symmetry

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Asymptotically Stable Solutions of Infinite Systems of Quadratic Hammerstein Integral Equations

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Abstract: In this paper, we present a result on the existence of asymptotically stable solutions of infinite systems (IS) of quadratic Hammerstein integral equations (IEs). Our study will be conducted in the Banach space of functions, which are continuous and bounded on the half-real axis with values in the classical Banach sequence space consisting of real bounded sequences. The main tool used in our investigations is the technique associated with the measures of noncompactness (MNCs) and a fixed point theorem of Darbo type. The applicability of our result is illustrated by a suitable example at the end of the paper.

Keywords: space of continuous and bounded functions; MNC; fixed point theorem; IS of IEs; asymptotic stability

MSC: 45G15; 47H08

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Citation: Banaś, J.; Madej, J. Asymptotically Stable Solutions of Infinite Systems of Quadratic Hammerstein Integral Equations. *Symmetry* **2024**, *16*, 107. https://doi.org/10.3390/sym16010107

Academic Editor: Calogero Vetro

Received: 4 December 2023
Revised: 29 December 2023
Accepted: 31 December 2023
Published: 16 January 2024

1. Introduction

IEs play a significant role in several branches of nonlinear analysis. Obviously, IEs are closely connected with the theory of differential equations, both ordinary and partial (cf. Corduneanu [1], Pogorzelski [2]).

With the help of IEs we can represent mathematical models of a lot of events appearing in mathematical physics, engineering, mechanics, kinetic theory of gases, transport theory, economics, biology, etc. (see Burton [3], Busbridge [4], Cahlon and Eskin [5], Case and Zweigel [6], Chandrasekhar [7], Deimling [8] and references therein). Let us mention that a lot of real-world problems can be described with the help of IEs (see Deimling [8] and Zabrejko et al. [9], for example).

Obviously, the theory of ISs of IEs represents both a generalization of the classical theory of IEs and simultaneously an advanced part of nonlinear analysis. That theory is a young branch of the theory of IEs since papers studying problems of ISs of IEs have only appeared in recent decades.

It is worthwhile mentioning that the investigations concerning solutions of ISs of IEs defined on an unbounded interval are quite new (cf. Banaś and Chlebowicz [10], Banaś and Madej [11] and references therein). Let us pay attention to the paper of Banaś and Madej [11], where we examined an IS of quadratic Urysohn IEs with integral taken as improper one defined on the real half-axis $\mathbb{R}_+$.

As far as we know, the above-quoted paper of Banaś and Madej is the first one of such a type. In that paper, we examined conditions ensuring the existence of ISs solutions for quadratic Urysohn IEs, which are converging to zero at infinity at the same rate. More precisely, we examined conditions guaranteeing that solutions of the mentioned ISs of quadratic Urysohn IEs being function sequence $(x_n(t))$ defined on the interval $\mathbb{R}_+$ are such that $\lim_{t\to\infty} x_n(t) = 0$ uniformly with respect to n belonging to the set of natural numbers $\mathbb{N}$.

In the present paper, we are going to study the IS of quadratic Hammerstein IEs having the form

$$x_n(t) = a_n(t) + f_n(t, x_1(t), x_2(t), \dots) \int_0^\infty g_n(t,\tau) h_n(\tau, x_1(\tau), x_2(\tau), \dots) d\tau, \qquad (1)$$

for $t \in \mathbb{R}_+ = [0, \infty)$ and for $n = 1, 2, \dots$.

The definitions of functions f_n and g_n appearing in IS of IEs (1) will be given in Section 3.

Our aim is to prove that under suitable conditions imposed on components of IS of IEs (1) there exists a solution $x(t) = (x_n(t))$ of this IS defined on $\mathbb{R}_+$ which is asymptotically stable. The main tool used in the proof is the technique of MNCs applied in a suitable Banach space. Due to MNCs, we are in a position to achieve the result for the existence of asymptotically stable solutions of IS of IEs (1).

The results obtained in the paper for IS of IEs (1) create the first step in investigations of conditions guaranteeing the existence of asymptotically stable solutions of IS of IEs. Indeed, we expect that, based on these results, we will be able to obtain similar results for IS of IEs of Urysohn type.

