(0, T; \mathcal{S}(\mathbb{R}^n)))$*. Although it showed that if* $\bar{x}$ *is an optimal state, then* $\Lambda\bar{x} \in L^2_{\mathcal{F}_t}(0, T; \mathbb{R}^n)$ *is the desired optimal control, and such feedback control is not robust, even with regard to some small perturbation, i.e., one cannot conclude that* $\Lambda(\bar{x} + \varepsilon x)$ $(\varepsilon x \in L^2_{\mathcal{F}_t}(\Omega; C(0, T, \mathbb{R}^n))$ *is a small perturbation) is an admissible control because the second part of solution* Q *will be involved in the feedback representation, which is not regular enough. Only when the optimal feedback operator has the sharp regularity* $\Lambda \in L^{\infty}_{\mathcal{F}_t}(\Omega; L^2(0, T; \mathcal{S}(\mathbb{R}^n)))$*, does it become a robust optimal feedback.*

Remark 4. *In Problem (SDG), because of the closed relationship with SLQ, it also meets the same difficulty stated in Remark 3, which denies the robustness of the control-strategy pair. Actually, taking advantage of the same direction of Yu's [12] approach, i.e., Hamadѐ's linear transform, we can obtain the solvability of Equation (9) with the existence results of BSRE in Tang's paper [19]; nevertheless, the second part of the solution* $Q \in L^{p}_{\mathcal{F}_t}(\Omega; L^2(0, T; \mathcal{S}(\mathbb{R}^n)))$ *is not regular enough to design a robust feedback control strategy.*

Next, let us give the notions of robust optimal feedback control-strategy operators for **Problem (SDG)**.

Definition 1. *The pair of stochastic processes* $(\psi_1(\cdot), \Psi_2(\cdot))$ *is called a robust optimal feedback control-strategy operator of Player 1 for* ***Problem (SDG)*** *if*

(i) $\psi_1(\cdot) \in L^{\infty}_{\mathcal{F}_t}(\Omega; L^2(0, T; (\mathbb{R}^{m_1 \times n}))$, $\Psi_2(\psi_1(\cdot)) \in L^{\infty}_{\mathcal{F}_t}(\Omega; L^2(0, T; (\mathbb{R}^{m_2 \times n}))$.

(ii) For all $(s, \theta) \in [0, T) \times L^2_{\mathcal{F}_s}(\Omega; \mathbb{R}^n)$ *and* $u_1(\cdot) \in \mathcal{U}_1[s, T]$, $\gamma_2(\cdot), \bar{\gamma}_2(\cdot) \in \mathcal{A}_2[s, T]$*, it holds that*

$$J_\theta(u_1(\cdot), \bar{\gamma}_2(u_1(\cdot))) \le J_\theta(u_1(\cdot), \gamma_2(u_1(\cdot))), \tag{18}$$

and

$$J_\theta(\psi_1(\cdot)\bar{x}(\cdot), \Psi_2(\psi_1(\cdot))\bar{x}(\cdot)) \ge J_\theta(u_1(\cdot), \bar{\gamma}_2(u_1(\cdot))), \tag{19}$$

where $\bar{x}(\cdot)$ *is the solution of*

$$\begin{cases} dx(t) = [A(t) + B_1(t)\psi_1(t) + B_2(t)\Psi_2(\psi_1(t))]x(t)dt \\ \qquad\qquad + \sum_{i=1}^{d} [C_i(t) + D_{1i}(t)\psi_1(t) + D_{2i}(t)\Psi_2(\psi_1(t))]x(t)dW_i(t), \\ x(s) = \theta. \end{cases} \tag{20}$$

Similarly, we can give the same notion of the robust optimal feedback control-strategy operator of Player 2.

Definition 2. *The pair of stochastic processes* $(\psi_2(\cdot), \Psi_1(\cdot))$ *is called a robust optimal feedback control-strategy operator of Player 2 for* ***Problem (SDG)*** *if*

(i) $\psi_2(\cdot) \in L^{\infty}_{\mathcal{F}_t}(\Omega; L^2(0, T; (\mathbb{R}^{m_2 \times n}))$, $\Psi_1(\psi_2(\cdot)) \in L^{\infty}_{\mathcal{F}_t}(\Omega; L^2(0, T; (\mathbb{R}^{m_1 \times n}))$.

(ii) For all $(s, \theta) \in [0, T) \times L^2_{\mathcal{F}_s}(\Omega; \mathbb{R}^n)$ *and* $u_2(\cdot) \in \mathcal{U}_2[s, T]$, $\gamma_1(\cdot), \bar{\gamma}_1(\cdot) \in \mathcal{A}_1[s, T]$*, it holds that*

$$J_\theta(\bar{\gamma}_1(u_2(\cdot)), u_2(\cdot)) \le J_\theta(\gamma_1(u_2(\cdot)), u_2(\cdot)), \tag{21}$$

and

$$J_\theta(\Psi_1(\psi_2(\cdot))\bar{x}(\cdot), \psi_2(\cdot)\bar{x}(\cdot)) \ge J_\theta(\bar{\gamma}_1(u_2(\cdot)), u_2(\cdot)), \tag{22}$$

where $\bar{x}(\cdot)$ is the solution of

$$\begin{cases} dx(t) =[A(t) + B_1(t)\Psi_1(\psi_2(t)) + B_2(t)\psi_2(t)]x(t)dt \\ \qquad\qquad + \sum_{i=1}^{d}[C_i(t) + D_{1i}(t)\Psi_1(\psi_2(t)) + D_{2i}(t)\psi_2(t)]x(t)dW_i(t), \\ x(s) =\theta. \end{cases} \tag{23}$$

Remark 5. *In the Definitions above, $\bar{\gamma}_1(u_2(\cdot))$ and $\bar{\gamma}_2(u_1(\cdot))$ are the feedback strategies for Player 1 and Player 2, respectively, which means Player 1 (Player 2) in the game knows how to deal with the control for Player 2 (Player 1) in the following feedback control-strategy form:*

$$\bar{\gamma}_1(u_2(\cdot)) = -\widetilde{N}_{11}^{-1}\Big[(\widetilde{B}_1)'\bar{x}(\cdot) + \widetilde{N}_{12}u_2(\cdot)\Big],$$
$$\bar{\gamma}_2(u_1(\cdot)) = -\widetilde{N}_{22}^{-1}\Big[(\widetilde{B}_2)'\bar{x}(\cdot) + (\widetilde{N}_{12})'u_1(\cdot)\Big].$$

Remark 6. *Taking Definition 1, for example, the optimal feedback control-strategy operator $(\psi_1(\cdot), \Psi_2(\cdot))$ is required to be independent of the initial state $\theta \in L^2_{\mathcal{F}_s}(\Omega;\mathbb{R}^n)$. For a fixed $(s,\theta) \in [0,T) \times L^2_{\mathcal{F}_s}(\Omega;\mathbb{R}^n)$, Equations (18) and (19) imply that the control-strategy pair*

$$\begin{cases} \bar{u}_1(\cdot) = \psi_1(\cdot)\bar{x}(\cdot) = -\Big[\widetilde{N}_{11} - \widetilde{N}_{12}\widetilde{N}_{22}^{-1}(\widetilde{N}_{12})'\Big]^{-1}\Big[\widetilde{B}_1 - \widetilde{B}_2\widetilde{N}_{22}^{-1}(\widetilde{N}_{12})'\Big]'\bar{x}(\cdot) \\ \bar{\gamma}_2(\bar{u}_1(\cdot)) = \Psi_2(\psi_1(\cdot))\bar{x}(\cdot) = -\widetilde{N}_{22}^{-1}\Big[(\widetilde{B}_2)' - (\widetilde{N}_{12})'\psi_1\Big]\bar{x}(\cdot) \end{cases}$$

*is optimal for Player 1 for **Problem (SDG)**. Therefore, the existence of a robust optimal feedback operator implies the existence of an optimal control-strategy pair $(\bar{u}_1(\cdot), \bar{\gamma}_2(\bar{u}_1(\cdot)))$ for Player 1. Player 2's situation can be stated similarly.*

3. Main Results

In this section, we state the main results of this paper, inspired by the idea from [12,16,22].

Theorem 1. *Suppose that the assumptions (AS1)–(AS2) hold. Then, **Problem (SDG)** admits a robust optimal feedback control-strategy operator $(\psi_1(\cdot), \Psi_2(\cdot))$ for Player 1 if and only if:*

(i) *The Equation (9) admits a $\mathcal{F}_t$-adapted solution, for $p \in [1,\infty)$,*

$$(P(\cdot), Q(\cdot)) \in L^\infty_{\mathcal{F}_t}(0,T;\mathcal{S}(\mathbb{R}^n)) \times L^p_{\mathcal{F}_t}(\Omega;L^2(0,T;(\mathcal{S}(\mathbb{R}^n))^d))$$

;

(ii) *The matrices $\widetilde{N}_{22}$ are positive, and $\widetilde{N}_{11} - \widetilde{N}_{12}\widetilde{N}_{22}^{-1}(\widetilde{N}_{12})'$ are negative—we denote this condition Condition (I);*

(iii) *The feedback operator of Player 1 can be written as:*

$$\psi_1 = -\Big[\widetilde{N}_{11} - \widetilde{N}_{12}\widetilde{N}_{22}^{-1}(\widetilde{N}_{12})'\Big]^{-1}\Big[\widetilde{B}_1 - \widetilde{B}_2\widetilde{N}_{22}^{-1}(\widetilde{N}_{12})'\Big]' \in L^\infty_{\mathcal{F}_t}(\Omega;L^2(0,T;(\mathbb{R}^{m_1\times n}))),$$
$$\Psi_2(\psi_1) = -\widetilde{N}_{22}^{-1}\Big[(\widetilde{B}_2)' - (\widetilde{N}_{12})'\psi_1\Big] \in L^\infty_{\mathcal{F}_t}(\Omega;L^2(0,T;(\mathbb{R}^{m_2\times n})));\tag{24}$$

(iv) *The value of Player 1 is:*

$$\inf_{\gamma_2\in\mathcal{A}_2[s,T]}\sup_{u_1\in\mathcal{U}_1[s,T]} J_\theta(u_1,\gamma_2(u_1)) = \frac{1}{2}E\langle P(s)\theta,\theta\rangle. \tag{25}$$

Proof. "$\Leftarrow$" Suppose the Equation (9) admits a unique solution,

$$(P(\cdot), Q(\cdot)) \in L^{\infty}_{\mathcal{F}_t}(0, T; \mathcal{S}(\mathbb{R}^n)) \times L^{p}_{\mathcal{F}_t}(\Omega; L^2(0, T; (\mathcal{S}(\mathbb{R}^n))^d)), \ \ p \in [1, \infty),$$

such that Conditions (ii) and (iii) hold. Then, the function pair $(\psi_1(\cdot), \Psi_2(\cdot))$ is a robust feedback operator. We only need to show it is an optimal one.

