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Optimizing the Monotonic Properties of Fourth-Order Neutral Differential Equations and Their Applications

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Abstract: We investigate the oscillation of the fourth-order differential equation for a class of functional differential equations of the neutral type. We obtain a new single-oscillation criterion for the oscillation of all the solutions of our equation. We establish new monotonic properties for some cases of positive solutions of the studied equation. Moreover, we improve these properties by using an iterative method. This development of monotonic properties contributes to obtaining new and more efficient criteria for verifying the oscillation of the equation. The results obtained extend and improve previous findings in the literature by using an Euler-type equation as an example. The importance of the results was clarified by applying them to some special cases of the studied equation. The fourth-order delay differential equations have great practical importance due to their wide applications in civil, mechanical, and aeronautical engineering. Research on this type of equation is still ongoing due to its remarkable importance in many fields.

Keywords: differential equation; monotonic properties; neutral; oscillation; fourth-order

MSC: 34C10; 34K11

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1. Introduction

Differential equations have been a significant area of pure and applied mathematics since their establishment in the middle of the 17th century. Despite their extensive study in the past, it remains an important field for research with the arrival of new connections with other branches of mathematics, the fruitful interaction with applied fields, the interesting reformulation of fundamental issues and theories in various eras, the new perspectives in the twentieth century, and so on. Ordinary differential equations (ODEs) have several applications in mathematics and other fields, but when they are used to explain certain phenomena, including natural phenomena, we find that they contain delay times in their modeling, which leads to the so-called delay differential equations (DDEs). DDEs are a type of differential equation that takes into account time delays in the dynamics of the system. This indicates that the delay differential equation can directly represent any event that happened in the past, which gives it the ability to capture and analyze the behavior of systems where time delays play a critical role. Therefore, it is easy to see how these equations are utilized in physics, engineering, biology, and other sciences (see references [1,2]). A subtype of delay differential equations is known as neutral delay differential equations (NDDEs), where the highest-order derivative of the unknown function appears on the solution both with and without delay, and the development of NDDEs involves past values of the time and state variables. The delay differential equation solution requires information about the state at a certain time in the past in addition to the current state. There are numerous applications for neutral delay differential equations (NDDEs) in science and engineering. They are employed in the modeling of systems with delayed feedback, including control systems, neural networks, chemical reactions, and populations, as highlighted in references [3,4]. One of the fundamental goals of oscillation theory is to find sufficient conditions to ensure that all differential equation solutions oscillate. The first monograph that dealt with oscillation theory was that of Ladas et al. [5], which covered the results until 1984. There has been a lot of research done in the last few years on the oscillation and the oscillatory properties of differential equations (see references [6–10]). In recent years, there have been numerous studies on the oscillation and non-oscillation of solutions to various kinds of neutral functional differential equations (see references [11,12]). Numerous authors have examined the oscillations of fourth-order differential equations, and a number of methods for generating oscillatory criteria for these equations [13,14].

In this paper, we pay particular attention to the oscillatory behavior of solutions to the fourth-order neutral differential equation

$$z^{(4)}(s) + q(s)x(\tau(s)) = 0, \tag{1}$$

where $s \geq s_0$, $z(s) = x(s) + p(s)x(\sigma(s))$ is called the corresponding function of the solution x. We will assume the following conditions:

(H_1) $p, q \in C([s_0, \infty), [0, \infty))$, $0 \leq p(s) \leq p_0 < 1$;
(H_2) $\tau, \sigma \in C([s_0, \infty), \mathbb{R})$, $\tau(s) \leq s$, $\sigma(s) \leq s$ and $\lim_{s\to\infty} \tau(s) = \infty$, $\lim_{s\to\infty} \sigma(s) = \infty$.

Via a solution of (1), we mean a function $x \in C([s_x, \infty), \mathbb{R})$ for $s_x \geq s_0$, which has the property $z \in C^4([s_x, \infty))$, and satisfies (1) on $[s_x, \infty)$. We only take into account the solutions x of (1) that satisfy $\mathrm{Sup}\{|x(s)| : s \geq T\} > 0$ for all $T \geq s_x$ oscillatory.

Definition 1. *If solution x for (1) is ultimately positive or negative, it is said to be non-oscillatory; if not, it is said to be oscillatory. If all of an equation's solutions oscillate, the equation itself is said to be oscillatory.*

One of the most important motivations for conducting this research is the importance of neutral differential equations, which have many uses in technology and natural science. They are often employed, for instance, in the study of distributed networks with lossless transmission lines (see reference [15]), therefore, their qualitative characteristics are crucial. Besides the importance of fourth-order differential equations in mathematical representations of several physical, biological, and chemical phenomena, fourth-order differential equations are frequently encountered. Their applications include, for example, elasticity issues, structure distortion, or soil settlement (see reference [16]). Complementary to the motives behind this paper is the fact that one of the conditions for oscillation is to find a condition in the form of a Kneser-type oscillation. The Kneser oscillation theorem states that a second-order linear differential equation of the form

$$x''(s) + q(s)x(s) = 0,$$

is oscillatory if

$$\liminf_{s\longrightarrow\infty} s^2 q(s) > \frac{1}{4},$$

where one finds that the above condition ensures the oscillatory behavior of all solutions while there is a positive solution in the case

$$\limsup_{s\longrightarrow\infty} s^2 q(s) < \frac{1}{4}.$$

Thus, such conditions are more accurate and effective for the oscillation test. Therefore, the aim of this study was to extend the results obtained in the second- and fourth-order delay equations to neutral. Moreover, there are a number of related results that inspired our study in particular; Chatzarakis et al. [17] analyzed the oscillation behavior of the following fourth-order differential equations

$$\left[r(s)\left([x(s)+p(s)x(\tau(s))]'''\right)^{\alpha}\right]' + \int_a^b q(s,\nu)f\,(x(\sigma(s,\nu)))d\nu = 0, \tag{2}$$

under the canonical case $\int_{s_0}^{\infty} r^{-1/\alpha}(\zeta)d\zeta = \infty$.

Li et al. [11] studied the oscillatory behavior of the fourth-order nonlinear differential equation:

$$[r(s)z(s)]^{(4)} + q(s)x(\sigma(s)) = 0. \tag{3}$$

Bazighifan et al. [18] study the oscillatory properties of solutions of the following equation:

$$\left(r(s)\left(z'''(s)\right)^{\beta}\right)' + \sum_{i=1}^{j} q_i(s)x^k(\tau_i(s)) = 0,$$

for $s \geq s_0$, under the canonical case $\int_{s_0}^{\infty} r^{-1/\alpha}(\zeta)d\zeta = \infty$.

In this work, we investigate the oscillatory behavior of solutions of fourth-order differential equations with neutral-delay arguments. We establish new monotonic properties for some cases of positive solutions of the studied equation and improve these properties by using an iterative method. This development of monotonic properties contributes to obtaining new and more efficient criteria for verifying the oscillation of the equation. The solutions of any equation are classified as positive, negative, and oscillatory. Most of the techniques used in studying oscillation to find oscillation standards are based on excluding positive and negative solutions. In this paper, we are interested in finding conditions that exclude positive solutions only, and this is based on the fact that every negative value of a positive solution to the studied equation is also considered a solution, or what is called symmetry between positive and negative solutions. As usual, Euler-type differential equations are used to highlight the improvement over the previous results from the literature. We will organize our paper as follows. In Section 2.1, we introduce the essential notations and the base of the method established in the sequel. In Section 2.2, we introduce the we introduce a number of lemmas that iteratively enhance the monotonic properties of the positive solutions. In Section 2.3, we present the main results and our main oscillations, that is, a single-oscillation criterion for (1) based on a series of lemmas. In the end, we highlight the importance of our results by comparing them with previous results in the literature.

