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Generalized Split Quaternions and Their Applications on Non-Parabolic Conical Rotations

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Abstract: In this study, we first generalize the Lorentzian inner and vector products, and then we define the generalized split quaternions by means of the generalized Lorentzian inner and vector products. Next, on any hyperboloid of one or two sheets, which is a generalized Lorentzian sphere, non-parabolic conical rotations with nonnull axes are expressed using the generalized split quaternions with supporting numerical examples.

Keywords: Lorentzian inner product; Lorentzian vector product; Lorentzian rotation matrix; Lorentzian symmetry; split quaternion

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1. Introduction

Quaternions were defined by W. R. Hamilton in 1843 to generalize complex numbers. Since rotations and reflections in three-dimensional Euclidean space can be expressed with quaternions, there are many application areas for real quaternions in various fields of science. In addition to real quaternions, numerous different kinds of quaternions and their applications have been investigated over the years. For instance, rotations in three-dimensional generalized Euclidean spaces have been expressed using elliptical quaternions [1–3]. Some studies on the topic are given in the reference section [4–7]. However, generalizations of quaternions and their applications have also attracted researchers' attention [8–11]. As is known, using generalized scalar products is one way to generalize quaternions [3,12]. See [13–18] for information on generalized scalar products and related concepts.

In this study, we deal specifically with split quaternions as defined by Inoguchi [19]. Split quaternions comprise a number system that is closely related to three-dimensional Lorentzian geometry, which is a geometric framework that is used to study the structure of spacetime in special relativity. Split quaternions can be used to represent conical rotations on standard hyperboloids, which are spheres in three-dimensional Lorentzian geometry [12,20,21]. Some recent studies on split quaternions are also given in the reference section [22–24].

The aim of our study is to provide a generalization for three-dimensional Lorentzian geometry and split quaternions by generalizing the Lorentzian inner and vector products, and to determine non-parabolic conical rotations in three-dimensional space using generalized split quaternions. Due to these generalized split quaternions, any non-parabolic conical rotation in three-dimensional space can be easily expressed without long calculations with affine transformations.

The paper is organized as follows. First, we describe the basics of three-dimensional Lorentzian geometry and define a three-dimensional generalized Lorentzian inner product whose sphere is any given hyperboloid of one or two sheets, having the equation

$$Ax^2 + By^2 + Cz^2 + 2Dxy + 2Exz + 2Fyz = \pm r^2 \quad (1)$$

and a three-dimensional generalized Lorentzian vector product. Then, we describe the basics of split quaternions and generalize them using generalized Lorentzian products. We also show how generalized split quaternions can be used to represent non-parabolic conical rotations, which are elliptic and hyperbolic rotational motions on any hyperboloids. Finally, we provide some numerical examples.

2. Generalized Lorentzian Inner and Vector Products

Three-dimensional Lorentzian geometry, also known as the Minkowski 3-space $\mathbb{R}^3_1$, is the Euclidean space $\mathbb{R}^3$ endowed with the Lorentzian inner product

$$\langle \mathbf{u}, \mathbf{w} \rangle_L = -u_1 w_1 + u_2 w_2 + u_3 w_3, \tag{2}$$

where $\mathbf{u} = (u_1, u_2, u_3), \mathbf{w} = (w_1, w_2, w_3) \in \mathbb{R}^3$. The associated matrix of the Lorentzian inner product is $diag(-1, 1, 1)$. Since it is indefinite, the vectors of $\mathbb{R}^3_1$ are classified as follows. A vector $\mathbf{u} \in \mathbb{R}^3_1$ is called spacelike, timelike, or lightlike if $\langle \mathbf{u}, \mathbf{u} \rangle_L > 0$, $\langle \mathbf{u}, \mathbf{u} \rangle_L < 0$, or $\langle \mathbf{u}, \mathbf{u} \rangle_L = 0$, respectively. The norm of the vector $\mathbf{u} \in \mathbb{R}^3_1$ is defined by $\|\mathbf{u}\|_L = \sqrt{|\langle \mathbf{u}, \mathbf{u} \rangle_L|}$, and the Lorentzian vector product of $\mathbf{u}$ and $\mathbf{w}$ is defined by

$$\mathbf{u} \times_L \mathbf{w} = \begin{vmatrix} -\mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ w_1 & w_2 & w_3 \end{vmatrix}, \tag{3}$$

where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are standard unit vectors. For further information on Lorentzian geometry, see [25–27].

Similar to the generalization of the three-dimensional Euclidean inner product [1,28], one can generalize the Lorentzian inner product by using a suitable general matrix instead of $diag(-1, 1, 1)$, as in the following example.

For vectors $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{w} = (w_1, w_2, w_3) \in \mathbb{R}^3$, and the real symmetric matrix

$$\Omega = \begin{bmatrix} A & D & E \\ D & B & F \\ E & F & C \end{bmatrix} \tag{4}$$

having a negative determinant, whose eigenvalues are not of the same sign, the map $\mathcal{B}_\Omega(\mathbf{u}, \mathbf{w}) = \mathbf{u}^T \Omega \mathbf{w} = Au_1w_1 + Bu_2w_2 + Cu_3w_3 + D(u_1w_2 + u_2w_1) + E(u_1w_3 + u_3w_1) + F(u_2w_3 + u_3w_2)$ is called the generalized Lorentzian (or $\mathcal{B}_\Omega$-) inner product, and the real vector space $\mathbb{R}^3$ with the generalized Lorentzian inner product is called the three-dimensional generalized Lorentzian space, which is denoted by $\mathbb{R}^3_{\mathcal{B}_\Omega}$.

Here, Ω is the matrix associated with the symmetric bilinear form, and

$$\Delta = \sqrt{|\det \Omega|} = \sqrt{|ABC + 2FDE - AF^2 - CD^2 - BE^2|}. \tag{5}$$

is the constant of it. As usual, the $\mathcal{B}_\Omega$-norm of the vector $\mathbf{u}$ is defined as

$$\|\mathbf{u}\|_{\mathcal{B}_\Omega} = \sqrt{|\mathcal{B}_\Omega(\mathbf{u}, \mathbf{u})|} = \sqrt{|Au_1^2 + Bu_2^2 + Cu_3^2 + 2Du_1u_2 + 2Eu_1u_3 + 2Fu_2u_3|},$$

and the vectors $\mathbf{u}$ and $\mathbf{w}$ are called $\mathcal{B}_\Omega$-orthogonal if $\mathcal{B}_\Omega(\mathbf{u}, \mathbf{w}) = 0$. As with the Lorentzian space, since the generalized Lorentzian inner product is indefinite, the vectors of $\mathbb{R}^3_{\mathcal{B}_\Omega}$ can be classified as follows:

(i) If $\mathcal{B}_\Omega(\mathbf{u}, \mathbf{u}) > 0$ or $\mathbf{u} = \mathbf{0}$, then $\mathbf{u}$ is called a $\mathcal{B}_\Omega$-spacelike vector.
(ii) If $\mathcal{B}_\Omega(\mathbf{u}, \mathbf{u}) < 0$, then $\mathbf{u}$ is called a $\mathcal{B}_\Omega$-timelike vector.
(iii) If $\mathcal{B}_\Omega(\mathbf{u}, \mathbf{u}) = 0$ and $\mathbf{u} \neq \mathbf{0}$, then $\mathbf{u}$ is called a $\mathcal{B}_\Omega$-lightlike or $\mathcal{B}_\Omega$-null vector.

