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Some Fixed Point Theorems for α-Admissible Mappings in Complex-Valued Fuzzy Metric Spaces

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Abstract: This paper discusses some properties of complex-valued fuzzy metric spaces and introduces the α-admissible mappings in the setting of complex-valued fuzzy metric spaces. We establish fixed point theorems for mappings satisfying symmetric contractive conditions with control functions. The results of this paper generalize, extend, and improve several results from metric, fuzzy metric, and complex-valued fuzzy metric spaces. Several examples are presented that verify and illustrate the new concepts, claims, and results.

Keywords: complex-valued fuzzy metric space; α-admissible mapping; fixed point; α-(ψ,ϕ)-contraction

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1. Introduction

Zadeh [1] introduced fuzzy sets, which are an extension of classical sets that allow for a degree of membership. This approach is particularly useful for systems with vague or incomplete data. Kramosil and Michálek [2] first proposed the idea of fuzzy metrics, which was later utilized by Grabiec [3] to introduce the fixed point theory in the context of fuzzy metric spaces. George and Veeramani [4] modified the definition of fuzzy metric spaces and discussed some topological properties in these spaces, demonstrating that the topology generated by the modified fuzzy metric spaces is Hausdorff.

Definition 1 (Schweizer and Sklar [5])**.** *A binary operation* $\star\colon [0,\ 1]\times[0,\ 1]\to[0,\ 1]$ *is called a t-norm if:*

(T1) $a\star b=b\star a$;
(T2) $a\star b\le c\star d$ *for* $a\le c, b\le d$;
(T3) $(a\star b)\star c=a\star(b\star c)$;
(T4) $a\star 0=0, a\star 1=a$, *for all* $a,b,c,d\in[0,\ 1]$.

Definition 2 (George and Veeramani [4])**.** *A triplet* $(\mathsf{F},M,\star)$ *is called a fuzzy metric space if* F *is a nonempty set,* $\star$ *is a continuous t-norm and* $M:\mathsf{F}\times\mathsf{F}\times(0,\infty)\to[0,\ 1]$ *is a fuzzy set satisfying the following conditions:*

(GV1) $M(\flat_1,\flat_2,t)>0$;
(GV2) $M(\flat_1,\flat_2,t)=1$ *if and only if* $\flat_1=\flat_2$;
(GV3) $M(\flat_1,\flat_2,t)=M(\flat_2,\flat_1,t)$;
(GV4) $M(\flat_1,\flat_3,t+s)\ge M(\flat_1,\flat_2,t)\star M(\flat_2,\flat_3,s)$;
(GV5) $M(\flat_1,\flat_2,.):(0,\infty)\to[0,\ 1]$ *is a continuous mapping.*

For all $\flat_1,\flat_2,\flat_3\in\mathsf{F}$ *and* $s,t>0$.

The notion of complex spaces is of great relevance in science; see, e.g., [6]. Azam et al. [7] introduced the notion of complex-valued metric spaces and allowed the metric function to take the values in the set of complex numbers instead of real numbers. They proved some common fixed point theorems in complex-valued metric spaces. Several researchers extended and generalized the results of Azam et al. [7] in several ways; see e.g., [8–12] and the references therein. In fuzzy metric spaces, the fuzzy metric is a fuzzy set which attains its values in the real interval $[0, 1]$. Shukla et al. [13] extended the fuzzy sets to their complex-valued version and introduced the notion of complex fuzzy sets. A complex fuzzy set attains complex values with some particular bounds on it. With the help of this extended notion, Shukla et al. [13] generalized and extended the notion of fuzzy metric spaces due to George and Veeramani [4] and introduced the notion of complex-valued fuzzy metric spaces. They also proved some fixed point theorems in this setting.

On the other hand, Samet et al. [14] introduced the α-admissible mappings in metric spaces and generalized several fixed point results from metric spaces.

Definition 3 (Samet et al. [14])**.** *Let $\digamma$ be a nonempty set and $\alpha\colon \digamma \times \digamma \to [0,\infty)$ be a function. A mapping $\top\colon \digamma \to \digamma$ is called α-admissible if*

$$\flat_1, \flat_2 \in \digamma, \alpha(\flat_1, \flat_2) \geq 1 \textit{ implies } \alpha(\top\flat_1, \top\flat_2) \geq 1.$$

The function α can be chosen in several ways, so that one can obtain several useful forms of a mapping defined on a space. Further, such functions can be used to weaken and generalize the contractive constraints on mappings which are utilized to establish fixed point results for the mappings under consideration. For generalization and extension of α-admissible mappings and its application, we refer to [15–18] and the references therein.

The use of control functions in the contractive conditions is one of the most popular ways to generalize the contractive conditions (see, e.g., [19,20] and the references therein). Recently, Humaira et al. [21] used control functions and proved some fixed point results in complex-valued fuzzy metric spaces. They illustrated the applicability of fixed point theorems to the existence of unique solution of a nonlinear mixed Volterr–Fredholm–Hammerstein integral equations.

In Section 2, we state some known definitions and concepts related to complex-valued fuzzy metric spaces and prove some topological properties in such spaces. We point out some flaws in the proof of the main results of Humaira et al. [21] and present some counterexamples of those results. In Section 3, we show that with some appropriate assumptions and some improvements in the methods of the proof of [21], one can draw all the conclusions of theorems of [21]. We have introduced some new contractive-type mappings and established some new fixed point results. Our results are not only an improvement to Humaira et al. [21], but at the same time, our results extend, generalize, improve and unify the results of Shukla et al. [13], Humaira et al. [21], Samet et al. [14], Ran and Reurings [22], Jachymski [23], and several other results into complex-valued fuzzy metric spaces. We present several examples to verify our claims and illustrate our conclusions. In Section 4, we have concluded the research work and provided the future scope of the research.

This paper provides a critical analysis of existing research, identifying and highlighting flaws, as well as presenting a new perspective and novel results that correct inaccuracies in the literature.

2. Complex-Valued Fuzzy Metric Spaces, Some Examples and Discussion

In this section, we state some definitions about complex-valued fuzzy metric spaces, establish some properties, discuss some concepts and results given by Humaira et al. [21], and point out some flaws therein with justification through examples. First, we state some definitions which will be needed in the sequel.

In what follows, $\mathbb{C}$ denotes the complex number system over the field of real numbers (see, Shukla et al. [13]). Denote $P = \{(a,b)\colon 0 \leq a < \infty, 0 \leq b < \infty\} \subset \mathbb{C}$, $\theta = (0,0)$,

$\ell = (1,1)$. Define a partial ordering $\preceq$ on $\mathbb{C}$ by $\varsigma_1 \preceq \varsigma_2$ (or, equivalently, $\varsigma_2 \succeq \varsigma_1$) if and only if $\varsigma_2 - \varsigma_1 \in P$. We write $\varsigma_1 \prec \varsigma_2$ (or, equivalently, $\varsigma_2 \succ \varsigma_1$) to indicate that $\mathrm{Re}(\varsigma_1) < \mathrm{Re}(\varsigma_2)$ and $\mathrm{Im}(\varsigma_1) < \mathrm{Im}(\varsigma_2)$ (see, also, Azam et al. [7]). If $\{\varsigma_n\}$ is a sequence in $\mathbb{C}$, then it is said to be monotonic (or monotonic with respect to $\preceq$) if either $\varsigma_n \preceq \varsigma_{n+1}$ for all $n \in \mathbb{N}$ or $\varsigma_{n+1} \preceq \varsigma_n$ for all $n \in \mathbb{N}$.

The closed unit complex interval I is defined by $I = \{(a,b) \colon 0 \le a \le 1, 0 \le b \le 1\}$, and the open unit complex interval by $I_o = \{(a,b) \colon 0 < a < 1, 0 < b < 1\}$. P_θ represents the set $\{(a,b) \colon 0 < a < \infty, 0 < b < \infty\}$. It is obvious that, for $\varsigma_1, \varsigma_2 \in \mathbb{C}$, $\varsigma_1 \prec \varsigma_2$ if and only if $\varsigma_2 - \varsigma_1 \in P_\theta$.

For $A \subset \mathbb{C}$, if there exists an element $\inf A \in \mathbb{C}$ such that it is a lower bound of A, that is, $\inf A \preceq a$ for all $a \in A$ and $u \preceq \inf A$ for every lower bound $u \in \mathbb{C}$ of A, then $\inf A$ is called the greatest lower bound or infimum of A. Similarly, we define $\sup A$, the least upper bound or supremum of A, in the usual manner.

Remark 1. *Humaira et al. [21] defined the open unit complex interval by $I_o = \{(a,b) \colon 0 \le a < 1, 0 \le b < 1\}$. We point out that this way of defining I_o is not appropriate, because the elements $r \in I_o$ are used in defining the convergence and Cauchyness of sequences (see Definition 7), and if we take $r = (0,0) \in I_o$, then these notions (which were used by Humaira et al. [21]) become inexpedient in context of the definition of complex-valued fuzzy metric spaces.*

Remark 2 (Shukla et al. [13])**.** *Let $\varsigma_n, \varsigma'_n, \omega \in P$ for all $n \in \mathbb{N}$, then:*

(a) *If the sequence $\{\varsigma_n\}$ is monotonic with respect to $\preceq$ and there exists $\alpha, \beta \in P$ such that $\alpha \preceq \varsigma_n \preceq \beta$, for all $n \in \mathbb{N}$, then there exists $\varsigma \in P$ such that $\lim_{n\to\infty} \varsigma_n = \varsigma$.*
(b) *Although the partial ordering $\preceq$ is not a linear (total) order on $\mathbb{C}$, the pair $(\mathbb{C}, \preceq)$ is a lattice.*
(c) *If $S \subset \mathbb{C}$ is such that there exist $\alpha, \beta \in \mathbb{C}$ with $\alpha \preceq s \preceq \beta$ for all $s \in S$, then $\inf S$ and $\sup S$ both exist.*
(d) *If $\varsigma_n \preceq \varsigma'_n \preceq \ell$ for all $n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} \varsigma_n = \ell$, then $\lim\limits_{n\to\infty} \varsigma'_n = \ell$.*
(e) *If $\varsigma_n \preceq \omega$ for all $n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} \varsigma_n = \varsigma \in P$, then $\varsigma \preceq \omega$.*
(f) *If $\omega \preceq \varsigma_n$ for all $n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} \varsigma_n = \varsigma \in P$, then $\omega \preceq \varsigma$.*

Definition 4 (Shukla et al. [13])**.** *Let $\digamma$ be a nonempty set. A complex fuzzy set M on $\digamma$ is characterized by a mapping with domain $\digamma$ and values in the closed unit complex interval I.*

Definition 5 (Shukla et al. [13])**.** *A binary operation $* \colon I \times I \to I$ is called a complex-valued t-norm if:*

1. $\varsigma_1 * \varsigma_2 = \varsigma_2 * \varsigma_1$;
2. $\varsigma_1 * \varsigma_2 \preceq \varsigma_3 * \varsigma_4$ *whenever* $\varsigma_1 \preceq \varsigma_3, \varsigma_2 \preceq \varsigma_4$;
3. $\varsigma_1 * (\varsigma_2 * \varsigma_3) = (\varsigma_1 * \varsigma_2) * \varsigma_3$;
4. $\varsigma * \theta = \theta, \varsigma * \ell = \varsigma$.

for all $\varsigma, \varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4 \in I$.

Example 1. *Let the binary operations $*_m, *_p, *_L, *^{sw}_\lambda \colon I \times I \to I$ be defined, respectively by the following: for all $\varsigma_1 = (a_1, b_1), \varsigma_2 = (a_2, b_2) \in I$*

1. $\varsigma_1 *_m \varsigma_2 = (\min\{a_1, a_2\}, \min\{b_1, b_2\})$ *(minimum of the corresponding coordinates).*
2. $\varsigma_1 *_p \varsigma_2 = (a_1 a_2, b_1 b_2)$ *(product of the corresponding coordinates).*
3. $\varsigma_1 *_L \varsigma_2 = (\max\{a_1 + a_2 - 1, 0\}, \max\{b_1 + b_2 - 1, 0\})$ *(Lukasiewicz t-norm of the corresponding coordinates).*
4. $\varsigma_1 *^{sw}_\lambda \varsigma_2 = \left(\max\left\{\frac{a_1+a_2-1+\lambda a_1 a_2}{1+\lambda}, 0\right\}, \max\left\{\frac{b_1+b_2-1+\lambda b_1 b_2}{1+\lambda}, 0\right\}\right)$ *(Sugeno–Weber t-norm of the corresponding coordinates).*

Then, $*_m, *_p, *_L$ and $*^{sw}_\lambda$ are complex-valued t-norms. It is obvious that $*_L$ is a particular case of $*^{sw}_\lambda$ when $\lambda = 0$.

