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A Semi-Discretization Method Based on Finite Difference and Differential Transform Methods to Solve the Time-Fractional Telegraph Equation

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Abstract: The telegraph equation is a hyperbolic partial differential equation that has many applications in symmetric and asymmetric problems. In this paper, the solution of the time-fractional telegraph equation is obtained using a hybrid method. The numerical simulation is performed based on a combination of the finite difference and differential transform methods, such that at first, the equation is semi-discretized along the spatial ordinate, and then the resulting system of ordinary differential equations is solved using the fractional differential transform method. This hybrid technique is tested for some prominent linear and nonlinear examples. It is very simple and has a very small computation time; also, the obtained results demonstrate that the exact solutions are exactly symmetric with approximate solutions. The results of our scheme are compared with the two-dimensional differential transform method. The numerical results show that the proposed method is more accurate and effective than the two-dimensional fractional differential transform technique. Also, the implementation process of this method is very simple, so its computer programming is very fast.

Keywords: time-fractional telegraph equation; finite difference method; fractional differential transform method; convergence

JEL Classification: 26A33; 35R11; 35Q60; 65M06

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1. Introduction

The differential transform method (DTM) is an iterative method based on Taylor's series. DTM has been used to solve various differential equations. It was first applied to solve electrical circuit problems. After that, it was used to solve ordinary differential equations (ODEs), partial differential equations (PDEs), fuzzy PDEs, fractional-order ODEs and PDEs, systems of ODEs, systems of PDEs, differential-algebraic equations, and eigenvalue problems [1–7]. Additionally, fractional DTM, which is based on a generalized Taylor's series, has been applied to solve various differential, differential-algebraic, and integral equations of fractional order [8–14]. In this paper, we intend to apply a combination of the finite difference (FD) and fractional differential transform (FDT) methods (FD-FDTM) to solve the one-dimensional time-fractional telegraph equation (FTE).

We consider the FTE in the following form [15–17]:

$$\frac{\partial^{2\gamma} v(x,t)}{\partial t^{2\gamma}} + 2\lambda \frac{\partial^{\gamma} v(x,t)}{\partial t^{\gamma}} + \mu v(x,t) = \nu \frac{\partial^2 v(x,t)}{\partial x^2} + q(x,t),\ a < x < b, t \geq 0, \tag{1}$$

with initial conditions

$$v(x,0) = f_1(x), v_t(x,0) = f_2(x),\ a < x < b, \tag{2}$$

and boundary conditions

$$v(a,t) = g_1(t), v(b,t) = g_2(t),\ t \geq 0. \tag{3}$$

where $0 < \gamma < 1$, λ, μ, and ν are arbitrary positive constants and f_1, f_2, g_1, and g_2 are known functions.

Also, $\frac{\partial^{2\gamma} v(x,t)}{\partial t^{2\gamma}}$ and $\frac{\partial^{\gamma} v(x,t)}{\partial t^{\gamma}}$ denote the Caputo fractional derivative of order 2γ and γ, respectively. The γ-order Caputo fractional derivative of the function f for $\gamma > 0$, $n-1 < \gamma < n$, is defined as follows:

$$D_0^{\gamma} f(x) = \frac{1}{\Gamma(n-\gamma)} \int_0^x (x-t)^{n-\gamma-1} f^{(n)}(t) dt.$$

The telegraph equation is a hyperbolic PDE that has many applications in physics and engineering, for example, in signal analysis, random walk theory, anomalous diffusion processes, wave phenomena, and wave propagation of the electrical signal in the cable of a transmission line. Different numerical and analytical techniques have been used to solve fractional-order telegraph equations [15–33].

This work aims to obtain an approximate solution to the FTE (1) using a hybrid method. In 2008, a hybrid method based on the combination of DTM and FDM was presented to solve a nonlinear heat conduction differential equation [34]. Also, in 2012, some nonlinear PDEs were solved with the hybrid method [35]. Arsalan (2020) applied a hybrid scheme to solve the one-dimensional integer-order telegraph equations [36,37].

We organize the rest of the paper as follows. In Section 2, we present the FDTM and related theorems. We propose our hybrid method to solve FTE in Section 3 and prove its convergence in Section 4. Also, in Section 5, we give some examples and solve them with the proposed method, we draw a conclusion in Section 6.

2. Fractional Differential Transform Method

The fractional differential transform method (or generalized differential transform method) is based on the fractional Taylor's formula. The α-order fractional Taylor expansion of function $u(t)$ about point $t = t_0$ is defined as [38]

$$u(t) = \sum_{k=0}^{\infty} \frac{(t-t_0)^{k\alpha}}{\Gamma(k\alpha+1)} (\frac{d^{\alpha}}{dt^{\alpha}})^k u(t) \mid_{t=t_0}, \tag{4}$$

where $\frac{d^{\alpha}}{dt^{\alpha}}$ is the α-order Caputo fractional derivative and $(\frac{d^{\alpha}}{dt^{\alpha}})^k = \underbrace{\frac{d^{\alpha}}{dt^{\alpha}} \cdots \frac{d^{\alpha}}{dt^{\alpha}}}_{k-times}$.

The α-order FDT of the function $u(t)$ about $t = t_0$ is denoted by $U^{\alpha}(k)$ and defined as $U^{\alpha}(k) = \frac{1}{\Gamma(k\alpha+1)} (\frac{d^{\alpha}}{dt^{\alpha}})^k u(t) \mid_{t=t_0}$, and the inverse transform is $u(t) = \sum_{k=0}^{\infty} U^{\alpha}(k)(t-t_0)^{k\alpha}$ [9]. Therefore, at $t = 0$, we have

$$u(t) = \sum_{k=0}^{\infty} U^{\alpha}(k) t^{k\alpha} = \sum_{k=0}^{\infty} \varphi_k(t), \tag{5}$$

where $\varphi_k(t) = U^{\alpha}(k) t^{k\alpha}$.

Also, the m-approximation fractional differential transform of $u(t)$ is defined as

$$U^m(t) = \sum_{k=0}^{m} U^{\alpha}(k) t^{k\alpha} = \sum_{k=0}^{m} \varphi_k(t).$$

Theorem 1 ([39])**.** *Suppose that $F^{\alpha}(k)$, $G^{\alpha}(k)$, and $H^{\alpha}(k)$ are the differential transformations of the functions $f(t)$, $g(t)$, and $h(t)$, respectively. Then we have*

(a) *if* $f(t) = g(t) \pm h(t)$, *then* $F^{\alpha}(k) = G^{\alpha}(k) \pm H^{\alpha}(k)$,

(b) *if* $f(t) = (t - t_0)^q$, *then* $F^{\alpha}(k) = \delta(k - \frac{q}{\alpha})$, *where* $\delta(k) = \begin{cases} 1, & \text{if } k = 0 \\ 0. & \text{if } k \neq 0 \end{cases}$

(c) *if* $f(t) = g(t)h(t)$, *then* $F^{\alpha}(k) = \sum_{l=0}^{k} G^{\alpha}(l) H^{\alpha}(k - l)$,

Theorem 2 ([39])**.** *Suppose that* $f(t) = t^{\lambda} g(t)$, *where* $\lambda > -1$ *and* $g(t)$ *has the generalized power series expansion* $g(t) = \sum_{n=0}^{\infty} a_n (t - t_0)^{n\alpha}$ *with radius of convergence* $R > 0$, $0 < \alpha \leq 1$. *Then*

$$D_a^{\gamma} D_a^{\beta} f(t) = D_a^{\gamma+\beta} f(t),$$

for all $t \in (0, R)$ *if:*

(a) $\beta < \lambda + 1$ *and* α *arbitrary or*

(b) $\beta \geq \lambda + 1$, γ *arbitrary, and* $a_k = 0$ *for* $k = 0, 1, \ldots, m - 1$, *where* $m - 1 < \beta \leq m$.