Normally,

$$\cos^{-1}\left(\frac{\mathcal{B}_{\Omega}(\mathbf{u},\mathbf{w})}{\|\mathbf{u}\|_{\mathcal{B}_{\Omega}}\|\mathbf{w}\|_{\mathcal{B}_{\Omega}}}\right) \text{ and } \cosh^{-1}\left(\frac{-\mathcal{B}_{\Omega}(\mathbf{u},\mathbf{w})}{\|\mathbf{u}\|_{\mathcal{B}_{\Omega}}\|\mathbf{w}\|_{\mathcal{B}_{\Omega}}}\right) \tag{6}$$

determine the $\mathcal{B}_{\Omega}$-measure of the angle between $\mathbf{u}$ and $\mathbf{w}$, if they are $\mathcal{B}_{\Omega}$-spacelike and $\mathcal{B}_{\Omega}$-timelike vectors, respectively. For a positive real number r, the set

$$S^2_{\mathcal{B}_{\Omega}}(r) = \left\{\mathbf{u} \in \mathbb{R}^3_{\mathcal{B}_{\Omega}}:\ \mathcal{B}_{\Omega}(\mathbf{u},\mathbf{u}) = r^2\right\} \tag{7}$$

is called a $\mathcal{B}_{\Omega}$-pseudosphere with the radius r, and the set

$$H^2_{\mathcal{B}_{\Omega}}(r) = \left\{\mathbf{u} \in \mathbb{R}^3_{\mathcal{B}_{\Omega}}:\ \mathcal{B}_{\Omega}(\mathbf{u},\mathbf{u}) = -r^2\right\} \tag{8}$$

is called a $\mathcal{B}_{\Omega}$-hyperbolic sphere with the radius r. The $\mathcal{B}_{\Omega}$-pseudosphere and the $\mathcal{B}_{\Omega}$-hyperbolic sphere are both called the generalized Lorentzian (or $\mathcal{B}_{\Omega}$-) sphere. In addition, the set

$$L_{\mathcal{B}_{\Omega}} = \left\{\mathbf{u} \in \mathbb{R}^3_{\mathcal{B}_{\Omega}}:\ \mathcal{B}_{\Omega}(\mathbf{u},\mathbf{u}) = 0\right\} \tag{9}$$

is called a $\mathcal{B}_{\Omega}$-lightcone. Using the classification conditions for quadrics (see [29]), one can easily see that the $\mathcal{B}_{\Omega}$-pseudosphere is a general hyperboloid of one sheet, the $\mathcal{B}_{\Omega}$-hyperbolic sphere is a general hyperboloid of two sheets, and the $\mathcal{B}_{\Omega}$-lightcone is a general cone.

It is known that in three-dimensional space, the vector product can be defined by a skew symmetric matrix. Now, we define and then determine the $\mathcal{B}_{\Omega}$-skew symmetric matrices in $\mathbb{R}^3_{\mathcal{B}_{\Omega}}$.

Definition 1. *Let T be a 3×3 real matrix. If*

$$\mathcal{B}_{\Omega}(S\mathbf{u},\mathbf{w}) = -\mathcal{B}_{\Omega}(\mathbf{u},S\mathbf{w}) \tag{10}$$

for every $\mathbf{u},\mathbf{w} \in \mathbb{R}^3_{\mathcal{B}_{\Omega}}$, then T is called $\mathcal{B}_{\Omega}$-skew symmetric.

One can derive that the matrix S is $\mathcal{B}_{\Omega}$-skew symmetric if and only if $S^t\Omega = -\Omega S$. Thus, we obtain the following theorem:

Theorem 1. *$\mathcal{B}_{\Omega}$-skew symmetric matrices are as follows*

$$\lambda \begin{bmatrix} \frac{(EF-CD)u_3-(DF-BE)u_2}{\det\Omega} & \frac{(DF-BE)u_1-(BC-F^2)u_3}{\det\Omega} & \frac{(BC-F^2)u_2-(EF-CD)u_1}{\det\Omega} \\ \frac{(AC-E^2)u_3-(DE-AF)u_2}{\det\Omega} & \frac{(DE-AF)u_1-(EF-CD)u_3}{\det\Omega} & \frac{(EF-CD)u_2-(AC-E^2)u_1}{\det\Omega} \\ \frac{(DE-AF)u_3-(AB-D^2)u_2}{\det\Omega} & \frac{(AB-D^2)u_1-(DF-BE)u_3}{\det\Omega} & \frac{(DF-BE)u_2-(DE-AF)u_1}{\det\Omega} \end{bmatrix} \tag{11}$$

where $u_1, u_2, u_3, \lambda \in \mathbb{R}$.

Proof. We need to find the matrix

$$S = \begin{bmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{bmatrix} \tag{12}$$

having the equality $S^t\Omega = -\Omega S$, so we have the following system of equations:

$$\begin{cases} As_{11} + Ds_{21} + Es_{31} = 0 \\ Bs_{22} + Fs_{32} + Ds_{12} = 0 \\ Cs_{33} + Fs_{23} + Es_{13} = 0 \\ As_{12} + Bs_{21} + Fs_{31} + Ds_{11} + Ds_{22} + Es_{32} = 0 \\ As_{13} + Cs_{31} + Fs_{21} + Es_{11} + Ds_{23} + Es_{33} = 0 \\ Bs_{23} + Cs_{32} + Fs_{22} + Fs_{33} + Ds_{13} + Es_{12} = 0 \end{cases}. \tag{13}$$

Using the substitutions

$$\begin{aligned} u_1 &= Et_{12} + Ft_{22} + Ct_{32} = -Dt_{13} - Bt_{23} - Ft_{33} \\ u_2 &= At_{13} + Dt_{23} + Et_{33} = -Et_{11} - Ft_{21} - Ct_{31} \\ u_3 &= Dt_{11} + Bt_{21} + Ft_{31} = -At_{12} - Dt_{22} - Et_{32}, \end{aligned} \tag{14}$$

one derives three systems of equations

$$\begin{cases} As_{11} + Ds_{21} + Es_{31} = 0 \\ Ds_{11} + Bs_{21} + Fs_{31} = u_3 \\ Es_{11} + Fs_{21} + Cs_{31} = -u_2 \end{cases} \quad \begin{cases} As_{12} + Ds_{22} + Es_{32} = -u_3 \\ Ds_{12} + Bs_{22} + Fs_{32} = 0 \\ Es_{12} + Fs_{22} + Cs_{32} = u_1 \end{cases} \quad \begin{cases} As_{13} + Ds_{23} + Es_{33} = u_2 \\ Ds_{13} + Bs_{23} + Fs_{33} = -u_1 \\ Es_{13} + Fs_{23} + Cs_{33} = 0 \end{cases}. \tag{15}$$

Solving them with Cramer's rule, one obtains matrix (11). □

Considering the following $\mathcal{B}_\Omega$-skew symmetric matrix

$$S = \begin{bmatrix} \Delta_5 u_2 - \Delta_6 u_3 & \Delta_3 u_3 - \Delta_5 u_1 & \Delta_6 u_1 - \Delta_3 u_2 \\ \Delta_4 u_2 - \Delta_2 u_3 & \Delta_6 u_3 - \Delta_4 u_1 & \Delta_2 u_1 - \Delta_6 u_2 \\ \Delta_1 u_2 - \Delta_4 u_3 & \Delta_5 u_3 - \Delta_1 u_1 & \Delta_4 u_1 - \Delta_5 u_2 \end{bmatrix}, \tag{16}$$

where $\Delta = \sqrt{|\det\Omega|}$, $\Delta_1 = (AB - D^2)/\Delta$, $\Delta_2 = (AC - E^2)/\Delta$, $\Delta_3 = (BC - F^2)/\Delta$, $\Delta_4 = (DE - AF)/\Delta$, $\Delta_5 = (DF - BE)/\Delta$, and $\Delta_6 = (EF - CD)/\Delta$, one has the following definition:

Definition 2. *Let* $\mathbf{u} = (u_1, u_2, u_3)$ *and* $\mathbf{w} = (w_1, w_2, w_3)$ *be any two vectors in* $\mathbb{R}^3_{\mathcal{B}_\Omega}$. *The generalized Lorentzian vector product is the function*

$$\times_{GL} : \mathbb{R}^3_{\mathcal{B}_\Omega} \times \mathbb{R}^3_{\mathcal{B}_\Omega} \to \mathbb{R}^3_{\mathcal{B}_\Omega}, \ (\mathbf{u}, \mathbf{w}) \to \mathbf{u} \times_{GL} \mathbf{w}, \tag{17}$$

defined by

$$\begin{aligned}
\mathbf{u} \times_{GL} \mathbf{w} &= \begin{bmatrix} \Delta_5 u_2 - \Delta_6 u_3 & \Delta_3 u_3 - \Delta_5 u_1 & \Delta_6 u_1 - \Delta_3 u_2 \\ \Delta_4 u_2 - \Delta_2 u_3 & \Delta_6 u_3 - \Delta_4 u_1 & \Delta_2 u_1 - \Delta_6 u_2 \\ \Delta_1 u_2 - \Delta_4 u_3 & \Delta_5 u_3 - \Delta_1 u_1 & \Delta_4 u_1 - \Delta_5 u_2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \\
&= \begin{bmatrix} -\Delta_3(u_2 w_3 - u_3 w_2) - \Delta_6(u_3 w_1 - u_1 w_3) - \Delta_5(u_1 w_2 - u_2 w_1) \\ -\Delta_6(u_2 w_3 - u_3 w_2) - \Delta_2(u_3 w_1 - u_1 w_3) - \Delta_4(u_1 w_2 - u_2 w_1) \\ -\Delta_5(u_2 w_3 - u_3 w_2) - \Delta_4(u_3 w_1 - u_1 w_3) - \Delta_1(u_1 w_2 - u_2 w_1) \end{bmatrix} \\
&= \begin{vmatrix} -\Delta_3 \mathbf{i} - \Delta_6 \mathbf{j} - \Delta_5 \mathbf{k} & -\Delta_6 \mathbf{i} - \Delta_2 \mathbf{j} - \Delta_4 \mathbf{k} & -\Delta_5 \mathbf{i} - \Delta_4 \mathbf{j} - \Delta_1 \mathbf{k} \\ u_1 & u_2 & u_3 \\ w_1 & w_2 & w_3 \end{vmatrix},
\end{aligned}$$

where $\Delta_1 = (AB - D^2)/\Delta$, $\Delta_2 = (AC - E^2)/\Delta$, $\Delta_3 = (BC - F^2)/\Delta$, $\Delta_4 = (DE - AF)/\Delta$, $\Delta_5 = (DF - BE)/\Delta$, $\Delta_6 = (EF - CD)/\Delta$, *and* $\mathbf{i}, \mathbf{j}, \mathbf{k}$ *are the usual standard unit vectors.*