Let $u_1(\cdot) \in \mathcal{U}_1[s, T]$, $u_2(\cdot) = \gamma_2(u_1(\cdot)) \in \mathcal{A}_2[s, T]$ be any given admissible control for Player 1 and strategy for Player 2, respectively, and also suppose $x(\cdot) \in L^2_{\mathcal{F}_t}(\Omega; C(s, T; \mathbb{R}^n))$ is the corresponding state trajectory of state Equation (1). By using state Equation (1) and Equation (16), and applying the Itô formula to $Px(t)$, we obtain:

$$\begin{aligned} d(Px(t)) =& \Big\{ - A'Px(t) - Rx(t) - \sum_{i=1}^{d}[(C_i)'PC_i + Q_iC_i]x(t) + PB_1u_1(t) + PB_2u_2(t) \\ &+ \tilde{B}_2\tilde{N}_{22}^{-1}(\tilde{B}_2)'x(t) + \sum_{i=1}^{d}[Q_iD_{1i}u_1(t) + Q_iD_{2i}u_2(t)] \\ &+ \Big[\widetilde{B}_1 - \widetilde{B}_2\widetilde{N}_{22}^{-1}(\widetilde{N}_{12})'\Big]\Big[\widetilde{N}_{11} - \widetilde{N}_{12}\widetilde{N}_{22}^{-1}(\widetilde{N}_{12})'\Big]^{-1}\Big[\widetilde{B}_1 - \widetilde{B}_2\widetilde{N}_{22}^{-1}(\widetilde{N}_{12})'\Big]'x(t)\Big\}dt \\ &+ \sum_{i=1}^{d}[P(C_ix(t) + D_{1i}u_1(t) + D_{2i}u_2(t)) + Q_ix(t)]dW_i(t). \end{aligned}$$

Further, applying the Itô formula to $\langle Px(t), x(t)\rangle$, and noticing the Equation (11) for simple computing, we obtain:

$$\begin{aligned} & d\langle Px(t), x(t)\rangle \\ =& \Big\{ - \langle Rx(t), x(t)\rangle + 2\langle\tilde{B}_1u_1(t), x(t)\rangle + 2\langle\tilde{B}_2u_2(t), x(t)\rangle + \langle\tilde{B}_2\tilde{N}_{22}^{-1}(\tilde{B}_2)'x(t), x(t)\rangle \\ &+ \sum_{i=1}^{d}\langle(D_{1i})'PD_{1i}u_1(t), u_1(t)\rangle + \sum_{i=1}^{d}\langle(D_{2i})'PD_{2i}u_2(t), u_2(t)\rangle + \langle\tilde{N}_{12}u_2(t), u_1(t)\rangle \\ &+ \Big\langle\Big[\widetilde{B}_1 - \widetilde{B}_2\widetilde{N}_{22}^{-1}(\widetilde{N}_{12})'\Big]\Big[\widetilde{N}_{11} - \widetilde{N}_{12}\widetilde{N}_{22}^{-1}(\widetilde{N}_{12})'\Big]^{-1}\Big[\widetilde{B}_1 - \widetilde{B}_2\widetilde{N}_{22}^{-1}(\widetilde{N}_{12})'\Big]'x(t), x(t)\Big\rangle\Big\}dt \\ &+ \sum_{i=1}^{d}[2\langle P(C_ix(t) + D_{1i}u_1(t) + D_{2i}u_2(t)), x(t)\rangle + \langle Q_ix(t), x(t)\rangle]dW_i(t). \end{aligned} \tag{26}$$

From Equation (24), we have:

$$\psi_1(\cdot) = -\Big[\widetilde{N}_{11} - \widetilde{N}_{12}\widetilde{N}_{22}^{-1}(\widetilde{N}_{12})'\Big]^{-1}\Big[\widetilde{B}_1 - \widetilde{B}_2\widetilde{N}_{22}^{-1}(\widetilde{N}_{12})'\Big]',$$

together with the symmetry of $[\widetilde{N}_{11} - \widetilde{N}_{12}\widetilde{N}_{22}^{-1}(\widetilde{N}_{12})']$, which implies that:

$$\Big[\widetilde{B}_1 - \widetilde{B}_2\widetilde{N}_{22}^{-1}(\widetilde{N}_{12})'\Big] = -(\psi_1)'\Big[\widetilde{N}_{11} - \widetilde{N}_{12}\widetilde{N}_{22}^{-1}(\widetilde{N}_{12})'\Big].$$

Since $\psi_1(\cdot) \in L^\infty_{\mathcal{F}_t}(\Omega; L^2(0,T;(\mathbb{R}^{m_1\times n}))$ and $[\widetilde{N}_{11} - \widetilde{N}_{12}\widetilde{N}_{22}^{-1}(\widetilde{N}_{12})']$ are bounded, we have $[\widetilde{B}_1 - \widetilde{B}_2\widetilde{N}_{22}^{-1}(\widetilde{N}_{12})'] \in L^\infty_{\mathcal{F}_t}(\Omega; L^2(0,T;(\mathbb{R}^{m_1\times n}))$. Moreover, from Equation (24), we derive that:

$$\begin{aligned}
&\Big[\widetilde{B}_1 - \widetilde{B}_2\widetilde{N}_{22}^{-1}(\widetilde{N}_{12})'\Big]\Big[\widetilde{N}_{11} - \widetilde{N}_{12}\widetilde{N}_{22}^{-1}(\widetilde{N}_{12})'\Big]^{-1}\Big[\widetilde{B}_1 - \widetilde{B}_2\widetilde{N}_{22}^{-1}(\widetilde{N}_{12})'\Big]'\\
&= -(\psi_1)'\Big[\widetilde{N}_{11} - \widetilde{N}_{12}\widetilde{N}_{22}^{-1}(\widetilde{N}_{12})'\Big]\Big[\widetilde{N}_{11} - \widetilde{N}_{12}\widetilde{N}_{22}^{-1}(\widetilde{N}_{12})'\Big]^{-1}\Big[\widetilde{B}_1 - \widetilde{B}_2\widetilde{N}_{22}^{-1}(\widetilde{N}_{12})'\Big]'\\
&= (\psi_1)'\Big[\widetilde{N}_{11} - \widetilde{N}_{12}\widetilde{N}_{22}^{-1}(\widetilde{N}_{12})'\Big]\psi_1.
\end{aligned}$$

Therefore, we rewrite Equation (26) as:

$$\begin{aligned}
&d\langle Px(t), x(t)\rangle\\
=&\Big\{ -\langle Rx(t), x(t)\rangle + 2\langle \tilde{B}_1 u_1(t), x(t)\rangle + 2\langle \tilde{B}_2 u_2(t), x(t)\rangle + \langle \tilde{B}_2\tilde{N}_{22}^{-1}(\tilde{B}_2)'x(t), x(t)\rangle\\
&+\sum_{i=1}^d \langle (D_{1i})'PD_{1i}u_1(t), u_1(t)\rangle + \sum_{i=1}^d \langle (D_{2i})'PD_{2i}u_2(t), u_2(t)\rangle + \langle \tilde{N}_{12}u_2(t), u_1(t)\rangle\\
&+\Big\langle (\psi_1)'\Big[\widetilde{N}_{11} - \widetilde{N}_{12}\widetilde{N}_{22}^{-1}(\widetilde{N}_{12})'\Big]\psi_1 x(t), x(t)\Big\rangle\Big\}dt\\
&+\sum_{i=1}^d [2\langle P(C_i x(t) + D_{1i}u_1(t) + D_{2i}u_2(t)), x(t)\rangle + \langle Q_i x(t), x(t)\rangle]dW_i(t).
\end{aligned} \tag{27}$$

In order to ensure that the stochastic integral above makes sense, we introduce the following stopping times τ_k as, for any $s \in [0,T)$,

$$\tau_k := \inf\left\{ r \geq s \,|\, \sum_{i=1}^d \int_s^r |Q_i(t)|^2 dt \geq k \right\} \wedge T.$$

Obviously, when $k \to \infty$, then $\tau_k \to T$ *a.s.*. Integrating from s to T and having an expectation on both sides of Equation (27), we have:

$$\begin{aligned}
&E\langle P(\tau_k)x(\tau_k), x(\tau_k)\rangle - \langle P(s)\theta, \theta\rangle\\
=&E\int_s^T \mathcal{X}[s,\tau_k]\Big\{ -\langle Rx(t), x(t)\rangle + 2\langle \tilde{B}_1 u_1(t), x(t)\rangle + 2\langle \tilde{B}_2 u_2(t), x(t)\rangle\\
&+\langle \tilde{B}_2\tilde{N}_{22}^{-1}(\tilde{B}_2)'x(t), x(t)\rangle + \sum_{i=1}^d \langle (D_{1i})'PD_{1i}u_1(t), u_1(t)\rangle + \sum_{i=1}^d \langle (D_{2i})'PD_{2i}u_2(t), u_2(t)\rangle\\
&+\langle \tilde{N}_{12}u_2(t), u_1(t)\rangle + \Big\langle (\psi_1)'\Big[\widetilde{N}_{11} - \widetilde{N}_{12}\widetilde{N}_{22}^{-1}(\widetilde{N}_{12})'\Big]\psi_1 x(t), x(t)\Big\rangle\Big\}dt.
\end{aligned} \tag{28}$$