Theorem 3 ([39])**.** *If* $f(t) = D_{t_0}^{\gamma} g(t)$, $m - 1 < \gamma \leq m$, *and the function* $g(t)$ *satisfies the conditions in theorem* (2), *then*

$$F^{\alpha}(k) = \frac{\Gamma(k\alpha + \gamma + 1)}{\Gamma(k\alpha + 1)} G^{\alpha}(k + \frac{\gamma}{\alpha}).$$

3. FD-FDTM for Solving the FTE

Consider the FTE (1). If the x-derivative at (x, t) is replaced by $\frac{1}{h^2}\{v(x - h, t) - 2v(x, t) + v(x + h, t)\} + O(h^2)$ and x is considered as a constant, Equation (1) can be written as the following ordinary differential equation

$$\frac{d^{2\gamma} v(t)}{dt^{2\gamma}} + 2\lambda \frac{d^{\gamma} v(t)}{dt^{\gamma}} + \mu v(t) = \nu \frac{1}{h^2}\{v(x - h, t) - 2v(x, t) + v(x + h, t)\} + O(h^2) + q(x, t). \tag{6}$$

We subdivide the interval $[a, b]$ into N equal subintervals of step-length $h = \frac{b - a}{N}$. Thus, the mesh points $x_i = a + ih$, $i = 0, 1, \ldots, N$ are obtained. Now, we write Equation (6) at the mesh point $x_i, i = 1, \ldots, N - 1$, along with time level t. If we discard the local truncation error $O(h^2)$ and denote $u_i(t)$ as the approximate solution of $v_i(t) = v(x_i, t)$, we have the following system of ODEs:

$$\frac{d^{2\gamma} u_i(t)}{dt^{2\gamma}} + 2\lambda \frac{d^{\gamma} u_i(t)}{dt^{\gamma}} + \mu u_i(t) = \frac{\nu}{h^2}\{u_{i-1}(t) - 2u_i(t) + u_{i+1}(t)\} + q_i(t),\ i = 1, \ldots, N - 1. \tag{7}$$

We solve the system (7) using FDTM. For this purpose, we consider the solution of equation, $u_i(t)$, as follows:

$$u_i(t) = \sum_{k=0}^{\infty} U_i^{\alpha}(k) t^{k\alpha}, \tag{8}$$

where the unknown coefficients $U_i^{\alpha}(k)$ are the FDT of $u_i(t)$ and should be obtained.

By choosing a suitable value for α, assuming $q_i^{\alpha}(k)$ as the α-fractional differential transform of $q_i(t)$, and by using Theorems 2 and 3, the fractional differential transform of Equation (7) leads to the following relation:

$$\begin{aligned} \frac{\Gamma(k\alpha + 2\gamma + 1)}{\Gamma(k\alpha + 1)} U_i^{\alpha}(k + \frac{2\gamma}{\alpha}) + 2\lambda \frac{\Gamma(k\alpha + \gamma + 1)}{\Gamma(k\alpha + 1)} U_i^{\alpha}(k + \frac{\gamma}{\alpha}) + \mu U_i^{\alpha}(k) = \\ \frac{\nu}{h^2}\{U_{i-1}^{\alpha}(k) - 2U_i^{\alpha}(k) + U_{i+1}^{\alpha}(k)\} + q_i^{\alpha}(k). \end{aligned} \tag{9}$$

Now, suppose that $G_1^\alpha(k)$ and $G_2^\alpha(k)$ are the fractional differential transforms of the functions $g_1(t)$ and $g_2(t)$, respectively. Therefore, by applying the FDTM to conditions (2) and (3), we have the initial conditions

$$\begin{aligned} U_i^\alpha(0) &= f_1(x_i), \\ U_i^\alpha(\frac{1}{\alpha}) &= f_2(x_i), \end{aligned} \tag{10}$$

and the boundary conditions

$$\begin{aligned} U_0^\alpha(k) &= G_1^\alpha(k), \\ U_N^\alpha(k) &= G_2^\alpha(k), \end{aligned} \tag{11}$$

We rewrite relations (9)–(11) as follows:

$$\begin{aligned} U_i^\alpha(k+\frac{2\gamma}{\alpha}) = \frac{\Gamma(k\alpha+1)}{\Gamma(k\alpha+2\gamma+1)}&\left[\nu\frac{U_{i-1}^\alpha(k)-2U_i^\alpha(k)+U_{i+1}^\alpha(k)}{h^2}-\mu U_i^\alpha(k)+q_i^\alpha(k)\right] \\ &-2\lambda\frac{\Gamma(k\alpha+\gamma+1)}{\Gamma(k\alpha+2\gamma+1)}U_i^\alpha(k+\frac{\gamma}{\alpha}), \qquad \forall k\geq 0 \end{aligned} \tag{12}$$

$$\begin{aligned} U_i^\alpha(0) &= f_1(ih), \\ U_i^\alpha(\frac{1}{\alpha}) &= f_2(ih), \end{aligned} \tag{13}$$

$$\begin{aligned} U_0^\alpha(k) &= G_1^\alpha(k), \\ U_N^\alpha(k) &= G_2^\alpha(k). \end{aligned} \tag{14}$$

Also, according to [9], the unknown coefficients $U_i^\alpha(1), U_i^\alpha(2), \ldots, U_i^\alpha(\frac{2\gamma}{\alpha}-1)$, will be available as follows:

$$U_i^\alpha(k) = \begin{cases} \dfrac{1}{\Gamma(k\alpha+1)}[\dfrac{d^{k\alpha}}{dt^{k\alpha}}u_i(t)]_{t=0}, & \text{if} \quad k\alpha\in\mathbb{Z}^+ \\ 0, & \text{if} \quad k\alpha\notin\mathbb{Z}^+ \end{cases} \qquad \forall k=0,1,\ldots(\frac{2\gamma}{\alpha}-1). \tag{15}$$

Therefore, all the unknown coefficients $U_i^\alpha(k), i=0,1,2,\ldots,N, \forall k\geq 0$ are calculated according to the recursive formula (12) and relations (13)–(15).

4. Convergence of FD-FDTM for FTE

Here, we discuss the convergence of FD-FDTM for solving the FTE (1). First, we present the following lemma [40].

Lemma 1. *Suppose that for some $k_0\in\mathbb{N}_0$ and for every $j\geq k_0$, there exist $0<\delta_j<1$ such that $\|\varphi_{j+1}\|\leq\delta_{j+1}\|\varphi_j\|$, $(\|\varphi_j\| = max_t|\varphi_j(t)|)$. Then the series $\sum_{k=0}^\infty \varphi_k(t)$ converges to $u(t)$.*

Proof. Consider the sequence $s_0, s_1, s_2, \ldots$, where $s_n = \sum_{k=0}^n \varphi_k(t)$. To prove the lemma, we show the sequence s_n is a Cauchy sequence in $(C[[0,1]], \|.\|_\infty)$.

