symmetry

Article

Scaled-Invariant Extended Quasi-Lindley Model: Properties, Estimation, and Application

Mohamed Kayid [1], Abdulrahman Abouammoh [1] and Ghadah Alomani [2,*]

[1] Department of Statistics and Operations Research, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia; drkayid@ksu.edu.sa (M.K.); abuammoh@ksu.edu.sa (A.A.)
[2] Department of Mathematical Sciences, College of Science, Princess Nourah bint Abdulrahman University, P.O. Box 84428, Riyadh 11671, Saudi Arabia
* Correspondence: gaalomani@pnu.edu.sa

Abstract: In many research fields, statistical probability models are often used to analyze real-world data. However, data from many fields, such as the environment, economics, and health care, do not necessarily fit traditional models. New empirical models need to be developed to improve the fit. In this study, we investigated a further extension of the quasi-Lindley model. This extension was asymmetrically distributed on the positive real number line. Maximum likelihood, least square error, Anderson–Darling, and expectation maximization algorithms were used to estimate the parameters studied. All techniques provided accurate and reliable estimates of the parameters. However, the mean square error of the expectation-maximization approach was lower. The usefulness of the proposed model was demonstrated by analyzing a reliability data set, and the analysis showed that it outperformed all other alternative models.

Keywords: quasi-Lindley model; maximum likelihood estimator; expectation maximization algorithm

check for updates

Citation: Kayid, M.; Abouammoh, A.; Alomani, G. Scaled-Invariant Extended Quasi-Lindley Model: Properties, Estimation, and Application. *Symmetry* **2023**, *15*, 1780. https://doi.org/10.3390/sym15091780

Academic Editors: Dário Ferreira and Calogero Vetro

Received: 13 July 2023
Revised: 13 September 2023
Accepted: 15 September 2023
Published: 18 September 2023

Copyright: © 2023 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (https://creativecommons.org/licenses/by/4.0/).

1. Introduction

Probability models can be classified as symmetric and asymmetric models. An asymmetric model is a type of model in which the probability density or mass function is symmetric about its mean. The shape of the asymmetric model is not symmetric. Both symmetric and asymmetric models have received considerable attention in the probability and statistics literature. A Lindley model, which is simple and remarkably flexible in application, was proposed by [1]. It is characterized by the probability density function (pdf):

$$f(x) = \frac{\xi^2}{\xi+1}(1+x)e^{-\xi x}, \;\; \xi > 0, \;\; x \geq 0 \tag{1}$$

which is a mixture of two gamma models $G(1,\xi)$ and $G(2,\xi)$ with weights $\xi/\xi+1$ and $1/\xi+1$, respectively. Numerous studies have been conducted on the Lindley model. For example, many properties, extensions, and applications of the model have been studied in [2–21]. A scale-invariant version of the Lindley model, namely the quasi-Lindley (QL), with the pdf:

$$f(x) = \frac{\xi}{\alpha+1}(\alpha+\xi x)e^{-\xi x}, \; \alpha > 0\,, \xi > 0, \; x \geq 0, \tag{2}$$

was proposed by [22]. It is a mixture of two gamma models $G(1,\xi)$ and $G(2,\xi)$ with weights $\alpha/\alpha+1$ and $1/\alpha+1$, respectively.

A family of models characterized by $f_\theta(x)$ is said to be scale-invariant if the transformation from x to kx lies within the family. In other words, $f_\theta(kx) = Jf_{\tilde{\theta}}(x)$ for every x where J is the transformation Jacobian. Thus, if you change the scale of measurement or the unit of x, the fit remains invariant. For instance, a lifetime can be measured in days, hours, or minutes, and the unit of measurement does not affect inferences about lifetimes.

Since scale invariance is an essential property of lifetime models, this model has attracted considerable interest. A comparison of the maximum likelihood estimator (MLE) and the expectation-maximization (EM) algorithm for estimating the parameters of the QL model was studied by [23] and new scale-invariant extensions of the Lindley model were proposed by [24,25].

Many data sets are composed of multiple populations or sources, and the subpopulation associated with each data point is usually unknown or not recorded. For example, the lifetime of a device or system may be available, but the manufacturer is unknown or an event associated with a living being lacks its geographic location. Such data sets are mixtures because information about some covariates, such as the manufacturer or geographic location, that significantly affect the observations is unknown. For detailed information on mixture models, see [26,27]. The Lindley model and its extensions are examples of mixture models of the gamma distribution that can be useful for describing many real-world applications.

The various needs for mixture models motivate us to propose a new extension of the scaled-invariant QL model, a mixture of three gamma models. Some statistical and reliability properties, such as failure rate (FR), mean residual life (MRL), and p-quantile residual life (p-QRL) functions, are discussed. The problem of estimating the parameters are discussed using the maximum likelihood (ML) method, the least-squares error (LSE) method, the weighted LSE method, and one innovative (EM) algorithm. It is examined that all methods provide consistent and efficient estimates of the parameters. However, the EM algorithm yields a lower mean square error.

The rest of this article is organized as follows. The scaled-invariant extended quasi-Lindley (EQL) model is explained in Section 2 along with some of its basic properties. In Section 3, an innovative EM algorithm is presented for estimating the model parameters, along with ML, LSE, and weighted LSE methods. In Section 4, a simulation study is conducted to investigate and compare the behavior of the estimators. In Section 5, the proposed model is fitted to a reliability data set of intervals between successive air conditioning failures in a Boeing 720 aircraft to demonstrate the usefulness of the model in practice. Finally, Section 6 concludes this paper.