Clearly,

$$|\langle P(\tau_k)x(\tau_k), x(\tau_k)\rangle| \leq |P|_{L^\infty_{\mathcal{F}_t}(0,T;\mathcal{S}(\mathbb{R}^n))}|x|^2_{L^2_{\mathcal{F}_t}(\Omega;C(0,T;\mathbb{R}^n))}$$

by virtue of the Dominated Convergence Theorem, we have:

$$\lim_{k\to\infty}\langle P(\tau_k)x(\tau_k), x(\tau_k)\rangle = \langle P(T)x(T), x(T)\rangle = \langle Gx(T), x(T)\rangle.$$

Similarly, since all the terms in the integral can be dominated by some bounded functions, we can prove the right hand side of Equation (28):

$$
\begin{aligned}
&\lim_{k\to\infty} E\int_s^T \mathcal{X}[s,\tau_k]\Big\{-\langle Rx(t),x(t)\rangle+2\langle\tilde{B}_1u_1(t),x(t)\rangle+2\langle\tilde{B}_2u_2(t),x(t)\rangle\\
&+\langle\tilde{B}_2\tilde{N}_{22}^{-1}(\tilde{B}_2)'x(t),x(t)\rangle+\sum_{i=1}^{d}\langle(D_{1i})'PD_{1i}u_1(t),u_1(t)\rangle+\sum_{i=1}^{d}\langle(D_{2i})'PD_{2i}u_2(t),u_2(t)\rangle\\
&+\langle\tilde{N}_{12}u_2(t),u_1(t)\rangle+\Big\langle(\psi_1)'\Big[\widetilde{N}_{11}-\widetilde{N}_{12}\widetilde{N}_{22}^{-1}(\widetilde{N}_{12})'\Big]\psi_1x(t),x(t)\Big\rangle\Big\}dt\\
=&E\int_s^T\Big\{-\langle Rx(t),x(t)\rangle+2\langle\tilde{B}_1u_1(t),x(t)\rangle+2\langle\tilde{B}_2u_2(t),x(t)\rangle+\langle\tilde{B}_2\tilde{N}_{22}^{-1}(\tilde{B}_2)'x(t),x(t)\rangle\\
&+\sum_{i=1}^{d}\langle(D_{1i})'PD_{1i}u_1(t),u_1(t)\rangle+\sum_{i=1}^{d}\langle(D_{2i})'PD_{2i}u_2(t),u_2(t)\rangle+\langle\tilde{N}_{12}u_2(t),u_1(t)\rangle\\
&+\Big\langle(\psi_1)'\Big[\widetilde{N}_{11}-\widetilde{N}_{12}\widetilde{N}_{22}^{-1}(\widetilde{N}_{12})'\Big]\psi_1x(t),x(t)\Big\rangle\Big\}dt.
\end{aligned}
\tag{29}
$$

Then, letting $k\to\infty$, and adding the same term:

$$
E\int_s^T[\langle Rx(t),x(t)\rangle+\langle N_1u_1(t),u_1(t)\rangle+\langle N_2u_2(t),u_2(t)\rangle]dt
$$

on both sides of Equation (28), and by the Equation (11), it follows that:

$$
\begin{aligned}
&2J_\theta(u_1,u_2)-\langle P(s)\theta,\theta\rangle\\
=&E\int_s^T\Big\{-\langle Rx(t),x(t)\rangle+2\langle\tilde{B}_1u_1(t),x(t)\rangle+2\langle\tilde{B}_2u_2(t),x(t)\rangle\\
&+\langle\tilde{B}_2\tilde{N}_{22}^{-1}(\tilde{B}_2)'x(t),x(t)\rangle+\sum_{i=1}^{d}\langle(D_{1i})'PD_{1i}u_1(t),u_1(t)\rangle+\sum_{i=1}^{d}\langle(D_{2i})'PD_{2i}u_2(t),u_2(t)\rangle\\
&+2\langle\tilde{N}_{12}u_2(t),u_1(t)\rangle+\Big\langle(\psi_1)'\Big[\widetilde{N}_{11}-\widetilde{N}_{12}\widetilde{N}_{22}^{-1}(\widetilde{N}_{12})'\Big]\psi_1x(t),x(t)\Big\rangle\Big\}dt\\
&+E\int_s^T[\langle Rx(t),x(t)\rangle+\langle N_1u_1(t),u_1(t)\rangle+\langle N_2u_2(t),u_2(t)\rangle]dt\\
=&E\int_s^T\Big\{2\langle\tilde{B}_1u_1(t),x(t)\rangle+2\langle\tilde{B}_2u_2(t),x(t)\rangle+\langle\tilde{B}_2\tilde{N}_{22}^{-1}(\tilde{B}_2)'x(t),x(t)\rangle\\
&+\langle\tilde{N}_{11}u_1(t),u_1(t)\rangle+\langle\tilde{N}_{22}u_2(t),u_2(t)\rangle+2\langle\tilde{N}_{12}u_2(t),u_1(t)\rangle\\
&+\Big\langle(\psi_1)'\Big[\widetilde{N}_{11}-\widetilde{N}_{12}\widetilde{N}_{22}^{-1}(\widetilde{N}_{12})'\Big]\psi_1x(t),x(t)\Big\rangle\Big\}dt,
\end{aligned}
$$

from the notation $\bar{\gamma}_2(u_1)$ in Remark 5, using the completion of a square for u_2, we get:

$$
\begin{aligned}
&2J_\theta(u_1,u_2)-\langle P(s)\theta,\theta\rangle\\
=&E\int_s^T\Big\{\langle\tilde{N}_{22}u_2(t),u_2(t)\rangle-2\langle\tilde{N}_{22}u_2(t),\bar{\gamma}_2(u_1(t))\rangle+\langle\tilde{N}_{22}\bar{\gamma}_2(u_1(t)),\bar{\gamma}_2(u_1(t))\rangle\\
&+2\langle\tilde{B}_1u_1(t),x(t)\rangle+\langle\tilde{B}_2\tilde{N}_{22}^{-1}(\tilde{B}_2)'x(t),x(t)\rangle+\langle\tilde{N}_{11}u_1(t),u_1(t)\rangle-\langle\tilde{N}_{22}\bar{\gamma}_2(u_1(t)),\bar{\gamma}_2(u_1(t))\rangle\\
&+\Big\langle(\psi_1)'\Big[\widetilde{N}_{11}-\widetilde{N}_{12}\widetilde{N}_{22}^{-1}(\widetilde{N}_{12})'\Big]\psi_1x(t),x(t)\Big\rangle\Big\}dt\\
=&E\int_s^T\Big\{\Big\langle\tilde{N}_{22}(u_2(t)-\bar{\gamma}_2(u_1(t))),(u_2(t)-\bar{\gamma}_2(u_1(t)))\Big\rangle+\Big\langle\Big[\widetilde{N}_{11}-\widetilde{N}_{12}\widetilde{N}_{22}^{-1}(\widetilde{N}_{12})'\Big]u_1(t),u_1(t)\Big\rangle\\
&+2\Big\langle\Big[\widetilde{B}_1-\widetilde{B}_2\widetilde{N}_{22}^{-1}(\widetilde{N}_{12})'\Big]u_1(t),x(t)\Big\rangle+\Big\langle(\psi_1)'\Big[\widetilde{N}_{11}-\widetilde{N}_{12}\widetilde{N}_{22}^{-1}(\widetilde{N}_{12})'\Big]\psi_1x(t),x(t)\Big\rangle\Big\}dt,
\end{aligned}
$$

noticing that:

$$
\Big[\widetilde{B}_1-\widetilde{B}_2\widetilde{N}_{22}^{-1}(\widetilde{N}_{12})'\Big]=-(\psi_1)'\Big[\widetilde{N}_{11}-\widetilde{N}_{12}\widetilde{N}_{22}^{-1}(\widetilde{N}_{12})'\Big],
$$

and using the completion square for u_1, we have:

$$2J_\theta(u_1,\gamma_2(u_1)) = 2J_\theta(\psi_1\bar{x},\Psi_2(\psi_1)\bar{x}) + E\int_s^T \Big\{\Big\langle \tilde{N}_{22}(u_2(t)-\bar{\gamma}_2(u_1(t))),(u_2-\bar{\gamma}_2(u_1(t)))\Big\rangle + \Big\langle\Big[\tilde{N}_{11}-\tilde{N}_{12}\tilde{N}_{22}^{-1}(\tilde{N}_{12})'\Big](u_1(t)-\psi_1x(t)),(u_1(t)-\psi_1x(t))\Big\rangle\Big\}dt. \quad (30)$$

From Condition (ii), we obtain

$$J_\theta(u_1,\bar{\gamma}_2(u_1)) \le J_\theta(u_1,\gamma_2(u_1)),\ \ \forall u_1\in\mathcal{U}_1[s,T],\ u_2=\gamma_2(u_1)\in\mathcal{A}_2[s,T],$$

and

$$J_\theta(\psi_1\bar{x},\Psi_2(\psi_1)\bar{x}) \ge J_\theta(u_1,\bar{\gamma}_2(u_1)),\ \ \forall u_1\in\mathcal{U}_1[s,T],$$

which means that $(\psi_1(\cdot),\Psi_2(\cdot))$ is an optimal operator for Player 1 by Definition 1.

"$\Rightarrow$" In this part, we prove the necessity in Theorem 1 and we divide the proof into three steps.