Indeed, if $I_{\mathbb{R}} = [0,\ 1]$ is the closed unit real interval and $\star_1, \star_2 \colon I_{\mathbb{R}} \times I_{\mathbb{R}} \to I_{\mathbb{R}}$ are two t-norms, then $* \colon I \times I \to I$ defined by

$$\varsigma_1 * \varsigma_2 = (a_1 \star_1 a_2, b_1 \star_2 b_2) \text{ for all } \varsigma_1 = (a_1, b_1), \varsigma_2 = (a_2, b_2) \in I$$

is a complex-valued t-norm.

Example 2 (Shukla et al. [13]). *Define $* \colon I \times I \to I$ as follows:*

$$\varsigma_1 * \varsigma_2 = \begin{cases} (a_1, b_1), & \text{if } (a_2, b_2) = \ell; \\ (a_2, b_2), & \text{if } (a_1, b_1) = \ell; \\ \theta, & \text{otherwise}, \end{cases}$$

for all $\varsigma_1 = (a_1, b_1), \varsigma_2 = (a_2, b_2) \in I$. Then, $*$ is a complex-valued t-norm. Note that, $\varsigma_1 * \varsigma_2$ cannot be expressed as $(a_1 \star_1 a_2, b_1 \star_2 b_2)$, where $\star_1, \star_2 \colon I_{\mathbb{R}} \times I_{\mathbb{R}} \to I_{\mathbb{R}}$ are two t-norms.

Definition 6 (Shukla et al. [13]). *Let $\digamma$ be a nonempty set, $*$ a continuous complex-valued t-norm and M be a complex fuzzy set on $\digamma \times \digamma \times P_\theta$ satisfying the following conditions:*

(CFMS1) $\theta \prec M(\flat_1, \flat_2, \varsigma)$;
(CFMS2) $M(\flat_1, \flat_2, \varsigma) = \ell$ *if and only if* $\flat_1 = \flat_2$;
(CFMS3) $M(\flat_1, \flat_2, \varsigma) = M(\flat_2, \flat_1, \varsigma)$;
(CFMS4) $M(\flat_1, \flat_2, \varsigma) * M(\flat_2, \flat_3, \varsigma') \preceq M(\flat_1, \flat_3, \varsigma + \varsigma')$;
(CFMS5) $M(\flat_1, \flat_2, \cdot) \colon P_\theta \to I$ is continuous;

for all $\flat_1, \flat_2, \flat_3 \in \digamma$ and $\varsigma, \varsigma' \in P_\theta$. Then, the triplet $(\digamma, M, *)$ is called a complex-valued fuzzy metric space and M is called a complex-valued fuzzy metric on $\digamma$. A complex-valued fuzzy metric can be thought of as the degree of nearness between two points of $\digamma$ with respect to a complex parameter $\varsigma \in P_\theta$.

Remark 3. *In the definition of complex-valued fuzzy metric spaces, Humaira et al. [21] used the condition "$\theta \preceq M(\flat_1, \flat_2, \varsigma)$" instead of (CFMS1), i.e., they allowed the value of $M(\flat_1, \flat_2, \varsigma)$ to be θ. In view of the fact that a complex-valued fuzzy metric space is an extension of the concept of George and Veeramani [4], the condition (CFMS1) is more natural than the condition as used in [21]. Hence, we will use (CFMS1) instead of "$\theta \preceq M(\flat_1, \flat_2, \varsigma)$."*

The following remark follows directly from the continuity of $*$ and the definitions of P, P_θ, I and I_o.

Remark 4. (I) *If $\varsigma_1, \varsigma_2 \in I$, then $\varsigma_1 *_m \varsigma_2 = \inf\{\varsigma_1, \varsigma_2\}$, and $\varsigma_1 *_L \varsigma_2 = \sup\{\varsigma_1 + \varsigma_2 - \ell, \theta\}$.*

(II) *For $\flat_1, \flat_2 \in \digamma, \varsigma \in P_\theta, r \in I_o$, if $M(\flat_1, \flat_2, \varsigma) \succ \ell - r$, then there exists a ς_0 such that $\theta \prec \varsigma_0 \prec \varsigma$ and $M(\flat_1, \flat_2, \varsigma_0) \succ \ell - r$.*

(III) *If $r_1, r_2 \in I_o$ and $r_1 \succ r_2$, then there exists an $r_3 \in I_o$ such that $r_1 * r_3 \succeq r_2$ and for any $r_4 \in I_o$ there exist a $r_5 \in I_o$ such that $r_5 * r_5 \succeq r_4$.*

(IV) *If $\{\alpha_n\}, \{\beta_n\}$ are two convergent sequences in $\mathbb{C}$ such that $\alpha_n \preceq \beta_n$ for all $n \in \mathbb{N}$, then $\lim\limits_{n\to\infty} \alpha_n \preceq \lim\limits_{n\to\infty} \beta_n$.*

(V) *If $\{\alpha_n\}$ is a convergent sequence in $\mathbb{C}$ such that $\alpha_n \preceq \alpha_{n+1}$ for all $n \in \mathbb{N}$, then $\lim\limits_{n\to\infty} \alpha_n = \sup\limits_{n\geq 1} \alpha_n$.*

Several examples of complex-valued fuzzy metric spaces are given in [13]. With the help of the following propositions one can construct several more examples of complex-valued fuzzy metric spaces.

Proposition 1. *If $(\digamma, d)$ is a complex-valued metric space (see, Azam et al. [7]) such that $\theta \preceq d(\flat_1, \flat_2) \prec \ell$ for all $\flat_1, \flat_2 \in \digamma$, then $(\digamma, M, *_L)$ is a complex-valued fuzzy metric space, where $M(\flat_1, \flat_2, \varsigma) = \ell - d(\flat_1, \flat_2)$ for all $\varsigma \in P_\theta$ and $\flat_1, \flat_2 \in \digamma$.*

Proof. The properties (CFMS1)–(CFMS3) and (CFMS5) of Definition 6 are obvious. To prove (CFMS4), suppose $\flat_1, \flat_2, \flat_3 \in \digamma$ and $\varsigma_1, \varsigma_2 \in P_\theta$. Then, we have

$$\begin{aligned} M(\flat_1, \flat_3, \varsigma_1) *_L M(\flat_3, \flat_2, \varsigma_2) &= \sup\{M(\flat_1, \flat_3, \varsigma_1) + M(\flat_3, \flat_2, \varsigma_2) - \ell, \theta\} \\ &= \sup\{\ell - d(\flat_1, \flat_3) - d(\flat_3, \flat_2), \theta\}. \end{aligned}$$

Since $\theta \preceq d(\flat_1, \flat_2) \prec \ell$, we have $M(\flat_1, \flat_2, \varsigma_1 + \varsigma_2) \succ \theta$. Also, since $d(\flat_1, \flat_2) \preceq d(\flat_1, \flat_3) + d(\flat_3, \flat_2)$, we have $M(\flat_1, \flat_2, \varsigma_1 + \varsigma_2) = \ell - d(\flat_1, \flat_2) \succeq \ell - d(\flat_1, \flat_3) - d(\flat_3, \flat_2)$. Therefore,

$$M(\flat_1, \flat_2, \varsigma_1 + \varsigma_2) \succeq \sup\{\ell - d(\flat_1, \flat_3) - d(\flat_3, \flat_2), \theta\} = M(\flat_1, \flat_3, \varsigma_1) *_L M(\flat_3, \flat_2, \varsigma_2).$$

This proves the result. □

In the next proposition, we show that every pair of fuzzy metrics on the same set produces a complex-valued fuzzy metric space.

Proposition 2. *If $(\digamma, M_1, \star_1)$ and $(\digamma, M_2, \star_2)$ are two fuzzy metric spaces, then $(\digamma, M, *)$ is a complex-valued fuzzy metric space, where $M(\flat_1, \flat_2, \varsigma) = (M_1(\flat_1, \flat_2, a), M_2(\flat_1, \flat_2, b))$ for all $\varsigma = (a, b) \in P_\theta$ and $\flat_1, \flat_2 \in \digamma$, where $*: I \times I \to I$ is defined by $\varsigma_1 * \varsigma_2 = (a_1 \star_1 a_2, b_1 \star_2 b_2)$ for all $\varsigma_1 = (a_1, b_1), \varsigma_2 = (a_2, b_2) \in I$.*

Proof. The proof follows directly from the definition of $*$ and the fact that $(\digamma, M_1, \star_1)$ and $(\digamma, M_2, \star_2)$ are fuzzy metric spaces. □

Lemma 1 (Shukla et al. [13])**.** *Let $(\digamma, M, *)$ be a complex-valued fuzzy metric space. If $\varsigma, \varsigma' \in P_\theta$ and $\varsigma \prec \varsigma'$, then $M(\flat_1, \flat_2, \varsigma) \preceq M(\flat_1, \flat_2, \varsigma')$ for all $\flat_1, \flat_2 \in \digamma$.*

Definition 7 (Shukla et al. [13])**.** *Let $(\digamma, M, *)$ be a complex-valued fuzzy metric space. A sequence $\{\flat_n\}$ in $\digamma$ converges to some $\flat \in \digamma$ if for each $r \in I_o$ and $\varsigma \in P_\theta$ there exists $n_0 \in \mathbb{N}$ such that $\ell - r \prec M(\flat_n, \flat, \varsigma)$ for all $n > n_0$. The sequence $\{\flat_n\}$ is called a Cauchy sequence if $\lim_{n\to\infty} \inf_{m>n} M(\flat_n, \flat_m, \varsigma) = \ell$ for all $\varsigma \in P_\theta$. The complex-valued fuzzy metric space $(\digamma, M, *)$ is called complete if every Cauchy sequence in $\digamma$ converges in $\digamma$.*

Lemma 2 (Shukla et al. [13])**.** *Let $(\digamma, M, *)$ be a complex-valued fuzzy metric space and $\{\flat_n\}$ be a sequence in $\digamma$. Then:*

(A) *The sequence $\{\flat_n\}$ is convergent to $\flat \in \digamma$ if and only if $\lim\limits_{n\to\infty} M(\flat_n, \flat, \varsigma) = \ell$ holds for all $\varsigma \in P_\theta$.*

(B) *The sequence $\{\flat_n\}$ is a Cauchy sequence if and only if for each $r \in I_o$ and $\varsigma \in P_\theta$ there exists $n_0 \in \mathbb{N}$ such that $\ell - r \prec M(\flat_n, \flat_m, \varsigma)$ for all $n, m > n_0$.*

Definition 8. *Let $(\digamma, M, *)$ be a complex-valued fuzzy metric space. A sequence $\{\flat_n\}$ in $\digamma$ is called a G-Cauchy sequence if for each $p \in \mathbb{N}$ we have $\lim_{n\to\infty} M(\flat_n, \flat_{n+p}, \varsigma) = \ell$ for all $\varsigma \in P_\theta$. The complex-valued fuzzy metric space $(\digamma, M, *)$ is called G-complete if every G-Cauchy sequence in $\digamma$ converges in $\digamma$.*

Remark 5. *From Definition 7 and Lemma 2, it is clear that every Cauchy sequence in a complex-valued fuzzy metric space is a G-Cauchy sequence and every G-complete complex-valued fuzzy metric space is complete. Also, with the help of Proposition 2 and Note 3.13 of [4], one can easily show that the the converse of these facts is not true.*

Definition 9. *Let $(\digamma, M, *)$ be a complex-valued fuzzy metric space. An open ball $B(\flat_1, r, \varsigma)$ with center $\flat_1 \in \digamma$ and radius $r \in I_o, \varsigma \in P_\theta$ is defined by*

$$B(\flat_1, r, \varsigma) = \{\flat_2 \in \digamma : M(\flat_1, \flat_2, \varsigma) \succ \ell - r\}$$

The collection $\{B(\flat_1, r, \varsigma) \colon \flat \in \digamma, r \in I_o, \varsigma \in P_\theta\}$ *is a neighborhood system for the topology* τ *on* $\digamma$ *induced by the complex-valued fuzzy metric* M.