Using the substitutions $\widehat{\mathbf{i}} = \Delta_3 \mathbf{i} + \Delta_6 \mathbf{j} + \Delta_5 \mathbf{k}$, $\widehat{\mathbf{j}} = \Delta_6 \mathbf{i} + \Delta_2 \mathbf{j} + \Delta_4 \mathbf{k}$, and $\widehat{\mathbf{k}} = \Delta_5 \mathbf{i} + \Delta_4 \mathbf{j} + \Delta_1 \mathbf{k}$, one has the generalized Lorentzian vector product as follows

$$\mathbf{u} \times_{GL} \mathbf{w} = \begin{vmatrix} -\widehat{\mathbf{i}} & -\widehat{\mathbf{j}} & -\widehat{\mathbf{k}} \\ u_1 & u_2 & u_3 \\ w_1 & w_2 & w_3 \end{vmatrix}. \tag{18}$$

3. Generalized Split Quaternions

For the real numbers q_0, q_1, q_2, q_3, and the basic elements $\mathbf{i}, \mathbf{j}, \mathbf{k}$ satisfying the equalities $\mathbf{i}^2 = -1$, $\mathbf{j}^2 = \mathbf{k}^2 = 1$, $\mathbf{ij} = \mathbf{k}, \mathbf{jk} = -\mathbf{i}, \mathbf{ki} = \mathbf{j}, \mathbf{ji} = -\mathbf{k}, \mathbf{kj} = \mathbf{i}, \mathbf{ik} = -\mathbf{j}$, a split quaternion q is represented by $q = (q_0, q_1, q_2, q_3) = q_0 + q_1 \mathbf{i} + q_2 \mathbf{j} + q_3 \mathbf{k}$, or $q = s_q + \mathbf{v}_q$, where $s_q = q_0$ and $\mathbf{v}_q = q_1 \mathbf{i} + q_2 \mathbf{j} + q_3 \mathbf{k}$ denote the scalar and vector parts of q, respectively. The conjugate and norm of q are defined as $\overline{q} = s_q - \mathbf{v}_q$ and $N_q = \sqrt{|q\overline{q}|} = \sqrt{|q_0^2 + q_1^2 - q_2^2 - q_3^2|}$. A split quaternion is called spacelike, timelike, or lightlike (null), if $I_q < 0$, $I_q > 0$, or $I_q = 0$, respectively, where $I_q = q_0^2 + q_1^2 - q_2^2 - q_3^2$. In addition, if $N_q = 1$, then q is called the unit split quaternion. The split quaternion multiplication of $p = (p_1, p_2, p_3, p_4)$ and $q = (q_1, q_2, q_3, q_4)$ is defined by

$$pq = p_0 q_0 + \langle \mathbf{u}, \mathbf{w} \rangle_L + p_0 \mathbf{v}_q + q_0 \mathbf{v}_p + \mathbf{v}_p \times_L \mathbf{v}_q, \tag{19}$$

and the algebra of the split quaternion set $\mathbb{H}$ is an associative, non-commutative, and non-division ring [19,21].

In this section, we generalize the split quaternions by using the generalized Lorentzian inner and vector products. The generalized split quaternions will be the number system related to the three-dimensional generalized Lorentzian geometry.

Consider the real value entries A, B, C, D, E, F of the matrix Ω and the basic elements $\mathbf{i}, \mathbf{j}, \mathbf{k}$ satisfying the equalities

$$\begin{aligned}
\mathbf{i}^2 &= A,\ \ \mathbf{j}^2 = B,\ \ \mathbf{k}^2 = C \\
\mathbf{ij} &= D - \mathbf{i}\Delta_5 - \mathbf{j}\Delta_4 - \mathbf{k}\Delta_1 \\
\mathbf{ji} &= D + \mathbf{i}\Delta_5 + \mathbf{j}\Delta_4 + \mathbf{k}\Delta_1 \\
\mathbf{jk} &= F - \mathbf{i}\Delta_3 - \mathbf{j}\Delta_6 - \mathbf{k}\Delta_5 \\
\mathbf{kj} &= F + \mathbf{i}\Delta_3 + \mathbf{j}\Delta_6 + \mathbf{k}\Delta_5 \\
\mathbf{ki} &= E - \mathbf{i}\Delta_6 - \mathbf{j}\Delta_2 - \mathbf{k}\Delta_4 \\
\mathbf{ik} &= E + \mathbf{i}\Delta_6 + \mathbf{j}\Delta_2 + \mathbf{k}\Delta_4,
\end{aligned} \tag{20}$$

where $\Delta = \sqrt{|\det \Omega|}$, $\Delta_1 = (AB - D^2)/\Delta$, $\Delta_2 = (AC - E^2)/\Delta$, $\Delta_3 = (BC - F^2)/\Delta$, $\Delta_4 = (DE - AF)/\Delta$, $\Delta_5 = (DF - BE)/\Delta$, and $\Delta_6 = (EF - CD)/\Delta$. Then, for $q_0, q_1, q_2, q_3 \in \mathbb{R}$, $q = (q_0, q_1, q_2, q_3) = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} = s_p + \mathbf{v}_p$ is called a generalized split (or g-split) quaternion, where $s_p = q_0$ and $\mathbf{v}_p = q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$. Here, s_q is called the scalar part of q, and $\mathbf{v}_q$ is called the vector part of q, considering $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the usual standard unit vectors. The set of all g-split quaternions is denoted by $\mathbb{H}_\Omega$. For every Ω, this set is an associative, non-commutative, and non-division ring. The g-split quaternion multiplication table is as follows:

$\cdot$	1	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
1	1	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
$\mathbf{i}$	$\mathbf{i}$	A	$D - \mathbf{i}\Delta_5 - \mathbf{j}\Delta_4 - \mathbf{k}\Delta_1$	$E + \mathbf{i}\Delta_6 + \mathbf{j}\Delta_2 + \mathbf{k}\Delta_4$
$\mathbf{j}$	$\mathbf{j}$	$D + \mathbf{i}\Delta_5 + \mathbf{j}\Delta_4 + \mathbf{k}\Delta_1$	B	$F - \mathbf{i}\Delta_3 - \mathbf{j}\Delta_6 - \mathbf{k}\Delta_5$
$\mathbf{k}$	$\mathbf{k}$	$E - \mathbf{i}\Delta_6 - \mathbf{j}\Delta_2 - \mathbf{k}\Delta_4$	$F + \mathbf{i}\Delta_3 + \mathbf{j}\Delta_6 + \mathbf{k}\Delta_5$	C

(21)

The g-split quaternion multiplication of two quaternions $p = p_0 + p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k} = s_p + \mathbf{v}_p$ and $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} = s_q + \mathbf{v}_q$ is defined by

$$pq = p_0q_0 + \mathcal{B}_\Omega(\mathbf{v}_p, \mathbf{v}_q) + p_0\mathbf{v}_q + q_0\mathbf{v}_p + \mathbf{v}_p \times_{GL} \mathbf{v}_q. \tag{22}$$

Then, one can derive that

$$s_{pq} = p_0q_0 + Ap_1q_1 + Bp_2q_2 + Cp_3q_3 + Dp_1q_2 + Dp_2q_1 + Ep_1q_3 + Ep_3q_1 + Fp_2q_3 + Fp_3q_2$$

and $\mathbf{v}_{pq} = \begin{pmatrix} p_0q_1 + q_0p_1 - \Delta_3(p_2q_3 - p_3q_2) - \Delta_6(p_3q_1 - p_1q_3) - \Delta_5(p_1q_2 - p_2q_1), \\ p_0q_2 + q_0p_2 - \Delta_6(p_2q_3 - p_3q_2) - \Delta_2(p_3q_1 - p_1q_3) - \Delta_4(p_1q_2 - p_2q_1), \\ p_0q_3 + q_0p_3 - \Delta_5(p_2q_3 - p_3q_2) - \Delta_4(p_3q_1 - p_1q_3) - \Delta_1(p_1q_2 - p_2q_1) \end{pmatrix}$.