2. Scaled-Invariant Extended QL Model

In this section, a new model is proposed and some of its basic statistical properties are examined. A random variable X follows from $EQL(\alpha, \xi)$ if its PDF is

$$f(x) = \frac{\xi}{1+\alpha+\alpha^2}\left(1+\alpha\xi x+\frac{1}{2}\alpha^2\xi^2x^2\right)e^{-\xi x}, \quad \alpha \geq 0, \quad \xi > 0, \quad x \geq 0. \tag{3}$$

It is a mixture of $G(1,\xi)$, $G(2,\xi)$, and $G(3,\xi)$ with weights $1/\left(1+\alpha+\alpha^2\right)$, $\alpha/\left(1+\alpha+\alpha^2\right)$, and $\alpha^2/\left(1+\alpha+\alpha^2\right)$, respectively, and presents an asymmetric form on the positive real line. When $\alpha = 0$, it reduces to the exponential model. The reliability function is an important yet very simple measure in reliability theory and survival analysis. For the EQL model, it is

$$R(x) = \frac{1}{1+\alpha+\alpha^2}\left(1+\alpha+\alpha^2+\alpha\xi x+\alpha^2\xi x+\frac{1}{2}\alpha^2\xi^2x^2\right)e^{-\xi x}. \tag{4}$$

The distribution function is simply related to the reliability function by $F(x) = 1 - R(x)$ and the quantile function, which is in fact the inverse of the distribution function:

$$q(p) = F^{-1}(p) = \min\{x : F(x) = p\}, \; 0 < p < 1.$$

The quantile function could be used for simulating random samples, estimating the parameters, and computing the skewness of the model.

In addition, for the EQL, the k-th moment is finite and equal to

$$E(X^k) = \frac{1}{1+\alpha+\alpha^2}\frac{1}{\xi^k}\left[\Gamma(k+1)+\alpha\Gamma(k+2)+\frac{\alpha^2}{2}\Gamma(k+3)\right]. \quad (5)$$

Reliability Properties

The proposed model represents a lifetime model. Thus, it is important to study the main reliability measures, such as the FR, MRL, and p-QRL functions, for the proposed model. The FR function at time x expresses the instantaneous risk of fail at x given survival up to x. Mathematically, it is defined by

$$\lambda(x) = \lim_{\delta\to 0} P(x < X < x+\delta \mid X > x) = \frac{f(x)}{R(x)}.$$

For more information about the FR function, refer to Lai and Xie [28]. In the case of the EQL model, we have

$$\lambda(x) = \frac{1+\alpha\xi x+\frac{1}{2}\alpha^2\xi^2x^2}{1+\alpha+\alpha^2+\alpha\xi x+\alpha^2\xi x+\frac{1}{2}\alpha^2\xi^2x^2}\xi. \quad (6)$$

Using simple algebra, we can see that the FR function increases from $\lambda(0) = \xi/\left(1+\alpha+\alpha^2\right)$ to $\lim_{x\to\infty}\lambda(x) = \xi$. Figure 1 shows the shape of the pdf and the FR function for some parameter values.

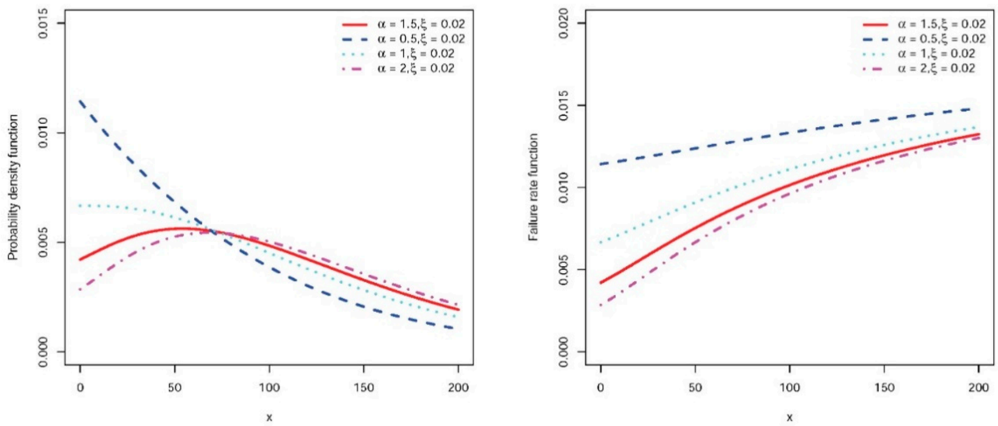

Figure 1. The PDF (**left**) and FR (**right**) of EQL for some parameter values.

Two other useful and well-known measures in reliability theory and survival analysis are the MRL and p-QRL functions. At time x, they describe the mean and p-quantile of the remaining life for survival to x. In practice, the MRL function is an attractive alternative to the survival or hazard function of survival. It provides the remaining life expectancy of a subject surviving up to time x. For EQL, the MRL is obtained by

$$m(x) = \frac{1+2\alpha+3\alpha^2+\left(\alpha\xi+2\alpha^2\xi\right)x+\frac{1}{2}\alpha^2\xi^2x^2}{1+\alpha+\alpha^2+(\alpha\xi+\alpha^2\xi)x+\frac{1}{2}\alpha^2\xi^2x^2}\frac{1}{\xi}. \quad (7)$$

Since the FR function is increasing, it follows that the MRL function decreases from

$$m(0) = \frac{1+2\alpha+3\alpha^2}{1+\alpha+\alpha^2}\frac{1}{\xi} \text{ to } \frac{1}{\xi} \text{ at infinity.}$$

The p-QRL reads

$$q_p(x) = R^{-1}((1-p)R(x)) - x, \quad (8)$$

which can be calculated numerically (refer to Lai and Xie [28] for more details). Like the MRL, this measure is a decreasing function of x. When $p = 0.5$, it is called the median residual life, which is a good alternative to the MRL. In Figure 2, the MRL and the median residual lifetime are plotted for some parameter values and show their similar behavior.