Theorem 1. *In a complex-valued fuzzy metric space every open ball is an open set.*

Proof. Let $(\digamma, M, *)$ be a complex-valued fuzzy metric space and $\flat_1 \in \digamma, r \in I_o, \varsigma \in P_\theta$ be fixed. Consider the open ball $B(\flat, r, \varsigma)$; then, we shall show that for every $\flat_2 \in B(\flat_1, r, \varsigma)$ there exists an open ball with center $\flat_2$ contained in $B(\flat_1, r, \varsigma)$. Obviously, $B(\flat, r, \varsigma) \neq \emptyset$. If $\flat_2 \in B(\flat_1, r, \varsigma)$, we have $M(\flat_1, \flat_2, \varsigma) \succ \ell - r$. Using (II) of Remark 4, we can choose a number $\varsigma_0, \theta \prec \varsigma_0 \prec \varsigma$ such that $M(\flat_1, \flat_2, \varsigma_0) \succ \ell - r$. Assume that $r_0 = M(\flat_1, \flat_2, \varsigma_0)$; then, there exists a complex number s such that $\theta \prec s \prec \ell$ and $r_0 \succ \ell - s \succ \ell - r$. As $r_0 \succ \ell - s$, again using (III) of Remark 4, we can find $r_1, \theta \prec r_1 \prec \ell$ such that $r_0 * r_1 \succeq \ell - s$. If $\flat_3 \in B(\flat_2, \ell - r_1, \varsigma - \varsigma_0)$, then $M(\flat_2, \flat_3, \varsigma - \varsigma_0) \succ r_1$. Therefore:

$$M(\flat_1, \flat_3, \varsigma) \succeq M(\flat_1, \flat_2, \varsigma_0) * M(\flat_2, \flat_3, \varsigma - \varsigma_0) \succeq r_0 * r_1 \succeq \ell - s \succ \ell - r.$$

This shows that $\flat_3 \in B(\flat_1, r, \varsigma)$, and so, $B(\flat_2, \ell - r_1, \varsigma - \varsigma_0) \subset B(\flat_1, r, \varsigma)$. This completes the proof. □

Remark 6. *If* $\varsigma_1, \varsigma_2 \in P_\theta$ *are such that* $\varsigma_1 \preceq \varsigma_2$*, then* $B(\flat, r, \varsigma_1) \subseteq B(\flat, r, \varsigma_2)$ *for all* $\flat \in \digamma$ *and* $r \in I_o$*. Also, if* $r_1, r_2 \in I_o$ *are such that* $r_1 \preceq r_2$*, then* $B(\flat, r_1, \varsigma) \subseteq B(\flat, r_2, \varsigma)$ *for all* $\flat \in \digamma$ *and* $\varsigma \in P_\theta$.

Remark 7. *For each* $\flat \in \digamma$*, if a sequence* $\{\varsigma_n\}$ *in* I_o *is such that* $\lim_{n\to\infty} \varsigma_n = \theta$*, then the collection* $\{B(\flat, \varsigma_n, \varsigma_n) : n \in \mathbb{N}\}$ *forms a local base at* $\flat$*. Indeed, if* $N(\flat)$ *is a neighborhood of* $\flat$*, then by definition there exists* $r \in I_o$*,* $\varsigma \in P_\theta$ *such that* $B(\flat, r, \varsigma) \subseteq N(\flat)$*, and by choice of* ς_n*, there exists* $m \in \mathbb{N}$ *such that* $\varsigma_m \preceq \inf\{r, \varsigma\}$*. Now, using Remark 6, one can show that* $B(\flat, \varsigma_m, \varsigma_m) \subset B(\flat, r, \varsigma)$*. Hence, the topology* τ *is first countable.*

Theorem 2. *The topology* τ *induced by a complex-valued fuzzy metric is Hausdorff.*

Proof. Let $(\digamma, M, *)$ be a complex-valued fuzzy metric space and $\flat_1, \flat_2 \in \digamma$ be two distinct points, i.e., $\theta \prec M(\flat_1, \flat_2, \varsigma) \prec \ell$ for all $\varsigma \in P_\theta$. If $\varsigma \in P_\theta$, assume that $M(\flat_1, \flat_2, \varsigma) = r$, then $r \in I_o$. If $r \prec r_0 \prec \ell$, then using Remark 4 one can choose an $r_1 \in I_o$ such that $r_1 * r_1 \succeq r_0$. We claim that $B(\flat_1, \ell - r_1, \varsigma/2) \cap B(\flat_2, \ell - r_1, \varsigma/2) = \emptyset$.

On contrary, suppose that there exists $\flat_3 \in \digamma$ such that $\flat_3 \in B(\flat_1, \ell - r_1, \varsigma/2) \cap B(\flat_2, \ell - r_1, \varsigma/2)$. Then,

$$r = M(\flat_1, \flat_2, \varsigma) \succeq M(\flat_1, \flat_3, \varsigma/2) * M(\flat_3, \flat_2, \varsigma/2) \succeq r_1 * r_1 \succeq r_0 \succ r.$$

This contradiction proves the claim, and so, the topology τ is Hausdorff. □

Remark 8. *The above theorem shows that the limit of a convergent sequence in a complex-valued fuzzy metric space is unique.*

Humaira et al. [21] proved the following theorem:

Theorem 3 (Theorem 3.1 of Humaira et al. [21])**.** *Let* $(\digamma, M, *)$ *be a complete complex-valued fuzzy metric space and let* $\top \colon \digamma \to \digamma$ *be a mapping satisfying the inequality:*

$$\psi(\ell - M(\top\flat_1, \top\flat_2, \varsigma)) \preceq \psi(\ell - M(\flat_1, \flat_2, \varsigma)) - \phi(\ell - M(\flat_1, \flat_2, \varsigma))$$

for all $\flat_1, \flat_2 \in \digamma, \varsigma \in P_\theta$*, where* $\psi, \phi \colon P \to P$ *both are continuous, monotonic nondecreasing functions with* $\psi(\varsigma), \phi(\varsigma) \succ \theta$ *for* $\varsigma \in P_\theta$ *and* $\psi(\theta) = \phi(\theta) = \theta$*. Then,* $\top$ *has a unique fixed point.*

Remark 9. *In the proof of the above theorem, from the monotonicity of function ψ the authors draw the following conclusion (see the proof of Theorem 3.1 of [21]):*

$$\psi(\ell - M(\omega_q, \omega_{q+1}, \varsigma)) \prec \psi(\ell - M(\omega_{q-1}, \omega_q, \varsigma)) \implies M(\omega_q, \omega_{q+1}, \varsigma) \succ M(\omega_{q-1}, \omega_q, \varsigma)$$

for $\varsigma \in P_\theta$. Note, this implication is not correct. In fact, the authors have drawn the above conclusion on the basis that if $\theta \preceq a \preceq \ell, \theta \preceq b \preceq \ell$ and $\psi(\ell - a) \prec \psi(\ell - b)$, then $b \prec a$, which is not true. To justify our claim we give the following example.

Example 3. *Let $\psi \colon P \to P$ be defined by $\psi(\varsigma) = \varsigma|\varsigma|$ for all $\varsigma \in P$, where $|\cdot|$ is used for the magnitude (modulus) of complex numbers. Then, ψ is a continuous and monotonic nondecreasing function with $\psi(\varsigma) \succ \theta$ for $\varsigma \in P_\theta$ and $\psi(\theta) = \theta$. Consider $a = (0.9, 0.1), b = (0.1, 0.15) \in P$, then $\theta \preceq a \preceq \ell, \theta \preceq b \preceq \ell$ and*

$$\psi(\ell - a) = \psi(0.1, 0.9) \prec \psi(0.9, 0.85) = \psi(\ell - b).$$

But, note that $b \not\prec a$. Indeed, a and b are not comparable with respect to $\preceq$.

The above implication was used to prove the Cauchyness of the sequence $\{\omega_n\}$ in the proof of Theorem 3.1 as well as in several other places (in the proof of Corollary 3.4 and Theorem 3.5) of [21]. Therefore, we conclude that the proofs for these results provided in [21] are not appropriate.

Remark 10. *In the proof of the above theorem (i.e., Theorem 3.1 of [21]), the authors constructed a sequence $\{\omega_n\}$ in the complex-valued fuzzy metric space $(\digamma, M, *)$ such that $\lim_{q\to\infty} M(\omega_q, \omega_{q+1}, \varsigma) = \ell$ for all $\varsigma \in P_\theta$, and then used (CFMS4) to obtain the following:*

$$M(\omega_q, \omega_{q+s}, \varsigma) \succeq M\Big(\omega_q, \omega_{q+1}, \frac{\varsigma}{s}\Big) * M\Big(\omega_{q+1}, \omega_{q+2}, \frac{\varsigma}{s}\Big) * \ldots * M\Big(\omega_{q+s-1}, \omega_{q+s}, \frac{\varsigma}{s}\Big) \quad (1)$$

for each $s \in \mathbb{N}$. As a consequence of the above inequality, they concluded that

$$\lim_{q\to\infty} \inf_{q+s>q} M(\omega_q, \omega_{q+s}, \varsigma) \succeq \ell * \ell * \ldots * \ell = \ell \quad (2)$$

hence, $\lim_{q\to\infty} M(\omega_q, \omega_{q+s}, \varsigma) = \ell$ for each $s \in \mathbb{N}$. This conclusion is drawn from (1) on the basis of the following:

(I) $\lim_{q\to\infty} M(\omega_q, \omega_{q+1}, \varsigma) = \ell$ *for all $\varsigma \in P_\theta$;*

(II) *the right hand side of (1) precedes or equal to the left hand side and so the infimum of right hand side over $s \in \mathbb{N}$ (i.e., $q + s > q$) also precedes or equal to the infimum of left hand side, and so, in limiting case (i.e., as $q \to \infty$), the infimum of the right hand side over $s \in \mathbb{N}$ will tend to ℓ.*

But, as the value of each $M(\omega_{q+i-1}, \omega_{q+i}, \frac{\varsigma}{s})$ (in the right hand side of (1)) depends on the parameter $\frac{\varsigma}{s}$ which further depends on $s \in \mathbb{N}$, therefore, although $\lim_{q\to\infty} M(\omega_q, \omega_{q+s}, \varsigma) = \ell$ for each $s \in \mathbb{N}$ may hold for such sequence, it is not fair to conclude (2). In particular, if the infimum over $s \in \mathbb{N}$ of the right hand side of the inequality (1) is θ, then as a conclusion, one can have only that $\lim_{q\to\infty} \inf_{q+s>q} M(\omega_q, \omega_{q+s}, \varsigma) \succeq \theta$ for each $s \in \mathbb{N}$; not $\lim_{q\to\infty} \inf_{q+s>q} M(\omega_q, \omega_{q+s}, \varsigma) = \ell$ for each $s \in \mathbb{N}$ (see the example below). So, the method used in [21] is not adequate. Also, by the process which is adopted in [21], it is clear that the authors actually proved that the sequence $\{\omega_q\}$ is G-Cauchy (not Cauchy), and since they have assumed the space $(\digamma, M, *)$ complete (not G-complete), therefore the convergence of the sequence $\{\omega_q\}$ can not be concluded (as the authors have done in [21]).

Example 4. *Let* $(\mathbb{R}, M_d, \star)$ *be the standard fuzzy metric space with* $a \star b = \min\{a, b\}$ *for all* $a, b \in [0,\ 1]$, *where d is the usual metric on* $\mathbb{R}$ *(see, [4]); then, by Proposition 2,* $(\mathbb{R}, M, *_m)$ *is a complex-valued fuzzy metric space, where*

$$M(\flat_1, \flat_2, \varsigma) = (M(\flat_1, \flat_2, a), M(\flat_1, \flat_2, b)) \text{ for all } \varsigma = (a, b) \in P_\theta \text{ and } \flat_1, \flat_2 \in \digamma.$$

Define a sequence $\{\omega_q\}$ *in* $\digamma$ *as follows: let* $\omega_0 \in \digamma$ *and* $\omega_q = \dfrac{\omega_0}{2^q}$ *for all* $q \in \mathbb{N}$. *Since* $\lim_{q\to\infty} d(\omega_q, \omega_{q+1}) = 0$, *hence we must have* $\lim_{q\to\infty} M(\omega_q, \omega_{q+1}, \varsigma) = \ell$ *for all* $\varsigma \in P_\theta$. *But note that for* $s \in \mathbb{N}$, *we have*

$$\begin{aligned}
\inf_{q+s>q} M(\omega_q, \omega_{q+s}, \varsigma) &\succeq \inf_{s\in\mathbb{N}} \Big[M\Big(\omega_q, \omega_{q+1}, \frac{\varsigma}{s}\Big) *_m M\Big(\omega_{q+1}, \omega_{q+2}, \frac{\varsigma}{s}\Big) \\
&\quad *_m \ldots *_m M\Big(\omega_{q+s-1}, \omega_{q+s}, \frac{\varsigma}{s}\Big)\Big] \\
&= \inf_{s\in\mathbb{N}} \left(\frac{\frac{a}{s}}{\frac{a}{s} + \frac{\omega_0}{2^{q+1}}}, \frac{\frac{b}{s}}{\frac{b}{s} + \frac{\omega_0}{2^{q+1}}} \right) \\
&= \theta
\end{aligned}$$

for all $\varsigma = (a, b) \in P_\theta$. *Hence, from the above, one cannot conclude that* $\lim\limits_{q\to\infty} \inf\limits_{q+s>q} M(\omega_q, \omega_{q+s}, \varsigma) = \ell$ *for each* $s \in \mathbb{N}$.