For $0<\delta_j<1$, we can write

$$\|s_j - s_{j-1}\| = \|\varphi_j\| \leq \delta_j\|\varphi_{j-1}\| \leq \delta_j\delta_{j-1}\|\varphi_{j-2}\| \leq \ldots \leq \delta_j\delta_{j-1}\ldots\delta_{k_0}\|\varphi_{k_0}\|,$$

Thus, for $n \geq m \geq k_0$, we have

$$\|s_n - s_m\| = \| \sum_{j=m+1}^{n} (s_j - s_{j-1})\| \leq \sum_{j=m+1}^{n} \|s_j - s_{j-1}\| \leq \sum_{j=m+1}^{n} \delta_j \delta_{j-1} \dots \delta_{k_0} \|\varphi_{k_0}\|,$$

If we let $\delta = max\{\delta_{k_0}, \delta_{k_0+1}, \dots, \delta_m, \delta_{m+1}, \dots, \delta_n\}$, the following relation is obtained:

$$\|s_n - s_m\| \leq \sum_{j=m+1}^{n} \delta^{j-k_0+1} \|\varphi_{k_0}\| \leq \|\varphi_{k_0}\| \delta^{m-k_0+2} \left[1 + \delta + \delta^2 + \dots + \delta^{n-m-1}\right] = \frac{1-\delta^{n-m}}{1-\delta} \delta^{m-k_0+2} \|\varphi_{k_0}\|. \tag{16}$$

Since $0 \leq \delta < 1$, we can derive $\lim_{n,m\to\infty} \|s_n - s_m\| = 0$, which means$\{s_n\}_{n=0}^{\infty}$ is a Cauchy sequence in $(C[I], \|.\|_\infty)$. Since the space $C[I]$ with $\|.\|_\infty$ is a Banach space, we can derive that the series $\sum_{k=0}^{\infty} \varphi_k(t)$ is convergent to $u(t)$. □

Theorem 4. *Suppose that $v(x_i, t)$ is the exact solution of the FTE at point $(x_i, t), u_i(t)$ is the exact solution of Equation (7), and $U_i^m(t) = \sum_{k=0}^{m} \varphi_k(t)$, is the m-approximation of $u_i(t)$ as the approximate solution of FTE at point (x_i, t). Also, suppose for some $k_0 \in \mathbb{N}_0$ and for every $n \geq m \geq k_0, \exists 0 < \delta_i < 1$, such that $\|\varphi_{i+1}\| \leq \delta_{i+1}\|\varphi_i\|$, where $\|\varphi_i\| = max_t|\varphi_i(t)|$. Then the solution $U_i^m(t)$ converges to the exact solution, $v_i(t)$, as $m \to \infty$. Furthermore, for some $a < \xi < b$ the maximum absolute error of the m-series, $U_i^m(t)$, as an approximation of the FTE's exact solution satisfies the following relation:*

$$\|v(x_i, t) - U_i^m(t)\| \leq \frac{h^2}{12} \frac{\partial^4 u(\xi, t)}{\partial x^4} + \frac{1}{1-\delta} \delta^{m-k_0+2} \|\varphi_{k_0}\|,$$

where $\delta = max\{\delta_{k_0}, \delta_{k_0+1}, \dots, \delta_n\}$.

Proof. We can write

$$\|v(x_i, t) - U_i^m(t)\| = \|v(x_i, t) - u_i(t) + u_i(t) - U_i^m(t)\| \leq \|v(x_i, t) - u_i(t)\| + \|u_i(t) - U_i^m(t)\|, \tag{17}$$

where $v(x_i, t)$ and $u_i(t)$ are the solutions of Equations (6) and (7), respectively.

Also, Equation (6) has been obtained by replacing the second x-derivative of $v(x, t)$ with the central FD formula in Equation (1). Therefore, for some $a < \xi < b$ we can write

$$\|v(x_i, t) - u_i(t)\| \leq \frac{h^2}{12} \frac{\partial^4 u(\xi, t)}{\partial x^4} \tag{18}$$

From relation (16), for $n \geq m \geq k_0$, we have

$$\|s_n - s_m\| \leq \frac{1-\delta^{n-m}}{1-\delta} \delta^{m-k_0+2} \|\varphi_{k_0}\|,$$

and since $0 \leq \delta < 1$, then $1 - \delta^{n-m} < 1$, so we have

$$\|s_n - s_m\| \leq \frac{1}{1-\delta} \delta^{m-k_0+2} \|\varphi_{k_0}\|.$$

If n approaches ∞, then $s_n \to u_i(t)$ and we have

$$\|u_i(t) - s_m\| \leq \frac{1}{1-\delta} \delta^{m-k_0+2} \|\varphi_{k_0}\|,$$

in the other words,

$$\|u_i(t) - U_i^m(t)\| \leq \frac{1}{1-\delta} \delta^{m-k_0+2} \|\varphi_{k_0}\|. \tag{19}$$

By replacing relations (18) and (19) in relation (17), the theorem is proved. □

5. Numerical Examples

In this section, we give some examples to show the efficiency and convenience of the mentioned method. The examples include linear and non-linear FTEs. We present the results of FD-FDTM for solving the examples and calculate the maximum absolute error (MAE) for different values of N using the following formula:

$$E(N) = \max_{1 \leq i \leq N} |v(x_i, t) - U_i^m(t)|.$$

Also, we compare the results of FD-FDTM with two-dimensional FDTM (2D-FDTM). Moreover, we obtain the rate of convergence (ROC) of FD-FDTM with the following formula:

$$ROC = \log_2\left(\frac{E(N)}{E(2N)}\right).$$

Example 1. *Consider the following linear FTE [30]:*

$$\frac{\partial^{2\gamma} v(x,t)}{\partial t^{2\gamma}} + 2\frac{\partial^{\gamma} v(x,t)}{\partial t^{\gamma}} + v(x,t) = \frac{\partial^2 v(x,t)}{\partial x^2},\ 0 < \gamma < 1,\ 0 < x < 1,\ t > 0, \tag{20}$$

with the following conditions

$$v(x,0) = e^x, v_t(x,0) = -2e^x, \tag{21}$$

$$v(0,t) = e^{-2t}, v(1,t) = e^{1-2t}, \tag{22}$$

which has the exact solution $v(x,t) = e^{x-2t}$ for $\gamma = 1$.

We describe the mentioned method to solve Equations (20)–(22) for $\gamma = 0.75$, $\alpha = 0.25$, and $h = 0.1$ ($N = 10$). Using Theorem 3, we can obtain the differential transform of the derivatives in Equation (20) as follows:

$$\begin{aligned}
\frac{\partial^{1.5} v(x,t)}{\partial t^{1.5}} &\to \frac{\Gamma(0.25k + 2.5)}{\Gamma(0.25k + 1)} U_i^{0.25}(k+6), \\
\frac{\partial^{0.75} v}{\partial t^{0.75}} &\to \frac{\Gamma(0.25k + 1.75)}{\Gamma(0.25k + 1)} U_i^{0.25}(k+3), \\
v(x,t) &\to U_i^{0.25}(k), \\
\frac{\partial^2 v}{\partial x^2} &\to \frac{U_{i-1}^{0.25}(k) - 2U_i^{0.25}(k) + U_{i+1}^{0.25}(k)}{h^2}.
\end{aligned}$$

According to relation (13), for the initial conditions we have

$$\begin{aligned}
v(x,0) = e^x &\to U_i^{0.25}(0) = e^{x_i}, \forall i = 0, 1, \ldots, 10, \\
v_t(x,0) = -2e^x &\to U_i^{0.25}(4) = -2e^{x_i}, \forall i = 0, 1, \ldots, 10,
\end{aligned}$$

and according to relation (15), we have

$$U_i^{0.25}(1) = U_i^{0.25}(2) = U_i^{0.25}(3) = U_i^{0.25}(5) = 0,\ \forall i = 0, 1, 2, \ldots, 10.$$

Also, we use relation (14) for the boundary conditions, and for the right side of these relations, we use the FDT of the e^{-2t}, e^{1-2t} functions, so we obtain

$$v(0,t) = e^{-2t} \to U_0^{0.25}(k) = \begin{cases} \dfrac{(-2)^{\frac{k}{4}}}{\Gamma(\frac{k}{4}+1)}, & \text{if} \quad \dfrac{k}{4} \in \mathbb{Z}^+ \\ & \qquad\qquad \forall k \geq 0 \\ 0. & \text{if} \quad \dfrac{k}{4} \notin \mathbb{Z}^+ \end{cases}$$

$$v(1,t) = e^{1-2t} \to U_{10}^{0.25}(k) = \begin{cases} e \times \dfrac{(-2)^{\frac{k}{4}}}{\Gamma(\frac{k}{4}+1)}, & \text{if} \quad \dfrac{k}{4} \in \mathbb{Z}^+ \\ & \qquad\qquad \forall k \geq 0 \\ 0. & \text{if} \quad \dfrac{k}{4} \notin \mathbb{Z}^+ \end{cases}$$