Step 1. We provide some existence and uniqueness results of SDEs. Suppose that $(\psi_1(\cdot),\Psi_2(\cdot))$ is a robust optimal feedback control-strategy operator for Player 1. Then, from Lemmas 2–4, for any $\mu\in\mathbb{R}^n$, there is the following forward–backward SDE:

$$\begin{cases} dx(t) = [A+B_1\psi_1+B_2\Psi_2(\psi_1)]x(t)dt+\sum_{i=1}^d[C_i+D_{1i}\psi_1+D_{2i}\Psi_2(\psi_1)]x(t)dW_i(t),\\ dy(t) = -\Big[A'y(t)+\sum_{i=1}^d(C_i)'z_i(t)+Rx(t)\Big]dt+\sum_{i=1}^d z_i(t)dW_i(t),\ \ t\in[0,T],\\ x(0)=\mu,\ \ y(T)=Gx(T), \end{cases} \quad (31)$$

which admits a unique solution:

$$(x(t),y(t),z(t))\in L^2_{\mathcal{F}_t}(\Omega;C(0,T;\mathbb{R}^n))\times L^2_{\mathcal{F}_t}(\Omega;C(0,T;\mathbb{R}^n))\times L^2_{\mathcal{F}_t}(0,T;\mathbb{R}^{n\times d})$$

such that:

$$\begin{cases} (B_1)'y(t)+\sum_{i=1}^d(D_{1i})'z_i(t)+N_1\psi_1x(t)=0,\\ (B_2)'y(t)+\sum_{i=1}^d(D_{2i})'z_i(t)+N_2\Psi_2(\psi_1)x(t)=0. \end{cases} \quad (32)$$

Additionally, we introduce another SDE:

$$\begin{cases} d\tilde{x}(t) = [-A-B_1\psi_1-B_2\Psi_2(\psi_1)+\sum_{i=1}^d(C_i+D_{1i}\psi_1+D_{2i}\Psi_2(\psi_1))^2]'\tilde{x}(t)dt\\ \qquad -\sum_{i=1}^d[C_i+D_{1i}\psi_1+D_{2i}\Psi_2(\psi_1)]'\tilde{x}(t)dW_i(t),\ \ t\in[0,T],\\ x(0)=\mu. \end{cases} \quad (33)$$

From Lemma 3, Equation (33) admits a unique solution $\tilde{x}(t)\in L^2_{\mathcal{F}_t}(\Omega;C(0,T;\mathbb{R}^n))$.

Moreover, consider the following $\mathbb{R}^{n\times n}$-valued SDEs:

$$\begin{cases} dX(t) = [A+B_1\psi_1+B_2\Psi_2(\psi_1)]X(t)dt+\sum_{i=1}^d[C_i+D_{1i}\psi_1+D_{2i}\Psi_2(\psi_1)]X(t)dW_i(t),\\ dY(t) = -\Big[A'Y(t)+\sum_{i=1}^d(C_i)'Z_i(t)+RX(t)\Big]dt+\sum_{i=1}^d Z_i(t)dW_i(t),\ \ t\in[0,T],\\ X(0)=I_n,\ \ Y(T)=GX(T), \end{cases} \quad (34)$$

and,

$$\begin{cases} d\tilde{X}(t) =[-A - B_1\psi_1 - B_2\Psi_2(\psi_1) + \sum_{i=1}^{d}(C_i + D_{1i}\psi_1 + D_{2i}\Psi_2(\psi_1))^2]'\tilde{X}(t)dt \\ \qquad\qquad - \sum_{i=1}^{d}[C_i + D_{1i}\psi_1 + D_{2i}\Psi_2(\psi_1)]'\tilde{X}(t)dW_i(t), \ \ t \in [0, T], \\ \tilde{X}(0) =\mu. \end{cases} \tag{35}$$

Equations (34) and (35) above also admit unique solutions $\tilde{X}(t) \in L^2_{\mathcal{F}_t}(\Omega; C(0, T; \mathbb{R}^{n\times n}))$ and $(X(t), Y(t), Z(t)) \in L^2_{\mathcal{F}_t}(\Omega;\ C(0, T;\ \mathbb{R}^{n\times n})) \times L^2_{\mathcal{F}_t}(\Omega;\ C(0, T;\ \mathbb{R}^{n\times n})) \times L^2_{\mathcal{F}_t}(0,\ T;\ (\mathbb{R}^{n\times n})^d)$. Obviously, from Equations (32) and (37), we obtain:

$$\begin{cases} (B_1)'Y(t) + \sum_{i=1}^{d}(D_{1i})'Z_i(t) + N_1\psi_1 X(t) = 0, \\ (B_2)'Y(t) + \sum_{i=1}^{d}(D_{2i})'Z_i(t) + N_2\Psi_2(\psi_1)X(t) = 0. \end{cases} \tag{36}$$

Also from Equations (31) to (35), for any $\mu \in \mathbb{R}^n$, it easily follows that:

$$x(t; \mu) = X(t)\mu, \ \ y(t; \mu) = Y(t)\mu, \ \ z(t; \mu) = Z(t)\mu, \ \ \tilde{x}(t; \mu) = \tilde{X}(t)\mu, \ \ a.e. \ \ t \in [0, T], \tag{37}$$

where $Z(\cdot) = (Z_1(\cdot), \dots, Z_d(\cdot))$ and $z(\cdot) = (z_1(\cdot), \dots, z_d(\cdot))$. Further, for any $\mu, \nu \in \mathbb{R}^n$ and $t \in [0, T]$, applying the Itô formula to $\langle x(t; \mu), \tilde{x}(t; \nu)\rangle$, we have:

$$\begin{aligned} &\langle x(t; \mu), \tilde{x}(t; \nu)\rangle - \langle \mu, \nu\rangle \\ =& \int_0^t \langle [A + B_1\psi_1 + B_2\Psi_2(\psi_1)]x(r; \mu), \tilde{x}(r; \nu)\rangle dr \\ &+ \int_0^t \left\langle \sum_{i=1}^{d}[C_i + D_{1i}\psi_1 + D_{2i}\Psi_2(\psi_1)]x(r; \mu), \tilde{x}(r; \nu) \right\rangle dW_i(r) \\ &+ \int_0^t \left\langle x(r; \mu), [-A - B_1\psi_1 - B_2\Psi_2(\psi_1) + \sum_{i=1}^{d}(C_i + D_{1i}\psi_1 + D_{2i}\Psi_2(\psi_1))^2]'\tilde{x}(r; \nu) \right\rangle dr \\ &- \int_0^t \left\langle x(r; \mu), \sum_{i=1}^{d}[C_i + D_{1i}\psi_1 + D_{2i}\Psi_2(\psi_1)]'\tilde{x}(r; \nu) \right\rangle dW_i(r) \\ &- \int_0^t \left\langle \sum_{i=1}^{d}[C_i + D_{1i}\psi_1 + D_{2i}\Psi_2(\psi_1)]^2 x(r; \mu), \tilde{x}(r; \nu) \right\rangle dr = 0. \end{aligned}$$

Therefore, from Equation (37), it follows that:

$$\langle x(t; \mu), \tilde{x}(t; \nu)\rangle = \langle X(t)\mu, \tilde{X}(t)\nu\rangle = \langle \mu, \nu\rangle, \ \ a.s.,$$

which implies that $X(t)[\tilde{X}(t)]' = I_n$, i.e., $[\tilde{X}(t)]' = X(t)^{-1}$.

Step 2. We try to prove that Equation (9) admits a solution through the idea of making the connection between the solvable SDG and the solvability of the Riccati equations. We assume that:

$$P(t) := Y(t)[\tilde{X}(t)]', \ \ \Upsilon_i(t) = Z_i(t)[\tilde{X}(t)]'. \tag{38}$$

Applying the Itô formula to $P(t)$, we have:

$$
\begin{aligned}
dP =& \Big\{ - \Big[A'Y + \sum_{i=1}^{d}(C_i)'Z_i + RX\Big]X^{-1} - \sum_{i=1}^{d} Z_iX^{-1}[C_i + D_{1i}\psi_1 + D_{2i}\Psi_2(\psi_1)] \\
&+ YX^{-1}\Big[- A - B_1\psi_1 - B_2\Psi_2(\psi_1) + \sum_{i=1}^{d}(C_i + D_{1i}\psi_1 + D_{2i}\Psi_2(\psi_1))^2\Big]\Big\}dt \\
&+ \Big[\sum_{i=1}^{d} Z_iX^{-1} - \sum_{i=1}^{d} YX^{-1}[C_i + D_{1i}\psi_1 + D_{2i}\Psi_2(\psi_1)]\Big]dW_i(t) \\
=& \Big\{ - A'P - \sum_{i=1}^{d}(C_i)'\Upsilon_i - R - \sum_{i=1}^{d}\Upsilon_i[C_i + D_{1i}\psi_1 + D_{2i}\Psi_2(\psi_1)] \\
&+ P\Big[- A - B_1\psi_1 - B_2\Psi_2(\psi_1) + \sum_{i=1}^{d}(C_i + D_{1i}\psi_1 + D_{2i}\Psi_2(\psi_1))^2\Big]\Big\}dt \\
&+ \sum_{i=1}^{d}\Big[\Upsilon_i - P[C_i + D_{1i}\psi_1 + D_{2i}\Psi_2(\psi_1)]\Big]dW_i(t).
\end{aligned}
$$

Put $Q_i := \Upsilon_i - P[C_i + D_{1i}\psi_1 + D_{2i}\Psi_2(\psi_1)]$, then $(P(\cdot), Q(\cdot))$ solves the following SDE:

$$
\begin{cases}
dP = - \Big\{A'P + PA + \sum_{i=1}^{d}(C_i)'PC_i + \sum_{i=1}^{d}[(C_i)'Q_i + Q_iC_i] \\
\qquad + \tilde{B}_1\psi_1 + \tilde{B}_2\Psi_2(\psi_1) + R\Big\}dt + \sum_{i=1}^{d} Q_i dW_i \\
P(T) = G. \ \ t \in [0, T].
\end{cases} \tag{39}
$$

From Lemma 3, we deduce that $(P, Q) \in L^{\infty}_{\mathcal{F}_t}(\Omega; C(0, T; \mathbb{R}^{n\times n})) \times L^{p}_{\mathcal{F}_t}(\Omega; L^2(0, T; (\mathbb{R}^{n\times n})^d))$.