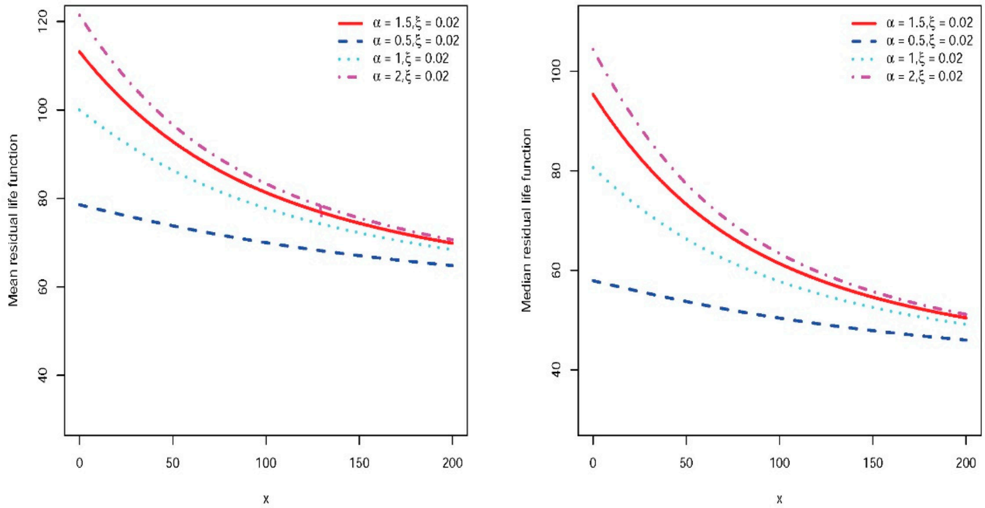

Figure 2. The MRL (**left**) and median residual life (**right**) of EQL for some parameter values.

An important concept in reliability theory and survival analysis is orderings between lifetimes. For two lifetimes X_1 and X_2 following reliability functions R_1 and R_2, respectively, we say that X_2 is greater than X_1, $X_2 \geq X_1$, in stochastic if $R_2(x) \geq R_1(x)$ for every x (refer to Lai and Xie [28] for more details about lifetime orderings). Equivalently, we may write $R_2 \geq R_1$ in stochastic. There are other useful orderings, e.g., by means of the FR function, $X_2 \geq X_1$ in FR if $h_1(x) \geq h_2(x)$ for every x. Moreover, $X_2 \geq X_1$ in MRL and p-QRL if $m_2(x) \geq m_1(x)$ and $q_{p,2}(x) \geq q_{p,1}(x)$ for every x, respectively. The following result shows that EQL is internally ordered in terms of α.

Proposition 1. *Let X_i, $i = 1, 2$ follow from $EQL(\alpha,\ \xi)$ and $\alpha_2 \geq \alpha_1$; then, $X_2 \geq X_1$ in stochastic, FR, MRL, and p-QRL.*

Proof. See Appendix A. □

3. Estimation

In this section, to estimate the model parameters, three well-known methods, ML, LSE, and weighted LSE, are first discussed. Then, an innovative EM algorithm for this purpose is presented.

3.1. ML Method

Let $x_1, x_2, \ldots, x_n$ represent independent and identically distributed (iid) instances from $EQL(\alpha, \xi)$. Then, the log-likelihood function is

$$l(\alpha, \xi; \mathbf{x}) = n\ln\xi - n\ln(1 + \alpha + \alpha^2) + \sum_{i=1}^{n} \ln(1 + \alpha\xi x_i + \frac{1}{2}\alpha^2\xi^2 x_i^2) - \xi\sum_{i=1}^{n} x_i. \quad (9)$$

The ML estimator of (α, ξ) denoted by $(\hat{\alpha}, \hat{\xi})$ maximizes the log-likelihood function and can be computed directly using numerical methods or by solving the following likelihood equations.

$$\frac{\partial}{\partial\alpha} l(\alpha, \xi; \mathbf{x}) = -n\frac{1 + 2\alpha}{1 + \alpha + \alpha^2} + \sum_{i=1}^{n} \frac{\xi x_i + \alpha\xi^2 x_i^2}{1 + \alpha\xi x_i + \frac{1}{2}\alpha^2 x\xi^2 x_i^2} = 0,$$

and

$$\frac{\partial}{\partial \xi} l(\alpha, \xi; \mathbf{x}) = \frac{n}{\xi} + \sum_{i=1}^{n} \frac{\alpha x_i + \alpha^2 \xi x_i^2}{1 + \alpha \xi x_i + \frac{1}{2}\alpha^2 \xi^2 x_i^2} - \sum_{i=1}^{n} x_i = 0.$$

The observed Fisher information matrix can be calculated by replacing $\hat{\alpha}$ and $\hat{\xi}$ for α and ξ in the following Fisher information matrix.

$$O = \begin{bmatrix} -\frac{\partial^2}{\partial \alpha^2} & -\frac{\partial^2}{\partial \alpha \partial \xi} \\ -\frac{\partial^2}{\partial \xi \partial \alpha} & -\frac{\partial^2}{\partial \xi^2} \end{bmatrix} l(\alpha, \xi; \mathbf{x}). \tag{10}$$

Then, the asymptotic distribution of $\left(\hat{\alpha}, \hat{\xi}\right)$ is approximately the bivariate normal distribution with mean (α, ξ) and variance-covariance matrix O^{-1}.