By replacing the above relations in Equation (20), we obtain the following recursive relationship:

$$U_i^{0.25}(k+6) = \frac{\Gamma(0.25k+1)}{\Gamma(0.25k+2.5)}\left[\frac{U_{i-1}^{0.25}(k) - 2U_i^{0.25}(k) + U_{i+1}^{0.25}(k)}{h^2} - U_i^{0.25}(k)\right] - 2\frac{\Gamma(0.25k+1.75)}{\Gamma(0.25k+2.5)}U_i^{0.25}(k+3). \qquad \forall i \geq 1$$

Thus, the solution in (x_i, t), $i = 0, 1, \ldots, 10$ is obtained as follows:

$$\begin{aligned} x_0 = 0, \quad u_0(t) &= \sum_{k=0}^{\infty} U_0^{0.25}(k)t^{0.25k} = U_0^{0.25}(0) + U_0^{0.25}(1)t^{0.25} + U_0^{0.25}(2)t^{0.5} + U_0^{0.25}(3)t^{0.75} + U_0^{0.25}(4)t + \ldots, \\ &= 1 + 0t^{0.25} + 0t^{0.5} + 0t^{0.75} - 2t + \ldots, \\ x_1 = 0.1, \quad u_1(t) &= \sum_{k=0}^{\infty} U_1^{0.25}(k)t^{0.25k} = U_1^{0.25}(0) + U_1^{0.25}(1)t^{0.25} + U_1^{0.25}(2)t^{0.5} + U_1^{0.25}(3)t^{0.75} + U_1^{0.25}(4)t + \ldots \\ &= e^{0.1} + 0t^{0.25} + 0t^{0.5} + 0t^{0.75} - 2e^{0.1}t + \ldots, \\ &\vdots \\ x_{10} = 1, \quad u_{10}(t) &= \sum_{k=0}^{\infty} U_{10}^{0.25}(k)t^{0.25k} = U_{10}^{0.25}(0) + U_{10}^{0.25}(1)t^{0.25} + U_{10}^{0.25}(2)t^{0.5} + U_{10}^{0.25}(3)t^{0.75} + U_{10}^{0.25}(4)t + \ldots, \\ &= e + 0t^{0.25} + 0t^{0.5} + 0t^{0.75} - 2et + \ldots. \end{aligned}$$

We show the results of our method for solving Example 1 in Tables 1 and 2. Table 1 contains the maximum absolute error of the obtained solution using the FD-FDTM for $\gamma = 1$, $m = 3$, and different values of N at $t = 0.001$. Also, we compared the results of FD-FDTM with two-dimensional FDTM. Table 1 shows the our method is more accurate than the two-dimensional DTM. Also, we can see that as N increases, the error decreases and the numerical ROC confirms the theoretical ROC. Table 2 compares the approximate solution of FD-FDTM and 2D-FDTM at $t = 0.01$, for $\gamma = 0.75$, $m = 10$, and $N = 10$. Figure 1 shows the comparison between the exact solution for $\gamma = 1$ and the results of FD-FDTM for $\gamma = 0.5, 0.7, 0.8, 0.9$.

Table 1. The MAE for Example 1 at $t = 0.001$, for $\gamma = 1$, $m = 3$, and different values of N.

N	MAE of FD-FDTM $(\gamma = 1)$ for $\alpha = 1$	ROC	CPU Time	MAE of 2D-DTM $(\gamma = 1)$ for $\alpha = 1$	CPU Time
10	1×10^{-9}	—	0.11	3.3×10^{-2}	0.09
20	2.7×10^{-10}	1.89	0.11	4.1×10^{-2}	0.09
40	6.7×10^{-11}	2.01	0.11	4.6×10^{-2}	0.09
80	1.5×10^{-11}	2.16	0.11	4.8×10^{-2}	0.11
160	2.6×10^{-12}	2.51	0.16	5.01×10^{-2}	0.14

Table 2. The approximate solution of Example 1 obtained with FD-FDTM and 2D-FDTM at $t = 0.01$, for $\gamma = 0.75$, $\alpha = 0.25$, $m = 10$, and $N = 10$.

x_i	FD-FDTM $\gamma = 0.75$	2D-FDTM $\gamma = 0.75$	Exact Solution $\gamma = 1$
0.1	1.08391073378	1.08391006915	1.08328706767
0.2	1.19790662077	1.19790588623	1.19721736312
0.3	1.32389155984	1.32389074805	1.32312981233
0.4	1.46312645063	1.46312555346	1.46228458943
0.5	1.617004802	1.61700381117	1.61607440219
0.6	1.7870666823	1.7870655864	1.78603843075
0.7	1.9750141259	1.9750129144	1.9738777322
0.8	2.1827281748	2.1827268341	2.1814722654
0.9	2.41228770087	2.412286213	2.4108997064

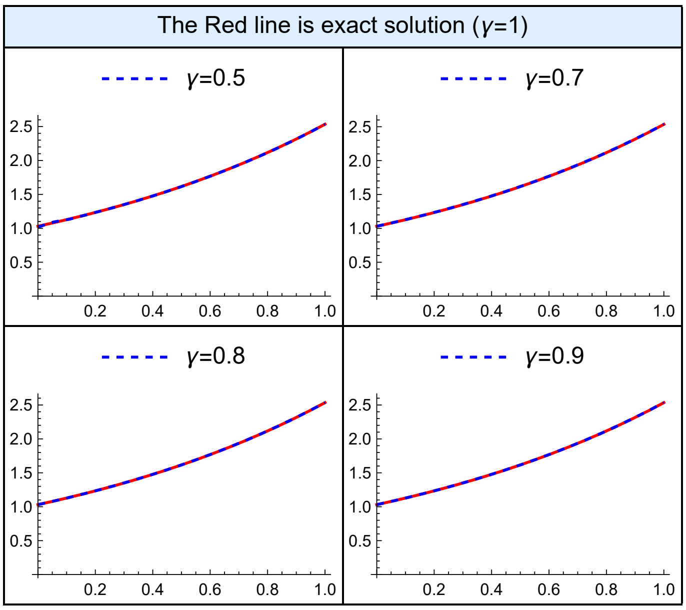

Figure 1. Comparison between the exact solution of Example 1 for $\gamma = 1$ and numerical solutions for $\gamma = 0.5, 0.7, 0.8, 0.9$.

Example 2. *In this example, we consider a non-homogeneous FTE [17]*

$$\frac{\partial^{2\gamma} v(x,t)}{\partial t^{2\gamma}} + 40\frac{\partial^{\gamma} v(x,t)}{\partial t^{\gamma}} + 100v(x,t) = \frac{\partial^2 v(x,t)}{\partial x^2} + 23e^{-2t}sinh(x),\ 0 < \gamma < 1,\ 0 < x < 1,\ t > 0, \tag{23}$$

with the following conditions:

$$v(x,0) = sinh(x), v_t(x,0) = -2sinh(x), \tag{24}$$

$$v(0,t) = 0, v(1,t) = e^{-2t} sinh(1). \tag{25}$$

The exact solution of (23) for $\gamma = 1$ with conditions (24) and (25) is $v(x,t) = e^{-2t} sinh(x)$.