Next, we show that (P, Q) are symmetric. We introduce the following forward–backward SDEs, for any $t \in [0, T)$ and $\alpha \in L^2_{\mathcal{F}_t}(\Omega; \mathbb{R}^n)$,

$$
\begin{cases}
dx^t(r) = [A + B_1\psi_1 + B_2\Psi_2(\psi_1)]x^t(r)dr + \sum_{i=1}^{d}[C_i + D_{1i}\psi_1 + D_{2i}\Psi_2(\psi_1)]x^t(r)dW_i(r), \\
dy^t(r) = - \Big[A'y^t(r) + \sum_{i=1}^{d}(C_i)'z^t_i(r) + Rx^t(r)\Big]dr + \sum_{i=1}^{d} z^t_i(r)dW_i(r), \ \ r \in [t, T], \\
x^t(t) = \alpha, \ \ y^t(T) = Gx^t(T),
\end{cases} \tag{40}
$$

and,

$$
\begin{cases}
dX^t(r) = [A + B_1\psi_1 + B_2\Psi_2(\psi_1)]X^t dr + \sum_{i=1}^{d}[C_i + D_{1i}\psi_1 + D_{2i}\Psi_2(\psi_1)]X^t(r)dW_i(r), \\
dY^t(r) = - \Big[A'Y^t(r) + \sum_{i=1}^{d}(C_i)'Z^t_i(r) + RX^t(r)\Big]dr + \sum_{i=1}^{d} Z^t_i(r)dW_i(r), \ \ r \in [t, T], \\
X^t(t) = I_n, \ \ Y^t(T) = GX^t(T).
\end{cases} \tag{41}
$$

Clearly, the above two SDEs admit unique solutions:

$$
(x^t, y^t, z^t) \in L^2_{\mathcal{F}_t}(\Omega; C(t, T; \mathbb{R}^n)) \times L^2_{\mathcal{F}_t}(\Omega; C(t, T; \mathbb{R}^n)) \times L^2_{\mathcal{F}_t}(t, T; \mathbb{R}^{n\times d})
$$

and,

$$
(X^t, Y^t, Z^t) \in L^2_{\mathcal{F}_t}(\Omega; C(t, T; \mathbb{R}^{n\times n})) \times L^2_{\mathcal{F}_t}(\Omega; C(t, T; \mathbb{R}^{n\times n})) \times L^2_{\mathcal{F}_t}(t, T; (\mathbb{R}^{n\times n})^d).
$$

Obviously, from Equations (40) and (41), we have:

$$x^t(r;\alpha) = X^t(r)\alpha,\;\; y^t(r;\alpha) = Y^t(r)\alpha,\;\; z^t(r;\alpha) = Z^t(r)\alpha,\;\; a.e.\;\; r \in [t,T]. \tag{42}$$

Due to the uniqueness of the solution to Equation (31), for any $\mu \in \mathbb{R}^n$ and $t \in [0,T]$, it follows that:

$$x^t(r;X(t)\mu) = X^t(r)X(t)\mu = x(r;\mu),\;\; a.s.$$

therefore,

$$y^t(t;X(t)\mu) = Y^t(t)X(t)\mu = Y(t)\mu,\;\; a.s.$$

which implies that:

$$Y^t(t) = Y(t)[\tilde{X}(t)]' = P(t).\;\; a.s.$$

Further, let $\alpha,\beta \in L^2_{\mathcal{F}_t}(\Omega;\mathbb{R}^n)$ and apply the Itô formula to $\langle x^t(r;\alpha), y^t(r;\beta)\rangle$, noticing $x^t(r;\alpha) = X^t(r)\alpha, y^t(r;\beta) = Y^t(r)\beta$; then, we have:

$$\begin{aligned} E\langle \alpha, P(t)\beta\rangle =& E\int_t^T \Big\{ \langle X^t(r)\alpha, RX^t(r)\beta\rangle - \langle [B_1\psi_1 + B_2\Psi_2(\psi_1)]X^t(r)\alpha, Y^t(r)\beta\rangle \\ & - \sum_{i=1}^d \langle [D_{1i}\psi_1 + D_{2i}\Psi_2(\psi_1)]X^t(r)\alpha, Z^t_i(r)\beta\rangle \Big\} dr + E\langle X^t(T)\alpha, GX^t(T)\beta\rangle. \end{aligned}$$

Noticing that Equation (36), we obtain:

$$\begin{aligned} E\langle P(t)\beta,\alpha\rangle =& E\int_t^T \Big\{ \langle RX^t(r)\beta, X^t(r)\alpha\rangle + \langle N_1\psi_1 X^t(r)\beta, \psi_1 X^t(r)\alpha\rangle \\ & + \langle N_2\Psi_2(\psi_1)X^t(r)\beta, \Psi_2(\psi_1)X^t(r)\alpha\rangle \Big\} dr + E\langle GX^t(T)\beta, X^t(T)\alpha\rangle \\ =& E\Big\langle [X^t(r)]'GX^t(T)\beta + \int_t^T \Big\{ [X^t(r)]'RX^t(r)\beta + [X^t(r)]'[\psi_1]'N_1\psi_1 X^t(r)\beta \\ & + [X^t(r)]'[\Psi_2(\psi_1)]'N_2\Psi_2(\psi_1)X^t(r)\beta \Big\} dr, \alpha \Big\rangle, \end{aligned}$$

which means:

$$\begin{aligned} P(t) =& E\Big\{ [X^t(r)]'GX^t(T) + \int_t^T \Big\{ [X^t(r)]'RX^t(r) + [X^t(r)]'[\psi_1]'N_1\psi_1 X^t(r) \\ & + [X^t(r)]'[\Psi_2(\psi_1)]'N_2\Psi_2(\psi_1)X^t(r) \Big\} dr \Big| \mathcal{F}_t \Big\}; \end{aligned}$$

therefore, $P(t)$ is symmetric, since $G, R(\cdot), N_1(\cdot), N_2(\cdot)$ are symmetric in **(AS2)**. On the other hand, (P', Q') satisfies that:

$$\begin{cases} dP' = - \Big\{ A'P' + P'A + \sum_{i=1}^d (C_i)'P'C_i + \sum_{i=1}^d [(C_i)'(Q_i)' + (Q_i)'C_i] \\ \qquad + (\psi_1)'(\tilde{B}_1)' + [\Psi_2(\psi_1)]'(\tilde{B}_2)' + R \Big\} dt + \sum_{i=1}^d (Q_i)'dW_i(t), \\ [P(T)]' = G.\;\; t \in [0,T] \end{cases} \tag{43}$$

Let Equation (39) minus Equation (43) and integrate from 0 to t; noticing that $P(t)$ is symmetric, we obtain the diffusion term:

$$\sum_{i=1}^d \int_0^t [Q_i - (Q_i)']dW_i(r) = 0$$

and then we conclude that $Q(t)$ is symmetric due to the uniqueness of the decomposition of semi-martingale.

Next we show that (P, Q) is the solution of Equation (9) in which F is in the Equation (16) and the matrices $\widetilde{N}_{22}$ and $\widetilde{N}_{11} - \widetilde{N}_{12}\widetilde{N}_{22}^{-1}(\widetilde{N}_{12})'$ are invertible. Actually, multiply X^{-1} on both sides of Equation (36); it follows that:

$$\begin{cases} (B_1)'P + \sum_{i=1}^{d}(D_{1i})'\Upsilon_i + N_1\psi_1 = 0, \\ (B_2)'P + \sum_{i=1}^{d}(D_{2i})'\Upsilon_i + N_2\Psi_2(\psi_1) = 0. \end{cases} \tag{44}$$

By the definition of Q and by Equations (11) and (44), we obtain:

$$\begin{cases} (\widetilde{B}_1)' + \widetilde{N}_{12}\Psi_2(\psi_1) + \widetilde{N}_{11}\psi_1 = 0, \\ (\widetilde{B}_2)' + (\widetilde{N}_{12})'\psi_1 + \widetilde{N}_{22}\Psi_2(\psi_1) = 0, \end{cases} \tag{45}$$

from which we have:

$$\begin{cases} \psi_1 = -\left[\widetilde{N}_{11} - \widetilde{N}_{12}\widetilde{N}_{22}^{-1}(\widetilde{N}_{12})'\right]^{-1}\left[\widetilde{B}_1 - \widetilde{B}_2\widetilde{N}_{22}^{-1}(\widetilde{N}_{12})'\right]' \\ \Psi_2(\psi_1) = -\widetilde{N}_{22}^{-1}\left[(\widetilde{B}_2)' - (\widetilde{N}_{12})'\psi_1\right]. \end{cases} \tag{46}$$

Putting these two equalities into Equation (39), for simplicity of computing, we have:

$$\begin{aligned} &\widetilde{B}_1\psi_1 + \widetilde{B}_2\Psi_2(\psi_1) \\ =& -\widetilde{B}_2\widetilde{N}_{22}^{-1}(\widetilde{B}_2)' - \left[\widetilde{B}_1 - \widetilde{B}_2\widetilde{N}_{22}^{-1}(\widetilde{N}_{12})'\right]\left[\widetilde{N}_{11} - \widetilde{N}_{12}\widetilde{N}_{22}^{-1}(\widetilde{N}_{12})'\right]^{-1}\left[\widetilde{B}_1 - \widetilde{B}_2\widetilde{N}_{22}^{-1}(\widetilde{N}_{12})'\right]', \end{aligned}$$

which implies that $(P, Q) \in L^{\infty}_{\mathcal{F}_t}(\Omega; C(0, T; \mathcal{S}(\mathbb{R}^n))) \times L^{p}_{\mathcal{F}_t}(\Omega; L^2(0, T; (\mathcal{S}(\mathbb{R}^n))^d))$ solves the Equation (9) and ends the proof of Condition (i).