3.2. LSE Method

Suppose that $x_1 \leq x_2 \leq \ldots \leq x_n$ represents the ordered sample. In this approach, we search for parameter values that minimize the sum of squared distances between the empirical distribution and the estimated distribution functions. More precisely, we minimize the following expression in terms of the parameters.

$$S^2 = \sum_{i=1}^{n} (F(x_i) - \hat{F}(x_i))^2,$$

where $\hat{F}(x_i) = \frac{i}{n}$ is the well-known empirical distribution function at x_i and provides a common estimate of $F(x_i)$. By substituting the distribution function, we have

$$S^2 = \sum_{i=1}^{n} \left(\frac{1}{(1 + \alpha + \alpha^2)} \left(1 + \alpha + \alpha^2 + \alpha \xi x_i + \alpha^2 \xi x_i + \frac{1}{2} \alpha^2 \xi^2 x_i^2 \right) e^{-\xi x_i} - \frac{i}{n} \right)^2.$$

Then, the estimates could be computed as follows:

$$(\hat{\alpha}, \hat{\xi}) = \arg \min_{(\alpha,\beta,\lambda)} \sum_{i=1}^{n} \left(\frac{1}{1 + \alpha + \alpha^2} \left(1 + \alpha + \alpha^2 + \alpha \xi x_i + \alpha^2 \xi x_i + \frac{1}{2} \alpha^2 \xi^2 x_i^2 \right) e^{-\xi x_i} - \frac{i}{n} \right)^2.$$

3.3. Weighted LSE Method

A well-known weight that could improve the LSE estimate is $\frac{1}{F(x_i)(1-F(x_i))}$. With this idea, the weighted LSE estimate is computed by minimizing the following expression in terms of the parameters.

$$S^2 = \sum_{i=1}^{n} \frac{1}{F(x_i)(1 - F(x_i))} \left(F(x_i) - \hat{F}(x_i) \right)^2.$$

This method is well known as the Anderson–Darling (AD) method.

3.4. EM Algorithm

The EM algorithm takes advantage of the fact that we have a mixed model and creates a more informative likelihood function. The parameters are then estimated iteratively. Suppose that X_i, $i = 1, 2, \ldots, n$ is an iid sample from $EQL(\alpha, \xi)$. For a short exposition, take $\theta = (\alpha, \xi)$. Since EQL is a mixture of three gamma models $G(j, \xi)$, $j = 1, 2, 3$, we consider a latent random variable V_i such that $V_i = j$ when X_i comes from $G(j, \xi)$. Thus, $(X_i | V_i = j, \theta) \sim G(j, \xi)$ and $P(V_i = j | \theta) = \frac{\alpha^{j-1}}{1+\alpha+\alpha^2}$, $j = 1, 2, 3$. However, the latent variable V_i will not be observed, but applying it helps to improve the estimation of the parameters in an iterative process. With the evidence X_i and V_i, $i = 1, 2, \ldots, n$, the likelihood function can be written as follows:

$$L(\theta; \mathbf{x}, \mathbf{v}) = \prod_{i=1}^{n} \prod_{j=1}^{3} \left(g(x_i|\theta)P(V_i = j|\theta)\right)^{I(v_i=j)}, \tag{11}$$

where $I(v_i = j)$ equals 1 when $v_i = j$ and 0 otherwise, and $g_j(x_i|\theta)$ represents the PDF of gamma $G(j, \xi)$. Then, the log-likelihood function is

$$l(\theta; \mathbf{x}, \mathbf{v}) = \sum_{i=1}^{n} \sum_{j=1}^{3} I(V_i = j) \ln\left(\frac{\xi^j x_i^{j-1}}{\Gamma(j)} e^{-\xi x_i} \frac{\alpha^{j-1}}{1 + \alpha + \alpha^2}\right). \tag{12}$$

Since this function depends on the unobserved random variable V_i, we cannot estimate the parameters by maximizing them directly. One approach is to implement an iterative process with expectation (E) and maximization (M) steps. In the E step, the expected log-likelihood function is constructed with respect to the conditional latent variable. In the M step, the expected log-likelihood function is maximized to estimate the parameters. See Appendix B for the implementations of the E-step and M-step.

4. Simulations

The goal of this section is to investigate and compare the behavior of the discussed estimators through simulations. To this end, we calculate the empirical bias (B) and mean square error (MSE) of the estimators. We generate a random sample of $EQL(\alpha, \xi)$ using the following steps:

1. First, drive one random instance from a multinomial model with parameters (p_1, p_2, p_3, n), where $p_1 = 1/\left(1 + \alpha + \alpha^2\right)$, $p_2 = \alpha/\left(1 + \alpha + \alpha^2\right)$, and $p_3 = 1 - p_1 - p_2$. Assume the derived instance is (k_1, k_2, k_3).
2. Generate and mix three identical and independent (iid) random samples from $G(1, \xi)$, $G(2, \xi)$, and $G(3, \xi)$ with sizes k_1, k_2, and k_3 respectively.

In each simulation run, $r = 1000$ samples are generated with a size of $n = 80$ or 150. Then, the parameters are estimated for each instance using the ML, LSE, and AD methods or EM algorithm. For the calculation of the optimum values of the parameters, the integrated function "optim" of R is used. The initial values needed for computing all estimators are randomly generated from a uniform distribution, e.g., the initial values for α are randomly and uniformly derived from the interval (0.9α, 1.1α). Table 1 shows the bias (B) and mean square error (MSE) for estimators and for some parameter values calculated using the following relations:

$$B_\alpha = \frac{1}{r} \sum_{i=1}^{n} (\hat{\alpha}_i - \alpha),$$

and

$$MSE_\alpha = \frac{1}{r} \sum_{i=1}^{n} (\hat{\alpha}_i - \alpha)^2,$$

with a similar approach for ξ. Small values of MSE reported in Table 1 show that all estimators are consistent and sufficiently efficient but the EM algorithm outperforms others for all selected parameters.