Lemma 1 ([19])**.** *Let $F \in C^m([s_0,\infty),\mathbb{R}^+)$. If $F^{(m)}$ is eventually of one sign for all large s, say, $s_1 \geq s_0$, then there exists a $s_x \geq s_0$ and an integer l, $0 \leq l \leq m$, with $m+l$ even for $F^{(m)}(s) \geq 0$, or $m+l$ odd for $F^{(m)}(s) \leq 0$ such that*

$$l \geq 0 \text{ implies that } F^{(k)}(s) > 0 \text{ for } s \geq s_x,\ k = 0,1,\ldots\ldots.l-1,$$

and $l \leq m-1$ implies that $(-1)^{l+k}F^{(k)}(s) > 0$ for $s \geq s_x$, $k = l, l+1, \ldots\ldots.m-1$.

2. Main Results

In this section, we will establish some important lemmas that we will use in the proof to illustrate the main results of the research.

Notation 1. *Firstly, we will display the important notation used in this paper. Our results are dependent on the necessity of positive β_* stated by*

$$\beta_* = \liminf_{s\longrightarrow\infty} \frac{\tau^3(s)sq(s)(1-p(\tau(s)))}{3!},$$

also, let us define

$$\gamma_* = \liminf_{s\longrightarrow\infty} \frac{\tau(s)s^3 q(s)(1-p(\tau(s)))}{3!},$$
$$\delta_* = \liminf_{s\longrightarrow\infty} \frac{s}{\tau(s)},$$

where $\{\beta_*, \gamma_*\}$ are positive because of (H_1) and (H_2). We will use, in our proof, the statement that there is a sufficiently large $s_1 \geq s_0$, such that

$$\frac{\tau^3(s)sq(s)(1-p(\tau(s)))}{3!} \geq \beta, \ \frac{\tau(s)s^3 q(s)(1-p(\tau(s)))}{3!} \geq \gamma \text{ and } \frac{s}{\tau(s)} \geq \delta, \tag{4}$$

on $[s_1, \infty)$, where for arbitrary but fixed

$$\beta \in (0, \beta_*), \ \gamma \in (0, \gamma_*) \text{ and } \delta \in (1, \delta_*),$$

for $\delta_* > 1$, and $\delta = \delta_*$ for $\delta_* = 1$.

In the following lemma, we classify the signs of the derivatives of non-oscillatory solutions to study the oscillatory features of solutions.

Lemma 2. *Assume that x is a positive solution of (1), then there are eventually only two possible cases for z*

$$\begin{array}{lll} Case(1) & z(s) > & 0,\ z'(s) > 0,\ z''(s) > 0,\ z'''(s) > 0,\ z^{(4)}(s) < 0, \\ Case(2) & z(s) > & 0,\ z'(s) > 0,\ z''(s) < 0,\ z'''(s) > 0. \end{array}$$

Proof. Let x be a positive solution of (1), we get $z^{(4)}(s) \leq 0$ from (1). By using Lemma 1, we obtain case (1), case (2), and their derivatives. □

Notation 2. *We will eventually refer to the class of positive solutions whose corresponding function to Case (1) by ρ_1, and whose corresponding function to Case (2) by ρ_2.*

Lemma 3. *Suppose that x is a positive solution of (1), then*

$$z^{(4)}(s) + q(s)(1-p(\tau(s)))z(\tau(s)) \leq 0. \tag{5}$$

Proof. Suppose that x is a positive solution of (1), it follows that there exists $s_1 \geq s_0$ such that $x(s) > 0$, $x(\tau(s)) > 0$ and $x(\sigma(s)) > 0$ for $s \geq s_1$. From the definition of z, we obtain

$$\begin{aligned} x(s) &\geq z(s) - p(s)x(\sigma(s)) \geq z(s) - p(s)z(\sigma(s)) \\ &\geq (1-p(s))z(s), \end{aligned} \tag{6}$$

with which with (1), we obtain (5). The proof is achieved. □

2.1. The Properties of the Solution in ρ_1

We will proceed to the first lemma, which analyses and provides details regarding the behavior of the positive solutions ρ_1.

Lemma 4. *Let $\beta_* > 0$ and x is a positive solution of (1) belonging to the class ρ_1. Then, eventually:*
($\boldsymbol{A}_1$) $\lim_{s\to\infty} z^{(i)}(s)/s^{3-i}$ converges to 0 for $i = 0, 1, 2, 3$;
($\boldsymbol{A}_2$) $z''(s)/s$ is decreasing;
($\boldsymbol{A}_3$) $z'(s)/s^2$ is decreasing;
($\boldsymbol{A}_4$) $z(s)/s^3$ is decreasing;

Proof. Suppose x to the class ρ_1, then for $s_1 \geq s_0$ there is $x(s) > 0$, $x(\tau(s)) > 0$ and $x(\sigma(s)) > 0$ for $s \geq s_1$.

(A_1): Since we have $z'''(s)$, which is a non-increasing positive function, then $z'''(s) \longrightarrow \ell \geq 0$ as $s \longrightarrow \infty$, if $l > 0$ then $z'''(s) \geq l > 0$, so

$$z(s) \geq \frac{\ell(s-s_1)^3}{3!}, \tag{7}$$

for $s \geq s_2 \geq s_1$. From (4) and (5), we see that

$$z^{(4)}(s) \leq -\frac{3!\beta}{\tau^3(s)s} z(\tau(s)). \tag{8}$$

From (7) into (8), we get

$$z^{(4)}(s) \leq -\frac{\beta\ell(\tau(s)-s_1)^3}{\tau^3(s)s}. \tag{9}$$

It is obvious that there exists $s_3 > s_2$ such that $(\tau(s)-s_1)^3 \geq \frac{1}{2}\tau^3(s)$ for $s \geq s_3$, so we find from (9)

$$-z^{(4)}(s) \geq \frac{\beta\ell}{2s},$$

for $s \geq s_3$. By integrating the above inequality from s_3 to s, we obtain

$$\begin{aligned} z'''(s_3) &\geq z'''(s) + \frac{\beta\ell}{2}\ln\frac{s}{s_3} \\ &\geq \ell + \frac{\beta\ell}{2}\ln\frac{s}{s_3} \to \infty \text{ as } s \to \infty, \end{aligned} \tag{10}$$

We find that there is a contradiction, therefore $\ell = 0$. We see when $z \in \rho_1$ that $z(s) \longrightarrow \infty$, $z'(s) \longrightarrow \infty$ as $s \to \infty$, and also $z''(s) > 0$ for $i = 2$, is increasing such that $z''(s)/s \longrightarrow 0$ as $s \to \infty$. Then, according to L'Hôpital's rule, we find that (A_1) is satisfied.

(A_2): Since $z'''(s)$ is non-increasing in ρ_1, we see that

$$z''(s) = z''(s_1) + \int_{s_1}^{s} z'''(\zeta)d\zeta \geq z''(s_1) + z'''(s)(s-s_1) \geq tz'''(s),$$

where by (A_1) there is $s_4 > s_3$ such that $z''(s_1) \geq s_1 z'''(s)$ for $s \geq s_4$. So

$$\left(\frac{z''(s)}{s}\right)' = \frac{z'''(s)s - z''(s)}{s^2} < 0,$$

for $s \geq s_4$, then $z''(s)/s$ is decreasing, which proves (A_2).

(A_3): From (A_1) and (A_2), $z''(s)/s$ decreases and tends to zero. Then, we find

$$\begin{aligned} z'(s) &= z'(s_4) + \int_{s_4}^{s} z''(\zeta)d\zeta \geq z'(s_4) + \frac{z''(s)}{s}\left(\frac{s^2}{2} - \frac{s_4^2}{2}\right) \\ &= z'(s_4) + \frac{z''(s)s}{2} - \frac{z''(s)s_4}{2t} > \frac{z''(s)s}{2}, \quad s \geq s_5, \end{aligned}$$

for $s_5 > s_4$. Hence

$$\left(\frac{z'(s)}{s^2}\right)' = \frac{z''(s)s - 2z'(s)}{s^3} < 0,$$

for $s \geq s_5$. We arrive at (A_3).