Another version of the fixed point theorem proved by Humaira et al. [21] is as follows:

Theorem 4 (Theorem 3.5 of Humaira et al. [21])**.** *Let* $(\digamma, M, *)$ *be a complete complex-valued fuzzy metric space such that for any sequence* $\{\varsigma_q\}$ *in* P_θ *with* $\lim_{q\to\infty} \varsigma_q = \infty$, *we have* $\lim_{q\to\infty} \inf_{\flat_2\in\digamma} M(\flat_1, \flat_2, \varsigma_q) = \ell$ *for all* $x \in \digamma$. *If* $\top\colon \digamma \to \digamma$ *satisfies*

$$\psi(M(\top\flat_1, \top\flat_2, \varsigma)) \succeq \psi(M(\flat_1, \flat_2, \varsigma)) - \phi(M(\flat_1, \flat_2, \varsigma)) \tag{3}$$

for all $\flat_1, \flat_2 \in \digamma, \varsigma \in P_\theta$, *where* $\psi, \phi\colon P \to P$ *both are continuous, monotonic nondecreasing functions with* $\psi(\varsigma), \phi(\varsigma) \succ \theta$ *for* $\varsigma \in P_\theta$ *and* $\psi(\theta) = \phi(\theta) = \theta$, *then* $\top$ *has a unique fixed point.*

We provide a counterexample to prove that the above theorem is flawed.

Example 5 (Counterexample)**.** *Let* $\digamma = [0,\ 1]$; *then* $(\digamma, M, *_L)$ *is a complete complex-valued fuzzy metric space, where*

$$M(\flat_1, \flat_2, \varsigma) = \left[1 - \frac{|\flat_1 - \flat_2|}{1 + ab}\right] \ell \text{ for all } \flat_1, \flat_2 \in \digamma, \varsigma = (a, b) \in P_\theta.$$

We note that, if $\{\varsigma_q\}$ *is a sequence in* P_θ *such that* $\varsigma_q = (a_q, b_q)$ *for all* $q \in \mathbb{N}$ *and* $\lim_{q\to\infty} \varsigma_q = \infty$, *then for all* $\flat_1 \in \digamma$ *we have*

$$\lim_{q\to\infty} \inf_{\flat_2\in\digamma} M(\flat_1, \flat_2, \varsigma_q) = \lim_{q\to\infty} \inf_{\flat_2\in\digamma} \left[1 - \frac{|\flat_1 - \flat_2|}{1 + a_q b_q}\right] \ell = \ell.$$

Define a mapping $\top\colon \digamma \to \digamma$ *by* $\top\flat = \flat/2$ *if* $\flat \neq 0$ *and* $\top 0 = 1$. *Let* $\psi, \phi\colon P \to P$ *be two mappings defined by* $\psi(\varsigma) = \phi(\varsigma) = \varsigma$ *for all* $\varsigma \in P$. *Then, it is clear that* $M(\flat_1, \flat_2, \varsigma) \succ \theta$ *for all* $\flat_1, \flat_2 \in \digamma$ *and as* $\psi(\varsigma) = \phi(\varsigma)$ *for all* $\varsigma \in P_\theta$; *hence, the condition (3) is satisfied trivially. Note that,* ψ *and* ϕ *both are continuous and monotonic nondecreasing functions. Thus, all the conditions of the above theorem are satisfied but* $\top$ *has no fixed point in* $\digamma$. *Hence, the existence of fixed point cannot be concluded. Also, if we choose* $\top$ *as the identity mapping on* $\digamma$, *then again (3) is satisfied trivially for any arbitrary function* $\phi\colon P \to P$, *and hence the uniqueness of fixed point is also an incorrect conclusion of Theorem 3.5 of Humaira et al. [21].*

In the next section, the results of Humaira et al. [21] are improved and generalized with suitable control functions and associated contractive conditions.

3. Fixed Point Theorems

We first state some definitions which will be needed in the sequel.

Definition 10. *Let F be a nonempty set, $\alpha\colon \mathsf{F}\times\mathsf{F}\to[0,\infty)$ a function and $\{\flat_n\}$ be a sequence in F. Then, the sequence $\{\flat_n\}$ is called an α-sequence if $\alpha(\flat_n,\flat_m)\geq 1$ for all $n,m\in\mathbb{N}$ with $m>n$. By $\mathfrak{A}_\alpha$, we denote the class of all α-sequences in F. By F_α, we denote the set $\{(\flat_1,\flat_2)\in\mathsf{F}\times\mathsf{F}\colon \alpha(\flat_1,\flat_2)\geq 1\}$. Then, a mapping $\top\colon\mathsf{F}\to\mathsf{F}$ will be α-admissible if and only if $(\flat_1,\flat_2)\in\mathsf{F}_\alpha$ implies $(\top\flat_1,\top\flat_2)\in\mathsf{F}_\alpha$ for all $\flat_1,\flat_2\in\mathsf{F}$. The set F_α is called transitive if $(\flat_1,\flat_2),(\flat_2,\flat_3)\in\mathsf{F}_\alpha$ implies that $(\flat_1,\flat_3)\in\mathsf{F}_\alpha$. A sequence $\{\flat_n\}_{n\geq 0}$ is said to be a $\top$-Picard sequence with initial value $\flat_0\in\mathsf{F}$ if $\flat_n=\top^n\flat_0$ for all $n\in\mathbb{N}$. The set of all $\top$-Picard sequences in F is denoted by $P_\top$, i.e.,*

$$P_\top=\{\{\flat_n\}\colon \flat_n=\top^n\flat_0 \text{ for some } \flat_0\in\mathsf{F}\}.$$

Definition 11. *Let $(\mathsf{F},M,*)$ be a complex-valued fuzzy metric space and $\alpha\colon\mathsf{F}\times\mathsf{F}\to[0,\infty)$ be a function. Then:*

1. *By $\mathfrak{C}$, we denote the class of all Cauchy sequences in F. A sequence $\{\flat_n\}$ in F is called an α-Cauchy sequence if $\{\flat_n\}\in\mathfrak{A}_\alpha\cap\mathfrak{C}$.*
2. *The space $(\mathsf{F},M,*)$ is called α-complete if every sequence of the class $\mathfrak{A}_\alpha\cap\mathfrak{C}$ converges to some $\flat\in\mathsf{F}$.*
3. *A mapping $\top$ is said to be continuous (respectively, α-continuous) at $u\in$ if for every convergent sequence $\{\flat_n\}$ in (respectively, $\{\flat_n\}\in\mathfrak{A}_\alpha$), the sequence $\{\top\flat_n\}$ converges to $\top u$, where $u\in$ is the limit of $\{\flat_n\}$. The mapping is said to be continuous (respectively, α-continuous) on $A\subseteq$ if it is continuous (respectively, α-continuous) at each point of A.*

It is easy to see that the completeness implies α-completeness and the continuity implies α-continuity, but the converse is not true, as shown in the following example.

Example 6. *Let $\mathsf{F}=[0,1)$ and d be the usual metric on F; then, $(\mathsf{F},M,*_L)$ is a complex-valued fuzzy metric space, where $M(\flat_1,\flat_2,\varsigma)=\ell-d(\flat_1,\flat_2)\ell$ for all $\flat_1,\flat_2\in\mathsf{F}$ and $\varsigma\in P_\theta$. Define a function $\alpha\colon\mathsf{F}\times\mathsf{F}\to[0,\infty)$ by*

$$\alpha(\flat_1,\flat_2)=\begin{cases}1, & \text{if } \flat_2\leq\flat_1<1/2;\\ 0, & \text{otherwise.}\end{cases}$$

Then, it is clear that a sequence is Cauchy (convergent) in (F,d) if and only if it is Cauchy (convergent) in $(\mathsf{F},M,*_L)$. As (F,d) is not complete, $(\mathsf{F},M,*_L)$ is not complete. On the other hand, if a sequence $\{\flat_n\}\in\mathfrak{A}_\alpha\cap\mathfrak{C}$, then by definition of α, the sequence $\{\flat_n\}$ must be a nonincreasing sequence in the interval $[0,1/2)$, and hence it must be convergent to some $\flat\in[0,1/2)$. Hence, $(\mathsf{F},M,*_L)$ is α-complete.

Consider a mapping $\top\colon\mathsf{F}\to\mathsf{F}$ defined by

$$\top\flat=\begin{cases}\flat, & \text{if } 0\leq\flat\leq 1/2;\\ 0, & \text{otherwise.}\end{cases}$$

Then, it is obvious that $\top$ is not continuous on F. On the other hand, since every sequence $\{\flat_n\}\in\mathfrak{A}_\alpha$ is a nonincreasing sequence in the interval $[0,1/2)$, and $\top$ is an identity mapping in $[0,1/2)$, therefore, if $\{\flat_n\}\in\mathfrak{A}_\alpha$ and converges to u, then $u\in[0,1/2)$ and $\{\top\flat_n\}=\{\flat_n\}$ converges to $\top u=u$. Thus, $\top$ is α-continuous.

By Θ we denote the class of all functions $\psi\colon I\to I$ such that for any double sequence $\{\varsigma_{n,m}\}$ in I we have $\lim_{n,m\to\infty}\varsigma_{n,m}=\theta$ if and only if $\lim_{n,m\to\infty}\psi(\varsigma_{n,m})=\theta$.

Example 7. *The following functions $\psi\colon I \to I$ are members of the class Θ:*

(a) $\psi(\varsigma) = \varsigma$ *for all* $\varsigma \in I$;
(b) $\psi(\varsigma) = k\varsigma$ *for all* $\varsigma \in I$, *where* $k \in (0,1)$ *is fixed;*
(c) $\psi(\varsigma) = \dfrac{a+b}{1+a+b}\ell$ *for all* $\varsigma = (a,b) \in I$;
(d) $\psi(\varsigma) = \varsigma^{cor(k)}$ *for all* $\varsigma = (a,b) \in I$, *where* $\varsigma^{cor(k)} = (a^k, b^k)$ *and* $k \in (0,\infty)$ *is fixed.*

Definition 12. *Let $(\digamma, M, *)$ be a complex-valued fuzzy metric space, $\alpha\colon \digamma \times \digamma \to [0,\infty)$ and $\top\colon \digamma \to \digamma$ be a mapping. Then:*

(A) *$\top$ is said to be a (ψ,ϕ)-contraction if there exist $\psi, \phi \in \Theta$ such that*

$$\psi(\ell - M(\top\flat_1, \top\flat_2, \varsigma)) \preceq \psi(\ell - M(\flat_1, \flat_2, \varsigma)) - \phi(\ell - M(\flat_1, \flat_2, \varsigma))$$

for all $\flat_1, \flat_2 \in \digamma$ and $\varsigma \in P_\theta$.

(B) *$\top$ is said to be an α-(ψ,ϕ)-contraction if there exist $\psi, \phi \in \Theta$ such that*

$$\alpha(\flat_1, \flat_2)\psi(\ell - M(\top\flat_1, \top\flat_2, \varsigma)) \preceq \psi(\ell - M(\flat_1, \flat_2, \varsigma)) - \phi(\ell - M(\flat_1, \flat_2, \varsigma))$$

for all $\flat_1, \flat_2 \in \digamma$ and $\varsigma \in P_\theta$.

Let $(\digamma, M, *)$ be a complex-valued fuzzy metric space and $\top\colon \digamma \to \digamma$ be a mapping. Denote by $P_\top^\digamma$ the set of all limits of $\top$-Picard sequences in $\digamma$, i.e.,

$$P_\top^\digamma = \{u \in \digamma\colon \top^n\flat_0 \text{ converges to } u \text{ for } \flat_0 \in \digamma\}.$$

The set of all fixed points of $\top$ is denoted by Fix($\top$), i.e., Fix($\top$)$= \{\flat \in \digamma\colon \top\flat = \flat\}$. We write $s(\flat_1, \flat_2) \in A \subset \digamma \times \digamma$ if at least one of the pairs $(\flat_1, \flat_2)$ and $(\flat_2, \flat_1)$ is an element of A. If $\alpha\colon \digamma \times \digamma \to [0,\infty)$ is a function, then we define the property (S) as follows:

$$\textbf{(S)}\text{: for every } u, v \in P_\top^\digamma \text{ there exists } z \in \digamma \text{ such that } s(u,z), s(v,z) \in \digamma_\alpha. \tag{4}$$

Next, the theorem ensures the existence of fixed point of an α-(ψ,ϕ)-contraction.

Theorem 5. *Let $(\digamma, M, *)$ be an α-complete complex-valued fuzzy metric space, $\alpha\colon \digamma \times \digamma \to [0,\infty)$ a function, and $\top\colon \digamma \to \digamma$ be an α-(ψ,ϕ)-contraction. Suppose $P_\top \cap \mathfrak{A}_\alpha \neq \emptyset$ and at least one of the following conditions is satisfied:*

(A) *If $\{\flat_n\} \in P_\top \cap \mathfrak{A}_\alpha \cap \mathfrak{C}$ and converges to $u \in \digamma$, then there exists $n_0 \in \mathbb{N}$ such that $s(\flat_n, u) \in \digamma_\alpha$ for all $n > n_0$;*
(B) *$\top$ is α-continuous on $P_\top^\digamma$.*

Then, $\top$ has a fixed point in $\digamma$.