For example, we put $\gamma = 0.6$ in (23) and consider $h = 0.1$ and $\alpha = 0.1$. For these values of γ, h, and α, we can write

$$\begin{aligned}
\frac{\partial^{1.2} v(x,t)}{\partial t^{1.2}} &\to \frac{\Gamma(0.1k + 2.2)}{\Gamma(0.1k+1)} U_i^{0.1}(k+12), \\
\frac{\partial^{0.6} v}{\partial t^{0.6}} &\to \frac{\Gamma(0.1k + 1.6)}{\Gamma(0.1k+1)} U_i^{0.1}(k+6), \\
v(x,t) &\to U_i^{0.1}(k), \\
\frac{\partial^2 v}{\partial x^2} &\to \frac{U_{i-1}^{0.1}(k) - 2U_i^{0.1}(k) + U_{i+1}^{0.1}(k)}{h^2}.
\end{aligned}$$

According to relation (13), for the initial conditions we have:

$$\begin{aligned}
v(x,0) = sinh(x) &\to U_i^{0.1}(0) = sinh(x_i), \ \forall i = 0,1,2,\dots,10, \\
v_t(x,0) = -2sinh(x) &\to U_i^{0.1}(10) = -2sinh(x_i), \ \forall i = 0,1,2,\dots,10,
\end{aligned}$$

and according to (15), we have

$$U_i^{0.1}(1) = U_i^{0.1}(2) = \dots = U_i^{0.1}(9) = U_i^{0.1}(11) = 0, \ \forall i = 0,1,\dots,10.$$

We use relation (14) for the boundary conditions, and for the right side of the boundary conditions, we use the FDT of the $e^{-2t} sinh(1)$ function. Therefore, we have

$$v(0,t) = 0 \to U_0^{0.1}(k) = 0, \forall k \geq 0$$

$$U_{10}^{0.1}(k) = \begin{cases} sinh(1) \times \dfrac{(-2)^{\frac{k}{10}}}{\Gamma(\frac{k}{10}+1)}, & if \quad \dfrac{k}{10} \in \mathbb{Z}^+ \\ 0. & if \quad \dfrac{k}{10} \notin \mathbb{Z}^+ \end{cases} \quad \forall k \geq 0$$

By putting the above relations in Equation (23), we conclude the following recursive relationship:

$$\begin{aligned}
U_i^{0.1}(k+12) = \frac{\Gamma(0.1k+1)}{\Gamma(0.1k+2.2)} &\left[23E^{0.1}(k) sinh(x_i) - 100U_i^{0.1}(k) + \frac{U_{i-1}^{0.1}(k) - 2U_i^{0.1}(k) + U_{i+1}^{0.1}(k)}{h^2} \right] \\
&- 40 \frac{\Gamma(0.1k+1.6)}{\Gamma(0.1k+2.2)} U_i^{0.1}(k+6),
\end{aligned}$$

where $E^{0.1}(k)$ is the fractional differential transform of e^{-2t} and can be obtained as follows:

$$E^{0.1}(k) = \begin{cases} \dfrac{(-2)^{\frac{k}{10}}}{\Gamma(\frac{k}{10}+1)}, & if \quad \dfrac{k}{10} \in \mathbb{Z}^+ \\ 0. & if \quad \dfrac{k}{10} \notin \mathbb{Z}^+ \end{cases} \quad \forall k \geq 0$$

Thus, in the points $(x_i, t), i = 0,1,\dots,N$, we can write:

$$
\begin{aligned}
x_0 = 0, \quad u_0(t) &= \sum_{k=0}^{\infty} U_0^{0.1}(k) t^{0.1k} = U_0^{0.1}(0) + U_0^{0.1}(1) t^{0.1} + U_0^{0.1}(2) t^{0.2} + U_0^{0.1}(3) t^{0.3} + U_0^{0.1}(4) t^{0.4} + \dots, \\
&= 0 + 0t^{0.1} + 0t^{0.2} + 0t^{0.3} + 0t^{0.4} + \dots,
\end{aligned}
$$

$$
\begin{aligned}
x_1 = 0.1, \quad u_1(t) &= \sum_{k=0}^{\infty} U_1^{0.1}(k) t^{0.1k} = U_1^{0.1}(0) + U_1^{0.1}(1) t^{0.1} + U_1^{0.1}(2) t^{0.2} + \dots + U_1^{0.1}(9) t^{0.9} + U_1^{0.1}(10) t + \dots, \\
&= sinh(0.1) + 0t^{0.1} + 0t^{0.2} + \dots + 0t^{0.9} - 2sinh(0.1)t + \dots, \\
&\vdots \\
x_{10} = 1, \quad u_{10}(t) &= \sum_{k=0}^{\infty} U_{10}^{0.1}(k) t^{0.1k} = U_{10}^{0.1}(0) + U_{10}^{0.1}(1) t^{0.1} + U_{10}^{0.1}(2) t^{0.2} + \dots + U_{10}^{0.1}(9) t^{0.9} + U_{10}^{0.1}(10) t + \dots, \\
&= sinh(1) + 0t^{0.1} + 0t^{0.2} + \dots + 0t^{0.9} - 2sinh(1)t + \dots.
\end{aligned}
$$

To compare the exact and numerical solution of Example 2 obtained using FD-FDTM and 2D-FDTM, we compute the MAE of the obtained solution at $t = 0.001$, and present them in Table 3. Also, we show the results of the FD-FDTM and 2D-FDTM at $t = 0.001$, for $\gamma = 0.6$, $m = 15$, and $N = 10$ in Table 4. Figure 2 shows the comparison between the exact solution for $\gamma = 1$ and the numerical solution for $\gamma = 0.5, 0.7, 0.8, 0.9$.

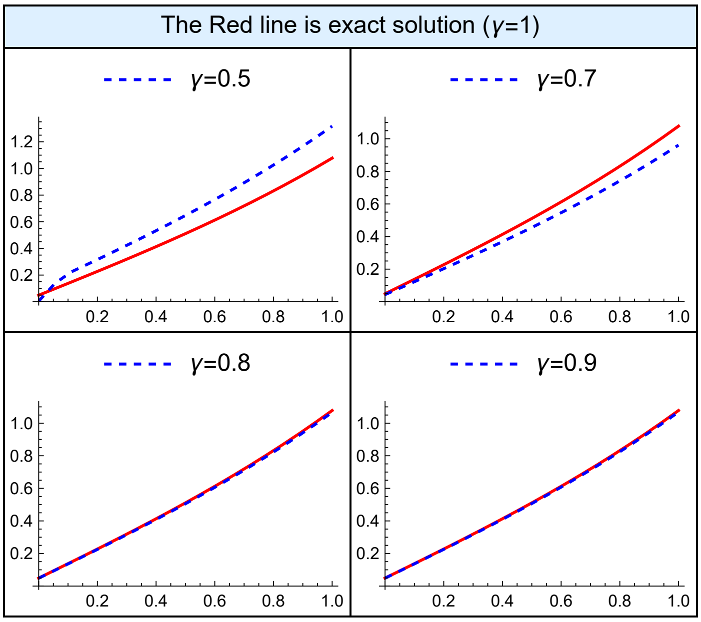

Figure 2. Comparison between the exact solution of Example 2 for $\gamma = 1$ and numerical solutions for $\gamma = 0.5, 0.7, 0.8, 0.9$.

Table 3. The MAE for Example 2 at $t = 0.001$, for $\gamma = 1$, $m = 3$, and different values of N.

N	MAE of FD-FDTM for $\alpha = 1$	ROC	CPU Time	MAE of 2D-FDTM for $\alpha = 1$	CPU Time
10	4.2×10^{-10}	—	0.11	9.1×10^{-1}	0.14
20	1.1×10^{-10}	1.8	0.12	9.5×10^{-1}	0.12
40	2.8×10^{-11}	1.97	0.11	9.7×10^{-1}	0.11
80	6.7×10^{-12}	2.06	0.14	9.8×10^{-1}	0.11
160	1.09×10^{-12}	2.61	0.17	9.9×10^{-1}	0.17

Table 4. The approximate solution for Example 2 at $t = 0.001$, for $\gamma = 0.6$, $\alpha = 0.1$, $m = 15$, and values of $N = 10$.

x_i	FD-FDTM $\gamma = 0.6$	2D-FDTM $\gamma = 0.6$	Exact Solution $\gamma = 1$
0.1	0.097705	0.0164227	0.099966
0.2	0.19633389	0.0330099	0.2009337
0.3	0.297038	0.049763	0.30391186
0.4	0.40066	0.0666829	0.409931642
0.5	0.508292	0.083771	0.52005415
0.6	0.621011	0.101028	0.63538154
0.7	0.739945	0.118456	0.757068
0.8	0.866285	0.136056	0.8863315
0.9	1.0013	0.153829	1.024465743

Example 3. *In this example, we solve the following nonlinear FTE [23]:*

$$\frac{\partial^{2\gamma} v(x,t)}{\partial t^{2\gamma}} + 2\frac{\partial^{\gamma} v(x,t)}{\partial t^{\gamma}} + u^2(x,t) = \frac{\partial^2 v(x,t)}{\partial x^2} + e^{2x-4t} - e^{x-2t},\ 0 < \gamma < 1,\ 0 < x < 1,\ t > 0, \tag{26}$$

with the following conditions

$$v(x,0) = e^x, v_t(x,0) = -2e^x, \tag{27}$$

$$v(0,t) = e^{-2t}, v(1,t) = e^{1-2t}. \tag{28}$$

The exact solution of Equation (26) for $\gamma = 1$ with conditions (27) and (28) is $v(x,t) = e^{x-2t}$.