(A_4): Likewise, since $z''(s)/s^2$ is decreasing and tends to zero, we obtain

$$z(s) = z(s_5) + \int_{s_5}^{s} z'(\zeta)d\zeta \geq z(s_5) + \frac{z'(s)}{s^2}\left(\frac{s^3}{3} - \frac{s_5^3}{3}\right)$$

$$= \frac{z'(s)s}{3},\ s \geq s_6,$$

for $s_6 > s_5$, so

$$\left(\frac{z(s)}{s^3}\right)' = \frac{z'(s)s - 3z(s)}{s^4} < 0,$$

for $s \geq s_6$. That proves (A_4).

As a result, the proof of the lemma is complete. □

In this lemma, we will establish some additional properties of the behavior of positive solutions in ρ_1

Lemma 5. *Assume x is a solution a positive of (1) belonging to the class ρ_1 and let $\beta_* > 0$. Then, for s large enough and every $\beta \in (0, \beta_*)$:*

(A_5) $z''(s)/s^{1-\beta}$ is decreasing;
(A_6) $\beta < 1$;
(A_7) $\lim_{s\to\infty} z^{(i)}(s)/s^{3-i-\beta} = 0$, $i = 0, 1, 2$;
(A_8) $z'(s)/s^{2-\beta}$ is decreasing;
(A_9) $z(s)/s^{3-\beta}$ is decreasing.

Proof. Assume x is a positive solution of (1) to the class ρ_1, then for $s_1 \geq s_0$ there is $x(s) > 0$, $x(\tau(s)) > 0$ and $x(\sigma(s)) > 0$ for $s \geq s_1$, and (A_1)–(A_2) of Lemma 4 holds for $s \geq s_1$.

(A_5): Define the positive function

$$\omega(s) = z''(s) - sz'''(s), \tag{11}$$

by differentiating $\omega(s)$ and employing (8), in addition to having $z(s)/s^3$ decrease in (A_4), we get

$$\omega'(s) = -sz^{(4)}(s) \geq 3!\beta\frac{z(\tau(s))}{\tau^3(s)} > 3!\beta\frac{z(s)}{s^3}. \tag{12}$$

From (A_4) and (A_3) respectively, in the above inequality, we see

$$\omega'(s) > 2\beta\frac{z'(s)}{s^2} > \beta\frac{z''(s)}{s}. \tag{13}$$

By integrating the above inequality from s_1 to s, and using (A_1) and (A_2), we obtain

$$\begin{aligned} \omega(s) &> \omega(s_1) + \beta\int_{s_1}^{s}\frac{z''(\zeta)}{\zeta}d\zeta \\ &\geq \omega(s_1) + \beta\frac{z''(s)}{s}\int_{s_1}^{s}d\zeta, \end{aligned}$$

for $s \geq s_2$, which is

$$\omega(s) \geq \beta z''(s),$$

it follows from (11) that $(1-\beta)z''(s) \geq sz'''(s)$ for $s \geq s_2$, and hence

$$\left(\frac{z''(s)}{s^{1-\beta}}\right)' < 0, \quad s \geq s_2. \tag{14}$$

We observe that $z''(s)/s^{1-\beta}$ is decreasing, thus (A_5) holds.

(A_6): Since $z''(s)$ is increasing, and from (A_5) we find that $\beta < 1$, (A_6) thus holds.

(A_7): For $i = 2$, to display that

$$\lim_{s\to\infty}\frac{z^{('')}(s)}{s^{1-\beta}} = 0, \tag{15}$$

it suffices to prove that there is $\varepsilon > 1$ such that, for large s,

$$\left(\frac{z''(s)}{s^{1-\varepsilon\beta}}\right)' < 0, \tag{16}$$

which if $\frac{z''(s)}{s^{1-\beta}} \geq c > 0$, then

$$\frac{z''(s)}{s^{1-\beta-\beta(\varepsilon-1)}} \geq cs^{\beta(\varepsilon-1)} \to \infty, \text{ as } s \to \infty.$$

We notice that there is a contradiction. We find that by using (14) for any $\eta \in (2-\beta/2, 1)$, then there is $s_3 > s_2$ large enough to obtain

$$\begin{aligned} z'(s) &= z'(s_2) + \int_{s_2}^{s} \frac{z''(\zeta)\zeta^{1-\beta}}{\zeta^{1-\beta}} d\zeta \geq z'(s_2) + \frac{z''(s)}{s^{1-\beta}} \int_{s_2}^{s} \zeta^{1-\beta} d\zeta \\ &= z'(s_2) + \frac{z''(s)}{s^{1-\beta}} \frac{\left(s^{2-\beta} - s_2^{2-\beta}\right)}{2-\beta} > \frac{kz''(s)s}{2-\beta}, \end{aligned} \tag{17}$$

for $s \geq s_3$, by employing this in (13), we have

$$\omega'(s) > 2\beta \frac{z'(s)}{s^2} > \frac{2\beta kz''(s)}{(2-\beta)s}.$$

Integrating the above inequality from s_3 to s and from (A_1) this forms

$$\omega(s) > \omega(s_3) + \frac{2\beta kz''(s)}{(2-\beta)s}(s - s_3) > \frac{2\beta\mu}{(2-\beta)} z''(s),$$

for $s \geq s_4$, from (11) in the above inequality; for $s_4 > s_3$ we arrive at

$$\left(1 - \frac{2\beta k}{(2-\beta)}\right) z''(s) > -sz'''(s),$$

so, from this, we can see that (16) is satisfied with

$$\varepsilon = \frac{2k}{(2-\beta)} > 1.$$

For $i = 0, 1$ the other limits in (A_7) are obtained from (15) and from using L'Hôpital's rule.

(A_8): From (16) into (17), we see

$$z'(s) \geq z'(s_2) + \frac{z''(s)s}{(2-\beta)} - \frac{z''(s)s_2^{2-\beta}}{s^{1-\beta}(2-\beta)} > \frac{z''(s)s}{(2-\beta)},$$

for $s \geq s_5$, and for $s_5 > s_4$ we arrive at

$$\left(\frac{z'(s)}{s^{2-\beta}}\right)' = \frac{sz''(s) - (2-\beta)z'(s)}{s^{3-\beta}} < 0,$$

for $s \geq s_5$, when it is obvious that $z'(s)/s^{2-\beta}$ is decreasing , then (A_8) holds.

(A_9): We notice from (A_7) and (A_8) that $z'(s)/s^{2-\beta}$ is decreasing and tends to zero, then

$$z(s) \geq z(s_5) + \int_{s_5}^{s} \frac{z'(\zeta)\zeta^{2-\beta}}{\zeta^{2-\beta}} d\zeta \geq z(s_5) + \frac{z'(s)}{s^{2-\beta}} \frac{\left(s^{3-\beta} - s_5^{3-\beta}\right)}{3-\beta}$$

$$= \quad z(s_5) + \frac{z'(s)s}{3-\beta} - \frac{z'(s)s_5^{3-\beta}}{s^{2-\beta}(3-\beta)} > \frac{z'(s)s}{3-\beta},$$

for $s \geq s_6$, and for $s_6 > s_5$, we obtain

$$\left(\frac{z(s)}{s^{3-\beta}}\right)' = \frac{sz'(s) - (3-\beta)z(s)}{s^{4-\beta}} < 0,$$

for $s \geq s_6$. It follows that (A_9) holds, thus Lemma 5 is proved. □

The next lemma is a result of (A_9).