2. Auxiliary Facts

In this section, we collect notations and auxiliary facts that will be utilized in our investigations of this paper. By the symbol $\mathbb{R}$ we denote the set of real numbers and we put $\mathbb{R}_+ = [0, \infty)$. Moreover, we denote by $\mathbb{N}$ the set of natural numbers.

Further, assume that E is a given Banach space with the norm $\|\cdot\|_E$ and the zero element θ. In our considerations, we will also write $\|\cdot\|$ instead of $\|\cdot\|_E$ if it does not lead to misunderstanding.

The symbol $B(x, r)$ denotes the closed ball centered at x and with radius r. We write B_r to denote the ball $B(\theta, r)$. If X is a subset of E then the symbols $\overline{X}$ and $ConvX$ stand for the closure and convex closure of X, respectively. Moreover, we will use the standard notation $X + Y$, λX to denote the classical algebraic operations on subsets of E.

Next, let $\mathfrak{M}_E$ denote the family of all nonempty and bounded subsets of E while its subfamily consisting of all relatively compact sets will be denoted by $\mathfrak{N}_E$.

The most important concept used in our paper is the concept of a MNC. We will accept the axiomatic definition of that concept taken from Banaś and Goebel [12].

Definition 1. *A function $\mu : \mathfrak{M}_E \to \mathbb{R}_+$ is said to be a MNC in the space E if it satisfies the following conditions:*

(i) *The family $ker\mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ is nonempty and $ker\mu \subset \mathfrak{N}_E$.*
(ii) $X \subset Y \Rightarrow \mu(X) \leqslant \mu(Y)$.
(iii) $\mu(\overline{X}) = \mu(X)$.
(iv) $\mu(ConvX) = \mu(X)$.
(v) $\mu(\lambda X + (1-\lambda)Y) \leqslant \lambda\mu(X) + (1-\lambda)\mu(Y)$ *for* $\lambda \in [0, 1]$.
(vi) *If $(X_n)_{n \geqslant 1}$ is a sequence of closed sets from $\mathfrak{M}_E$ such that $X_{n+1} \subset X_n$ for $n = 1, 2, \dots$ and $\lim_{n\to\infty} \mu(X_n) = 0$, then the set $X_\infty = \bigcap_{n=1}^\infty X_n$ is nonempty.*

The family $ker\mu$ appearing in (i) is called the kernel of the MNC μ.

If $ker\mu = \mathfrak{N}_E$ then the MNC μ is called full.

Let us note that the set X_∞ from axiom (vi) is an element of the kernel $ker\mu$. It follows immediately from the inequality $\mu(X_\infty) \leqslant \mu(X_n)$ for $n = 1, 2, \dots$. Hence we infer that $\mu(X_\infty) = 0$ and consequently $X_\infty \in ker\mu$. This simple observation plays a significant role in applications of the technique connected with MNCs.

Further, assume that μ is a MNC in the space E. The measure μ is called sublinear [12] if it satisfies the following conditions:

(vii) $\mu(X + Y) \leqslant \mu(X) + \mu(Y)$.

(viii) $\mu(\lambda X) = |\lambda|\mu(X)$, for $\lambda \in \mathbb{R}$.

If μ satisfies the condition

(ix) $\mu(X \cup Y) = max\{\mu(X), \mu(Y)\}$

then it is called the MNC with the maximum property. If μ is a full and sublinear MNC, which has the maximum property, then it is called regular [12].

It is worthwhile mentioning that the first MNC was defined by K. Kuratowski [13]. Nevertheless, the most important and useful MNC is the so-called Hausdorff (or ball) MNC defined in [14,15] with the help of the following formula

$$\chi(X) = \inf\{\varepsilon > 0 : \ X \text{ has a finite } \varepsilon - \text{net in } E\},$$

for $X \in \mathfrak{M}_E$. It can be shown that χ is a regular MNC [12]. Let us notice that in some Banach spaces such as c_0, l_p $(1 \leqslant p < \infty)$, $C([a,b])$ we can give formulas expressing χ in connection with the structure of these Banach spaces (cf. Akhmerov et al. [16], Ayerbe et al. [17], Banaś and Goebel [12]). On the other hand, there are Banach spaces such as c or $L^p(a,b)$ in which we know formulas for regular MNCs are equivalent to the Hausdorff MNC χ in the mentioned spaces (cf. Banaś and Goebel [12]).