To explain the FD-FDTM, we put $\gamma = 0.75$ in (26) and consider $h = 0.1$ and $\alpha = 0.25$. For these values of $\gamma, h,$ and α, we can write

$$\begin{aligned}
\frac{\partial^{1.5} v(x,t)}{\partial t^{1.5}} &\to \frac{\Gamma(0.25k+2.5)}{\Gamma(0.25k+1)} U_i^{0.25}(k+6), \\
\frac{\partial^{0.75} v}{\partial t^{0.75}} &\to \frac{\Gamma(0.25k+1.75)}{\Gamma(0.25k+1)} U_i^{0.25}(k+3), \\
v(x,t) &\to U_i^{0.25}(i,k), \\
\frac{\partial^2 v}{\partial x^2} &\to \frac{U_{i-1}^{0.25}(k) - 2U_i^{0.25}(k) + U_{i+1}^{0.25}(k)}{h^2}.
\end{aligned}$$

According to relation (13), for the initial conditions we have

$$\begin{aligned}
v(x,0) = e^x &\to U_i^{0.25}(0) = e^{x_i}, \forall i = 0,1,\ldots,10, \\
v_t(x,0) = -2e^x &\to U_i^{0.25}(4) = -2e^{x_i}, \forall i = 0,1,\ldots,10.
\end{aligned}$$

From relationship (15), we can write

$$U_i^{0.25}(1) = U_i^{0.25}(2) = U_i^{0.25}(3) = U_i^{0.25}(5) = 0,\ \forall i = 0,1,2,\ldots,10.$$

We use relation (14) for the boundary conditions and an FDT of e^{-2t}, e^{1-2t} for the right side of the boundary conditions. Therefore, we have

$$v(0,t)=e^{-2t}\to U_0^{0.25}(k)=\begin{cases}\dfrac{(-2)^{\frac{k}{4}}}{\Gamma(\frac{k}{4}+1)}, & \text{if} \quad \dfrac{k}{4}\in\mathbb{Z}^+\\ 0. & \text{if} \quad \dfrac{k}{4}\notin\mathbb{Z}^+\end{cases}\quad \forall k\geq 0$$

$$v(1,t)=e^{1-2t}\to U_{10}^{0.25}(k)=\begin{cases}e\times\dfrac{(-2)^{\frac{k}{4}}}{\Gamma(\frac{k}{4}+1)}, & \text{if} \quad \dfrac{k}{4}\in\mathbb{Z}^+\\ 0. & \text{if} \quad \dfrac{k}{4}\notin\mathbb{Z}^+\end{cases}\quad \forall k\geq 0$$

In Example 3, we have $q(x,t)=e^{2x-4t}-e^{x-2t}$, so by using FD-FDTM, we have:

$$q_i^{0.25}(k)=e^{2x_i}\times E_1^{0.25}(k)-e^{x_i}\times E_2^{0.25}(k),\forall i=0,1,\ldots,10,$$

where $E_1^{0.25}(k)$ and $E_2^{0.25}(k)$ are the FDT of the e^{-4t} and e^{-2t} functions, respectively, and are obtained as follows:

$$E_1^{0.25}(k)=\begin{cases}\dfrac{(-4)^{\frac{k}{4}}}{\Gamma(\frac{k}{4}+1)}, & \text{if} \quad \dfrac{k}{4}\in\mathbb{Z}^+\\ 0, & \text{if} \quad \dfrac{k}{4}\notin\mathbb{Z}^+\end{cases}\quad \forall k\geq 0$$

and

$$E_2^{0.25}(k)=\begin{cases}\dfrac{(-2)^{\frac{k}{4}}}{\Gamma(\frac{k}{4}+1)}, & \text{if} \quad \dfrac{k}{4}\in\mathbb{Z}^+\\ 0. & \text{if} \quad \dfrac{k}{4}\notin\mathbb{Z}^+\end{cases}\quad \forall k\geq 0$$

By substituting the above relations in Equation (26), we obtain the following recursive formula

$$U_i^{0.25}(k+6)=\frac{\Gamma(0.25k+1)}{\Gamma(0.25k+2.5)}\left[q_i^{0.25}(k)+\frac{U_{i-1}^{0.25}(k)-2U_i^{0.25}(k)+U_{i+1}^{0.25}(k)}{h^2}-\sum_{j=0}^{k}U_i^{0.25}(j)U_i^{0.25}(k-j)\right]$$
$$-2\frac{\Gamma(0.25k+1.75)}{\Gamma(0.25k+2.5)}U_i^{0.25}(k+3).\qquad \forall i\geq 1$$

In the points $(x_i,t),i=0,1,\ldots,N$, we have:

$$
\begin{aligned}
x_0 = 0, \quad u_0(t) &= \sum_{k=0}^{\infty} U_0^{0.25}(k)t^{0.25k} = U_0^{0.25}(0) + U_0^{0.25}(1)t^{0.25} + U_0^{0.25}(2)t^{0.5} + U_0^{0.25}(3)t^{0.75} + U_0^{0.25}(4)t + \dots \\
&= 1 + 0t^{0.25} + 0t^{0.5} + 0t^{0.75} - 2t + \dots, \\
x_1 = 0.1, \quad u_1(t) &= \sum_{k=0}^{\infty} U_1^{0.25}(k)t^{0.25k} = U_1^{0.25}(0) + U_1^{0.25}(1)t^{0.25} + U_1^{0.25}(2)t^{0.5} + U_1^{0.25}(3)t^{0.75} + U_1^{0.25}(4)t + \dots \\
&= e^{0.1} + 0t^{0.25} + 0t^{0.5} + 0t^{0.75} - 2e^{0.1}t + \dots, \\
&\vdots \\
x_{10} = 1, \quad u_{10}(t) &= \sum_{k=0}^{\infty} U_{10}^{0.25}(k)t^{0.25k} = U_{10}^{0.25}(0) + U_{10}^{0.25}(1)t^{0.25} + U_{10}^{0.25}(2)t^{0.5} + U_{10}^{0.25}(3)t^{0.75} + U_{10}^{0.25}(4)t + \dots \\
&= e + 0t^{0.25} + 0t^{0.5} + 0t^{0.75} - 2et + \dots.
\end{aligned}
$$

We show the results of our method for solving Example 3 in Table 5. Table 5 contains the MAE of the solution obtained using FD-FDTM, for $\gamma = 1$, $m = 3$, and different values of N at $t = 0.01$. Also, we compared the results of FD-FDTM with two-dimensional FDTM. Table 5 shows that the FD-FDTM is more accurate than the two-dimensional FDTM. Figure 3 compares the exact solution for $\gamma = 1$ with the numerical solution for $\gamma = 0.5, 0.7, 0.8, 0.9$.

Table 5. The MAE for Example 3 at $t = 0.01$, for $\gamma = 1$, $m = 3$, and different values of N.