Lemma 6. *Assume that $\beta_* > 0$ and $\delta_* = \infty$ then $\rho_1 = \varnothing$.*

Proof. Suppose the opposite is true and assume x is a positive solution of (1) to the class ρ_1; then, for $s_1 \geq s_0$ there is $x(s) > 0$, $x(\tau(s)) > 0$ and $x(\sigma(s)) > 0$ for $s \geq s_1$. From (4) in (12), and taking into consideration (A_9), we get

$$\omega'(s) = -sz^{(4)}(s) \geq 3!\beta\frac{z(\tau(s))}{\tau^{3-\beta}(s)\tau^{\beta}(s)} > 3!\beta\frac{z(s)}{s^3}\left(\frac{s}{\tau(s)}\right)^{\beta},$$

which is

$$\omega'(s) > 3!\beta\delta^{\beta}\frac{z(s)}{s^3}.$$

Furthermore, from (A_9) and (A_8), respectively, we obtain

$$\omega'(s) > 3!\beta\delta^{\beta}\frac{z'(s)}{(3-\beta)s^2} > 3!\beta\delta^{\beta}\frac{z''(s)}{(3-\beta)(2-\beta)s}.$$

By integrating the above inequality from s_1 to s and from (A_1) and (A_2) in Lemma 5, with the definition of $\omega(s)$ in (11) for $s_2 \geq s_1$, we arrive at

$$\left(1 - \frac{3!\beta\delta^{\beta}}{(3-\beta)(2-\beta)}\right)z''(s) > sz'''(s),$$

for $s \geq s_2$, where δ can be arbitrarily large, we chose it in such a way that

$$\delta^{\beta} > \frac{(3-\beta)(2-\beta)}{3!\beta}.$$

This indicates that $-z''(s) > sz'''(s)$, where $z''(s)$ and $z'''(s)$ are positive, this causes a contradiction and completes the proof of Lemma 6. □

From (**A**$_{10}$) and Lemma 6 we can assume that $\delta_* < \infty$, so $\rho_1 \neq \varnothing$.
The following lemma can be considered as an iterative version of Lemma 5.
Let us define a sequence $\{\beta_n\}$ (that is needed in the next lemma) as follows:

$$(J)_0 = (J)_*, \ (J)_m = \frac{3!(J)_0\delta_*^{(J)_{m-1}}}{\left(3-(J)_{m-1}\right)\left(2-(J)_{m-1}\right)\left(1-(J)_{m-1}\right)}, \tag{18}$$

for $m \in \mathbb{N}$, where (J) denotes β or γ.

By induction, it is simple to demonstrate that, if $(J)_i < 1$ for $i = 1, 2, ..., m$, then $(J)_{m+1}$ holds, such that

$$\frac{(J)_{m+1}}{(J)_m} = \ell_{(J)_m} > 1, \tag{19}$$

where the definitions of $\ell_{(J)_m}$ is as follows:

$$\ell_{(J)_0} = \frac{(J)_1}{(J)_0} = \frac{3!\delta_*^{(J)_0}}{(3-(J)_m)(2-(J)_m)(1-(J)_m)} > 1,$$

$$\ell_{(J)_m} = \frac{(J)_{m+1}}{(J)_m} = \frac{\delta_*^{(J)_m}\left(3-(J)_{m-1}\right)\left(2-(J)_{m-1}\right)\left(1-(J)_{m-1}\right)}{\delta_*^{(J)_{m-1}}(3-(J)_m)(2-(J)_m)(1-(J)_m)} > 1, \ \ m \in \mathbb{N}.$$

We need to specify the sequence $\left\{\varepsilon_{(J)_n}\right\}$ as follows:

$$\varepsilon_{(J)_0} = \frac{(J)}{(J)_*} < 1,$$

$$\varepsilon_{(J)_m} = \varepsilon_{(J)_0}\frac{\delta^{\varepsilon_{(J)_{m-1}}(J)_{m-1}}\left(3-(J)_{m-1}\right)\left(2-(J)_{m-1}\right)\left(1-(J)_{m-1}\right)}{\delta_*^{(J)_{m-1}}\left(3-\varepsilon_{(J)_{m-1}}(J)_{m-1}\right)\left(2-\varepsilon_{(J)_{m-1}}(J)_{m-1}\right)\left(1-\varepsilon_{(J)_{m-1}}(J)_{m-1}\right)}, \ m \in \mathbb{N}. \tag{20}$$

The value of $\varepsilon_{(J)_n}$ is arbitrary and determined by the value of β, where β is defined in (4). It is simple to show that

$$\begin{aligned} \lim_{(J)\longrightarrow(J)_*} \varepsilon_{(J)_0} &= 1, \\ \lim_{(\delta\longrightarrow\delta_*)((J)\longrightarrow(J)_*)} \varepsilon_{(J)_{m+1}} &= 1. \end{aligned}$$

Lemma 7. *Assume x is a positive solution of (1) belonging to the class ρ_1, and let $\beta_* > 0$. Then, for s large enough and $\varepsilon\beta_m \in (0,1)$*

($\boldsymbol{A}_{10}$) $_m$ $z''(s)/s^{1-\tilde{\beta}_m}$ is decreasing;
($\boldsymbol{A}_{11}$) $_m$ $\tilde{\beta}_m < 1$;
($\boldsymbol{A}_{12}$) $_m$ $\lim_{s\to\infty} z^{(i)}(s)/s^{3-i-\tilde{\beta}_m} = 0$, $i = 0,1,2$;
($\boldsymbol{A}_{13}$) $_m$ $z'(s)(s)/s^{2-\tilde{\beta}_m}$ is decreasing;
($\boldsymbol{A}_{14}$) $_m$ $z(s)/s^{3-\tilde{\beta}_m}$ is decreasing,

where $\tilde{\beta}_m = \varepsilon_{\beta_m}\beta_m$

Proof. Assume x is a positive solution of (1) belonging to the class ρ_1; then, for $s_1 \geq s_0$ there is $x(s) > 0$, $x(\tau(s)) > 0$ and $x(\sigma(s)) > 0$ for $s \geq s_1$. This Lemma will be proved by induction on m. For $m = 0$, it holds from Lemma 5 with $\beta = \tilde{\beta}_0$. After that, assume that $(A_{10})_m$−$(A_{13})_m$ hold for $m \geq 1$ and $s \geq s_m \geq s_1$. We will display that they all hold $m+1$.

$(A_{10})_{m+1}$; by using (4) and (A_8) in (12) we see

$$\begin{aligned} \omega'(s) &\geq 3!\tilde{\beta}_0\frac{z(\tau(s))}{\tau^3(s)} \\ &= 3!\tilde{\beta}_0\frac{z(\tau(s))}{\tau^{3-\tilde{\beta}_m}(s)\tau^{\tilde{\beta}_m}(s)} \\ &\geq 3!\tilde{\beta}_0\frac{z(s)}{s^3}\left(\frac{s}{\tau(s)}\right)^{\tilde{\beta}_m} \geq 3!\tilde{\beta}_0\delta^{\tilde{\beta}_m}\frac{z(s)}{s^3}, \end{aligned}$$

from $(A_{14})_m$ and $(A_{13})_m$ we obtain

$$\omega'(s) > \frac{3!\tilde{\beta}_0\delta^{\tilde{\beta}_m}z'(s)}{(3-\tilde{\beta}_m)s^2} > \frac{3!\tilde{\beta}_0\delta^{\tilde{\beta}_m}z''(s)}{(3-\tilde{\beta}_m)(2-\tilde{\beta}_m)s}. \tag{21}$$

By integrating the above inequality from s_m to s and from $(A_{10})_m$ and $(A_{12})_m$, we find that there exists $s'_m > s_m$ such that

$$\begin{aligned}\omega(s) &\geq \omega(s_m) + \frac{3!\tilde{\beta}_0\delta^{\tilde{\beta}_m}}{(3-\tilde{\beta}_m)(2-\tilde{\beta}_m)}\int_{s_m}^{s}\frac{z''(\zeta)}{\zeta^{1-\tilde{\beta}_m}\zeta^{\tilde{\beta}_m}}d\zeta \\ &\geq \omega(s_m) + \frac{3!\tilde{\beta}_0\delta^{\tilde{\beta}_m}z''(s)}{(1-\tilde{\beta}_m)(3-\tilde{\beta}_m)(2-\tilde{\beta}_m)s^{1-\tilde{\beta}_m}}\left(s^{1-\tilde{\beta}_m} - s_m^{1-\tilde{\beta}_m}\right) \\ &> \frac{3!\tilde{\beta}_0\delta^{\tilde{\beta}_m}}{(1-\tilde{\beta}_m)(3-\tilde{\beta}_m)(2-\tilde{\beta}_m)}z''(s),\end{aligned}$$

which is

$$\omega(s) > \tilde{\beta}_{m+1}z''(s),$$

from the definition of $\omega(s)$, it follows that

$$(1-\tilde{\beta}_{m+1})z''(s) > sz'''(s), \tag{22}$$

and we obtain that

$$\left(\frac{z''(s)}{s^{1-\tilde{\beta}_{m+1}}}\right)' < 0. \tag{23}$$

That is the prove of $(A_{10})_{m+1}$.