Table 1. Simulation results for ML, LSE, AD, and EM algorithm. The first and second lines of every cell correspond to α and ξ.

		n			
Method		**80**		**150**	
	α,ξ	B	MSE	B	MSE
MLE	0.1, 0.1	0.2093 0.0221	0.1413 0.0015	0.1486 0.0166	0.0853 0.0010
	0.3, 0.5	0.1009 0.0415	0.1222 0.0209	0.0466 0.0168	0.0771 0.0133
	0.8, 1	0.0598 0.0029	0.1716 0.0326	0.0273 −0.0043	0.0948 0.0201
EM	0.1, 0.1	0.0833 0.0097	0.0377 0.0004	0.0463 0.0054	0.0126 0.0002
	0.3, 0.5	0.1346 0.0530	0.1095 0.0181	0.0807 0.0329	0.0554 0.0100
	0.8, 1	0.1023 0.0223	0.1856 0.0299	0.0281 0.0015	0.0728 0.0156
LSE	0.1, 0.1	0.2350 0.0306	0.1970 0.0026	0.1734 0.0230	0.1185 0.0016
	0.3, 0.5	0.0906 0.0523	0.1726 0.0315	0.0466 0.0287	0.1083 0.0206
	0.8, 1	0.0609 0.0075	0.2960 0.0500	0.0143 −0.0038	0.1085 0.0270
Weighted LSE (AD)	0.1, 0.1	0.0283 0.0097	0.0592 0.0009	0.0249 0.0084	0.0392 0.0006
	0.3, 0.5	−0.1116 −0.0233	0.1157 0.0211	−0.1373 −0.0367	0.0778 0.0121
	0.8, 1	−0.3220 −0.1497	0.2795 0.0700	−0.2426 −0.1230	0.1963 0.0466

5. Application

In this section, the EQL and some alternative models are fitted to a data set of air conditioning systems of a Boeing 720 aircraft to verify the usefulness of the proposed model. Alternative models include gamma, exponentiated gamma (EG), Lehmann gamma (LG), Marshal–Olkin gamma (MOG), and QL.

Table 2 shows 29 time intervals, in terms of hours, between successive air conditioning failures in a Boeing 720 aircraft. For more details about the experiment and the data, see Proschan [29].

Table 2. Time interval, in terms of hours, between successive failures of air conditioner system of Boeing 720 aircraft.

59	20	68	67	25	13	5	79	76
127	117	100	52	189	398	60	117	263
143	39	194	128	160	88	74	66	199
180	156							

For this data set, the total time on test (TTT) is plotted in Figure 3 (left), which shows an increasing FR function. The TTT plot is really a nonparametric plot, which is very useful for determining the FR form of the data. Figure 3 (right) draws the histogram of the data and the calculated PDF of the EQL and gives a graphical investigation.

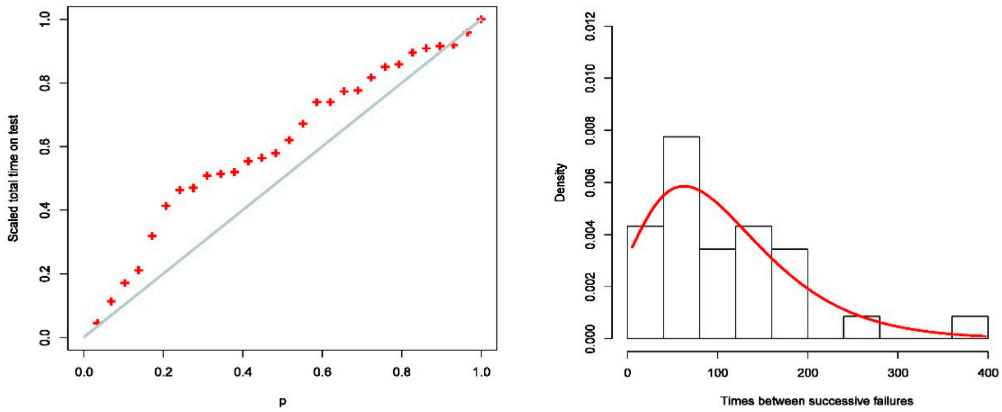

Figure 3. The TTT plot (**left**) and histogram along with estimated PDF (**right**) of times between failures of air conditioning system. The red plus in the left figure shows calculated TTT for each data entry.

For each model, parameters were estimated using ML. In addition, the parameters of the EQL were estimated using the ML method and the EM algorithm, and because the results were approximately the same, only the EM estimates are reported in Table 3. The Akaike information criterion (AIC), Cramer–von Mises (CVM) statistics, Anderson–Darling (AD) statistics, and Kolmogorov–Smirnov (KS) statistics were also calculated and are summarized in Table 3.

Table 3. Fitting the successive times between failures.

Model	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\xi}$	AIC	CVM	AD	KS
					p-Value	*p*-Value	*p*-Value
EQL	1.9668	—	0.0215	331.22	0.0278 0.9843	0.1833 0.9944	0.0801 0.9923
Gamma	1.7195	—	0.0153	331.55	0.0363 0.9539	0.2399 0.9754	0.1028 0.9190
EG	2.8250	0.0823	0.1459	334.57	0.0647 0.7882	0.3836 0.8638	0.1308 0.7037
LG	1.4504	1.1997	0.0142	333.59	0.0373 0.9682	0.2454 0.9727	0.1041 0.9120
MOG	1.6439	1.2563	0.0161	333.37	0.0322 0.9705	0.2169 0.9851	0.0965 0.9498
QL	0.1382	—	0.0167	331.35	0.0320 0.9712	0.2057 0.9888	0.0965 0.9499

In Figure 4, the empirical and fitted distribution functions for EQL and some alternatives are plotted, providing a graphical investigation.