Proof. Suppose $P_\top \cap \mathfrak{A}_\alpha \neq \emptyset$. Then, there exists $\flat_0 \in \digamma$ such that $\{\flat_n\} \in P_\top \cap \mathfrak{A}_\alpha$, $\flat_n = \top^n\flat_0 = \top\flat_{n-1}$ for all $n \in \mathbb{N}$ and $(\flat_n, \flat_m) \in \digamma_\alpha$ for all $n, m \in \mathbb{N}$ with $m > n$.

We shall show that $\{\flat_n\} \in \mathfrak{C}$. Since $(\flat_n, \flat_m) \in \digamma_\alpha$ for all $m > n$, and $\top$ is an α-(ψ,ϕ)-contraction, we have

$$\begin{aligned} \psi(\ell - M(\flat_{n+1}, \flat_{m+1}, \varsigma)) &= \psi(\ell - M(\top\flat_n, \top\flat_m, \varsigma)) \\ &\preceq \alpha(\flat_n, \flat_m)\psi(\ell - M(\top\flat_n, \top\flat_m, \varsigma)) \\ &\preceq \psi(\ell - M(\flat_n, \flat_m, \varsigma)) - \phi(\ell - M(\flat_n, \flat_m, \varsigma)) \end{aligned}$$

for all $m > n$ and $\varsigma \in P_\theta$. This shows that

$$\psi(\ell - M(\flat_{n+1}, \flat_{m+1}, \varsigma)) + \phi(\ell - M(\flat_n, \flat_m, \varsigma)) \preceq \psi(\ell - M(\flat_n, \flat_m, \varsigma)) \tag{5}$$

for all $m > n$ and $\varsigma \in P_\theta$. Since $\psi, \phi\colon I \to I$, hence we must have $\theta \preceq \psi(\ell - M(\flat_n, \flat_m, \varsigma)) \preceq \ell$ and $\theta \preceq \phi(\ell - M(\flat_n, \flat_m, \varsigma)) \preceq \ell$ for all $n, m \in \mathbb{N}$, $\varsigma \in P_\theta$; therefore, for each $\varsigma \in P_\theta$, we can define

$$\alpha_n = \sup_{m>n} \psi(\ell - M(\flat_n, \flat_m, \varsigma)), \beta_n = \sup_{m>n} \phi(\ell - M(\flat_n, \flat_m, \varsigma)).$$

Then, by definitions of α_n, β_n and the inequality (5), we obtain

$$\alpha_{n+1} + \beta_n \preceq \alpha_n \text{ for all } n \in \mathbb{N}. \tag{6}$$

Since $\theta \preceq \beta_n$ for all $n \in \mathbb{N}$, the above inequality yields $\alpha_{n+1} \preceq \alpha_n$ for all $n \in \mathbb{N}$. Also, since $\theta \preceq \alpha_n \preceq \ell$ for all $n \in \mathbb{N}$, hence by Remark 2 there exists $\alpha \in P$ such that $\theta \preceq \alpha \preceq \ell$ and

$$\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \sup_{m>n} \psi(\ell - M(\flat_n, \flat_m, \varsigma)) = \alpha. \tag{7}$$

From (6) we have $\beta_n \preceq \alpha_n - \alpha_{n+1}$ for all $n \in \mathbb{N}$, therefore

$$|\beta_n| \leq |\alpha_n - \alpha_{n+1}| \text{ for all } n \in \mathbb{N}.$$

For every given $\varepsilon > 0$, by (7) there exists $n_1 \in \mathbb{N}$ such that $|\alpha_n - \alpha_{n+1}| < \varepsilon$ for all $n > n_1$, which with the above inequality gives

$$|\sup_{m>n} \phi(\ell - M(\flat_n, \flat_m, \varsigma))| = |\beta_n| < \varepsilon \text{ for all } n > n_1.$$

Since $\phi(\ell - M(\flat_n, \flat_m, \varsigma)) \preceq \sup_{m>n} \phi(\ell - M(\flat_n, \flat_m, \varsigma))$ for all $m > n$, hence

$$|\phi(\ell - M(\flat_n, \flat_m, \varsigma))| \leq |\sup_{m>n} \phi(\ell - M(\flat_n, \flat_m, \varsigma))| < \varepsilon \text{ for all } m > n > n_1.$$

This shows that

$$\lim_{n,m\to\infty} \phi(\ell - M(\flat_n, \flat_m, \varsigma)) = \theta.$$

Since $\phi \in \Theta$, the above shows that $\lim_{n,m\to\infty}\{\ell - M(\flat_n, \flat_m, \varsigma)\} = \theta$, i.e.,

$$\lim_{n,m\to\infty} M(\flat_n, \flat_m, \varsigma) = \ell.$$

Hence, $\{\flat_n\}$ is a Cauchy sequence, i.e., $\{\flat_n\} \in \mathfrak{C}$.

Thus, $\{\flat_n\} \in P_\top \cap \mathfrak{A}_\alpha \cap \mathfrak{C}$ and by α-completeness of F it converges to some $u \in \mathsf{F}$. We shall show that u is a fixed point of $\top$.

Case I. Suppose (A) holds; then, there exists $n_0 \in \mathbb{N}$ such that $s(\flat_n, u) \in \mathsf{F}_\alpha$ for all $n > n_0$.

Assume that $(\flat_n, u) \in \mathsf{F}_\alpha$ for all $n > n_0$ (the proof for the second case is same). Then, as $\top$ is an α-(ψ, ϕ)-contraction, we have

$$\begin{aligned} \psi(\ell - M(\top\flat_n, \top u, \varsigma)) &\preceq \alpha(\flat_n, u)\psi(\ell - M(\top\flat_n, \top u, \varsigma)) \\ &\preceq \psi(\ell - M(\flat_n, u, \varsigma)) - \phi(\ell - M(\flat_n, u, \varsigma)). \end{aligned} \tag{8}$$

Since $\{\flat_n\}$ converges to u, we have $\lim_{n\to\infty} M(\flat_n, u, \varsigma) = \ell$, i.e., $\lim_{n\to\infty}\{\ell - M(\flat_n, u, \varsigma)\} = \theta$, and $\psi, \phi \in \Theta$, we must have

$$\lim_{n\to\infty} \psi(\ell - M(\flat_n, u, \varsigma)) = \lim_{n\to\infty} \phi(\ell - M(\flat_n, u, \varsigma)) = \theta \text{ for all } \varsigma \in P_\theta.$$

The above with (8) gives

$$\lim_{n\to\infty} \psi(\ell - M(\flat_{n+1}, \top u, \varsigma)) = \theta \text{ for all } \varsigma \in P_\theta.$$

Again, as $\psi \in \Theta$, we must have $\lim_{n\to\infty}\{\ell - M(\flat_{n+1}, \top u, \varsigma)\} = \theta$, i.e.,

$$\lim_{n\to\infty} M(\flat_{n+1}, \top u, \varsigma) = \ell \text{ for all } \varsigma \in P_\theta.$$

Now, for every $n \in \mathbb{N}$ and $\varsigma \in P_\theta$, we have

$$M(\top u, u, \varsigma) \;\; \succeq \;\; M(\top u, \flat_{n+1}, \varsigma/2) * M(\flat_{n+1}, u, \varsigma/2).$$

Letting $n \to \infty$ and using Remark 2 in the above inequality, we obtain $M(\top u, u, \varsigma) = \ell$ for all $\varsigma \in P_\theta$, i.e., $\top u = u$. Thus, u is a fixed point of $\top$.

Case II. Suppose (B) holds. Then, since $\{\flat_n\} \in P_\top \cap \mathfrak{A}_\alpha \cap \mathfrak{C}$ and converges to $u \in \digamma$, hence by α-continuity of $\top$ on $P_\top^{\digamma}$, the sequence $\{\top\flat_n\} = \{\flat_{n+1}\}$ converges to $\top u$. By the uniqueness of the limit of convergent sequence in complex-valued fuzzy metric spaces the limits of the sequences $\{\flat_n\}$ and $\{\flat_{n+1}\}$ must be same, i.e., $\top u = u$. Thus, u is a fixed point of $\top$. $\square$

Remark 11. *If $(\digamma, M, *)$ is a complex-valued fuzzy metric space, $\top\colon \digamma \to \digamma$ is a mapping and $\alpha\colon \digamma \times \digamma \to [0, \infty)$ is a function. Then, $\digamma$ is called $\top$-α-complete if every sequence of the class $P_\top \cap \mathfrak{A}_\alpha \cap \mathfrak{C}$ converges to some $\flat \in \digamma$. In the above theorem, even if we replace the α-completeness of $\digamma$ with the $\top$-α-completeness, it still ensures the existence of the fixed point of $\top$.*

We next establish a condition for the uniqueness of fixed point of $\top$.

Theorem 6. *Suppose that all the hypotheses of Theorem 5 are satisfied. In addition, if the property (S) is satisfied and $\top$ is α-admissible, then $\top$ has a unique fixed point in $\digamma$.*

Proof. By Theorem 5, $\top$ has a fixed point $u \in \digamma$. We notice that if $\flat \in \mathrm{Fix}(\top)$, then $\top^n\flat = \flat$ for all $n \in \mathbb{N}$; therefore, $\flat \in P_\top^{\digamma}$, i.e., Fix($\top$)$\subseteq P_\top^{\digamma}$. For uniqueness of fixed point u of $\top$, on the contrary, suppose that $v \in \mathrm{Fix}(\top)$ and $u \neq v$, and then $v \in P_\top^{\digamma}$; hence, by property (S), there exists $z \in \digamma$ such that $s(u,z), s(v,z) \in \digamma_\alpha$. Suppose $(u,z), (v,z) \in \digamma_\alpha$ (the proof for all other cases is same); then, as $\top$ is α-admissible, we have $(\top^n u, \top^n z) \in \digamma_\alpha$ for all $n \in \mathbb{N}$. As $\top$ is an α-(ψ, ϕ)-contraction, we have

$$\begin{aligned}
\psi(\ell - M(u, \top^n z, \varsigma)) &= \psi(\ell - M(\top^n u, \top^n z, \varsigma)) \\
&\preceq \alpha(\top^{n-1}u, \top^{n-1}z)\psi(\ell - M(\top^n u, \top^n z, \varsigma)) \\
&\preceq \psi(\ell - M(\top^{n-1}u, \top^{n-1}z, \varsigma)) - \phi(\ell - M(\top^{n-1}u, \top^{n-1}z, \varsigma)) \\
&= \psi(\ell - M(u, \top^{n-1}z, \varsigma)) - \phi(\ell - M(u, \top^{n-1}z, \varsigma)).
\end{aligned}$$

For each $\varsigma \in P_\theta$, let $\gamma_n = \psi(\ell - M(u, \top^n z, \varsigma)), \delta_n = \phi(\ell - M(u, \top^{n-1}z, \varsigma))$ for all $n \in \mathbb{N}$; then, from the above inequality we have

$$\gamma_n + \delta_n \preceq \gamma_{n-1} \text{ for all } n \in \mathbb{N}. \tag{9}$$

Since $\gamma_n, \delta_n \in I$, the above inequality shows that $\gamma_n \preceq \gamma_{n-1}$ for all $n \in \mathbb{N}$. By Remark 2, there exists $\gamma \in I$ such that $\lim_{n\to\infty} \gamma_n = \gamma$. Again, by (9) we have $\delta_n \preceq \gamma_{n-1} - \gamma_n$ for all $n \in \mathbb{N}$, and so $|\delta_n| \leq |\gamma_{n-1} - \gamma_n|$ for all $n \in \mathbb{N}$. As $\lim_{n\to\infty} \gamma_n = \gamma$, for every given $\varepsilon > 0$, there exists $n_1 \in \mathbb{N}$ such that

$$|\delta_n| \leq |\gamma_{n-1} - \gamma_n| < \varepsilon \text{ for all } n > n_1.$$

This shows that

$$\lim_{n\to\infty} \phi(\ell - M(u, \top^n z, \varsigma)) = \lim_{n\to\infty} \delta_n = \theta.$$

Since $\phi \in \Theta$, we must have $\lim_{n\to\infty}\{\ell - M(u, \top^n z, \varsigma)\} = \theta$, i.e., $\lim_{n\to\infty} M(u, \top^n z, \varsigma) = \ell$. Similarly, we obtain $\lim_{n\to\infty} M(v, \top^n z, \varsigma) = \ell$. Hence, $u = v$. This contradiction proves the uniqueness of the fixed point. □

Example 8. *Let $\digamma = [0,1)$; then, $(\digamma, M, *_L)$ is a complete complex-valued fuzzy metric space, where*

$$M(\flat_1, \flat_2, \varsigma) = \left[1 - \frac{|\flat_1 - \flat_2|}{1+ab}\right]\ell \text{ for all } \flat_1, \flat_2 \in \digamma, \varsigma = (a,b) \in P_\theta.$$

Let $a_n = \frac{1}{2^n}$ for all $n \in \mathbb{N}$ and define a function $\alpha\colon \digamma \times \digamma \to [0,\infty)$ by

$$\alpha(\flat_1, \flat_2) = \begin{cases} 1, & \flat_1 = a_n, \flat_2 = a_m, m > n \text{ or } \flat_1 = \flat_2 = 0; \\ 0, & \text{otherwise.} \end{cases}$$

Then, it is clear that if $\{\flat_n\} \in \mathfrak{A}_\alpha \cap \mathfrak{C}$, then $\flat_n \to 0$ (with respect to usual metric of $\mathbb{R}$), and hence $\{\flat_n\}$ must be convergent to 0 in $(\digamma, M, *_L)$. Therefore, $(\digamma, M, *_L)$ is α-complete. Consider the functions $\psi, \phi\colon I \to I$ and $\top\colon \digamma \to \digamma$ defined by $\psi(\varsigma) = \varsigma$, $\phi(\varsigma) = \varsigma/2$ for all $\varsigma \in I$ and

$$\top\flat = \begin{cases} a_{n+1}, & \text{if } \flat = a_n; \\ 0, & \text{otherwise.} \end{cases}$$

Then, $\top$ is α-admissible. For every fixed $k \in \mathbb{N}$ and $\flat_n = \top^n a_k = a_{n+k}$ for all $n \in \mathbb{N}$, we have $\{\flat_n\} \in P_\top \cap \mathfrak{A}_\alpha$, hence $P_\top \cap \mathfrak{A}_\alpha \neq \emptyset$. It is easy to verify that $\top$ is an α-(ψ,ϕ)-contraction. Also, note that $P_\top^\digamma = \{0\}$, and hence $\top$ is an α-continuous mapping on $P_\top^\digamma$ and property (S) is satisfied. Thus, all the conditions of Theorem 6 are satisfied and hence by Theorem 6 the mapping $\top$ must have a unique fixed point in $\digamma$. Indeed, Fix($\top$)=$\{0\}$.