$(A_{11})_{m+1}$: We have $z''(s)$ increasing and, from $(A_{10})_{m+1}$, we arrive at proving $(A_{11})_{m+1}$.

$(A_{12})_{m+1}$: To prove this case, it suffices to prove that there is $\varepsilon > 1$, as done in the case $m = 0$, such that for $i = 2$

$$\left(\frac{z^{(i)}(s)}{s^{3-i-\epsilon\tilde{\beta}_{m+1}}}\right)' < 0. \tag{24}$$

From (23), we find that there is $s''_m > s'_m$ sufficiently large such that

$$\begin{aligned}z'(s) &= z'(s'_m) + \int_{s'_m}^{s}\frac{z''(\zeta)}{\zeta^{1-\tilde{\beta}_{m+1}}}\zeta^{1-\tilde{\beta}_{m+1}}d\zeta \\ &\geq z'(s'_m) + \frac{z''(s)}{s^{1-\tilde{\beta}_{m+1}}}\int_{s'_m}^{s}\zeta^{1-\tilde{\beta}_{m+1}}d\zeta \\ &= z'(s'_m) + \frac{z''(s)}{s^{1-\tilde{\beta}_{m+1}}}\frac{\left(s^{2-\tilde{\beta}_{m+1}} - (s'_m)^{2-\tilde{\beta}_{m+1}}\right)}{(2-\tilde{\beta}_{m+1})},\end{aligned} \tag{25}$$

which is

$$z'(s) > \frac{\eta}{2-\tilde{\beta}_{m+1}}z''(s)s,\ s \geq s''_m,$$

for any $\eta \in (0,1)$. If we merge the above inequality with (21) we obtain

$$\omega'(s) > \frac{3!\tilde{\beta}_0\delta^{\tilde{\beta}_m}\eta}{(3-\tilde{\beta}_m)(2-\tilde{\beta}_{m+1})}\frac{z''(s)}{s},$$

by integration from s''_m to s and from $(A_{12})_m$ we receive

$$\begin{aligned}\omega(s) &> \omega(s''_m) + \frac{3!\tilde{\beta}_0\delta^{\tilde{\beta}_m}\eta}{(3-\tilde{\beta}_m)(2-\tilde{\beta}_{m+1})}\int_{s''_m}^{s}\frac{z''(\zeta)}{\zeta^{1-\tilde{\beta}_m}\zeta^{\tilde{\beta}_m}}d\zeta \\ &> \frac{3!\tilde{\beta}_0\delta^{\tilde{\beta}_m}\eta}{(1-\tilde{\beta}_m)(3-\tilde{\beta}_m)(2-\tilde{\beta}_{m+1})}z''(s) \\ &= \frac{\eta(2-\tilde{\beta}_m)}{(2-\tilde{\beta}_{m+1})}\tilde{\beta}_{m+1}z''(s) \\ &= \varepsilon\tilde{\beta}_{m+1}z''(s),\end{aligned}$$

for $s \geq s'''_m > s''_m$. Since $\tilde{\beta}_m < \tilde{\beta}_{m+1}$, we can choose η such that $\varepsilon > 1$, from (A_2) and the definition of ω we see that (16) is satisfied; the other limits are the same as those for $m = 0$.

$(A_{13})_{m+1}$: By using $(A_{12})_{m+1}$ in (25), we obtain

$$(2 - \tilde{\beta}_{m+1})z'(s) > z''(s)s.$$

Then, $(A_{13})_{m+1}$ is satisfied.

$(A_{14})_{m+1}$: From (A_7) and (A_8), we obtain

$$\begin{aligned} z(s) &> z(s'''_m) + \int_{s'''_m}^{s} \frac{z'(\zeta)}{\zeta^{2-\tilde{\beta}_{m+1}}}\zeta^{2-\tilde{\beta}_{m+1}}d\zeta \\ &> \frac{z'(s)}{(3-\tilde{\beta}_{m+1})s^{2-\tilde{\beta}_{m+1}}}\left(s^{3-\tilde{\beta}_{m+1}} - (s'''_m)^{3-\tilde{\beta}_{m+1}}\right) \\ &> \frac{z'(s)s}{3-\tilde{\beta}_{m+1}}, \end{aligned}$$

which indicates $(A_{14})_{m+1}$ holds and submits the lemma's proof. □

The following lemma can be easily deduced from the aforementioned arguments.

Lemma 8. *Suppose that $\delta_* < \infty$ and*

$$\liminf_{s \longrightarrow \infty} \tau^3(s)sq(s) > \varrho_0, \tag{26}$$

where

$$\varrho_0 = \frac{\max\ \{c(1-c)(2-c)(3-c)\delta_*^{-c} : 0 < c < 1\}}{(1-p_0)}. \tag{27}$$

Then $\rho_1 = \varnothing$.

Proof. Suppose the opposite is true, that $x \in \rho_1$. We claim that

$$\beta_{m-1} < 1, \quad m \in \mathbb{N}. \tag{28}$$

From the case $(A_{11})_m$ we have $\tilde{\beta}_m < 1$. Since $\varepsilon_{\beta_m} \in (0,1)$ can be picked arbitrarily, set $\varepsilon_{\beta_m} > 1/\ell_{\beta_m}$, where ℓ_{β_m} is given by (19). Then

$$1 > \tilde{\beta}_m = \varepsilon_{\beta_m}\ell_{\beta_m}\beta_{m-1} > \beta_{m-1},$$

which supports the claim. From (28) we conclude that the sequence $\{\beta_m\}_{m=0}^{\infty}$ is increasing and bounded from above, which is defined by (18), which means that

$$\lim_{s \longrightarrow \infty} \beta_m = c,$$

where $c \in (0,1)$ is a root of the following equation:

$$\frac{c(1-c)(2-c)(3-c)\delta_*^{-c}}{(1-p_0)} = 3!\beta_*. \tag{29}$$

However, from (26) we find that (29) has no positive solutions. As a result, $\rho_1 = \varnothing$, which is the end of the proof and the Lemma. □

2.2. The Properties of the Solution in ρ_2

In this section, we will show results similar to the previous results of the section of solutions in the class ρ_2.

Lemma 9. *Assume x is a positive solution of (1) relating to the class ρ_2 and let $\gamma_* > 0$. Then, for s large enough:*

(A_{15}) $\lim_{s\to\infty} z^{(i)}(s)/s^{1-i}$ *converges to* 0 *for* $i = 0, 1$;
(A_{16}) $z(s)/s$ *is decreasing.*

Proof. Suppose x is a positive solution of (1) belonging to the class ρ_2, then, for $s_1 \geq s_0$, there is $x(s) > 0$, $x(\tau(s)) > 0$ and $x(\sigma(s)) > 0$ for $s \geq s_1$.