The g-split quaternion left and right multiplications can be expressed as the following:

$$L_p(q) = pq = \begin{bmatrix} p_0 & Ap_1 + Dp_2 + Ep_3 & Dp_1 + Bp_2 + Fp_3 & Ep_1 + Fp_2 + Cp_3 \\ p_1 & p_0 + p_2\Delta_5 - p_3\Delta_6 & p_3\Delta_3 - p_1\Delta_5 & p_1\Delta_6 - p_2\Delta_3 \\ p_2 & p_2\Delta_4 - p_3\Delta_2 & p_0 + p_3\Delta_6 - p_1\Delta_4 & p_1\Delta_2 - p_2\Delta_6 \\ p_3 & p_2\Delta_1 - p_3\Delta_4 & p_3\Delta_5 - p_1\Delta_1 & p_0 + p_1\Delta_4 - p_2\Delta_5 \end{bmatrix} \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} \tag{23}$$

$$R_p(q) = qp = \begin{bmatrix} p_0 & Ap_1 + Dp_2 + Ep_3 & Dp_1 + Bp_2 + Fp_3 & Ep_1 + Fp_2 + Cp_3 \\ p_1 & p_0 + p_3\Delta_6 - p_2\Delta_5 & p_1\Delta_5 - p_3\Delta_3 & p_2\Delta_3 - p_1\Delta_6 \\ p_2 & p_3\Delta_2 - p_2\Delta_4 & p_0 + p_1\Delta_4 - p_3\Delta_6 & p_2\Delta_6 - p_1\Delta_2 \\ p_3 & p_3\Delta_4 - p_2\Delta_1 & p_1\Delta_1 - p_3\Delta_5 & p_0 + p_2\Delta_5 - p_1\Delta_4 \end{bmatrix} \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}. \tag{24}$$

One can see that the associative property for g-split quaternion multiplication is satisfied since $R_q L_p = L_p R_q$. As usual, we also have the following conjugate, norm, and inverse definitions for a g-split q:

$$\begin{aligned} \overline{q} &= s_q - \mathbf{v}_q \\ N_q &= \sqrt{|J_q|} \text{ where } J_q = q\overline{q} = \overline{q}q = s_q^2 - \mathcal{B}_\Omega(\mathbf{v}_q, \mathbf{v}_q) \\ q^{-1} &= \frac{\overline{q}}{J_q}. \end{aligned} \tag{25}$$

In addition, if $N_q = 1$, then q is called a unit g-split quaternion, and if $N_q \neq 0$, then q/N_q is a unit g-split quaternion. The g-split quaternions are classified according to the sign of J_q as the following split quaternions: a g-split quaternion is called g-timelike, g-spacelike, and g-lightlike (or g-null) if $J_q > 0$, $J_q < 0$ and $J_q = 0$, respectively. Here, one can also see with a long calculation that

$$s_{pq}^2 - \mathcal{B}_\Omega(\mathbf{v}_{pq}, \mathbf{v}_{pq}) - (s_p^2 - \mathcal{B}_\Omega(\mathbf{v}_p, \mathbf{v}_p))(s_q^2 - \mathcal{B}_\Omega(\mathbf{v}_q, \mathbf{v}_q)) = 0. \tag{26}$$

Then, we have $J_{pq} = J_p J_q$, and so $N_{pq} = N_p N_q$. Notice that the set of g-spacelike g-split quaternions and the set of g-lightlike g-split quaternions do not form a group under g-split quaternion multiplication. However, the set of g-timelike g-split quaternions forms a group under g-split quaternion multiplication.

The vector part of any g-timelike g-split quaternion can be $\mathcal{B}_\Omega$-spacelike, $\mathcal{B}_\Omega$-timelike, or $\mathcal{B}_\Omega$-null. However, the vector part of any g-spacelike g-split quaternion is $\mathcal{B}_\Omega$-spacelike necessarily. Any g-split quaternion $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$ without a $\mathcal{B}_\Omega$-null vector part can be expressed in the polar form similar to the split quaternions, as in the following:

(i) Every g-timelike g-split quaternion with a $\mathcal{B}_\Omega$-spacelike vector part can be written in the form

$$q = N_q(\cosh\theta_q + \varepsilon_q \sinh\theta_q), \tag{27}$$

where $\cosh\theta_q = \frac{q_0}{N_q}$, $\sinh\theta_q = \frac{\sqrt{\mathcal{B}_\Omega(\mathbf{v}_q,\mathbf{v}_q)}}{N_q}$, and $\varepsilon_q = \frac{q_1 i + q_2 j + q_3 k}{\sqrt{\mathcal{B}_\Omega(\mathbf{v}_q,\mathbf{v}_q)}}$, which is a unit $\mathcal{B}_\Omega$-spacelike vector in $\mathbb{R}^3_{\mathcal{B}_\Omega}$, satisfying the equality $\boldsymbol{\varepsilon}_q^2 = 1$.

(ii) Every g-timelike g-split quaternion with a $\mathcal{B}_\Omega$-timelike vector part can be written in the form

$$q = N_q(\cos\theta_q + \varepsilon_q \sin\theta_q), \tag{28}$$

where $\cos\theta_q = \frac{q_0}{N_q}$, $\sin\theta_q = \frac{\sqrt{|\mathcal{B}_\Omega(\mathbf{v}_q,\mathbf{v}_q)|}}{N_q}$, and $\varepsilon_q = \frac{q_1 i + q_2 j + q_3 k}{\sqrt{|\mathcal{B}_\Omega(\mathbf{v}_q,\mathbf{v}_q)|}}$ is a unit $\mathcal{B}_\Omega$-timelike vector in $\mathbb{R}^3_{\mathcal{B}_\Omega}$, satisfying the equality $\boldsymbol{\varepsilon}_q^2 = -1$.

(iii) Every g-spacelike g-split quaternion can be written in the form

$$q = N_q(\sinh\theta_q + \varepsilon_q \cosh\theta_q), \tag{29}$$

where $\sinh\theta_q = \frac{q_0}{N_q}$, $\cosh\theta_q = \frac{\sqrt{\mathcal{B}_\Omega(\mathbf{v}_q,\mathbf{v}_q)}}{N_q}$, and $\varepsilon_q = \frac{q_1 i + q_2 j + q_3 k}{\sqrt{\mathcal{B}_\Omega(\mathbf{v}_q,\mathbf{v}_q)}}$, which is a unit $\mathcal{B}_\Omega$-spacelike vector in $\mathbb{R}^3_{\mathcal{B}_\Omega}$, satisfying the equality $\boldsymbol{\varepsilon}_q^2 = 1$.

4. Generating Generalized Lorentzian Rotation Matrices

In this section, we first define $\mathcal{B}_\Omega$-rotations, which can be elliptic or hyperbolic rotations of the three-dimensional space, and then we use generalized split quaternions to generate $\mathcal{B}_\Omega$-rotation matrices, similar to the three-dimensional Lorentzian space.

It is known that orthogonal matrices whose determinants are 1 can be used to represent rotations since they are the only linear transformations preserving both the norm and the vector product. Therefore, we need to define the $\mathcal{B}_\Omega$-orthogonal matrices in $\mathbb{R}^3_{\mathcal{B}_\Omega}$:

Definition 3. *Let R be a 3×3 real matrix. If*

$$\mathcal{B}_{\Omega}(R\mathbf{u}, R\mathbf{w}) = \mathcal{B}_{\Omega}(\mathbf{u}, \mathbf{w}) \tag{30}$$

for all vectors $\mathbf{u}, \mathbf{w} \in \mathbb{R}^3_{\mathcal{B}_\Omega}$, then R is called $\mathcal{B}_\Omega$-orthogonal.