Step 3. In this part, we show that Conditions (ii)–(iv) hold. Obviously, from Equation (46) and $(\psi_1(\cdot), \Psi_2(\cdot))$ being robust, it easily follows that Condition (iii) holds. We only need to prove Conditions (ii) and (iv). Actually, from Equation (46), it follows that:

$$\begin{aligned} &\left[\widetilde{B}_1 - \widetilde{B}_2\widetilde{N}_{22}^{-1}(\widetilde{N}_{12})'\right]\left[\widetilde{N}_{11} - \widetilde{N}_{12}\widetilde{N}_{22}^{-1}(\widetilde{N}_{12})'\right]^{-1}\left[\widetilde{B}_1 - \widetilde{B}_2\widetilde{N}_{22}^{-1}(\widetilde{N}_{12})'\right]' \\ &= (\psi_1)'\left[\widetilde{N}_{11} - \widetilde{N}_{12}\widetilde{N}_{22}^{-1}(\widetilde{N}_{12})'\right]\psi_1, \end{aligned}$$

and repeating the procedures to derive (30), we have:

$$\begin{aligned} &2J_\theta(u_1, \gamma_2(u_1)) \\ =&\langle P(s)\theta, \theta\rangle + E\int_s^T \left\{\left\langle \widetilde{N}_{22}(u_2(t) - \bar{\gamma}_2(u_1(t))), (u_2(t) - \bar{\gamma}_2(u_1(t)))\right\rangle\right. \\ &\left. + \left\langle\left[\widetilde{N}_{11} - \widetilde{N}_{12}\widetilde{N}_{22}^{-1}(\widetilde{N}_{12})'\right](u_1(t) - \psi_1 x(t)), (u_1(t) - \psi_1 x(t))\right\rangle\right\}dt. \\ =&2J_\theta(\psi_1\bar{x}, \Psi_2(\psi_1)\bar{x}) + E\int_s^T \left\{\left\langle \widetilde{N}_{22}(u_2(t) - \bar{\gamma}_2(u_1(t))), (u_2(t) - \bar{\gamma}_2(u_1(t)))\right\rangle\right. \\ &\left. + \left\langle\left[\widetilde{N}_{11} - \widetilde{N}_{12}\widetilde{N}_{22}^{-1}(\widetilde{N}_{12})'\right](u_1(t) - \psi_1 x(t)), (u_1(t) - \psi_1 x(t))\right\rangle\right\}dt. \end{aligned}$$

Hence, from the optimality of feedback control-strategy operator $(\psi_1(\cdot), \Psi_2(\cdot))$, we obtain that Condition (iv) holds, and from Definition 1 we have:

$$E\int_s^T \Big\langle \widetilde{N}_{22}(u_2(t)-\bar{\gamma}_2(u_1(t))), (u_2(t)-\bar{\gamma}_2(u_1(t)))\Big\rangle dt \geq 0, \ \ \forall u_2(t)=\gamma_2(u_1(t)) \in \mathcal{A}_2[s,T],$$
$$E\int_s^T \Big\langle \Big[\widetilde{N}_{11}-\widetilde{N}_{12}\widetilde{N}_{22}^{-1}(\widetilde{N}_{12})'\Big](u_1(t)-\psi_1 x(t)), (u_1(t)-\psi_1 x(t))\Big\rangle dt \leq 0, \ \ \forall u_1(t) \in \mathcal{U}_1[s,T],$$

from which we obtain that Condition (ii) holds and we finish the proof of the necessity. □

A similar statement of Player 2 is made in the following sense.

Theorem 2. *Suppose that the Assumptions **(AS1)–(AS2)** hold. Then, **Problem (SDG)** admits a robust optimal feedback control-strategy operator $(\psi_2(\cdot), \Psi_1(\cdot))$ for Player 2 if and only if:*

(i) The Equation (9) admits a $\mathcal{F}_t$-adapted solution for $p \in [1,\infty)$,

$$(P(\cdot), Q(\cdot)) \in L^\infty_{\mathcal{F}_t}(0,T;\mathcal{S}(\mathbb{R}^n)) \times L^p_{\mathcal{F}_t}(\Omega; L^2(0,T;(\mathcal{S}(\mathbb{R}^n))^d))$$

;

(ii) The matrices $\widetilde{N}_{11}$ are negative, $\widetilde{N}_{22}-(\widetilde{N}_{12})'\widetilde{N}_{11}^{-1}\widetilde{N}_{12}$ is positive, we denote this condition by Condition (II).

(iii) The feedback operator of Player 2 can be written as:

$$\begin{aligned}
&\Psi_1(\psi_2) = -\widetilde{N}_{11}^{-1}\Big[(\widetilde{B}_1)' - \widetilde{N}_{12}\psi_2\Big] \in L^\infty_{\mathcal{F}_t}(\Omega; L^2(0,T;(\mathbb{R}^{m_1\times n}))).\\
&\psi_2 = -\Big[\widetilde{N}_{22}-(\widetilde{N}_{12})'\widetilde{N}_{11}^{-1}\widetilde{N}_{12}\Big]^{-1}\Big[\widetilde{B}_2-\widetilde{B}_1\widetilde{N}_{11}^{-1}\widetilde{N}_{12}\Big]' \in L^\infty_{\mathcal{F}_t}(\Omega; L^2(0,T;(\mathbb{R}^{m_2\times n}))).
\end{aligned} \tag{47}$$

(iv) The value of Player 2 is:

$$\sup_{\gamma_1\in\mathcal{A}_1[s,T]} \inf_{u_2\in\mathcal{U}_2[s,T]} J_\theta(\gamma_1(u_2), u_2) = \frac{1}{2}E\langle P(s)\theta, \theta\rangle. \tag{48}$$

Proof. This theorem can be proved with the same approach in Theorem 1, so we omit it here. □

Corollary 1. *Suppose that the Assumptions **(AS1)–(AS2)** hold. Then, **Problem (SDG)** admits robust optimal feedback control-strategy operators $(\psi_1(\cdot), \Psi_2(\cdot))$ and $(\psi_2(\cdot), \Psi_1(\cdot))$ with:*

$$\psi_1(\cdot) = \Psi_1(\psi_2(\cdot)), \ \ \psi_2(\cdot) = \Psi_2(\psi_1(\cdot)). \tag{49}$$

if and only if:

(i) The Equation (9) admits a unique $\mathcal{F}_t$-adapted solution $(P(\cdot), Q(\cdot)) \in L^\infty_{\mathcal{F}_t}(0,T;\mathcal{S}(\mathbb{R}^n)) \times L^p_{\mathcal{F}_t}(\Omega; L^2(0,T;(\mathcal{S}(\mathbb{R}^n))^d))$ (where $p \in [1,\infty)$);

(ii) The matrices $\widetilde{N}_{11}$ is negative, $\widetilde{N}_{22}$ is positive, we denote it by Condition (III);

(iii) $\psi_1(\cdot), \Psi_2(\psi_1(\cdot)), \Psi_1(\psi_2(\cdot)), \psi_2(\cdot)$ satisfy the condition given by Equations (24) and (47);

(iv) Further, the value of Problem (SDG) is:

$$\inf_{\gamma_2\in\mathcal{A}_2[s,T]} \sup_{u_1\in\mathcal{U}_1[s,T]} J_\theta(u_1, \gamma_2(u_1)) = \sup_{\gamma_1\in\mathcal{A}_1[s,T]} \inf_{u_2\in\mathcal{U}_2[s,T]} J_\theta(\gamma_1(u_2), u_2) = \frac{1}{2}E\langle P(s)\theta, \theta\rangle. \tag{50}$$

Proof. "⇒" If (i)–(iv) hold, we can obtain that robust optimal feedback control-strategy operators $(\psi_1(\cdot), \Psi_2(\cdot))$ and $(\psi_2(\cdot), \Psi_1(\cdot))$ exist from Theorems 1 and 2, because Condition (III) in this Corollary holds if and only if Condition (I) and Condition (II) hold, which are the conditions (ii) in Theorems 1 and 2, respectively. Furthermore, from Condition (III) in this Corollary,

$$(\bar{u}_1, \bar{\gamma}_2(\bar{u}_1)) = (\psi_1 \bar{x}, \Psi_2(\psi_1)\bar{x}), \;\; (\bar{\gamma}_1(\bar{u}_2), \bar{u}_2) = (\Psi_1(\psi_2)\bar{x}, \psi_2 \bar{x}) \tag{51}$$

are the unique solution of the algebra Equation (13), which leads to the Equation (49).

"$\Leftarrow$" If Problem (SDG) admits robust optimal feedback control-strategy operators $(\psi_1(\cdot), \Psi_2(\cdot))$ and $(\psi_2(\cdot), \Psi_1(\cdot))$ with:

$$\psi_1(\cdot) = \Psi_1(\psi_2(\cdot)), \;\; \psi_2(\cdot) = \Psi_2(\psi_1(\cdot)).$$

We can easily prove the results (i)–(iv) from Theorems 1 and 2, which ends the proof. □

Remark 7. *Equation (49) can be replaced by the following matrices form if the matrices* $\widetilde{N}_{11}$ *and* $\widetilde{N}_{22}$ *are invertible:*

$$\begin{aligned} (\widetilde{N}_{12}')^{-1}\widetilde{N}_{22}\widetilde{N}_{12}^{-1}(\widetilde{B}_1)' &= \widetilde{N}_{11}^{-1}\Big[I - \widetilde{N}_{12}\widetilde{N}_{22}^{-1}(\widetilde{N}_{12})'\widetilde{N}_{11}^{-1}\Big](\widetilde{B}_1)', \\ \widetilde{N}_{12}^{-1}\widetilde{N}_{11}(\widetilde{N}_{12}')^{-1}(\widetilde{B}_2)' &= \widetilde{N}_{22}^{-1}\Big[I - (\widetilde{N}_{12})'\widetilde{N}_{11}^{-1}\widetilde{N}_{12}\widetilde{N}_{22}^{-1}\Big](\widetilde{B}_2)'. \end{aligned} \tag{52}$$

These two conditions can be obtained by using Equations (24), (47) and (49) with the observation that $\widetilde{N}_{11}$ *and* $\widetilde{N}_{22}$ *are symmetric matrices.*

Remark 8. *Actually, the solution* (P, Q) *of the Equation (9) is also unique, because the Lemma 3 ensures the uniqueness of the solution for the Equation (39), which is another form of Equation (9).*

Remark 9. *In the Corollary 1, if the Equation* (9) *admits a solution* (P, Q) *with the feedback operator having the sharp regularity, i.e., Equation* (24) *and* (47) *hold, then* ***Problem (SDG)*** *admits robust optimal feedback control-strategy operators* $(\psi_1(\cdot), \Psi_2(\cdot))$ *and* $(\psi_2(\cdot), \Psi_1(\cdot))$*, and we can obtain that* ***Problem (SDG)*** *is closed-loop solvable and the saddle point exists in the following sense:*

$$(\bar{u}_1, \bar{u}_2) = (\Psi_1(\psi_2)\bar{x}, \Psi_2(\psi_1)\bar{x})$$

and,

$$\begin{aligned} J_\theta(\bar{u}_1, \bar{u}_2) =& \inf_{\gamma_2 \in \mathcal{A}_2[s,T]} \sup_{u_1 \in \mathcal{U}_1[s,T]} J_\theta(u_1, \gamma_2(u_1)) = \sup_{\gamma_1 \in \mathcal{A}_1[s,T]} \inf_{u_2 \in \mathcal{U}_2[s,T]} J_\theta(\gamma_1(u_2), u_2) \\ =& \frac{1}{2} E\langle P(s)\theta, \theta\rangle = J_\theta(\Psi_1(\psi_2)\bar{x}, \Psi_2(\psi_1)\bar{x}). \end{aligned}$$

However, if the solution (P, Q) *does not have the needed regularity,* ***Problem (SDG)*** *may fail to obtain a robust optimal feedback control-strategy; this phenomenon is different from the deterministic case.*

4. Examples

In this section, we provide two examples to illustrate our theoretical results. Example 1 shows that the solvability of the backward stochastic Riccati equation is not sufficient enough to guarantee that **Problem (SDG)** admits a robust optimal feedback control-strategy. Example 2 shows that there does exist a feedback control-strategy pair under some suitable conditions. These two examples are inspired by [16,22,26,32].