(A_{15}): As a result of the fact that $z'(s)$ is decreasing positive function, $z'(s) \longrightarrow \ell \geq 0$ as $s \longrightarrow \infty$, if $l > 0$ then $z'(s) \geq l > 0$, so

$$z(s) \geq \ell(s - s_1) > \ell s/3, \tag{30}$$

for $s \geq s_2 \geq s_1$. By (4) and (5) we obtain

$$-z^{(4)}(s) \geq \frac{3!\gamma}{\tau(s)s^3} z(\tau(s)). \tag{31}$$

From (30) in the above inequality we obtain

$$-z^{(4)}(s) \geq 2\gamma\ell/s^3,$$

we have obtained this by integrating twice from s to ∞

$$-z''(s) \geq \frac{\gamma\ell}{s}.$$

Integrating from s_2 to s we arrive at

$$z'(s_2) \geq z'(s) + \gamma\ell \ln\frac{s_2}{s} \longrightarrow \infty \text{ as } s \longrightarrow \infty,$$

there is, as we found, a contradiction. Therefore $\ell = 0$. We can show that (A_{15}) is satisfied by applying L'Hôpital's rule.

(A_{16}): we have $z'(s)$ non-increasing and by (A_{15}) we see that

$$z(s) = z(s_1) + \int_{s_1}^{s} z'(\zeta)d\zeta \geq z(s_1) + z'(s)(s - s_1) \geq tz'(s),$$

for $s \geq s_3 > s_1$, where s_3 is sufficiently large such that $z(s_1) - s_1 z'(s) > 0$ for $s \geq s_3$. Then

$$\left(\frac{z(s)}{s}\right)' = \frac{sz'(s) - z(s)}{s^2},$$

for $s \geq s_3$, the completion of the proof follows. □

Lemma 10. *Assume x is a positive solution of (1) belonging to the class ρ_2 and let $\gamma_* > 0$. Then, for s large enough and any $\gamma \in (0, \gamma_*)$:*

(A_{17}) $z(s)/s^{1-\gamma}$ *is decreasing;*
(A_{18}) $\gamma < 1$;
(A_{19}) $\lim_{s\to\infty} z(s)/s^{1-\gamma} = 0$;
(A_{20}) $z(s)/s^{\gamma}$ *is non-decreasing.*

Proof. Suppose x is a positive solution of (1) belonging to the class ρ_2, then for $s_1 \geq s_0$ there is $x(s) > 0$, $x(\tau(s)) > 0$ and $x(\sigma(s)) > 0$ for $s \geq s_1$.

(A_{17}): From (A_{16}) we have $z(s)/s$, which is decreasing; in (31) we obtain

$$-z^{(4)}(s) \geq \frac{3!\gamma}{\tau(s)s^3} z(\tau(s)) \geq 3!\gamma\frac{z(s)}{s^4},$$

by integrating from s to ∞ twice such that $z(s)$ is increasing, it goes as follows:

$$-z''(s) \geq \gamma \frac{z(s)}{s^2}. \tag{32}$$

Let us define a positive function

$$\mu(s) = z(s) - sz'(s),$$

by differentiating and from (32) we find

$$\mu'(s) = -sz''(s) \geq \gamma \frac{z(s)}{s}. \tag{33}$$

Integrating, again, s_1 to s, and we have $z(s)/s$ decreasing and tending to zero, and we see

$$\begin{aligned} \mu(s) &\geq \mu(s_1) + \gamma \int_{s_1}^{s} \frac{z(\zeta)}{\zeta} d\zeta \\ &\geq \mu(s_1) + \gamma \frac{z(s)}{s}(s - s_1) > z(s)\gamma, \end{aligned} \tag{34}$$

for $s \geq s_2$, where $s_2 > s_1$ is sufficiently large such that $\mu(s_1) - z(s)s_1/s > 0$ for $s \geq s_2$. From the definition of $\mu(s)$ we obtain

$$(1-\gamma)z(s) \geq sz'(s),$$

and

$$\left(\frac{z(s)}{s^{1-\gamma}}\right)' = \frac{sz'(s) - (1-\gamma)z(s)}{s^{2-\gamma}} < 0, \tag{35}$$

$s \geq s_2$, then (A_{17}) holds.

(A_{18}): This simply implies from (A_{17}), and from the case that x is increasing.

(A_{19}): The proof is identical to the proof for class ρ_1, and it is sufficient to show that

$$\left(\frac{z(s)}{s^{1-\varepsilon\gamma}}\right)' < 0. \tag{36}$$

For $\varepsilon > 1$, we can derive from (35) into (34), finding that there exists $s_3 \geq s_2$, such that

$$\begin{aligned} \mu(s) &\geq \mu(s_2) + \gamma \int_{s_2}^{s} \frac{z(\zeta)}{\zeta^{1-\gamma}\zeta^{\gamma}} d\zeta \\ &\geq \mu(s_2) + \frac{\gamma z(s)}{(1-\gamma)s^{1-\gamma}}\left(s^{1-\gamma} - s_2^{1-\gamma}\right) > \frac{\eta\gamma}{(1-\gamma)} z(s),\ s \geq s_3, \end{aligned} \tag{37}$$

for any $\eta \in (1-\gamma, 1)$, from that we deduce that

$$\left(1 - \frac{\eta\gamma}{(1-\gamma)}\right) z(s) \geq sz'(s).$$

It is now obvious that (36) holds with $\varepsilon = \eta/(1-\gamma) > 1$.

(A_{20}): By integrating (32) from s to ∞ we see

$$z'(s) \geq \gamma \int_{s}^{\infty} \frac{z(s)}{s^2} \geq \gamma \frac{z(s)}{s},$$

and so

$$\left(\frac{z(s)}{s^{\gamma}}\right)' \geq 0,$$

which is proof that $z(s)/s^{\gamma}$ is increasing. The lemma's proof is now accomplished. □

Lemma 11. *Assume that $\gamma_* > 0$ and $\delta_* = \infty$ then $\rho_2 = \varnothing$.*

Proof. Suppose x is a positive solution of (1) belonging to the class ρ_2, then for $s_1 \geq s_0$ there is $x(s) > 0$, $x(\tau(s)) > 0$ and $x(\sigma(s)) > 0$ for $s \geq s_1$. From (4) into (31), and taking into account (A_{17}), we arrive at

$$-z^{(4)}(s) \geq \frac{3!\gamma}{s^3}\frac{z(\tau(s))}{\tau^{1-\gamma}(s)\tau^{\gamma}(s)} \geq \frac{3!\gamma z(s)}{s^4}\left(\frac{s}{\tau(s)}\right)^{\gamma} \geq \frac{3!\gamma\delta^{\gamma}z(s)}{s^4}.$$

By twice integrating from s to ∞, with the assumption that x is increasing, we obtain

$$-z''(s) \geq \frac{\gamma\delta^{\gamma}z(s)}{s^2}. \tag{38}$$

From the above inequality into (33)

$$\mu'(s) = -sz''(s) \geq \frac{\gamma\delta^{\gamma}z(s)}{s}.$$

Using integration as in (34), and replacing γ by $\gamma\delta^{\gamma}$, we obtain

$$(1-\gamma\delta^{\gamma})z(s) \geq sz'(s),$$

where we can choose δ in such a way that it can be arbitrarily large, so that $\delta^{\gamma} > 1/\gamma$, this shows $-z(s) = sz'(s)$—a contradiction. This illustrates the lemma. □

Now, from Lemma 10, we obtain an iterative.

Lemma 12. *Assume x is a positive solution of (1) belonging to the class ρ_2 and let $\gamma_* > 0$. Then, for s large enough and any $\varepsilon_{\gamma_m} \in (0,1)$:*
(A_{21})$_m$ $z(s)/s^{1-\tilde{\gamma}_m}$ is decreasing;
(A_{22})$_m$ $\tilde{\gamma}_m < 1$;
(A_{23})$_m$ $\lim_{s\to\infty} z(s)/s^{1-\tilde{\gamma}_m} = 0$;
(A_{24})$_m$ $z(s)/s^{\tilde{\gamma}_m}$ is non-decreasing;
where $\tilde{\gamma}_m = \varepsilon_{\gamma_m}\gamma_m$.