One can derive that R is $\mathcal{B}_\Omega$-orthogonal if and only if $R^t \Omega R = \Omega$. It is clear that the determinant of a $\mathcal{B}_\Omega$-orthogonal matrix is either 1 or -1. If R is $\mathcal{B}_\Omega$-orthogonal and $\det(R) = 1$, then R is called a $\mathcal{B}_\Omega$-rotation matrix. We denote the $\mathcal{B}_\Omega$-rotation around the $\mathcal{B}_\Omega$-axis $\mathbf{u}$ by the $\mathcal{B}_\Omega$-angle θ, with $R^{\mathbf{u}}_\theta$. Note that a $\mathcal{B}_\Omega$-rotation occurs on a general hyperboloid of one or two sheets. By the following theorem, one can see that every unit g-timelike g-split quaternion defined by the matrix Ω, whose vector part is not $\mathcal{B}_\Omega$-null, determines a $\mathcal{B}_\Omega$-rotation in $\mathbb{R}^3_{\mathcal{B}_\Omega}$:

Theorem 2. *If $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$ is a unit g-timelike g-split quaternion whose vector part is not $\mathcal{B}_\Omega$-null, then $R_q(p) = qpq^{-1}$ determines a $\mathcal{B}_\Omega$-rotation, and the matrix R_q of the transformation is*

$$\begin{bmatrix} (2q_0^2-1)-2(\delta'_{q_1}-\Delta'_{q(5,6)}) & -2(\delta''_{q_1}-\Delta''_{q(3,5)}) & -2(\delta'''_{q_1}-\Delta'''_{q(6,3)}) \\ -2(\delta'_{q_2}-\Delta'_{q(4,2)}) & (2q_0^2-1)-2(\delta''_{q_2}-\Delta''_{q(6,4)}) & -2(\delta'''_{q_2}-\Delta'''_{q(2,6)}) \\ -2(\delta'_{q_3}-\Delta'_{q(1,4)}) & -2(\delta''_{q_3}-\Delta''_{q(5,1)}) & (2q_0^2-1)-2(\delta'''_{q_3}-\Delta'''_{q(4,5)}) \end{bmatrix}, \tag{31}$$

where $\delta'_{q_i} = q_i(Aq_1 + Dq_2 + Eq_3)$, $\delta''_{q_i} = q_i(Dq_1 + Bq_2 + Fq_3)$, $\delta'''_{q_i} = q_i(Eq_1 + Fq_2 + Cq_3)$, $\Delta'_{q(i,j)} = q_0(\Delta_i q_2 - \Delta_j q_3)$, $\Delta''_{q(i,j)} = q_0(\Delta_i q_3 - \Delta_j q_1)$, and $\Delta'''_{q(i,j)} = q_0(\Delta_i q_1 - \Delta_j q_2)$. In addition:

(i) If the vector part of q is $\mathcal{B}_\Omega$-spacelike, and the polar form of q is

$$\cosh\theta_q + \boldsymbol{\varepsilon}_q \sinh\theta_q, \tag{32}$$

then the $\mathcal{B}_\Omega$-axis of the $\mathcal{B}_\Omega$-rotation is $\boldsymbol{\varepsilon}_q$, and the $\mathcal{B}_\Omega$-angle of the $\mathcal{B}_\Omega$-rotation is $2\theta_q = 2\cosh^{-1} q_0$. Therefore, $R_q = R^{\boldsymbol{\varepsilon}_q}_{2\theta}$.

(ii) If the vector part of q is $\mathcal{B}_\Omega$-timelike, and the polar form of q is

$$\cos\theta_q + \boldsymbol{\varepsilon}_q \sin\theta_q, \tag{33}$$

then the $\mathcal{B}_\Omega$-axis of the $\mathcal{B}_\Omega$-rotation is $\boldsymbol{\varepsilon}_q$, and the $\mathcal{B}_\Omega$-angle of the $\mathcal{B}_\Omega$-rotation is $2\theta_q = 2\cos^{-1} q_0$. Therefore, $R_q = R^{\boldsymbol{\varepsilon}_q}_{2\theta}$.

Proof. Let q be a unit g-timelike g-split quaternion whose vector part is not $\mathcal{B}_\Omega$-null. Let us consider the map $R_q(p) = qpq^{-1} = qp\overline{q}$ for any g-split quaternion p. It is not difficult to see that R_q is linear, preserving the norm of the vector part and the scalar part of p. Then, we must consider only pure g-split quaternions whose maps are also pure g-split quaternions with the same norm. To determine the transformation matrix, it is enough to find $R_q(\mathbf{i})$, $R_q(\mathbf{j})$, and $R_q(\mathbf{k})$. Using $R_q(\mathbf{i}) = L_q(L_{\mathbf{i}}(\overline{q}))$, $R_q(\mathbf{j}) = L_q(L_{\mathbf{j}}(\overline{q}))$, $R_q(\mathbf{k}) = L_q(L_{\mathbf{k}}(\overline{q}))$, and $J_q = 1$, one obtains

$$\begin{aligned} R_q(\mathbf{i}) &= \left(2q_0^2 - 1 - 2(\delta'_{q_1} - \Delta'_{q(5,6)})\right)\mathbf{i} - 2(\delta'_{q_2} - \Delta'_{q(4,2)})\mathbf{j} - 2(\delta'_{q_3} - \Delta'_{q(1,4)})\mathbf{k} \\ R_q(\mathbf{j}) &= -2(\delta''_{q_1} - \Delta''_{q(3,5)})\mathbf{i} + \left(2q_0^2 - 1 - 2(\delta''_{q_2} - \Delta''_{q(6,4)})\right)\mathbf{j} - 2(\delta''_{q_3} - \Delta''_{q(5,1)})\mathbf{k} \\ R_q(\mathbf{k}) &= -2(\delta'''_{q_1} - \Delta'''_{q(6,3)})\mathbf{i} - 2(\delta'''_{q_2} - \Delta'''_{q(2,6)})\mathbf{j} + \left(2q_0^2 - 1 - 2(\delta'''_{q_3} - \Delta'''_{q(4,5)})\right)\mathbf{k}, \end{aligned}$$

and we obtain matrix (31). In addition, one can see that $(R_q)^t \Omega R_q = \Omega$ and $\det(R_q) = 1$. Therefore, R_q is a $\mathcal{B}_\Omega$-rotation matrix. In addition,

(i) Let the vector part of q be $\mathcal{B}_{\Omega}$-spacelike, and let the polar form of q be $\cosh\theta_q + \boldsymbol{\varepsilon}_q \sinh\theta_q$. Since $\mathbf{v}_q \parallel \boldsymbol{\varepsilon}_q$, we have $\mathbf{v}_q \times_{GL} \boldsymbol{\varepsilon}_q = \boldsymbol{\varepsilon}_q \times_{GL} \mathbf{v}_q = 0$ and $q\boldsymbol{\varepsilon}_q = \boldsymbol{\varepsilon}_q q$. Then, we have $R_q(\boldsymbol{\varepsilon}_q) = q\boldsymbol{\varepsilon}_q q^{-1} = \boldsymbol{\varepsilon}_q q q^{-1} = \boldsymbol{\varepsilon}_q$, which means that $\boldsymbol{\varepsilon}_q$ is the $\mathcal{B}_{\Omega}$-axis of the $\mathcal{B}_{\Omega}$-rotation. To determine the $\mathcal{B}_{\Omega}$-angle of the $\mathcal{B}_{\Omega}$-rotation, let us consider a $\mathcal{B}_{\Omega}$-orthonormal set $\{\boldsymbol{\varepsilon}_q, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2\}$ satisfying

$$\boldsymbol{\varepsilon}_1 \times_{GL} \boldsymbol{\varepsilon}_2 = \boldsymbol{\varepsilon}_q, \;\; \boldsymbol{\varepsilon}_2 \times_{GL} \boldsymbol{\varepsilon}_q = \boldsymbol{\varepsilon}_1 \text{ and } \;\boldsymbol{\varepsilon}_q \times_{GL} \boldsymbol{\varepsilon}_1 = \boldsymbol{\varepsilon}_2. \tag{34}$$

It is enough to determine how $\boldsymbol{\varepsilon}_1$ changes under R_q. Considering the g-split quaternion multiplication, one obtains

$$\begin{aligned} R_q(\boldsymbol{\varepsilon}_1) &= q\boldsymbol{\varepsilon}_1\overline{q} \\ &= \left(\cosh\theta_q + \boldsymbol{\varepsilon}_q \sinh\theta_q\right)\varepsilon_1\left(\cosh\theta_q - \boldsymbol{\varepsilon}_q \sinh\theta_q\right) \\ &= \left(\varepsilon_1 \cosh\theta_q + \varepsilon_2 \sinh\theta_q\right)\left(\cosh\theta_q - \boldsymbol{\varepsilon}_q \sinh\theta_q\right) \\ &= \boldsymbol{\varepsilon}_1 \cosh 2\theta_q + \boldsymbol{\varepsilon}_2 \sinh 2\theta_q. \end{aligned} \tag{35}$$

Thus, the $\mathcal{B}_{\Omega}$-angle of the $\mathcal{B}_{\Omega}$-rotation is $2\theta_q$. In addition, $2\theta_q = 2\cosh^{-1} q_0$, since $\cosh\theta_q = q_0$.