Example 1. *Firstly, we introduce the following stochastic processes* $\Gamma(\cdot), \sigma(\cdot)$ *and stopping time* τ_1*:*

$$\begin{cases} \Gamma(t) := \int_0^t \frac{1}{\sqrt{T-r}} dW(r), \ \ t \in [0,T), \\ \sigma(t) := \frac{\pi}{2\sqrt{2}\sqrt{T-t}} \chi_{[0,\tau_1]}(t), \ \ t \in [0,T), \\ \tau_1 := \inf\{t \in [0,T), |\Gamma(t)| > 1\} \wedge T. \end{cases} \tag{53}$$

From [22,32], we have:

$$\left| \int_0^T \sigma(r) dW(r) \right| = \frac{\pi}{2\sqrt{2}} \left| \int_0^{\tau_1} \frac{1}{\sqrt{T-r}} dW(r) \right| = \frac{\pi}{2\sqrt{2}} |\Gamma(\tau_1)| \leq \frac{\pi}{2\sqrt{2}}, \tag{54}$$

$$E\left[\exp\left(\int_0^T |\sigma(r)|^2 dr \right) \right] = \infty. \tag{55}$$

In this example, we consider ***Problem (SDG)*** *with the following form:*

$$\begin{cases} dx(r) = [u_1(r) + u_2(r)] dW(r), \ \ r \in [0,T] \\ x(0) = \theta, \end{cases}$$

and the objective functional:

$$J_\theta(u_1, u_2) = \frac{1}{2} \kappa^{-1} E x(T)^2 + \frac{1}{2} E \int_0^T u_2(r)^2 dr,$$

for any $\theta \in L^2_{\mathcal{F}_0}(\Omega; \mathbb{R})$*, where* $m = n = d = 1$*. Obviously, we have the following data:*

$$\begin{cases} A = 0, \ \ B_1 = 0, \ \ B_2 = 0, \ \ C = 0, \ \ D_1 = 1, \\ D_2 = 1, \ \ R = 0, \ \ N_1 = 0, \ \ N_2 = 1, \ \ G = \kappa^{-1}, \end{cases} \tag{56}$$

where $\kappa = \varphi(T)$ *and* $(\varphi(t), \phi(t))$ *satisfy the following backward stochastic differential equation:*

$$\varphi(t) = \int_0^T \sigma(r) dW(r) + \frac{\pi}{2\sqrt{2}} + 1 - \int_t^T \phi(r) dW(r), \ \ t \in [0,T].$$

It admits a unique solution $(\varphi(t), \phi(t))$ *in the following sense:*

$$\varphi(t) = \int_0^t \sigma(r) dW(r) + \frac{\pi}{2\sqrt{2}} + 1, \ \ \phi(t) = \sigma(t), \ \ t \in [0,T].$$

By Equations (53), (54), and (55) above, it has:

$$\begin{cases} 1 \leq \varphi(t) \leq \frac{\pi}{\sqrt{2}} + 1, \\ \phi(t) \notin L^\infty_{\mathcal{F}_t}(\Omega; L^2(0,T;\mathbb{R})). \end{cases} \tag{57}$$

Then, from Equations (11) and Data (56) above, for simple computing, it follows that:

$$\tilde{N}_{11} = P, \ \ \tilde{N}_{22} = P + 1, \ \ \tilde{N}_{12} = P, \ \ \tilde{B}_1 = Q, \ \ \tilde{B}_2 = Q,$$

and the Equation (9), in which F is in the Form (I) in (16), is given by:

$$\begin{cases} dP(r) = P(r)^{-1} Q(r)^2 dr + Q(r) dW(r), \ \ r \in [0,T] \\ P(T) = \kappa^{-1}. \end{cases} \tag{58}$$

Applying theItô formula to $\varphi(\cdot)^{-1}$, we deduce that $(P(\cdot), Q(\cdot)) = (\varphi(\cdot)^{-1}, -\varphi(\cdot)^{-2}\phi(\cdot))$ is the unique solution of Equation (58). Furthermore, from Theorem 1 and Equation (57), we obtain:

$$\begin{cases} \psi_1 = \varphi(\cdot)^{-1}\phi(\cdot) \notin L^{\infty}_{\mathcal{F}_t}(\Omega; L^2(0,T;\mathbb{R})), \\ \Psi_2(\psi_1) = \dfrac{2\varphi(\cdot)^{-2}\phi(\cdot)}{\varphi(\cdot)^{-1}+1} \notin L^{\infty}_{\mathcal{F}_t}(\Omega; L^2(0,T;\mathbb{R})). \end{cases} \tag{59}$$

*Therefore, there is no robust feedback control-strategy operator for Player 1, that is, **Problem (SDG)** is not closed-loop solvable.*

Example 2. *In this example, we shall apply the Itô formula to $\cos W(t)$, which shows that:*

$$\cos W(T) - \cos W(t) = \int_t^T -\sin W(s)dW(s) - \frac{1}{2}\int_t^T \cos W(s)ds. \tag{60}$$

From the above equality, we suppose that:

$$\begin{aligned} \delta &= \frac{3}{2} + \frac{T}{2} + \cos W(T) + \frac{1}{2}\int_0^T \cos W(s)ds, \\ \gamma(t) &= \frac{3}{2} + \frac{T}{2} + \cos W(t) + \frac{1}{2}\int_0^t \cos W(s)ds. \ \Upsilon(t) = -\sin W(t), \ \ t \in [0,T]. \end{aligned} \tag{61}$$

From Equation (61), we have:

$$\frac{1}{2} \le \gamma(\cdot) \le \frac{5}{2} + T, \ \ \frac{1}{2} \le \delta \le \frac{5}{2} + T.$$

And $(\gamma(\cdot), \Upsilon(\cdot))$ satisfies:

$$\gamma(t) = \delta - \int_t^T \Upsilon(s)dW(s), \ \ t \in [0,T]. \tag{62}$$

In this example, we suppose the following data:

$$\begin{cases} m = n = d = 1, \ \ A = 0, \ \ B_1 = 0, \ \ B_2 = 0, \ \ C = 0, \\ D_1 = 1, \ \ D_2 = 1, \ \ R = 0, \ \ N_1 = 0, \ \ N_2 = 1, \ \ G = \delta^{-1}. \end{cases} \tag{63}$$

Then, from Equations (11) and (63) above, for simple computing, it follows that:

$$\tilde{N}_{11} = P, \ \ \tilde{N}_{22} = P+1, \ \ \tilde{N}_{12} = P, \ \ \tilde{B}_1 = Q, \ \ \tilde{B}_2 = Q,$$

and the Equation (9), in which F is in the Form (I) in (16), is given by:

$$\begin{cases} dP(r) = P(r)^{-1}Q(r)^2dr + Q(r)dW(r), \ \ r \in [0,T] \\ P(T) = \delta^{-1}. \end{cases} \tag{64}$$

Applying the Itô formula to $\gamma(\cdot)^{-1}$, we deduce that $(P(\cdot), Q(\cdot)) = (\gamma(\cdot)^{-1}, -\gamma(\cdot)^{-2}\Upsilon(\cdot))$ is the unique solution of Equation (64). Furthermore, from Theorem 1, we obtain that:

$$\begin{cases} \psi_1 = \gamma(\cdot)^{-1}\Upsilon(\cdot) \in L^{\infty}_{\mathcal{F}_t}(\Omega; L^2(0,T;\mathbb{R})), \\ \Psi_2(\psi_1) = \dfrac{2\gamma(\cdot)^{-2}\Upsilon(\cdot)}{\gamma(\cdot)^{-1}+1} \in L^{\infty}_{\mathcal{F}_t}(\Omega; L^2(0,T;\mathbb{R})) \end{cases} \tag{65}$$

is a robust optimal feedback control-strategy operator. For clear presentation, we shall provide the following graphs with $T = 10$. Figure 1 shows one controlled sample path of the state process under the optimal control-strategy $(\bar{u}_1, \bar{u}_2)$. Figure 2 shows the optimal control-strategy $(\bar{u}_1, \bar{u}_2)$, which is the solution of the game.

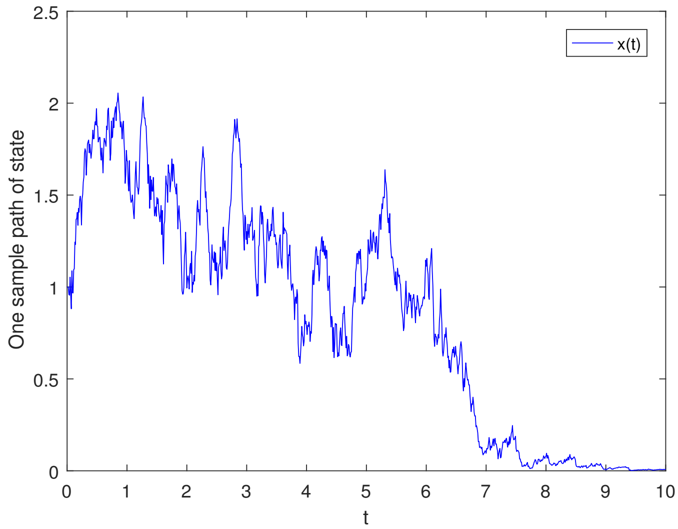

Figure 1. One controlled sample path of the state process under the optimal control-strategy $(\bar{u}_1, \bar{u}_2)$.