Proof. Assume x is a positive solution of (1) belonging to the class ρ_2, then for $s_1 \geq s_0$ there is $x(s) > 0$, $x(\tau(s)) > 0$ and $x(\sigma(s)) > 0$ for $s \geq s_1$ by induction on m. For $m = 0$, it holds from Lemma 5 that $\gamma = \tilde{\gamma}_0$. After that, assume that (A_{21})$_m$−(A_{24})$_m$ hold for $m \geq 1$ and $s \geq s_m \geq s_1$. We will show that (A_{21})$_{m+1}$ holds.

(A_{21})$_{m+1}$: From (31) and (A_{21})$_m$ we find

$$\begin{aligned}-z^{(4)}(s) &\geq \frac{3!\tilde{\gamma}_0}{s^3}\frac{z(\tau(s))}{\tau^{1-\tilde{\gamma}_m}(s)\tau^{\tilde{\gamma}_m}(s)} \geq \frac{3!\gamma z(s)}{s^4}\left(\frac{s}{\tau(s)}\right)^{\tilde{\gamma}_m}\\ &\geq \frac{3!\tilde{\gamma}_0\delta^{\tilde{\gamma}_m}z(s)}{s^4}.\end{aligned}$$

Integrate the above inequality from s to ∞, and taking into account that $z(s)/s^{\tilde{\gamma}_m}$ is increasing, yield

$$\begin{aligned}z'''(s) &\geq 3!\tilde{\gamma}_0\delta^{\tilde{\gamma}_m}\int_{s_2}^{s}\frac{z(\zeta)}{\zeta^{4-\tilde{\gamma}_m}\zeta^{\tilde{\gamma}_m}}d\zeta\\ &\geq \frac{3!\tilde{\gamma}_0\delta^{\tilde{\gamma}_m}z(\zeta)}{s^{\tilde{\gamma}_m}}\int_{s_2}^{s}\frac{d\zeta}{\zeta^{4-\tilde{\gamma}_m}} \geq \frac{3!\gamma\delta^{\tilde{\gamma}_m}z(s)}{(3-\tilde{\gamma}_m)s^3}.\end{aligned}$$

Repeating this process, we obtain

$$z''(s) \geq \frac{3!\gamma\delta^{\tilde{\gamma}_m} z(s)}{(3-\tilde{\gamma}_m)(2-\tilde{\gamma}_m)s^2},$$

By incorporating this into (33), we have

$$\mu'(s) = -sz''(s) \geq \frac{3!\gamma\delta^{\tilde{\gamma}_m} z(s)}{(3-\tilde{\gamma}_m)(2-\tilde{\gamma}_m)s},$$

By integrating from s_m to s and from $(A_{21})_m$ and $(A_{23})_m$, we find

$$\begin{aligned}
\mu(s_m) &\geq \mu(s) + \frac{3!\gamma_0\delta^{\tilde{\gamma}_m}}{(3-\tilde{\gamma}_m)(2-\tilde{\gamma}_m)} \int_{s_m}^{s} \frac{z(\zeta)}{\zeta^{1-\tilde{\gamma}_m}\zeta^{\tilde{\gamma}_m}} d\zeta \\
&\geq \frac{3!\gamma_0\delta^{\tilde{\gamma}_m} z(s)}{(3-\tilde{\gamma}_m)(2-\tilde{\gamma}_m)(1-\tilde{\gamma}_m)s^{1-\tilde{\gamma}_m}} \left(s^{1-\tilde{\gamma}_m} - s_m^{1-\tilde{\gamma}_m}\right) \\
&\geq \tilde{\gamma}_{m+1} z(s), \quad s \geq s'_m,
\end{aligned}$$

That is the proof of $(A_{21})_{m+1}$. The other parts' proofs of the lemma are the same as those in the case where $m = 0$. □

Lemma 13. *Suppose that $\delta_* < \infty$ and*

$$\liminf_{s\longrightarrow\infty} \tau(s)s^3 q(s) > \varrho_0, \tag{39}$$

where

$$\varrho_0 = \frac{\max\ \{c(1-c)(2-c)(3-c)\alpha^{c-3}\}}{(1-p_0)}, \tag{40}$$

for $0 < c < 1$, then $\rho_2 = \varnothing$.

Proof. The proof is similar to the proof of Lemma 8 and defines $\{\gamma_m\}$ as in (18). □

Now, in the next theorem, we offer the fundamental result in this work by combining the results from the previous two sections.

2.3. Oscillation Results

Theorem 1. *Suppose that*

$$\liminf_{s\longrightarrow\infty} \tau^3(s)sq(s) > \begin{cases} 0 & for\ \delta_* = \infty, \\ \varrho_0 & for\ \delta_* < \infty, \end{cases} \tag{41}$$

where

$$\varrho_0 = \frac{\max\ \{c(1-c)(2-c)(3-c)\delta_*^{-c}\}}{(1-p_0)},$$

where $0 < c < 1$, then (1) is oscillatory.

Proof. From (41) observe that $\beta_* > 0$, and since

$$\liminf_{s\longrightarrow\infty} \tau^3(s)sq(s) \leq \liminf_{s\longrightarrow\infty} \tau(s)s^3 q(s),$$

we find that $\gamma_* > 0$. Now, if $\delta_* = \infty$, then Lemmas 6 and 11 imply that $\rho_2 = \rho_1 = \varnothing$. For $\delta_* < \infty$, from Lemmas 8 and 13, the same conclusion is derived. This illustrates the theorem. □

Corollary 1. *Let $\tau(s) = \alpha s$ with $0 < \alpha \leq 1$. If*

$$\liminf_{s \longrightarrow \infty} s^4 q(s) > \frac{\max\{c(1-c)(2-c)(3-c)\alpha^{c-3} : 0 < c < 1\}}{(1-p_0)},$$

then (1) is oscillatory.

2.4. Application and Discussion

In the next section, we provide an example to highlight our study results.

Example 1. *Now, consider the fourth-order Euler delay differential equation*

$$(x(s) + p_0 x(\sigma(s)))^{(4)}(s) + \frac{q_0 \delta_*^3}{s^4} x\left(\frac{1}{\delta_*} s\right) = 0, \tag{42}$$

for $s > 1$, where $p_0 > 0$, $q_0 > 0$ and $\delta_* \geq 1$, By applying condition (41), we obtain

$$\liminf_{s \longrightarrow \infty} \left(\frac{s}{\delta_*}\right)^3 s\left(\frac{q_0 \delta_*^3}{s^4}\right) > \frac{\max\ \{c(1-c)(2-c)(3-c)\delta_*^{-c}\}}{(1-p_0)},$$

which is,

$$q_0 > \frac{\max\ \{c(1-c)(2-c)(3-c)\delta_*^{-c}\}}{(1-p_0)}, \tag{43}$$

where $0 < c < 1$. Thus, by applying Theorem 1, we can guarantee that all solutions of Equation (42) are oscillatory if condition (43) is satisfied.