The following example justifies the significance of the property (S) in Theorem 6.

Example 9. *Consider the the complex-valued fuzzy metric space $(\digamma, M, *_L)$ and the function $\alpha\colon \digamma \times \digamma \to [0,\infty)$ as defined in Example 6. Define a mapping $\top\colon \digamma \to \digamma$ by*

$$\top\flat = \begin{cases} \flat/2, & \text{if } 0 \le \flat \le 1/2; \\ \flat, & \text{otherwise.} \end{cases}$$

Define the functions $\psi, \phi\colon I \to I$ by $\psi(\varsigma) = \varsigma$, $\phi(\varsigma) = \varsigma/2$ for all $\varsigma \in I$. Then, one can verify easily that $\top$ is an α-(ψ,ϕ)-contraction. Consider a $\top$-Picard sequence $\{\flat_n\}$, where $\flat_0 \in (1/2, 1)$, then by definition of $\top$ we have $\flat_n = \flat$ for all $n \in \mathbb{N}$, therefore $(1/2,) \subset P_\top^\digamma$. Now, for any $u, v \in (1/2, 1)$ there exists no $z \in \digamma$ such that $s(u,z), s(v,z) \in \digamma_\alpha$. Hence, the property (S) is not satisfied. It is easy to verify that all other conditions of Theorem 6 are satisfied. Note that $\top$ has infinitely many fixed points in $\digamma$. Indeed, Fix($\top$) $= \{0\} \cup (1/2, 1)$.

The following corollary is a generalization of Theorem 3.1 of Shukla et al. [13].

Corollary 1. *Let $(\digamma, M, *)$ be an α-complete complex-valued fuzzy metric space, $\alpha\colon \digamma \times \digamma \to [0,\infty)$ a function, and $\top\colon \digamma \to \digamma$ be a mapping such that*

$$\alpha(\flat_1, \flat_2)[\ell - M(\top\flat_1, \top\flat_2, \varsigma)] \preceq k[\ell - M(\flat_1, \flat_2, \varsigma)] \text{ for all } \flat_1, \flat_2 \in \digamma, \varsigma \in P_\theta$$

where $k \in (0,1)$. Suppose $P_\top \cap \mathfrak{A}_\alpha \neq \emptyset$ and at least one of the following conditions is satisfied:

(A) *If $\{\flat_n\} \in P_\top \cap \mathfrak{A}_\alpha \cap \mathfrak{C}$ and converges to $u \in \digamma$, then there exists $n_0 \in \mathbb{N}$ such that $s(\flat_n, u) \in \digamma_\alpha$ for all $n > n_0$ for all $n > n_0$;*

(B) *$\top$ is α-continuous on $P_\top^\digamma$.*

Then, $\top$ has a fixed point in $\digamma$. If the property (S) is satisfied and $\top$ is α-admissible, then the fixed point of $\top$ is unique.

Proof. The proof follows from Theorem 6 with $\psi(\varsigma) = \varsigma$ and $\phi(\varsigma) = (1-k)\varsigma$. $\square$

Remark 12. *If we take the constant function $\alpha(\flat_1, \flat_2) = 1$ for all $\flat_1, \flat_2 \in \mathsf{F}$ in the above corollary, we obtain Theorem 3.1 of Shukla et al. [13].*

The following corollary is an improved version of Theorem 3.1 of Humaira et al. [21] in the sense that there is no constraint of continuity and nondecreasingness on the functions ψ and ϕ.

Corollary 2. *Let $(\mathsf{F}, M, *)$ be a complete complex-valued fuzzy metric space and let $\top : \mathsf{F} \to \mathsf{F}$ be a (ψ, ϕ)-contraction. Then, $\top$ has a unique fixed point.*

Proof. The proof follows from Theorem 6 with $\alpha(\flat_1, \flat_2) = 1$ for all $\flat_1, \flat_2 \in \mathsf{F}$. $\square$

We next prove that if F_α is transitive, then for an α-admissible mapping, the constraint $P_\top \cap \mathfrak{A}_\alpha \neq \emptyset$ of Theorem 5 always holds; hence, we establish the existence and uniqueness of fixed point of such mappings.

Theorem 7. *Let $(\mathsf{F}, M, *)$ be an α-complete complex-valued fuzzy metric space, $\alpha : \mathsf{F} \times \mathsf{F} \to [0, \infty)$ a function and $\top : \mathsf{F} \to \mathsf{F}$ be an α-(ψ, ϕ)-contraction. Suppose the following conditions are satisfied:*

(I) *F_α is transitive;*
(II) *$\top$ is α-admissible;*
(III) *There exists $z \in \mathsf{F}$ such that $(z, \top z) \in \mathsf{F}_\alpha$;*
(IV) *At least one of the following conditions is satisfied:*
 (A) *If $\{\flat_n\} \in P_\top \cap \mathfrak{A}_\alpha \cap \mathfrak{C}$ and converges to $u \in \mathsf{F}$, then there exists $n_0 \in \mathbb{N}$ such that $s(\flat_n, u) \in \mathsf{F}_\alpha$ for all $n > n_0$;*
 (B) *$\top$ is α-continuous on $P_\top^{\mathsf{F}}$.*

Then, $\top$ has a fixed point in F. In addition, if the property (S) is satisfied, then the fixed point of $\top$ is unique.

Proof. We shall show that there exists $\flat_0 \in \mathsf{F}$ such that $\{\flat_n\} \in P_\top \cap \mathfrak{A}_\alpha$. We define a sequence $\{\flat_n\}$ in F as follows: by (III) there exists $z \in \mathsf{F}$ such that $(z, \top z) \in \mathsf{F}_\alpha$, let $\flat_0 = z$ and $\flat_1 = \top \flat_0$ so that $(\flat_0, \flat_1) \in \mathsf{F}_\alpha$ Then, by (II) we obtain $(\top \flat_0, \top^2 \flat_0) = (\flat_1, \top \flat_1) \in \mathsf{F}_\alpha$. Let $\top \flat_1 = \flat_2$ so that $(\flat_1, \flat_2) \in \mathsf{F}_\alpha$. On continuing in a similar way we obtain a sequence $\{\flat_n\}$ such that

$$\flat_n = \top \flat_{n-1} \text{ and } (\flat_{n-1}, \flat_n) \in \mathsf{F}_\alpha \text{ for all } n \in \mathbb{N}.$$

Since F_α is transitive, it follows from the above inclusion that $(\flat_n, \flat_m) \in \mathsf{F}_\alpha$ for all $m > n$. Thus, $\{\flat_n\}$ is a $\top$-Picard sequence with initial value $\flat_0$, and so $\{\flat_n\} \in P_\top \cap \mathfrak{A}_\alpha$. Now, the proof follows from Theorem 6. $\square$

By Λ, we denote the class of all functions $\phi : I \to I$ such that for any double sequence $\{\varsigma_{n,m}\}$ in I we have $\lim_{n,m\to\infty} \varsigma_{n,m} = \ell$ if and only if $\lim_{n,m\to\infty} \phi(\varsigma_{n,m}) = \theta$. While $\mathfrak{L}$ denotes the class of all functions $\psi : I \to I$ such that for any sequence $\{\varsigma_n\}$ in I, we have $\lim_{n\to\infty} \varsigma_n = \ell$ if and only if $\lim_{n\to\infty} \psi(\varsigma_n) = \ell$.

Example 10. *If $\psi \in \Theta$, then $\phi \in \Lambda$, where the function $\phi : I \to I$ is defined by $\phi(\varsigma) = \psi(\ell - \varsigma)$ for all $\varsigma \in I$.*

Example 11. *The following functions $\psi : I \to I$ are members of the class $\mathfrak{L}$:*

(a) *$\psi(\varsigma) = \varsigma$ for all $\varsigma \in I$;*
(b) *$\psi(\varsigma) = \varsigma^{cor(k)}$ for all $\varsigma = (a, b) \in I$, where $\varsigma^{cor(k)} = (a^k, b^k)$ and $k \in (0, \infty)$ is fixed;*
(c) *$\psi(\varsigma) = \ell - f(\varsigma)\ell$ for all $\varsigma \in I$, where $f : I \to [0, 1]$ is a function such that for any sequence $\{\varsigma_n\}$ in I we have $\lim_{n\to\infty} f(\varsigma_n) = 0$ if and only if $\lim_{n\to\infty} \varsigma_n = \ell$.*

In the next theorem, an improved version of Theorem 3.5 of Humaira et al. [21] is presented.

Theorem 8. *Let $(\digamma, M, *)$ be an α-complete complex-valued fuzzy metric space, $\alpha \colon \digamma \times \digamma \to [0,\infty)$ a function, and $\top \colon \digamma \to \digamma$ be a mapping satisfying the following condition: there exist $\psi \in \mathfrak{L}, \phi \in \Lambda$ such that*

$$\psi(M(\top\flat_1, \top\flat_2, \varsigma)) \succeq \alpha(\flat_1, \flat_2)[\psi(M(\flat_1, \flat_2, \varsigma)) + \phi(M(\flat_1, \flat_2, \varsigma))] \tag{10}$$

for all $\flat_1, \flat_2 \in \digamma$ and $\varsigma \in P_\theta$. Suppose $P_\top \cap \mathfrak{A}_\alpha \neq \emptyset$ and at least one of the following conditions is satisfied:

(A) *If $\{\flat_n\} \in P_\top \cap \mathfrak{A}_\alpha \cap \mathfrak{C}$ and converges to $u \in \digamma$, then there exists $n_0 \in \mathbb{N}$ such that $s(\flat_n, u) \in \digamma_\alpha$ for all $n > n_0$;*

(B) *$\top$ is α-continuous on $P_\top^\digamma$.*

Then, $\top$ has a fixed point in $\digamma$.

Proof. Suppose $P_\top \cap \mathfrak{A}_\alpha \neq \emptyset$. Then, there exists $\flat_0 \in \digamma$ such that $\{\flat_n\} \in P_\top \cap \mathfrak{A}_\alpha$, $\flat_n = \top^n\flat_0 = \top\flat_{n-1}$ for all $n \in \mathbb{N}$ and $(\flat_n, \flat_m) \in \digamma_\alpha$ for all $n, m \in \mathbb{N}$ with $m > n$.