(ii) Let the vector part of q be $\mathcal{B}_{\Omega}$-timelike, and let the polar form of q be $\cos\theta_q + \boldsymbol{\varepsilon}_q \sin\theta_q$. Since $\mathbf{v}_q \parallel \boldsymbol{\varepsilon}_q$, we have $\mathbf{v}_q \times_{GL} \boldsymbol{\varepsilon}_q = \boldsymbol{\varepsilon}_q \times_{GL} \mathbf{v}_q = 0$, and so $q\boldsymbol{\varepsilon}_q = \boldsymbol{\varepsilon}_q q$. Then, we have $R_q\left(\boldsymbol{\varepsilon}_q\right) = q\boldsymbol{\varepsilon}_q q^{-1} = \boldsymbol{\varepsilon}_q q q^{-1} = \boldsymbol{\varepsilon}_q$. Thus, $\boldsymbol{\varepsilon}_q$ is the $\mathcal{B}_{\Omega}$-axis of the $\mathcal{B}_{\Omega}$-rotation. To determine the $\mathcal{B}_{\Omega}$-angle of the $\mathcal{B}_{\Omega}$-rotation, let us consider a $\mathcal{B}_{\Omega}$-orthonormal set $\{\boldsymbol{\varepsilon}_q, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2\}$ satisfying

$$\boldsymbol{\varepsilon}_1 \times_{GL} \boldsymbol{\varepsilon}_2 = \boldsymbol{\varepsilon}_q, \;\; \boldsymbol{\varepsilon}_2 \times_{GL} \boldsymbol{\varepsilon}_q = \boldsymbol{\varepsilon}_1, \text{ and } \;\boldsymbol{\varepsilon}_q \times_{GL} \boldsymbol{\varepsilon}_1 = \boldsymbol{\varepsilon}_2. \tag{36}$$

It is enough to determine how $\boldsymbol{\varepsilon}_1$ changes under R_q. Considering the g-split quaternion multiplication, one obtains

$$\begin{aligned} R^q(\boldsymbol{\varepsilon}_1) &= q\boldsymbol{\varepsilon}_1\overline{q} \\ &= \left(\cos\theta_q + \boldsymbol{\varepsilon}_0 \sin\theta_q\right)\varepsilon_1\left(\cos\theta_q - \boldsymbol{\varepsilon}_0 \sin\theta_q\right) \\ &= \left(\varepsilon_1 \cos\theta_q + \varepsilon_2 \sin\theta_q\right)\left(\cos\theta_q - \boldsymbol{\varepsilon}_q \sin\theta_q\right) \\ &= \boldsymbol{\varepsilon}_1 \cos 2\theta_q + \boldsymbol{\varepsilon}_2 \sin 2\theta_q. \end{aligned} \tag{37}$$

Thus, the $\mathcal{B}_{\Omega}$-angle of the $\mathcal{B}_{\Omega}$-rotation is $2\theta_q$. In addition, $2\theta_q = 2\cos^{-1} q_0$, since $\cos\theta = q_0$. □

By the following corollary, we see that all $\mathcal{B}_{\Omega}$-rotations whose axes are not $\mathcal{B}_{\Omega}$-null vectors can be represented by unit g-timelike g-split quaternions defined by the matrix Ω:

Corollary 1.

(i) For a unit $\mathcal{B}_{\Omega}$-spacelike vector $\mathbf{n} = (n_1, n_2, n_3)$ in $\mathbb{R}^3_{\mathcal{B}_{\Omega}}$, the $\mathcal{B}_{\Omega}$-rotation matrix $R^{\mathbf{n}}_{\theta}$ is

$$\begin{bmatrix} 1-\mu_s+\delta'_{n_1}\mu_s+\Delta'_{5,6}\rho_s & \delta''_{n_1}\mu_s+\Delta''_{3,5}\rho_s & \delta'''_{n_1}\mu_s+\Delta'''_{6,3}\rho_s \\ \delta'_{n_2}\mu_s+\Delta'_{4,2}\rho_s & 1-\mu_s+\delta''_{n_2}\mu_s+\Delta''_{6,4}\rho_s & \delta'''_{n_2}\mu_s+\Delta'''_{2,6}\rho_s \\ \delta'_{n_3}\mu_s+\Delta'_{1,4}\rho_s & \delta''_{n_3}\mu_s+\Delta''_{5,1}\rho_s & 1-\mu_s+\delta'''_{n_3}\mu_s+\Delta'''_{4,5}\rho_s \end{bmatrix}, \tag{38}$$

where $\mu_s = 1-\cosh\theta$, $\rho_s = \sinh\theta$, $\delta'_{n_i} = n_i(An_1 + Dn_2 + En_3)$, $\delta''_{n_i} = n_i(Dn_1 + Bn_2 + Fn_3)$, $\delta'''_{n_i} = n_i(En_1 + Fn_2 + Cn_3)$, $\Delta'_{i,j} = (\Delta_i n_2 - \Delta_j n_3)$, $\Delta''_{i,j} = (\Delta_i n_3 - \Delta_j n_1)$, and $\Delta'''_{i,j} = (\Delta_i n_1 - \Delta_j n_2)$.

(ii) For a unit $\mathcal{B}_{\Omega}$-timelike vector $\mathbf{n} = (n_1, n_2, n_3)$ in $\mathbb{R}^3_{\mathcal{B}_{\Omega}}$, the $\mathcal{B}_{\Omega}$-rotation matrix $R^{\mathbf{n}}_{\theta}$ is

$$\begin{bmatrix} 1+\mu_t+\delta'_{n_1}\mu_t+\underline{\Delta}'_{5,6}\rho_t & \delta''_{n_1}\mu_t+\underline{\Delta}''_{3,5}\rho_t & \delta'''_{n_1}\mu_t+\underline{\Delta}'''_{6,3}\rho_t \\ \delta'_{n_2}\mu_t+\underline{\Delta}'_{4,2}\rho_t & 1+\mu_t+\delta''_{n_2}\mu_t+\underline{\Delta}''_{6,4}\rho_t & \delta'''_{n_2}\mu_t+\underline{\Delta}'''_{2,6}\rho_t \\ \delta'_{n_3}\mu_t+\underline{\Delta}'_{1,4}\rho_t & \delta''_{n_3}\mu_t+\underline{\Delta}''_{5,1}\rho_t & 1+\mu_t+\delta'''_{n_3}\mu_t+\underline{\Delta}'''_{4,5}\rho_t \end{bmatrix}, \tag{39}$$

where $\delta'_{n_i}=n_i(An_1+Dn_2+En_3)$, $\delta''_{n_i}=n_i(Dn_1+Bn_2+Fn_3)$, $\delta'''_{n_i}=n_i(En_1+Fn_2+Cn_3)$, $\underline{\Delta}'_{i,j}=\sin\theta(\Delta_i n_2-\Delta_j n_3)$, $\underline{\Delta}''_{i,j}=\sin\theta(\Delta_i n_3-\Delta_j n_1)$, *and* $\underline{\Delta}'''_{i,j}=\sin\theta(\Delta_i n_1-\Delta_j n_2)$.

Proof.

(i) Let $\mathbf{n}$ be a unit $\mathcal{B}_\Omega$-spacelike vector. Then, by the theorem, the unit g-timelike g-split quaternion

$$q=q_0+q_1\mathbf{i}+q_2\mathbf{j}+q_3\mathbf{k}=\cosh(\theta/2)+(n_1,n_2,n_3)\sinh(\theta/2), \tag{40}$$

whose vector part is $\mathcal{B}_\Omega$-spacelike, determines the $\mathcal{B}_\Omega$-rotation around the $\mathcal{B}_\Omega$-axis $\mathbf{n}$ by the $\mathcal{B}_\Omega$-angle θ. Therefore, we have $q_0=\cosh(\theta/2)$, $q_1=n_1\sinh(\theta/2)$, $q_2=n_2\sinh(\theta/2)$, and $q_3=n_3\sinh(\theta/2)$. Substituting these values into matrix (31), we obtain matrix (38).