Considering Table 3, a smaller AIC indicates a better fit to the data. Here, the AIC of the proposed EQL model is smaller than that of all other selected models, indicating that it is preferred over the other models. In addition, the model with a smaller CVM (AD and KS) statistic better describes the data. Fortunately, the value of the CVM (AD and KS) statistic for EQL is smallest among all the selected models. This shows that EQL is preferred in terms of CVM, AD, and KS statistics.

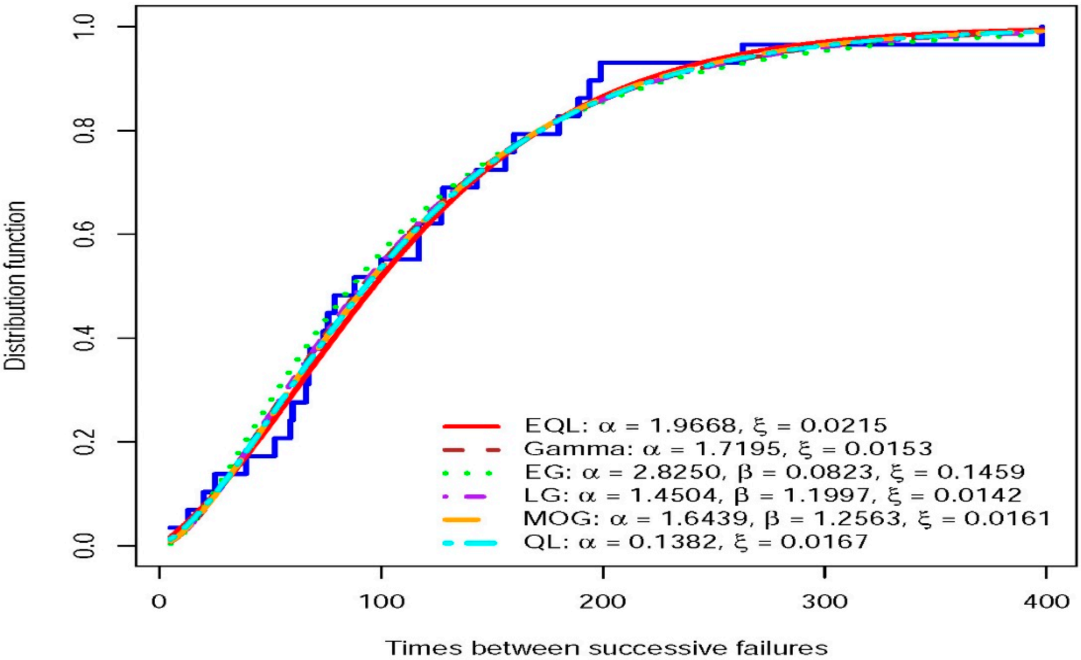

Figure 4. The empirical and estimated CDF for QL and some alternative models of times between failures of air conditioning system.

6. Conclusions

For data modeling and analysis, an appropriate statistical model must be used to draw more accurate conclusions. The EQL model, which combines three gamma distributions, is an extension of QL, which can be used in various scientific disciplines. In the context of reliability theory and survival analysis, it could be useful for data with increasing FR and decreasing MRL functions, for example, for modeling the lifetime of devices subject to depletion. The model can be useful in practice, as shown by the analysis of a data set consisting of the intervals between successive failures of the air conditioning system of a Boeing 720 aircraft. Based on the simulation results, the ML and EM algorithms provide accurate and consistent parameter estimates. However, the EM algorithm provides a more accurate approximation than the MLE. Some future research related to this study can be considered as follows:

- Estimate the unknown parameters of the proposed model, along with the reliability and hazard rate functions under different types of censoring schemes, such as progressive type II, hybrid, general progressive, and adaptive censoring schemes.
- Consider the maximum likelihood and maximum product-of-spacing methods to determine the point estimates and approximate confidence intervals of the various model parameters.
- Provide Bayesian estimates based on the likelihood function and product of the distance function of the unknown parameters using the quadratic error loss function with independent gamma priors.
- The methods investigated in this study can be extended to study estimation problems in more complex cases.

Author Contributions: Conceptualization, M.K. and A.A.; methodology, G.A.; software, G.A.; validation, A.A., M.K. and G.A.; formal analysis, M.K.; investigation, G.A.; resources, A.A.; data curation, G.A.; writing—original draft preparation, M.K.; writing—review and editing, A.A.; visualization, G.A.; supervision, M.K.; project administration, G.A.; funding acquisition, G.A. All authors have read and agreed to the published version of the manuscript.

Funding: Princess Nourah bint Abdulrahman University Researchers Supporting Project number (PNURSP2023R226), Princess Nourah bint Abdulrahman University, Riyadh, Saudi Arabia.

Data Availability Statement: The data is included in the paper.

Acknowledgments: The authors thank the editor and three anonymous reviewers for their suggestions and constructive comments that improved the presentation and readability of the article. This work is supported by Princess Nourah bint Abdulrahman University Researchers Supporting Project number (PNURSP2023R226), Princess Nourah bint Abdulrahman University, Riyadh, Saudi Arabia.

Conflicts of Interest: The authors declare no conflict of interest.