We shall show that $\{\flat_n\} \in \mathfrak{C}$. Since $(\flat_n, \flat_m) \in \digamma_\alpha$, $n, m \in \mathbb{N}$ with $m > n$, from (10) we have

$$\begin{aligned}
\psi(M(\flat_{n+1}, \flat_{m+1}, \varsigma)) &= \psi(M(\top\flat_n, \top\flat_m, \varsigma)) \\
&\succeq \alpha(\flat_n, \flat_m)[\psi(M(\flat_n, \flat_m, \varsigma)) + \phi(M(\flat_n, \flat_m, \varsigma))] \\
&\succeq \psi(M(\flat_n, \flat_m, \varsigma)) + \phi(M(\flat_n, \flat_m, \varsigma))
\end{aligned}$$

for all $\varsigma \in P_\theta$. Hence,

$$\psi(M(\flat_n, \flat_m, \varsigma)) + \phi(M(\flat_n, \flat_m, \varsigma)) \preceq \psi(M(\flat_{n+1}, \flat_{m+1}, \varsigma)) \text{ for all } \varsigma \in P_\theta. \tag{11}$$

for all $m > n$. Since $\psi, \phi \colon I \to I$, hence we must have $\theta \preceq \psi(M(\flat_n, \flat_m, \varsigma)) \preceq \ell$ and $\theta \preceq \phi(M(\flat_n, \flat_m, \varsigma)) \preceq \ell$ for all $n, m \in \mathbb{N}$ for all $\varsigma \in P_\theta$, and therefore for each $\varsigma \in P_\theta$ we can define

$$\alpha_n = \sup_{m>n} \psi(M(\flat_n, \flat_m, \varsigma)), \beta_n = \sup_{m>n} \phi(M(\flat_n, \flat_m, \varsigma)).$$

Then, by definitions of α_n, β_n and the inequality (11) we obtain:

$$\alpha_n + \beta_n \preceq \alpha_{n+1} \text{ for all } n \in \mathbb{N}. \tag{12}$$

Since $\theta \preceq \beta_n$ for all $n \in \mathbb{N}$, the above inequality yields $\alpha_n \preceq \alpha_{n+1}$ for all $n \in \mathbb{N}$. Also, since $\theta \preceq \alpha_n \preceq \ell$ for all $n \in \mathbb{N}$, hence by Remark 2 there exists $\alpha \in P$ such that $\theta \preceq \alpha \preceq \ell$ and

$$\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \sup_{m>n} \psi(M(\flat_n, \flat_m, \varsigma)) = \alpha. \tag{13}$$

From (12), we have $\beta_n \preceq \alpha_{n+1} - \alpha_n$ for all $n \in \mathbb{N}$, and therefore

$$|\beta_n| \leq |\alpha_{n+1} - \alpha_n| \text{ for all } n \in \mathbb{N}.$$

For every given $\varepsilon > 0$, by (13) there exists $n_1 \in \mathbb{N}$ such that $|\alpha_n - \alpha_{n+1}| < \varepsilon$ for all $n > n_1$, which with the above inequality gives

$$|\sup_{m>n} \phi(M(\flat_n, \flat_m, \varsigma))| = |\beta_n| < \varepsilon \text{ for all } n > n_1.$$

Since $\phi(M(\flat_n, \flat_m, \varsigma)) \preceq \sup_{m>n} \phi(M(\flat_n, \flat_m, \varsigma))$ for all $m > n$, hence

$$|\phi(M(\flat_n, \flat_m, \varsigma))| \leq |\sup_{m>n} \phi(M(\flat_n, \flat_m, \varsigma))| < \varepsilon \text{ for all } m > n > n_1.$$

This shows that
$$\lim_{n,m\to\infty}\phi(M(\flat_n,\flat_m,\varsigma))=\theta.$$
Since $\phi\in\Lambda$, the above equality shows that
$$\lim_{n,m\to\infty}M(\flat_n,\flat_m,\varsigma)=\ell.$$

Hence, $\{\flat_n\}$ is a Cauchy sequence, i.e., $\{\flat_n\}\in\mathfrak{C}$.

Thus, $\{\flat_n\}\in P_\top\cap\mathfrak{A}_\alpha\cap\mathfrak{C}$ and by α-completeness of $\digamma$ it converges to some $u\in\digamma$. We shall show that u is a fixed point of $\top$.

Case I. Suppose (A) holds; then, there exists $n_0\in\mathbb{N}$ such that $s(\flat_n,u)\in\digamma_\alpha$ for all $n>n_0$.

Assume that $(\flat_n,u)\in\digamma_\alpha$ for all $n>n_0$ (the proof for the second case is same). Then, from (10) we have
$$\begin{aligned}\psi(M(\flat_n,u,\varsigma))+\phi(M(\flat_n,u,\varsigma)) &\preceq \alpha(\flat_n,u)[\psi(M(\flat_n,u,\varsigma))+\phi(M(\flat_n,u,\varsigma))]\\ &\preceq \psi(M(\top\flat_n,\top u,\varsigma))\\ &= \psi(M(\flat_{n+1},\top u,\varsigma))\end{aligned}\tag{14}$$
for all $\varsigma\in P_\theta$. Since $\{\flat_n\}$ converges to u, we have $\lim_{n\to\infty}M(\flat_n,u,\varsigma)=\ell$ for all $\varsigma\in P_\theta$ and $\psi\in\mathfrak{L},\phi\in\Lambda$, and we must have
$$\lim_{n\to\infty}\psi(M(\flat_n,u,\varsigma))=\ell,\ \lim_{n\to\infty}\phi(M(\flat_n,u,\varsigma))=\theta \text{ for all } \varsigma\in P_\theta.$$

The above equality with (14) gives
$$\lim_{n\to\infty}\psi(M(\flat_{n+1},\top u,\varsigma))=\ell \text{ for all } \varsigma\in P_\theta.$$

Again, as $\psi\in\mathfrak{L}$, we must have
$$\lim_{n\to\infty}M(\flat_{n+1},\top u,\varsigma)=\ell \text{ for all } \varsigma\in P_\theta.$$

Now, for every $n\in\mathbb{N}$ and $\varsigma\in P_\theta$ we have
$$M(\top u,u,\varsigma)\ \succeq\ M(\top u,\flat_{n+1},\varsigma/2)*M(\flat_{n+1},u,\varsigma/2).$$

Letting $n\to\infty$ and using Remark 2 in the above inequality, we obtain $M(\top u,u,\varsigma)=\ell$ for all $\varsigma\in P_\theta$, i.e., $\top u=u$. Thus, u is a fixed point of $\top$.

Case II. The proof of this case is similar to the Case II of Theorem 5. □

Theorem 9. *Suppose that all the hypotheses of Theorem 8 are satisfied. In addition, if the property (S) is satisfied and $\top$ is α-admissible, then $\top$ has a unique fixed point in $\digamma$.*

Proof. By Theorem 8, $\top$ has a fixed point $u\in\digamma$. We notice that if $\flat\in\mathrm{Fix}(\top)$, then $\top^n\flat=\flat$ for all $n\in\mathbb{N}$, and therefore $\flat\in P_\top^\digamma$, i.e., Fix($\top$)$\subseteq P_\top^\digamma$. For uniqueness of fixed point u of $\top$, on contrary, suppose that $v\in\mathrm{Fix}(\top)$ and $u\neq v$, then $v\in P_\top^\digamma$, and hence by property (S), there exists $z\in\digamma$ such that $s(u,z),s(v,z)\in\digamma_\alpha$. Suppose $(u,z),(v,z)\in\digamma_\alpha$ (the proof for all other cases is same); then, as $\top$ is α-admissible, we have $(\top^n u,\top^n z)\in\digamma_\alpha$ for all $n\in\mathbb{N}$. By (10) we have

$$\begin{aligned}
\psi(M(u,\top^n z,\varsigma)) &= \psi(M(\top^n u,\top^n z,\varsigma))\\
&\succeq \alpha(\top^{n-1}u,\top^{n-1}z)\Big[\psi(M(\top^{n-1}u,\top^{n-1}z,\varsigma))+\phi(M(\top^{n-1}u,\top^{n-1}z,\varsigma))\Big]\\
&\succeq \psi(M(\top^{n-1}u,\top^{n-1}z,\varsigma))+\phi(M(\top^{n-1}u,\top^{n-1}z,\varsigma))\\
&= \psi(M(u,\top^{n-1}z,\varsigma))+\phi(M(u,\top^{n-1}z,\varsigma)).
\end{aligned}$$

For each $\varsigma \in P_\theta$, let $\gamma_n = \psi(M(u,\top^n z,\varsigma)), \delta_n = \phi(M(u,\top^{n-1}z,\varsigma))$ for all $n \in \mathbb{N}$; then, from the above inequality we have

$$\gamma_{n-1}+\delta_{n-1} \preceq \gamma_n \text{ for all } n \in \mathbb{N}. \tag{15}$$

Since $\gamma_n, \delta_n \in I$ for all $n \in \mathbb{N}$, the above inequality shows that $\gamma_{n-1} \preceq \gamma_n$ for all $n \in \mathbb{N}$. By Remark 2, there exists $\gamma \in I$ such that $\lim_{n\to\infty}\gamma_n = \gamma$. Again, by (15) we have $\delta_{n-1} \preceq \gamma_n - \gamma_{n-1}$ for all $n \in \mathbb{N}$, and so $|\delta_n| \leq |\gamma_n - \gamma_{n-1}|$ for all $n \in \mathbb{N}$. As $\lim_{n\to\infty}\gamma_n = \gamma$, for every given $\varepsilon > 0$ there exists $n_1 \in \mathbb{N}$ such that

$$|\delta_n| \leq |\gamma_n - \gamma_{n-1}| < \varepsilon \text{ for all } n > n_1.$$

This shows that

$$\lim_{n\to\infty}\phi(M(u,\top^n z,\varsigma)) = \lim_{n\to\infty}\delta_n = \theta.$$

Since $\phi \in \Lambda$, we must have $\lim_{n\to\infty} M(u,\top^n z,\varsigma) = \ell$. Similarly, we obtain $\lim_{n\to\infty} M(v,\top^n z,\varsigma) = \ell$. Hence, $u = v$. This contradiction proves the uniqueness of the fixed point. □

The proof of the following theorem is similar to the proof of Theorem 7.

Theorem 10. *Let $(\digamma, M, *)$ be an α-complete complex-valued fuzzy metric space, $\alpha\colon \digamma\times\digamma \to [0,\infty)$ a function, and $\top\colon \digamma \to \digamma$ be a mapping satisfying the following condition: there exist $\psi \in \mathfrak{L}, \phi \in \Lambda$ such that*

$$\psi(M(\top\flat_1,\top\flat_2,\varsigma)) \succeq \alpha(\flat_1,\flat_2)[\psi(M(\flat_1,\flat_2,\varsigma))+\phi(M(\flat_1,\flat_2,\varsigma))]$$

for all $\flat_1,\flat_2 \in \digamma$ and $\varsigma \in P_\theta$. Suppose the following conditions are satisfied:

(I) *$\digamma_\alpha$ is transitive;*
(II) *$\top$ is α-admissible;*
(III) *There exists $z \in \digamma$ such that $(z,\top z) \in \digamma_\alpha$;*
(IV) *At least one of the following conditions is satisfied:*
   (A) *If $\{\flat_n\} \in P_\top \cap \mathfrak{A}_\alpha \cap \mathfrak{C}$ and converges to $u \in \digamma$, then there exists $n_0 \in \mathbb{N}$ such that $s(\flat_n,u) \in \digamma_\alpha$ for all $n > n_0$ for all $n > n_0$;*
   (B) *$\top$ is α-continuous on $P_\top^\digamma$.*

Then, $\top$ has a fixed point in $\digamma$. In addition, if the property (S) is satisfied, then the fixed point of $\top$ is unique.

As a generalization and extension of the results of Ran and Reurings [22] and Jachymski [23] in complex-valued fuzzy metric spaces, we now present two consequences of our results.

Let $(\digamma,\sqsubseteq)$ be a poset and $(\digamma, M, *)$ be a complex-valued fuzzy metric space. A mapping $\top\colon \digamma \to \digamma$ is said to be an ordered-(ψ,ϕ)-contraction if there exist $\psi,\phi \in \Theta$ such that

$$\psi(\ell - M(\top\flat_1,\top\flat_2,\varsigma)) \preceq \psi(\ell - M(\flat_1,\flat_2,\varsigma)) - \phi(\ell - M(\flat_1,\flat_2,\varsigma))$$

for all $\varsigma \in P_\theta$ and for all $\flat_1,\flat_2 \in \digamma$ with $\flat_1 \sqsubseteq \flat_2$. The mapping $\top$ is called order preserving if $\flat_1 \sqsubseteq \flat_2$ implies $\top\flat_1 \sqsubseteq \top\flat_2$ for all $\flat_1,\flat_2 \in \digamma$. By $O_\sqsubseteq$, we denote the class of all sequence

$\{\flat_n\}$ such that $\flat_n \sqsubseteq \flat_{n+1}$. The space $(\digamma, M, *)$ is said to be O-complete if every sequence $\{\flat_n\} \in O_{\sqsubseteq} \cap \mathfrak{C}$ converges to some $\flat \in \digamma$. The mapping $\top$ is said to be O-continuous if for every convergent sequence $\{\flat_n\} \in O_{\sqsubseteq}$ the sequence $\{\top\flat_n\}$ converges to $\top u$, where $u \in \digamma$ is the limit of $\{\flat_n\}$. A pair $(\flat_1, \flat_2) \in \digamma \times \digamma$ is called $\sqsubseteq$-comparable if $\flat_1 \sqsubseteq \flat_2$ or $\flat_2 \sqsubseteq \flat_1$.