(ii) Let $\mathbf{n}$ be a unit $\mathcal{B}_\Omega$-timelike vector. Then, by the theorem, the unit g-timelike g-split quaternion

$$q=q_0+q_1\mathbf{i}+q_2\mathbf{j}+q_3\mathbf{k}=\cos(\theta/2)+(n_1,n_2,n_3)\sin(\theta/2), \tag{41}$$

whose vector part is $\mathcal{B}_\Omega$-timelike, determines the $\mathcal{B}_\Omega$-rotation around the $\mathcal{B}_\Omega$-axis $\mathbf{n}$ by the $\mathcal{B}_\Omega$-angle θ. Therefore, we have $q_0=\cos(\theta/2)$, $q_1=n_1\sin(\theta/2)$, $q_2=n_2\sin(\theta/2)$, and $q_3=n_3\sin(\theta/2)$. Substituting these values into matrix (31), we obtain matrix (39).

□

Remark 1. *Note that a $\mathcal{B}_\Omega$-rotation of a point P around a $\mathcal{B}_\Omega$-axis ℓ by the $\mathcal{B}_\Omega$-angle π, which is a $\mathcal{B}_\Omega$-half turn, is a point P', which is determined by the $\mathcal{B}_\Omega$-symmetry about the plane through the line ℓ and $\mathcal{B}_\Omega$-orthogonal to the line through the points P and P'.*

5. Numerical Results

In this section, we give some numerical examples as applications of our results. In the first example, we see that in three-dimensional Lorentzian space corresponding to a given matrix Ω, a unit g-timelike g-split quaternion determines a conical rotation on a hyperboloid of one sheet about its vector part by double its argument. This rotation is elliptic since the vector part is $\mathcal{B}_\Omega$-spacelike.

Example 1. *Given a matrix*

$$\Omega=\begin{bmatrix} 1 & -2 & 0 \\ -2 & 1 & -2 \\ 0 & -2 & 4 \end{bmatrix}, \tag{42}$$

which is suitable for the conditions, then we have $\Delta=\sqrt{|-16|}=4$, $\Delta_1=-\frac{3}{4}$, $\Delta_2=\Delta_5=1$, $\Delta_3=0$, $\Delta_4=\frac{1}{2}$, $\Delta_5=1$, *and* $\Delta_6=2$. *Let us consider the g-split quaternion* $p=(1/\sqrt{2},1/\sqrt{6},1/\sqrt{6},1/2\sqrt{6})$. *It is unit g-timelike with a $\mathcal{B}_\Omega$-timelike vector part, since* $N_p=1$, $J_p=1$, *and* $\mathcal{B}_\Omega(\mathbf{v}_p,\mathbf{v}_p)=-\frac{1}{2}$. *Then, its polar form is*

$$p=\cos(\pi/4)+\left(\tfrac{1}{\sqrt{3}}i+\tfrac{1}{\sqrt{3}}j+\tfrac{1}{2\sqrt{3}}k\right)\sin(\pi/4). \tag{43}$$

With a simple calculation, we obtain

$$R_p=\begin{bmatrix} 1/3 & (2-\sqrt{3})/3 & 2/\sqrt{3} \\ 1/3 & (\sqrt{3}+4)/6 & -1/\sqrt{3} \\ (1-2\sqrt{3})/6 & (5\sqrt{3}+4)/12 & -\sqrt{3}/6 \end{bmatrix}, \tag{44}$$

which is the $\mathcal{B}_{\Omega}$-rotation around $\boldsymbol{\varepsilon}_p = \left(1/\sqrt{3}, 1/\sqrt{3}, 1/2\sqrt{3}\right) = \boldsymbol{\varepsilon}$ by the $\mathcal{B}_{\Omega}$-angle $\pi/2$. One can check that $(R_p)^t \Omega R_p = \Omega$ and $\det R_p = 1$. Let us also consider the g-split quaternion $q = (1/2, 1/2, 1/2, 1/4)$, which is also unit g-timelike with a $\mathcal{B}_{\Omega}$-timelike vector part, since $N_q = 1$, $J_q = 1$ and $\mathcal{B}_{\Omega}(\mathbf{v}_q, \mathbf{v}_q) = -\frac{3}{4}$. Then, its polar form is

$$q = \cos(\pi/3) + \left(\tfrac{1}{\sqrt{3}} i + \tfrac{1}{\sqrt{3}} j + \tfrac{1}{2\sqrt{3}} k\right) \sin(\pi/3). \tag{45}$$

With a simple calculation again, one obtains

$$R_q = \begin{bmatrix} 0 & \frac{1}{2} & 1 \\ \frac{1}{2} & \frac{3}{4} & -\frac{1}{2} \\ -\frac{1}{4} & \frac{9}{8} & -\frac{3}{4} \end{bmatrix}, \tag{46}$$

which is the $\mathcal{B}_{\Omega}$-rotation $\boldsymbol{\varepsilon}_q = \boldsymbol{\varepsilon} = \left(1/\sqrt{3}, 1/\sqrt{3}, 1/2\sqrt{3}\right)$ by the $\mathcal{B}_{\Omega}$-angle $2\pi/3$. One can check that $(R_q)^t \Omega R_q = \Omega$ and $\det R_q = 1$. In addition, we have

$$\begin{aligned} pq &= \left(\tfrac{\sqrt{2}-\sqrt{6}}{4}, \tfrac{\sqrt{3}+1}{2\sqrt{6}}, \tfrac{\sqrt{3}+1}{2\sqrt{6}}, \tfrac{\sqrt{3}+1}{4\sqrt{6}}\right) \\ &= \cos(7\pi/12) + \left(\tfrac{1}{\sqrt{3}} i + \tfrac{1}{\sqrt{3}} j + \tfrac{1}{2\sqrt{3}} k\right) \sin(7\pi/12), \end{aligned} \tag{47}$$

since $\mathcal{B}_{\Omega}(\mathbf{v}_p, \mathbf{v}_q) = -\sqrt{6}/4$ and $\mathbf{v}_p \times_{GL} \mathbf{v}_q = \mathbf{0}$. Note also that $N_{pq} = 1$, and we obtain

$$R_{pq} = \begin{bmatrix} \frac{1-\sqrt{3}}{3} & \frac{3\sqrt{3}+4}{6} & -\frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}+2}{6} & \frac{8-3\sqrt{3}}{12} & \frac{\sqrt{3}}{6} \\ \frac{3\sqrt{3}+2}{12} & \frac{8-\sqrt{3}}{24} & -\frac{5\sqrt{3}}{12} \end{bmatrix}, \tag{48}$$

which is the matrix of the $\mathcal{B}_{\Omega}$-rotation around the $\mathcal{B}_{\Omega}$-axis $\boldsymbol{\varepsilon}_{pq} = \boldsymbol{\varepsilon} = \left(1/\sqrt{3}, 1/\sqrt{3}, 1/2\sqrt{3}\right)$ by the $\mathcal{B}_{\Omega}$-angle $7\pi/6$. Note that

$$R^{\varepsilon}_{\pi/2} R^{\varepsilon}_{2\pi/3} = R^{\varepsilon}_{2\pi/3} R^{\varepsilon}_{\pi/2} = R^{\varepsilon}_{7\pi/6}. \tag{49}$$

Now we have $\mathcal{B}_{\Omega}$-rotation matrices. Let us consider the vector $\mathbf{u} = (1, 2, 3)$ on the hyperboloid of one sheet with the equation

$$x^2 + y^2 + 4z^2 - 4xy - 4yz = 9. \tag{50}$$

One can see that

$$R^{\varepsilon}_{\pi/2}(\mathbf{u}) = \left(\tfrac{4\sqrt{3}+5}{3}, \tfrac{5-2\sqrt{3}}{3}, \tfrac{5}{6}\right) = \mathbf{v} \tag{51}$$

$$R^{\varepsilon}_{2\pi/3}(\mathbf{u}) = \left(4, \tfrac{1}{2}, -\tfrac{1}{4}\right) = \mathbf{v}' \tag{52}$$

$$R^{\varepsilon}_{2\pi/3}(\mathbf{v}) = R^{\varepsilon}_{\pi/2}(\mathbf{v}') = R^{\varepsilon}_{7\pi/6}(\mathbf{u}) = \left(\tfrac{5-\sqrt{3}}{3}, \tfrac{\sqrt{3}+10}{6}, \tfrac{10-13\sqrt{3}}{12}\right) = \mathbf{w}. \tag{53}$$

Note that these vectors are on the same hyperboloid. The rotation occurs on an ellipse that is the intersection curve of the hyperboloid and the plane Π_{ε} through the end point of the vector $\mathbf{u}$ and $\mathcal{B}_{\Omega}$-orthogonal to the vector $\boldsymbol{\varepsilon}$.