N	MAE of FD-FDTM for $\alpha = 1$	CPU Time	MAE of 2D-DTM for $\alpha = 1$	CPU Time
10	7.04×10^{-9}	0.11	3.1×10^{-1}	0.11
20	8.64×10^{-9}	0.12	4.1×10^{-2}	0.11
40	9.3×10^{-9}	0.12	4.62×10^{-2}	0.11
80	9.59×10^{-9}	0.16	4.88×10^{-2}	0.11
160	9.72×10^{-9}	0.17	5.01×10^{-2}	0.12

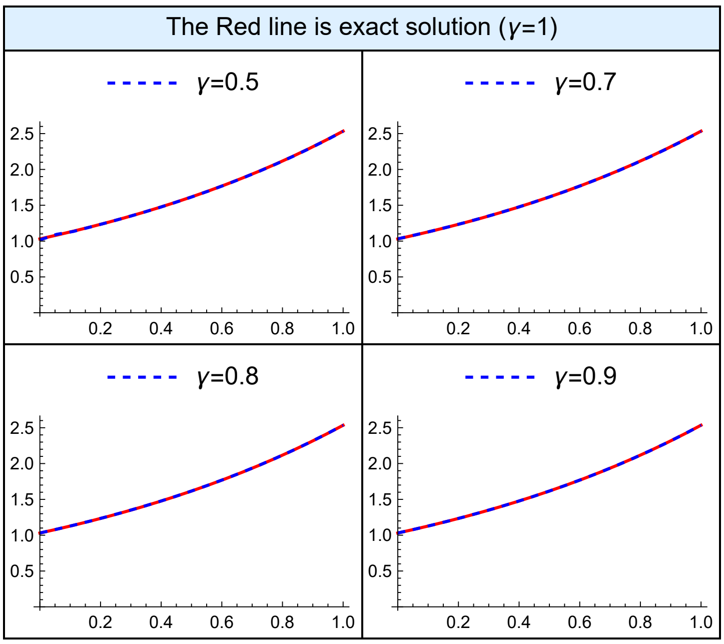

Figure 3. Comparison between the exact solution of Example 3 for $\gamma = 1$ and numerical solutions for $\gamma = 0.5, 0.7, 0.8, 0.9$.

Example 4. *Consider the following time-fractional multi-term wave equation [32]:*

$$\frac{\partial^{1.7} v(x,t)}{\partial t^{1.7}} - \frac{1}{2}\frac{\partial v(x,t)}{\partial t} = \frac{\partial^2 v(x,t)}{\partial x^2},\ 0 < x < 4,\ t > 0, \tag{29}$$

with initial conditions

$$v(x,0) = sin(\pi x), v_t(x,0) = 0, \tag{30}$$

and boundary conditions

$$v(0,t) = 0, v(4,t) = 0, \tag{31}$$

We describe the mentioned method to solve Equations (29)–(31) for $\alpha = 0.1$ By using theorem (3), we can obtain the differential transform of the derivatives in Equation (29) as follows:

$$\begin{aligned}
\frac{\partial^{1.7} v(x,t)}{\partial t^{1.7}} &\to \frac{\Gamma(0.1k+2.7)}{\Gamma(0.1k+1)} U_i^{0.1}(k+17), \\
\frac{\partial v}{\partial t} &\to \frac{\Gamma(0.1k+2)}{\Gamma(0.1k+1)} U_i^{0.1}(k+10), \\
\frac{\partial^2 v}{\partial x^2} &\to \frac{U_{i-1}^{0.1}(k) - 2U_i^{0.1}(k) + U_{i+1}^{0.1}(k)}{h^2}.
\end{aligned}$$

According to relation (13), for the initial conditions we have

$$\begin{aligned}
v(x,0) = sin(\pi x) &\to U_i^{0.1}(0) = sin(\pi x_i), \forall i = 0,1,\ldots,N, \\
v_t(x,0) = 0 &\to U_i^{0.1}(10) = 0, \forall i = 0,1,\ldots,N,
\end{aligned}$$

and according to relation (15) we have

$$U_i^{0.1}(1) = U_i^{0.1}(2) = \ldots = U_i^{0.1}(15) = U_i^{0.1}(16) = 0,\ \forall i = 0,1,\ldots,N,$$

and

$$\begin{aligned}
v(0,t) = 0 &\to U_0^{0.1}(k) = 0, \\
v(4,t) = 0 &\to U_N^{0.1}(k) = 0.
\end{aligned}$$

By replacing the above relations in Equation (29), we obtain the following recursive relationship:

$$\begin{aligned}
U_i^{0.1}(k+17) = \frac{\Gamma(0.1k+1)}{\Gamma(0.1k+2.7)} \Bigg(&\frac{U_{i-1}^{0.1}(k) - 2U_i^{0.1}(k) + U_{i+1}^{0.1}(k)}{h^2} \Bigg) \\
&+ \frac{1}{2}\frac{\Gamma(0.1k+2)}{\Gamma(0.1k+2.5)} U_i^{0.1}(k+10). \qquad \forall i \geq 1
\end{aligned}$$

We show the results of our method for solving Example 4 in Figures 4 and 5 for $t = 1$.

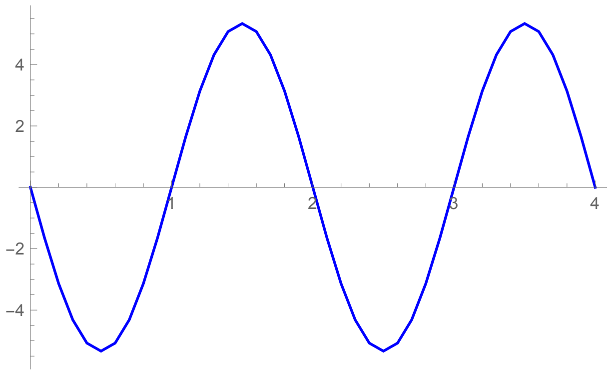

Figure 4. Solution of Example 4 for $N = 40,\ m = 20$.

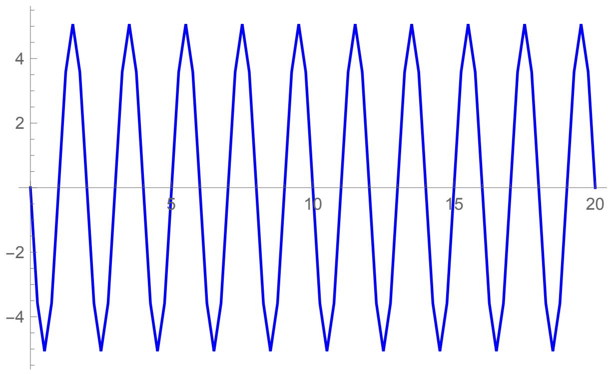

Figure 5. Solution of Example 4 for $N = 80,\ m = 20$.

6. Conclusions

In this work, a hybrid method has been used to solve the linear and non-linear time-fractional telegraph equation approximately. The central difference method has been applied to discretize the spatial derivative and the fractional differential transform method has been used to solve the obtained system of fractional ordinary equations. A convergence analysis of the mentioned method has been conducted. The numerical results show that the proposed hybrid method is more accurate and effective than two-dimensional FDTM. Also, the implementation process of this method is very simple, so its computer programming is very fast.