Appendix A

Proof of Proposition 1. To show the FR ordering, the derivative of the FR function in terms of α is proportional to

$$\frac{d\lambda}{d\alpha} \propto -\left(\frac{1}{2}\alpha^2\xi^2x^2 + \alpha^2\xi x + 2\alpha\xi x + 2\alpha + 1\right) < 0.$$

So, the FR ordering follows. The stochastic, MRL, and p-QRL orderings follow from the FR ordering. See Lai and Xie [28] for the relationship between orderings. □

Appendix B. E Step and M Step of EM Algorithm

E step:

Assume that the estimate of the parameters at iteration t, $\theta_t = (\alpha_t, \xi_t)$ is known. Then, through the well-known Bayes formula, the conditional probability of V_i is

$$\begin{aligned} p_{ij,t} &= P(V_i = j \mid X_i = x_i, \theta_t) = \frac{f(X_i = x_i \mid V_i = j, \theta_t)P(V_i = j \mid \theta_t)}{f(X_i = x_i|\theta_t)} \\ &= \frac{\frac{\xi_t^j}{\Gamma(j)}x_i^{j-1}e^{-\xi_t x_i}\alpha_t^{j-1}}{\sum\limits_{j=1}^{3}\frac{\xi_t^j}{\Gamma(j)}x_i^{j-1}e^{-\xi_t x_i}\alpha_t^{j-1}}, i = 1, 2, \ldots, n, j = 1, 2, 3. \end{aligned} \tag{A1}$$

So,

$$p_{i1,t} = \frac{1}{1 + \alpha_t\xi x_i + \frac{1}{2}\alpha_t^2 + \xi_t^2 x_i^2}, \tag{A2}$$

$$p_{i2,t} = \frac{\alpha_t\xi_t x_i}{1 + \alpha_t\xi_t x_i + \frac{1}{2}\alpha_t^2 + \xi_t^2 x_i^2}, \tag{A3}$$

and

$$p_{i3,t} = 1 - p_{i1,t} + p_{i2,t}$$

Now, applying these probabilities, we can write the expected log-likelihood function at iteration t.

$$Q(\theta|\theta_t) = E_{V|X,\theta_t}(l(\theta; \mathbf{x}, \mathbf{V})) = \sum_{i=1}^{n} E_{V_i|X_i,\theta_t}\sum_{j=1}^{3} I(V_i = j)\ln\left(\frac{\xi^j x_i^{j-1}}{\Gamma(j)}e^{-\xi x_i}\frac{\alpha^{j-1}}{1+\alpha+\alpha^2}\right)$$

$$= \sum\nolimits_{i=1}^{n} P(V_i = 1|X_i = x_i, \theta_t)\ln\left(\frac{\xi}{1+\alpha+\alpha^2}e^{-\xi x_i}\right)$$

$$
\begin{aligned}
&+\sum_{i=1}^{n} P(V_i = 2|X_i = x_i, \theta_t)\ln\left(\frac{\alpha\xi^2 x_i}{1+\alpha+\alpha^2}e^{-\xi x_i}\right)\\
&+\sum_{i=1}^{n} P(V_i = 3|X_i = x_i, \theta_t)\ln\left(\frac{1}{2}\frac{\alpha^2\xi^3 x_i^2}{1+\alpha+\alpha^2}e^{-\xi x_i}\right)\\
&=\sum_{i=1}^{n} (1 + p_{i2,t} + 2p_{i3,t})\ln\xi - \xi\sum_{i=1}^{n} x_i\\
&+\sum_{i=1}^{n} (p_{i2,t} + 2p_{i3,t})\ln(\alpha x_i) - n\ln(1+\alpha+\alpha^2)\\
&+\sum_{i=1}^{n} p_{i3,t}\ln\tfrac{1}{2}.
\end{aligned} \tag{A4}
$$

Clearly, $Q(\theta|\theta_t)$ consists of three expressions:

$$
Q_1(\xi) = \sum_{i=1}^{n} (1 + p_{i2,t} + 2p_{i3,t})\ln\xi - \xi\sum_{i=1}^{n} x_i,
$$

$$
Q_2(\alpha) = \textstyle\sum_{i=1}^{n} (p_{i2,t} + 2p_{i3,t})\ln(\alpha x_i) - n\ln(1+\alpha+\alpha^2), \tag{A5}
$$

depending solely on ξ and α, respectively, and $Q_3 = \ln\frac{1}{2}\sum_{i=1}^{n} p_{i3,t}$, which does not depend on ξ or α.

M step:

To estimate the parameters at iteration $t+1$, we should maximize $Q(\theta|\theta_t)$ in terms of $\theta = (\alpha, \xi)$. Thus, for estimating ξ at iteration $t+1$, we could simply solve the likelihood equation $\frac{\partial Q_1(\xi)}{\partial\xi} = 0$, which gives $\hat{\xi}_{t+1}$ as follows:

$$
\hat{\xi}_{t+1} = \frac{\sum_{i=1}^{n} 1 + p_{i2,t} + p_{i3,t}}{\sum_{i=1}^{n} x_i}.
$$

Similarly, by solving the likelihood equation $\frac{\partial Q_2(\alpha)}{\partial\alpha} = 0$, we could check that $\hat{\alpha}_{t+1}$ is the positive solution of the following quadratic equation in terms of α:

$$
\alpha^2(c - 2n) + \alpha(c - n) + c = 0,
$$

where $c = \sum_{i=1}^{n} p_{i2,t} + 2p_{i3,t}$. The sequence θ_t converges to θ and we could stop the iterations when $Q(\theta|\theta_t)$ does not improve significantly, i.e., for a predefined small value $\epsilon > 0$, $Q(\theta|\theta_{t+1}) < Q(\theta|\theta_t) + \epsilon$. See Wu [30] for more information about convergence of the EM algorithm.