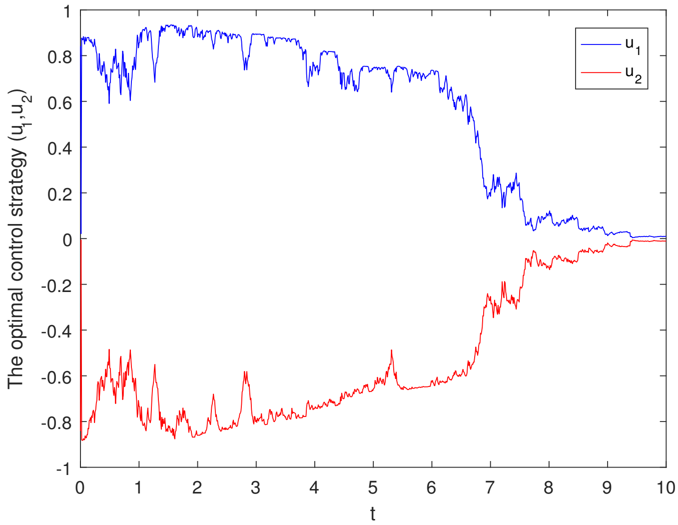

Figure 2. The trajectory of optimal control-strategy $(\bar{u}_1, \bar{u}_2)$.

Remark 10. *From the above two examples, although the form of the the Equation (9) is the same, it remarkably shows that, with different endpoints, the robustness of the feedback control-strategy operator will be different. Generally, it is interesting to find some suitable conditions to guarantee that Equation (9) admits a unique solution:*

$$(P(\cdot), Q(\cdot)) \in L^\infty_{\mathcal{F}_t}(0, T; \mathcal{S}(\mathbb{R}^n)) \times L^\infty_{\mathcal{F}_t}(\Omega; L^2(0, T; (\mathcal{S}(\mathbb{R}^n))^d)),$$

which is unsolved.

5. Concluding Remarks

In this paper, we consider a closed-loop two-person zero-sum LQ stochastic differential game with random coefficients. As we know, the feedback control-strategy contains the robustness when facing small perturbations; however, our results show that the same remarkable phenomenon appears in **Problem (SDG)** as appeared in the work of Lü-Wang-Zhang, that the regularity of the solutions for the corresponding backward stochastic Riccati equations is not enough to establish the closed-loop control-strategies; this phenomenon has not previously been mentioned in the literature. On the other hand, under some suitable conditions, we have established the equivalence between the existence of the robust optimal feedback control-strategy operators and the solvability of the corresponding backward stochastic Riccati equations in SDG, which shows the importance of the regularity of the solutions for the stochastic Riccati equations when trying to solve the closed-loop SDG. Additionally, the saddle points in the closed-loop case have been obtained by some sharp regularity assumptions on the control-strategy operators. Finally, from the examples, **Problem (SDG)** is not closed-loop solvable without the sharp regularity of the solution for the corresponding backward stochastic Riccati equations, which illustrate our theoretical results.

As with the generalization, there are several interesting problems that deserve further investigation. One is the solvability of the Equation (9), how to obtain a unique solution $(P(\cdot), Q(\cdot)) \in L^{\infty}_{\mathcal{F}_t}(0, T; \mathcal{S}(\mathbb{R}^n)) \times L^{\infty}_{\mathcal{F}_t}(\Omega; L^2(0, T; (\mathcal{S}(\mathbb{R}^n))^d))$ so that the feedback operator has the sharp regularity to guarantee the robustness, which is still open. Another one is to extend the results in this paper to other LQ games, such as the mean-field LQ game, the indefinite LQ game, and so on.

Author Contributions: Conceptualization, C.T.; Writing—original draft, J.L. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Not Applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Isaacs, R. *Differential Games*; John Wiley and Sons: New York, NY, USA, 1965.
2. Berkovitz, L.D. *Lectures on Differential Gams, Differential Games and Related Topics*; Kuhn, H.W., Szego, G.P., Eds.; North-Holland: Amsterdam, The Netherlands, 1971; pp. 3–45.
3. Bernhard, P. Linear-quadratic, two-person, zero-sum differential games: Necessary and sufficient conditions. *J. Optim. Theory Appl.* **1979**, *27*, 51–69. [CrossRef]
4. Ellot, R.J.; Kalton, N.J. Existence of value in differential games. *Mem. Amer. Math. Soc.* **1972**, *126*, 1–23. [CrossRef]
5. Erans, L.C.; Souganidis, P.E. Differential games and representation formulas for solutions of Hamilton-Jacobi-Isaacs equations. *Indiana Univ. Math. J.* **1984**, *33*, 773–797.
6. Fleming, W.H.; Souganidis, P.E. On the existence of value functions of two player, zero-sum stochastic differential games. *Indiana Univ. Math. J.* **1989**, *38*, 293–314. [CrossRef]
7. Ho, Y.C.; Bryson, A.E.; Baron, S. Differential games and optimal pursuit-evasion strategies. *IEEE Trans. AC* **1965**, *10*, 385–389. [CrossRef]
8. Hamadè, S.; Lepeltier, J.P. Zero-sum stochastic differential games and backward equations. *Syst. Control Lett.* **1995**, *24*, 259–263. [CrossRef]
9. Mou, L.; Yong, J. Two-person zero-sum linear quadratic stochastic differential games by a Hilbert space method. *J. Ind. Manag. Optim.* **2006**, *2*, 95–117. [CrossRef]
10. Yong, J. A leader-follower stochastic linear quadratic differential game. *SIAM J. Control Optim.* **2002**, *41*, 1015–1041. [CrossRef]
11. Buckdahn, R.; Li, J. Stochastic differential games and viscosity solutions of Hamilton-Jacobi-Bellman-Isaacs equations. *SIAM J. Control Optim.* **2008**, *47*, 444–475. [CrossRef]
12. Yu, Z.Y. An optimal feedback control-strategy pair for zero-sum linear-quadratic stochastic differential game: The Riccati equation approach. *SIAM J. Control Optim.* **2015**, *53*, 2141–2167. [CrossRef]
13. Moon, J. Linear quadratic stochastic stackelberg differential games for jump-diffusion systems. *SIAM J. Control Optim.* **2021**, *59*, 954–976. [CrossRef]
14. Wang, H.Y.; Wu, Z. Time-inconsistent linear-quadratic non-zero sum stochastic differential games with random jumps. *Int. J. Control.* **2022**, *95*, 1864–1874. [CrossRef]
15. Moon, J.; Kim, W. Explicit characterization of feedback Nash equilibria for indefinite, linear-Quadratic, mean-field-type stochastic zero-sum differential games with jump-diffusion models. *Mathematics* **2020**, *8*, 1669. [CrossRef]
16. Sun, J.R.; Yong, J. Linear quadratic stochastic differential games: Open-loop and closed-loop saddle points. *SIAM J. Control Optim.* **2014**, *52*, 4082–4121. [CrossRef]
17. Sun, J.R.; Yong, J. Linear quadratic stochastic two-person nonzero-sum differential games: Open-loop and closed-loop Nash equilibria. *Stoch. Process. Their Appl.* **2014**, *129*, 381–418. [CrossRef]
18. Yu, Z.Y.; Zhang, B.K.; Zhang, F. One kind of linear-quadratic zero-sum stochastic differential game with jumps. *Int. J. Control.* **2022**, *95*, 1470–1481. [CrossRef]

19. Tang, S. General linear quadratic optimal stochastic control problems with random coefficients: Linear stochastic Hamilton systems and backward stochastic Riccati equations, *SIAM J. Control Optim.* **2003**, *42*, 53–75. [CrossRef]
20. Bismut, J.M. *Contrôle des Systèmes Linéaires Quadratiques: Applications de L'IntÉGrale Stochastique*; Lecture Notes in Mathematics; Séminaire de Probabilités XII, Université de Strasbourg 1976/77, 180–264; Springer: Berlin/Heidelberg, Germany, 1978; p. 649.
21. Tang, S. Dynamic programming for general linear quadratic optimal stochastic control with random coefficients. *SIAM J. Control Optim.* **2015**, *53*, 1082–1106. [CrossRef]
22. Lü, Q.; Wang, T.; Zhang, X. Characterization of optimal feedback for stochastic linear quadratic control problem. *Probab. Uncertain. Quant. Risk* **2017**, *2*, 11. [CrossRef]
23. Zhang, F.; Dong, Y.C.; Meng, Q.X. Backward stochastic Riccati equation with jumps associated with stochastic linear quadratic optimal control with jumps and random coefficients. *SIAM J. Control Optim.* **2020**, *58*, 393–424. [CrossRef]
24. Ali, I.; KhanA, S.U. Dynamic Competition Analysis of Stochastic Fractional Differential Equation Arising in Finance via Pseudospectral Method. *Mathematics* **2023**, *11*, 1328. [CrossRef]
25. Butt, A.I.K.; Ahmad, W.; Rafiq, M.; Ahmad, N.; Imran, M. Computationally efficient optimal control analysis for the mathematical model of Coronavirus pandemic. *Expert Syst. Appl.* **2023**, *234*, 121094. [CrossRef]
26. Sun, J.R.; Li, X.; Yong, J.; Open-loop and closed-loop solvabilities for stochastic linear quadratic optimal control problem. *SIAM J. Control Optim.* **2016**, *54*, 2274–2308. [CrossRef]
27. Yong, J.; Zhou, X.Y. Stochastic controls: Hamiltonian systems and HJB equations. In *Applied Mathematics New York*; Springer: New York, NY, USA, 1999; p. 43.
28. Protter, P.E. Stochastic Integration and Differential Equations. In *Stochastic Modelling and Applied Probability*; Springer: Berlin/Heidelberg, Germany, 2005; p. 21.
29. Brian, P.; Delyon, B.; Hu, Y.; Pardoux, E.; Stoica, L. L^p solutions of backward stochastic differential equations. *Stochastic Process. Appl.* **2003**, *108*, 109–129. [CrossRef]
30. Wang, G.C.; Yu, Z.Y. A Pontryagin's Maximum principle for non-Zero sum differential games of BSDEs with applications. *IEEE Trans. AC* **2010**, *55*, 1742–1747. [CrossRef]
31. Bismut, J.M. Linear quadratic optimal stochastic control with random coefficients. *SIAM J. Control Optim.* **1976**, *14*, 419–444. [CrossRef]
32. Frei, C.; Reis, G.d. A financial market with interacting investors: does an equilibrium exist? *Math. Finan. Econ.* **2011**, *4*, 161–182. [CrossRef]