Remark 1. *If we consider the special case $p_0 = 0.5$ and $\delta_* = 2$, the condition (43) reduces to*

$$q_0 > 1.785. \tag{44}$$

By checking the result of the oscillation constants for Equation (42) in references [20] and [21], respectively, with $p_0 = 0.5$ and $\delta_* = 2$, we see

$$q_0 > \frac{96}{e \ln 2} \simeq 50.951, \tag{45}$$

and

$$q_0 > \frac{82}{9} \simeq 9.111. \tag{46}$$

Example 2. *Consider the NDE as the following:*

$$\left(x(s) + \frac{3}{4} x\left(\frac{5s}{2}\right)\right)^{(4)}(s) + \frac{(4)^3 q_0}{s^4} x\left(\frac{s}{4}\right) = 0, \tag{47}$$

where $s > 1$, $p_0 = \frac{3}{4}$, $q(s) = \frac{(4)^3 q_0}{s^4}$ and $\delta_* = 4$, $\sigma(s) = \frac{5s}{2}$. To check the oscillation of Equation (47), we will apply condition (41) of Theorem 1 in the previous section and see that

$$\liminf_{s \longrightarrow \infty} \left(\frac{s}{4}\right)^3 s\left(\frac{64 q_0}{s^4}\right) > \frac{\max\ \{c(1-c)(2-c)(3-c)4^{-c}\}}{(1-p_0)},$$

which is

$$q_0 > 2.561. \tag{48}$$

Applying results in both [22] and [23] to Equation (47), we get, respectively,

$$q_0 > 832, \tag{49}$$

and

$$q_0 > \frac{675}{14\mathrm{e}\ln 10} \simeq 8.627. \tag{50}$$

We notice that the condition (48) improves the condition (49) and (50). It also improves results (45) and (46).

Remark 2. *It can be easily observed that condition (44) and (48) improve conditions (45), (46), (49), and (50). In addition to this improvement, there is something that distinguishes our results from other results in [20] and [21] is that their results require constraints $\tau(s) < s$, $\tau'(s) \geq 0$ but Theorem 1 does not need them. This leads to the conclusion that Theorem 1 improves many previous results in the literature, even without the usual restrictive suppositions on the diverging argument.*

Remark 3. *We make a simulation experiment for Example 1, by considering the ODE of (42)*

$$(x(s) + p_0 x(s))^{(4)}(s) + \frac{q_0\delta_*^3}{s^4}x(s) = 0, \tag{51}$$

where $p_0 = 0.5$, $q_0 = \frac{q_0\delta_*^3}{s^4}$ *and* $\delta_* = 1$. *By using Theorem 1, Equation (51) is oscillatory if*

$$\liminf_{s\longrightarrow\infty}(s)^3 s\left(\frac{q_0}{s^4}\right) > \frac{\max\ \{c(1-c)(2-c)(3-c)\}}{(1-p_0)},$$

which is

$$q_0 > \frac{\max\{c(1-c)(2-c)(3-c)\}}{(1-p_0)},\ \text{where } 0 < c < 1. \tag{52}$$

3. Conclusions

The oscillatory behavior of the solutions of the equation of the neutral type was studied, where the positive solutions of the equation were classified as ρ_1 and ρ_2, and then we studied the monotonic properties of these positive solutions by providing a series of lemmas for each case to iteratively improve the monotonicity features of non-oscillatory solutions. We observe that the main difference between ρ_1 and ρ_2 is the change of the second derivative of the function z, but this simple change constitutes a change in many of the monotonic and asymptotic properties. For example, in case ρ_1, we can get new monotonic properties of the first, second, and third derivatives of the function z, while in case ρ_2 we can only get an ordinal behavior of z; there is no change except for the function z only. By using the iterative method based on these characteristics, we have provided a single criterion to eliminate the positive solutions to our equation that ensures the oscillatory nature of the solutions. The example improved upon previous results and showed the importance of the new properties. The most important thing that distinguishes our results is that they can be applied without some of the restrictions that other results require.

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References

1. Rihan, F.A. *Delay Differential Equations and Applications to Biology*; Springer Nature Singapore Pte Ltd.: Singapore, 2021. [CrossRef]
2. Hale, J.K. Functional Differential Equations. In *Oxford Applied Mathematical Sciences*; Springer: New York, NY, USA, 1971.
3. Gyori, I.; Ladas, G. *Oscillation Theory of Delay Differential Equations with Applications*; Clarendon Press: Oxford, UK, 1991.
4. Erbe, L.H.; Kong, Q.; Zhong, B.G. *Oscillation Theory for Functional Differential Equations*; Marcel Dekker: New York, NY, USA, 1995.
5. Ladde, G.S.; Lakshmikantham, V.; Zhang, B.G. *Oscillation Theory of Differential Equations with Deviating Arguments*; Marcel Dekker: New York, NY, USA, 1987.
6. Grace, S.R.; Jadlovska, I.; Zafer, A. On oscillation of second order delay differential equations with a sublinear neutral term. *Mediterr. J. Math.* **2020**, *17*, 116. [CrossRef]
7. Jadlovská, I. New criteria for sharp oscillation of second-order neutral delay differential equations. *Mathematics* **2021**, *9*, 2089. [CrossRef]
8. Agarwal, R.P.; Grace, S.R.; O'Regan, D. Oscillation criteria for certain nth order differential equations with deviating arguments. *J. Math. Anal. Appl.* **2001**, *262*, 601–622. [CrossRef]
9. Grace, S.R.; Agarwal, R.P.; Pavani, R.; Thandapani, E. On the oscillation of certain third order nonlinear functional differential equations. *Appl. Math. Comput.* **2008,** *202*, 102–112. [CrossRef]
10. Agarwal, R.P.; Zhang, C.; Li, T. Some remarks on oscillation of second order neutral differential equations. *Appl. Math. Comput.* **2016**, *274*, 178–181. [CrossRef]
11. Li, T.; Baculìková, B.; Džurina, J.; Zhang, C. Oscillation of fourth-order neutral differential equations with p-Laplacian like operators. *Bound. Value Probl.* **2014**, *2014*, 56. [CrossRef]
12. Grace, S.R.; Džurina, J.; Jadlovská, I.; Li, T. An improved approach for studying oscillation of second-order neutral delay differential equations. *J. Inequalities Appl.* **2018**, *2018*, 193. [CrossRef] [PubMed]
13. Agarwal, R.; Grace, S.; Manojlovic, J. Oscillation criteria for certain fourth order nonlinear functional differential equations. *Math. Comput. Model.* **2006**, *44*, 163–187. [CrossRef]
14. Cesarano, C.; Bazighifan, O. Oscillation of fourth-order functional differential equations with distributed delay. *Axioms* **2019**, *8*, 61. [CrossRef]
15. Hale, J.K. *Theory of Functional Differential Equations*; Springer: New York, NY, USA, 1977. [CrossRef]
16. Bartušek, M.; Cecchi, M.; Došlá, Z.; Marini, M. Fourth-order differential equation with deviating argument. *Abstr. Appl. Anal.* **2012**, *2012*, 185242. [CrossRef]
17. Chatzarakis, G.E.; Elabbasy, E.M.; Bazighifan, O. An oscillation criterion in 4th-order neutral differential equations with a continuously distributed delay. *Adv. Differ. Equ.* **2019**, *2019*, 336. [CrossRef]
18. Bazighifan, O.; Ruggieri, M.; Scapellato, A. An improved criterion for the oscillation of fourth-order differential equations. *Mathematics* **2020**, *8*, 610. [CrossRef]
19. Kiguradze, I.; Chanturia, T.A. *Asymptotic Properties of Solutions of Nonautonomous Ordinary Differential Equations*; Kluwer Academic, Publishers: Dordrecht, The Netherlands, 1993. [CrossRef]
20. Zafer, A. Oscillation criteria for even order neutral differential equations. *Appl. Math. Lett.* **1998,** *11*, 21–25. [CrossRef]
21. Agarwal, R.P.; Bohner, M.; Li, T.; Zhang, C. A new approach in the study of oscillatory behavior of even-order neutral delay differential equations. *Appl. Math. Comput.* **2013**, *225*, 787–794. [CrossRef]
22. Cesarano, C.; Moaaz, O.; Qaraad, B.; Muhib, A. Oscillatory and asymptotic properties of higher-order quasilinear neutral differential equations. *AIMS Math.* **2021**, *6*, 11124–11138. [CrossRef]
23. Moaaz, O.; Alhgilan, A.E. A Study of the Monotonic Properties of Solutions of Neutral Differential Equations and Their Applications. *Axioms* **2023**, *12*, 346. [CrossRef]