References

1. Lindley, D.V. Fiducial distributions and Bayes' theorem. *J. R. Stat. Soc. Ser.* **1958**, *20*, 102–107. [CrossRef]
2. Ghitany, M.E.; Atieh, B.; Nadarajah, S. Lindley distribution and its application. *Math. Comput. Simul.* **2008**, *78*, 493–506. [CrossRef]
3. Shanker, R.; Ghebretsadik, A.H. A New Quasi Lindley Distribution. *Int. J. Stat. Syst.* **2013**, *8*, 143–156.
4. Zakerzadeh, H.; Dolati, A. Generalized Lindley distribution. *J. Math. Ext.* **2009**, *3*, 13–25.
5. Shanker, R.; Shukla, K.K.; Shanker, R.; Leonida, T.A. A Three-Parameter Lindley Distribution. *Am. J. Math. Stat.* **2017**, *7*, 15–26.
6. Merovci, F.; Sharma, V.K. The Beta-Lindley Distribution: Properties and Applications. *J. Appl. Math.* **2014**, *2014*, 198951. [CrossRef]
7. Ibrahim, E.; Merovci, F.; Elgarhy, M. A new generalized Lindley distribution. *Math. Theory Model.* **2013**, *3*, 30–47.
8. Sankaran, M. The discrete poisson-Lindley distribution. *Biometrics* **1970**, *26*, 145–149. [CrossRef]
9. Ghitany, M.E.; Al-Mutairi, D.K.; Balakrishnan, N.; Al-Enezi, L.J. Power Lindley distribution and associated inference. *Comput. Stat. Data Anal.* **2013**, *64*, 20–33. [CrossRef]
10. Al-Mutairi, D.K.; Ghitany, M.E.; Kundu, D. Inferences on stress-strength reliability from Lindley distribution. *Commun. Stat.-Theory Methods* **2013**, *42*, 1443–1463. [CrossRef]
11. Zamani, H.; Ismail, N. Negative Binomial-Lindley Distribution and Its Application. *J. Math. Stat.* **2010**, *6*, 4–9. [CrossRef]
12. Al-babtain, A.A.; Eid, H.A.; A-Hadi, N.A.; Merovci, F. The five parameter Lindley distribution. *Pak. J. Stat.* **2014**, *31*, 363–384.
13. Ghitany, M.E.; Al-Mutairi, D.K.; Aboukhamseen, S.M. Estimation of the reliability of a stress-strength system from power Lindley distributions. *Commun. Stat.-Simul. Comput.* **2015**, *44*, 118–136. [CrossRef]

14. Abouammoh, A.M.; Alshangiti Arwa, M.; Ragab, I.E. A new generalized Lindley distribution. *J. Stat. Comput. Simul.* **2015**, *85*, 3662–3678. [CrossRef]
15. Ibrahim, M.; Singh Yadav, A.; Yousof, H.M.; Goual, H.; Hamedani, G.G. A new extension of Lindley distribution: Modified validation test, characterizations and different methods of estimation. *Commun. Stat. Appl. Methods* **2019**, *26*, 473–495. [CrossRef]
16. Marthin, P.; Rao, G.S. Generalized Weibull-Lindley (GWL) distribution in modeling lifetime Data. *J. Math.* **2020**, *2020*, 2049501. [CrossRef]
17. Al-Babtain, A.A.; Ahmed, A.H.N.; Afify, A.Z. A new discrete analog of the continuous Lindley distribution, with reliability applications. *Entropy* **2020**, *22*, 603. [CrossRef] [PubMed]
18. Joshi, R.K.; Kumar, V. Lindley Gompertz distribution with properties and applications. *Int. J. Appl. Math. Stat.* **2020**, *5*, 28–37. [CrossRef]
19. Afify, A.Z.; Nassar, M.; Cordeiro, G.M.; Kumar, D. The Weibull Marshall and Olkin Lindley distribution: Properties and estimation. *J. Taibah Univ. Sci.* **2020**, *14*, 192–204. [CrossRef]
20. Chesneau, C.; Tomy, L.; Gillariose, J.; Jamal, F. The Inverted Modified Lindley Distribution. *J. Stat. Theory Pract.* **2020**, *14*, 46. [CrossRef]
21. Algarni, A. On a new generalized lindley distribution: Properties, estimation and applications. *PLoS ONE* **2021**, *16*, e0244328.
22. Shanker, R.; Mishra, A. A quasi Lindley distribution. *Afr. J. Math. Comput. Sci. Res.* **2013**, *6*, 64–71.
23. Kayid, M.; Al-Maflehi, N.S. EM Algorithm for Estimating the Parameters of Quasi-Lindley Model with Application. *J. Math.* **2022**, *2022*, 8467291. [CrossRef]
24. Kayid, M.; Alskhabrah, R.; Alshangiti, A.M. A New Scale-Invariant Lindley Extension Distribution and Its Applications. *Math. Probl. Eng.* **2021**, *2021*, 3747753. [CrossRef]
25. Alrasheedi, A.; Abouammoh, A.; Kayid, M. A new flexible extension of the Lindley distribution with applications. *J. King Saud Univ.-Sci.* **2022**, *34*, 101714. [CrossRef]
26. Titterington, D.M.; Smith, A.F.M.; Makov, U.E. *Statistical Analysis of Finite Mixture Distributions*; John Wiley and Sons: Chichester, UK, 1985.
27. Ord, J.K. *Families of Frequency Distributions*; Charles Griffin: London, UK, 1972.
28. Lai, C.D.; Xie, M. *Stochastic Aging and Dependence for Reliability*; Springer: Berlin/Heidelberg, Germany, 2006; ISBN 978-0-387-29742-2.
29. Proschan, F. Theoretical Explanation of Observed Decreasing Failure Rate. *Technometrics* **1963**, *5*, 375–383. [CrossRef]
30. Wu, C.F.J. On the convergence properties of the EM algorithm. *Ann. Stat.* **1983**, *11*, 95–103. [CrossRef]

Disclaimer/Publisher's Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.