Theorem 11. *Let $(\digamma, \sqsubseteq)$ be a poset, $(\digamma, M, *)$ an O-complete complex-valued fuzzy metric space, and $\top\colon \digamma \to \digamma$ be an ordered-(ψ, ϕ)-contraction. Suppose $P_\top \cap O_{\sqsubseteq} \neq \emptyset$ and at least one of the following conditions is satisfied:*

(A) *If $\{\flat_n\} \in P_\top \cap O_{\sqsubseteq} \cap \mathfrak{C}$ and converges to $u \in \digamma$, then there exists $n_0 \in \mathbb{N}$ such that $\flat_n \sqsubseteq u$ for all $n > n_0$ or $u \sqsubseteq \flat_n$ for all $n > n_0$;*
(B) *$\top$ is O-continuous on $P_\top^\digamma$.*

Then, $\top$ has a fixed point in $\digamma$. In addition, if for every $u, v \in P_\digamma^\top$ there exists $z \in \digamma$ such that the pairs (z, u) and (z, v) both are $\sqsubseteq$-comparable, and $\top$ is order preserving, then $\top$ has a unique fixed point in $\digamma$.

Proof. Define a function $\alpha\colon \digamma \times \digamma \to [0, \infty)$ by

$$\alpha(\flat_1, \flat_2) = \begin{cases} 1, & \text{if } \flat_1 \sqsubseteq \flat_2; \\ 0, & \text{otherwise.} \end{cases}$$

Now, the proof follows from Theorem 6. □

Let $(\digamma, M, *)$ be a complex-valued fuzzy metric space and Γ be a graph with the set of vertices $V(\Gamma) = \digamma$ and the set of edges $E(\Gamma) \subseteq \digamma \times \digamma$. In this case, we say that $(\digamma, M, *)$ is endowed with graph Γ. A mapping $\top\colon \digamma \to \digamma$ is said to be a Γ-(ψ, ϕ)-contraction if there exist $\psi, \phi \in \Theta$ such that

$$\psi(\ell - M(\top\flat_1, \top\flat_2, \varsigma)) \preceq \psi(\ell - M(\flat_1, \flat_2, \varsigma)) - \phi(\ell - M(\flat_1, \flat_2, \varsigma))$$

for all $\varsigma \in P_\theta$ and for all $\flat_1, \flat_2 \in \digamma$ with $(\flat_1, \flat_2) \in E(\Gamma)$. The mapping $\top$ is called edge preserving if $(\flat_1, \flat_2) \in E(\Gamma)$ implies $(\top\flat_1, \top\flat_2) \in E(\Gamma)$ for all $\flat_1, \flat_2 \in \digamma$. By $\mathfrak{G}$, we denote the class of all sequence $\{\flat_n\}$ such that $(\flat_n, \flat_m) \in E(\Gamma)$ for all $m > n$, and then $(\digamma, M, *)$ is said to be G-complete if every sequence $\{\flat_n\} \in \mathfrak{G} \cap \mathfrak{C}$ converges to some $\flat \in \digamma$. The mapping $\top$ is said to be Γ-continuous if for every convergent sequence $\{\flat_n\} \in \mathfrak{G}$ the sequence $\{\top\flat_n\}$ converges to $\top u$, where $u \in \digamma$ is the limit of $\{\flat_n\}$. A pair $(\flat_1, \flat_2) \in \digamma \times \digamma$ is called edge connected in Γ if $(\flat_1, \flat_2) \in E(\Gamma)$ or $(\flat_2, \flat_1) \in E(\Gamma)$.

Theorem 12. *Let $(\digamma, M, *)$ be a complete complex-valued fuzzy metric space endowed with a graph Γ. Suppose $(\digamma, M, *)$ is G-complete and $\top\colon \digamma \to \digamma$ be a Γ-(ψ, ϕ)-contraction. Suppose $P_\top \cap \mathfrak{G} \neq \emptyset$ and at least one of the following conditions is satisfied:*

(A) *If $\{\flat_n\} \in P_\top \cap \mathfrak{G} \cap \mathfrak{C}$ and converges to $u \in \digamma$, then there exists $n_0 \in \mathbb{N}$ such that $(\flat_n, u) \in E(\Gamma)$ for all $n > n_0$ or $(u, \flat_n) \in E(\Gamma)$ for all $n > n_0$;*
(B) *$\top$ is Γ-continuous on $P_\top^\digamma$.*

Then, $\top$ has a fixed point in $\digamma$. In addition, if for every $u, v \in P_\top^\digamma$ there exists $z \in \digamma$ such that the pairs (u, z) and (v, z) are edge connected in Γ, and $\top$ is edge preserving, then $\top$ has a unique fixed point in $\digamma$.

Proof. Define a function $\alpha\colon \digamma \times \digamma \to [0, \infty)$ by

$$\alpha(\flat_1, \flat_2) = \begin{cases} 1, & \text{if } (\flat_1, \flat_2) \in E(\Gamma); \\ 0, & \text{otherwise.} \end{cases}$$

Now, the proof follows from Theorem 6. □

4. Conclusions

The fixed point theory has several applications in various branches of science. In particular, the metric fixed point theory influences the study of initial value problems, boundary value problems, integral equations, economic systems, biological systems, dynamical systems, and many more. The fuzzy metric structures with complex values draw the interest of the researchers working in the areas where the measurement has uncertainties and complexity. Consequently, the spaces in which the metric function has fuzzy and complex values are introduced, and fixed point results in such spaces are established. Because of the nature of partial order in complex numbers, one has to be careful when using the results in such spaces. The presented work consists of some observations on some recent concepts and results established in the complex-valued fuzzy metric spaces, as well as some new results related to the existence and uniqueness of fixed points of mappings. Some topological properties of such spaces are established. The fixed point results are proved with weaker constraints, e.g., α-continuity of mappings instead of continuity, T-α-completeness instead of completeness of underlying space are used, and the constraints of nondecreasingness and continuity of control functions in contractive conditions have been removed and so some recent results have been improved. We have introduced three new classes of control functions, namely Θ, Λ and $\mathfrak{L}$, so that our fixed point results can produce several fixed point results as consequences.

The set-valued mappings have several applications, e.g., in finding the solutions of differential inclusion, integral inclusion, difference inclusion, etc. (see, e.g., [24–26]). The presented work is related to single-valued mappings, so it will be interesting to find an analog of our results for set-valued mappings and their applications in inclusion problems. On the other hand, to obtain the coincidence and common fixed point results for two (or more than two) commuting and non-commuting mappings is a famous and interesting way to generalize the fixed point results (see, e.g., [27–30] and the references therein) as well as find some applications in solving nonlinear integral equations (see, ref. [31]) and in dynamical systems (see, ref. [32]). The results of this paper are still open for the investigation of coincidence and common fixed points.

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References

1. Zadeh, L. Fuzzy sets. *Inf. Control* **1965**, *89*, 338–353. [CrossRef]
2. Kramosil, I.; Michálek, J. Fuzzy metrics and statistical metric Spaces. *Kybernetika* **1975**, *11*, 336–344.
3. Grabiec, M. Fixed points in fuzzy metric spaces. *Fuzzy Sets Syst.* **1988**, 27, 385–389. [CrossRef]
4. George, A.; Veeramani, P. On some results in fuzzy metric spaces. *Fuzzy Sets Syst.* **1994**, *64*, 395–399. [CrossRef]
5. Schweizer, B.; Sklar, A. Statistical metric spaces. *Pac. J. Math.* **1960**, *10*, 313–334. [CrossRef]
6. Naschie, M.E. On the unification of the fundamental forces and complex time in the space. *Chaos Solitons Fractals* **2000**, *11*, 1149–1162. [CrossRef]
7. Azam, A.; Fisher, B.; Khan, M. Common fixed point theorems in complex valued metric spaces. *Appl. Anal. Discret. Math.* **2011**, *32*, 243–253. [CrossRef]
8. Verma, R.K.; Pathak, H.K. Common fixed point theorems using property (EA) in complex-valued metric spaces. *Thai J. Math.* **2012**, *11*, 347–355.

9. Nashine, H.K.; Imdad, M.; Hasan, M. Common fixed point theorems under rational contractions in complex valued metric spaces. *J. Nonlinear Sci.* **2014**, *7*, 42–50. [CrossRef]
10. Ege, O. Complex valued rectangular b-metric spaces and an application to linear equations. *J. Nonlinear Sci. Appl.* **2015**, *8*, 1014–1021. [CrossRef]
11. Beg, I.; Datta, S.K.; Pal, D. Fixed point in bicomplex valued metric spaces. *Int. J. Nonlinear Anal. Appl.* **2021**, *12*, 717–727.
12. Demir, I. Fixed Point Theorems in complex valued fuzzy b-metric spaces with application to integral equations. *Miskolc Math. Notes* **2021**, *22*, 153–171. [CrossRef]
13. Shukla, S.; Lopez, R.; Abbas, M. Fixed point results for contractive mappings in complex valued fuzzy metric spaces. *Fixed Point Theory* **2018**, *19*, 751–754. [CrossRef]
14. Samet, B.; Vetro, C.; Vetro, P. Fixed point theorem for α-ψ contractive type mappings. *Nonlinear Anal.* **2012**, *75*, 2154–2165. [CrossRef]
15. Karapinar, E.; Samet, B. Generalized α-ψ Contractive Type Mappings and Related Fixed Point Theorems with Applications. *Abstr. Appl. Anal.* **2012**, *2012*, 793486. [CrossRef]
16. Aydi, H.; Karapinar, E. Fixed point results for generalized α-ψ-contractions in metric-like spaces and applications. *Electron. J. Differ. Equ.* **2015**, *2015*, 1–15.
17. Shukla, S.; Shahzad, N. Fixed points of α-admissible Prešić type operators. *Nonlinear Anal. Model. Control* **2016**, *21*, 424–436. [CrossRef]
18. Aydi, H.; Karapinar, E.; Yazidi, H. Modified F-Contractions via α-Admissible Mappings and Application to Integral Equations. *Filomat* **2017**, *31*, 1141–1148. [CrossRef]
19. Matkowski, J. Fixed point theorems for mappings with a contractive iterate at a point. *Proc. Am. Math. Soc.* **1997**, *62*, 344–348. [CrossRef]
20. Shukla, S.; Gopal, D.; Roldán-López-de-Hierro, A.F. Some fixed point theorems in 1-M-complete fuzzy metric-like spaces. *Int. J. Gen. Syst.* **2016**, *45*, 815–829. [CrossRef]
21. Humaira Sarwar, M.; Abdeljawad, T. Existence of unique solution to nonlinear mixed Volterra Fredholm-Hammerstein integral equations in complex-valued fuzzy metric spaces. *J. Intell. Fuzzy Syst.* **2021**, *40*, 4065–4074. [CrossRef]
22. Ran, A.; Reurings, M. A fixed point theorem in partially ordered sets and some applications to matrix equations. *Proc. Am. Math. Soc.* **2004**, *132*, 1435–1443. [CrossRef]
23. Jachymski, J. The contraction principle for mappings on a metric space with a graph. *Proc. Am. Math. Soc.* **2008**, *136*, 1359–1373. [CrossRef]
24. Shukla, S.; Rai, S.; Shukla, R. A relation-theoretic set-valued version of Prešíc-Ćirić theorem and applications. *Bound. Value Probl.* **2023**, *2023*, 59. [CrossRef]
25. Shukla, S.; Rodríguez-López, R. Fixed points of multi-valued relation-theoretic contractions in metric spaces and application. *Quaest. Math.* **2020**, *43*, 409–424. [CrossRef]
26. Turkoglu, D.; Altun, I. A fixed point theorem for multi-valued mappings and its applications to integral inclusions. *Appl. Math. Lett.* **2007**, 20, 563–570. [CrossRef]
27. Jungck, G. Periodic and Fixed Points and Commuting Mappings. *Proc. Am. Math. Soc.* **1989**, *76*, 333–338. [CrossRef]
28. Jungck, G. Common fixed point for noncontinuous nonself maps on non-metric spaces. *Far East J. Math. Sci.* **1996**, *4*, 199–215.
29. Abbas, M.; Jungck, G. Common fixed point results for noncommuting mappings without continuity in cone metric spaces. *J. Math. Anal. Appl.* **2008**, *341*, 416–420. [CrossRef]
30. Jungck, G.; Radenović, S.; Radojević, S.; Rakočević V. Common fixed point theorems for weakly compatible pairs on cone metric spaces. *Fixed Point Theory Appl.* **2009**, *2009*, 643840. [CrossRef]
31. Pathak, H.K.; Khan, M.S.; Tiwari, R. A common fixed point theorem and its application to nonlinear integral equations. *Comput. Math. Appl.* **2007**, *53*, 961–971. [CrossRef]
32. Shukla, S.; Dubey, N.; Shukla, R.; Mezník, I. Coincidence point of Edelstein type mappings in fuzzy metric spaces and application to the stability of dynamic markets. *Axioms* **2023**, *12*, 854. [CrossRef]