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References

1. Chen, C.K.; Ho, S.H. Application of differential transformation to eigenvalue problems. *Appl. Math. Comput.* **1996**, *79*, 173–188. [CrossRef]
2. Zhou, J.K. *Differential Transformation and Its Applications for Electrical Circuits*; Huazhong University Press: Wuhan, China, 1986.
3. Jang, M.J.; Chen, C.L.; Liy, Y.C. On solving the initial-value problems using the differential transformation method. *Appl. Math. Comput.* **2000**, *115*, 145–160. [CrossRef]
4. Ayaz, F. Applications of differential transform method to differential-algebraic equations. *Appl. Math. Comput.* **2004**, *152*, 649–657. [CrossRef]

5. Hassan, I.A.H. On solving some eigenvalue problems by using a differential transformation. *Appl. Math. Comput.* **2002**, *127*, 1–22.
6. Alquran, M.T. Applying differential transform method to nonlinear partial differential equations: A modified approach. *Appl. Appl. Math. Int. J. (AAM)* **2012**, *7*, 10.
7. Mirzaee, F.; Yari, M.K. A novel computing three-dimensional differential transform method for solving fuzzy partial differential equations. *Ain Shams Eng. J.* **2016**, *7*, 695–708. [CrossRef]
8. Ertürk, V.S.; Momani, S. Solving systems of fractional differential equations using differential transform method. *J. Comput. Appl. Math.* **2008**, *215*, 142–151. [CrossRef]
9. Odibat, Z.; Momani, S. A generalized differential transform method for linear partial differential equations of fractional order. *Appl. Math. Lett.* **2008**, *21*, 194–199. [CrossRef]
10. Ibis, B.; Bayram, M.; Agargun, A.G. Applications of fractional differential transform method to fractional differential-algebraic equations. *Eur. J. Pure Appl. Math.* **2011**, *4*, 129–141.
11. Ertürk, V.S. Computing eigenelements of Sturm–Liouville problems of fractional order via fractional differential transform method. *Math. Comput. Appl.* **2011**, *16*, 712–720. [CrossRef]
12. Secer, A.; Akinlar, M.A.; Cevikel, A. Efficient solutions of systems of fractional PDEs by the differential transform method. *Adv. Differ. Equ.* **2012**, *2012*, 1–7. [CrossRef]
13. Zain, F.A.S. Comparison study between differential transform method and Adomian decomposition method for some delay differential equations. *Int. J. Phys. Sci.* **2013**, *8*, 744–749.
14. Rahimi, E.; Taghvafard, H.; Erjaee, G.H. Fractional differential transform method for solving a class of weakly singular Volterra integral equations. *Iran. J. Sci. Technol.* **2014**, *38*, 69.
15. Hosseini, V.R.; Chen, W.; Avazzadeh, Z. Numerical solution of fractional telegraph equation by using radial basis functions. *Eng. Anal. Bound. Elem.* **2014**, *38*, 31–39. [CrossRef]
16. Asghari, M.; Ezzati, R.; Allahviranloo, T. Numerical solutions of time-fractional order telegraph equation by Bernstein polynomials operational matrices. *Math. Probl. Eng.* **2016**, *2016*, 1683849. [CrossRef]
17. Akram, T.; Abbas, M.; Ismail, A.I.; Ali, N.H.M.; Baleanu, D. Extended cubic B-splines in the numerical solution of time fractional telegraph equation. *Adv. Differ. Equ.* **2019**, *2019*, 365. [CrossRef]
18. Orsingher, E.; Beghin, L. Time-fractional telegraph equations and telegraph processes with Brownian time. *Probab. Theory Relat. Fields* **2004**, *128*, 141–160. [CrossRef]
19. Chen, J.; Liu, F.; Anh, V. Analytical solution for the time-fractional telegraph equation by the method of separating variables. *J. Math. Anal. Appl.* **2008**, *338*, 1364–1377. [CrossRef]
20. Biazar, J.; Eslami, M. Analytic solution for telegraph equation by differential transform method. *Phys. Lett. A* **2010**, *374*, 2904–2906. [CrossRef]
21. Garg, M.; Manohar, P.; Kalla, S.L. Generalized differential transform method to space-time fractional telegraph equation. *Int. J. Differ. Equ.* **2011**, *2011*, 548982. [CrossRef]
22. Zhao, Z.; Li, C. Fractional difference/finite element approximations for the time-space fractional telegraph equation. *Appl. Math. Comput.* **2012**, *219*, 2975–2988. [CrossRef]
23. Srivastava, V.K.; Awasthi, M.K.; Tamsir, M. RDTM solution of Caputo time fractional-order hyperbolic telegraph equation. *AIP Adv.* **2013**, *3*, 032142. [CrossRef]
24. Kumar, S. A new analytical modelling for fractional telegraph equation via Laplace transform. *Appl. Math. Model.* **2014**, *38*, 3154–3163. [CrossRef]
25. Joice Nirmala, R.; Balachandran, K. Analysis of solutions of time fractional telegraph equation. *J. Korean Soc. Ind. Appl. Math.* **2014**, *18*, 209–224. [CrossRef]
26. Saadatmandi, A.; Mohabbati, M. Numerical solution of fractional telegraph equation via the tau method. *Math. Rep.* **2015**, *17*, 155–166.
27. Li, H. A new analytical modelling for fractional telegraph equation via Elzaki transform. *J. Adv. Math.* **2015**, *11*, 5617–5625. [CrossRef]
28. Dhunde, R.R.; Waghmare, G.L. Double Laplace transform method for solving space and time fractional telegraph equations. *Int. J. Math. Math. Sci.* **2016**, *2016*, 1414595. [CrossRef]
29. Eltayeb, H.; Abdalla, Y.T.; Bachar, I.; Khabir, M.H. Fractional telegraph equation and its solution by natural transform decomposition method. *Symmetry* **2019**, *11*, 334. [CrossRef]
30. Khan, H.; Shah, R.; Kumam, P.; Baleanu, D.; Arif, M. An efficient analytical technique, for the solution of fractional-order telegraph equations. *Mathematics* **2019**, *7*, 426. [CrossRef]
31. Abdel-Rehim, E.A.; El-Sayed, A.M.A.; Hashem, A.S. Simulation of the approximate solutions of the time-fractional multi-term wave equations. *Comput. Math. Appl.* **2017**, *73*, 1134–1154. [CrossRef]
32. Abdel-Rehim, E.A.; Hashem, A.S. Simulation of the Space-Time-Fractional Ultrasound Waves with Attenuation in Fractal Media. In *Fractional Calculus, Proceedings of the ICFDA 2018, Amman, Jordan, 16–18 July 2019*; Springer: Singapore, 2019; pp. 173–197.
33. Abdel-Rehim, E.A. The approximate and analytic solutions of the time-fractional intermediate diffusion wave equation associated with the fokker–planck operator and applications. *Axioms* **2021**, *10*, 230. [CrossRef]
34. Chu, H.-P.; Chen, C.-L. Hybrid differential transform and finite difference method to solve the nonlinear heat conduction problem. *Commun. Nonlinear Sci. Numer. Simul.* **2008**, *13*, 1605–1614. [CrossRef]

35. Süngü, I.Ç.; Demir, H. Application of the hybrid differential transform method to the nonlinear equations. *Sci. Res.* **2012**, *3*, 246–250. [CrossRef]
36. Arsalan, D. The numerical study of a hybrid method for solving telegraph equation. *Appl. Math. Nonlinear Sci.* **2020**, *5*, 293–302. [CrossRef]
37. Du, Y.; Qin, B.; Zhao, C.; Zhu, Y.; Cao, J.; Ji, Y. A novel spatio-temporal synchronization method of roadside asynchronous MMW radar-camera for sensor fusion. *IEEE Trans. Intell. Transp. Syst.* **2021**, *23*, 22278–22289. [CrossRef]
38. Odibat, Z.; Shawagfeh, N. Generalized Taylor's formula. *Appl. Math. Comput.* **2007**, *186*, 286–293. [CrossRef]
39. Erturk, V.S.; Momani, S.; Odibat, Z. Application of generalized differential transform method to multi-order fractional differential equations. *Commun. Nonlinear Sci. Numer. Simul.* **2008**, *13*, 1642–1654. [CrossRef]
40. Odibat, Z.; Kumar S.; Shawagfeh, N.; Alsaedi, A.; Hayat, T. A study on the convergence conditions of generalized differential transform method. *Math. Methods Appl. Sci.* **2016**, *40*, 40–48. [CrossRef]