In the next example, we take a unit g-timelike g-split quaternion having a $\mathcal{B}_{\Omega}$-timelike vector part, and then the movement of the $\mathcal{B}_{\Omega}$-rotation turns into a hyperbola.

Example 2. *Let us consider the g-split quaternions* $q = (1/2, 1/2, 1/2, 1/4)$ *and* $r = (0, 1/\sqrt{3}, 2/\sqrt{3}, 2/\sqrt{3})$*, instead of p and q. The g-split quaternion r is also unit g-timelike having a* $\mathcal{B}_\Omega$*-timelike vector part, since* $N_q = 1$*,* $J_q = 1$ *and* $\mathcal{B}_\Omega(\mathbf{v}_r, \mathbf{v}_r) = -1$*. Then, its polar form is*

$$r = \cos(\pi/2) + \left(\tfrac{1}{\sqrt{3}}i + \tfrac{2}{\sqrt{3}}j + \tfrac{2}{\sqrt{3}}k\right)\sin(\pi/2). \tag{54}$$

Then, one obtains

$$R_r = \begin{bmatrix} 1 & \frac{8}{3} & -\frac{8}{3} \\ 4 & \frac{13}{3} & -\frac{16}{3} \\ 4 & \frac{16}{3} & -\frac{19}{3} \end{bmatrix}, \tag{55}$$

which is the $\mathcal{B}_\Omega$*-rotation around* $\boldsymbol{\varepsilon}_r = \left(1/\sqrt{3}, 2/\sqrt{3}, 2/\sqrt{3}\right)$ *by the* $\mathcal{B}_\Omega$*-angle* π*. For this example, there are two different* $\mathcal{B}_\Omega$*-axes. In addition, since*

$$\mathcal{B}_\Omega(\mathbf{v}_q, \mathbf{v}_r) = -\frac{5\sqrt{3}}{6} \tag{56}$$

$$\mathbf{v}_q \times_{GL} \mathbf{v}_r = \left(\tfrac{\sqrt{3}}{3}, -\tfrac{\sqrt{3}}{6}, \tfrac{\sqrt{3}}{12}\right), \tag{57}$$

one obtains

$$qr = \left(-\tfrac{5\sqrt{3}}{6}, \tfrac{\sqrt{3}}{2}, \tfrac{\sqrt{3}}{6}, \tfrac{5\sqrt{3}}{12}\right), \tag{58}$$

which is a unit g-timelike g-split quaternion whose vector part is $\mathcal{B}_\Omega$*-spacelike. Then, it has the polar form*

$$qr = \cosh\theta + \left(\frac{3i + j + \frac{5}{2}k}{\sqrt{13}}\right) + \sinh\theta, \tag{59}$$

where $\theta = \sinh^{-1}(\sqrt{\tfrac{13}{12}})$*. Thus, one obtains*

$$R_{qr} = \begin{bmatrix} 6 & \frac{15}{2} & -9 \\ \frac{3}{2} & \frac{23}{12} & -\frac{13}{6} \\ \frac{5}{4} & \frac{5}{24} & -\frac{7}{12} \end{bmatrix}, \tag{60}$$

which is the $\mathcal{B}_\Omega$*-rotation around* $\boldsymbol{\varepsilon}_{qr} = \left(3/\sqrt{13}, 1/\sqrt{13}, 5/2\sqrt{13}\right)$ *by the* $\mathcal{B}_\Omega$*-angle* $2\sinh^{-1}(\sqrt{\tfrac{13}{12}})$*. Note that* $R_q R_r = R_{qr}$ *but* $R_r R_q \neq R_{qr}$*. In this case, for the vector* $\mathbf{u} = (1, 2, 3)$*,* $R_{qr}(\mathbf{u}) = (-6, -\tfrac{7}{6}, -\tfrac{1}{12}) = \mathbf{s}$*. Note that* $\mathcal{B}_\Omega$*-rotation occurs on a hyperbola that is the intersection curve of the hyperboloid and the plane* $\Pi_{\boldsymbol{\varepsilon}_{qr}}$ *through the end point of the vector* $\mathbf{u}$ *and* $\mathcal{B}_\Omega$*-orthogonal to the vector* $\boldsymbol{\varepsilon}_{qr} = \left(3/\sqrt{13}, 1/\sqrt{13}, 5/2\sqrt{13}\right)$*.*

Note that $\mathcal{B}_\Omega$-rotation matrices having the same coefficients are also valid for hyperboloids of two sheets. For example, take a vector $\mathbf{m} = (2, 2, \frac{\sqrt{3}}{2} + 1)$ on the hyperboloid of two sheets with the equation

$$x^2 + y^2 + 4z^2 - 4xy - 4yz = -9. \tag{61}$$

Then, one has $R_{qr}(\mathbf{m}) = \left(\frac{36 - 9\sqrt{3}}{2}, \frac{56 - 13\sqrt{3}}{12}, \frac{56 - 7\sqrt{3}}{24}\right) = \mathbf{n}$. The rotation occurs on the intersection hyperbola of this hyperboloid and the plane $\Pi_{\varepsilon_{qr}} : x - 10y + 8z = 4\sqrt{3} - 10$ that passes through the end point of the vector $\mathbf{m}$ and $\mathcal{B}_\Omega$-orthogonal to the vector $\boldsymbol{\varepsilon}_{qr} = \left(3/\sqrt{13}, 1/\sqrt{13}, 5/2\sqrt{13}\right)$.

In the last example, we determine $\mathcal{B}_\Omega$-rotation for a given axis and an angle, considering the same matrix Ω.

Example 3. *Let us determine the matrix of the $\mathcal{B}_\Omega$-rotation around the $\mathcal{B}_\Omega$-axis $\mathbf{u} = (1, 1, 1/2)$ by $\mathcal{B}_\Omega$-angle $2\pi/3$. Since we have the unit $\mathcal{B}_\Omega$-timelike vector $\mathbf{u}/N_{\mathbf{u}} = \left(1/\sqrt{3}, 1/\sqrt{3}, 1/2\sqrt{3}\right)$ and $\mathcal{B}_\Omega$-angle $\theta = \pi/3$, we need to consider the g-split quaternion q whose polar form is*

$$\cos(\pi/3) + \left(1/\sqrt{3}, 1/\sqrt{3}, 1/2\sqrt{3}\right)\sin(\pi/3). \tag{62}$$

This unit g-timelike g-split quaternion, whose vector part is $\mathcal{B}_\Omega$-timelike, determines the $\mathcal{B}_\Omega$-rotation around $\mathcal{B}_\Omega$-axis $\boldsymbol{\varepsilon}_q$ and $\mathcal{B}_\Omega$-angle $2\pi/3$. By a straightforward calculation, one obtains $\delta'_{q_1} = \delta'_{q_2} = \delta''_{q_3} = -\frac{1}{3}$, $\delta'_{q_3} = -\frac{1}{6}$, $\delta''_{q_1} = \delta''_{q_2} = -\frac{2}{3}$, $\delta'''_{q_1} = \delta'''_{q_2} = \delta'''_{q_3} = 0$, $\underline{\Delta}'_{5,6} = \underline{\Delta}'_{4,2} = 0$, $\underline{\Delta}'_{1,4} = \underline{\Delta}''_{3,5} = \underline{\Delta}'''_{2,6} = -\frac{1}{2}$, $\underline{\Delta}''_{6,4} = \frac{1}{4}$, $\underline{\Delta}''_{5,1} = \frac{5}{8}$, $\underline{\Delta}'''_{6,3} = 1$, $\underline{\Delta}'''_{4,5} = -\frac{1}{4}$, and

$$R^{\varepsilon_q}_{2\pi/3} = \begin{bmatrix} 0 & \frac{1}{2} & 1 \\ \frac{1}{2} & \frac{3}{4} & -\frac{1}{2} \\ -\frac{1}{4} & \frac{9}{8} & -\frac{3}{4} \end{bmatrix}, \tag{63}$$

which is the matrix of the $\mathcal{B}_\Omega$-rotation around the $\mathcal{B}_\Omega$-axis $\mathbf{u} = (1, 1, 1/2)$ by the $\mathcal{B}_\Omega$-angle $2\pi/3$ (see Example 1).

6. Discussion

In this paper, we generalized three-dimensional Lorentzian geometry and its associated number system, the split quaternions. Due to the results, the generalized split quaternions can easily express any non-parabolic conical rotation in three-dimensional space without long calculations with affine transformations. For future studies, one can consider the famous Rodrigues, Hausholder, and Cayley transformations to derive new simple formulas for non-parabolic conical rotations in three-dimensional space.

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