Moreover, let us pay attention to the fact that in some Banach spaces, there exist regular MNCs that are not equivalent to the Hausdorff MNC χ (cf. Ablet et al. [18], Mallet-Paret and Nussbaum [19]).

Let us point out that in a lot of Banach spaces, we are not in a position to construct formulas expressing the Hausdorff MNC χ or MNCs equivalent to χ. In such a situation, we have to restrict ourselves to MNCs in the sense of Definition 1, in which they are not even full.

Now, we recall the fixed point theorem of the Darbo type utilizing the concept of an MNC (cf. Banaś and Goebel [12], Darbo [20]). That theorem will be important in our further considerations.

Theorem 1. *Let Ω be a nonempty, bounded, closed, and convex subset of a Banach space E. Assume that $Q : \Omega \to \Omega$ is a continuous operator and there exists a constant $k \in [0,1)$ such that $\mu(QX) \leqslant k\mu(X)$ for any nonempty subset X of Ω, where μ is an MNC in the space E. Then, Q has at least one fixed point in the set Ω.*

Notice that it can be shown (see Banaś and Goebel [12]) that the set $FixQ$ of fixed points of the operator Q belonging to Ω is an element of the kernel $ker\mu$. This simple observation allows us to characterize solutions of considered operator equations.

In what follows, we will work in the Banach space $BC(\mathbb{R}_+, E)$ consisting of functions defined, continuous, and bounded on $\mathbb{R}_+$ with values in a given Banach space E. Here, we will assume that in the space E, there is given an MNC μ, which, in general, is not equivalent to the Hausdorff MNC χ. If $x \in BC(\mathbb{R}_+, E)$ then we define the norm of x as

$$\| x \|_\infty = \sup\{\| x(t) \|_E : \ t \in \mathbb{R}_+\},$$

where $\| \cdot \|_E$ is a norm in the Banach space E.

We will also consider the space $C_T = C([0,T], E)$ where $T > 0$ is arbitrarily fixed. Obviously the space C_T consists of functions $x : [0,T] \to E$ being continuous on the interval $[0,T]$ and normed by the formula

$$\| x \|_T = \sup\{\| x(t) \|_E : \ t \in [0,T]\}.$$

Notice that if we take a function $x \in BC(\mathbb{R}_+, E)$ then the restriction $x|_{[0,T]}$ of x to the interval $[0,T]$ is an element of the space C_T.

Now, we are going to present the construction of the MNC in the space $BC(\mathbb{R}_+, E)$ (cf. Banaś et al. [21]). This MNC will be utilized in considerations conducted in the paper. Let us indicate that the mentioned MNC is associated with the investigations of conditions ensuring the existence of solutions of the IS of IEs (1) which are asymptotically stable.

Thus, let us take an arbitrary nonempty and bounded subset X of the space $BC(\mathbb{R}_+, E)$. Fix a function $x \in X$. For $\varepsilon > 0$ we define the quantity $\omega^\infty(x, \varepsilon)$ by putting

$$\omega^\infty(x,\varepsilon) = \sup\{\| x(t) - x(s) \|_E \colon \; t, s \in \mathbb{R}_+, \; |t-s| \leqslant \varepsilon\}.$$

Notice that $\lim_{\varepsilon \to 0} \omega^\infty(x, \varepsilon) = 0$ if and only if the function $x = x(t)$ is uniformly continuous on $\mathbb{R}_+$. On the other hand observe that for any $T > 0$ we have

$$\omega^T(x,\varepsilon) \leqslant \omega^\infty(x,\varepsilon),$$

where $\omega^T(x, \varepsilon)$ denotes the modulus of the restriction $x|_{[0,T]}$ in the space C_T i.e.,

$$\omega^T(x,\varepsilon) = \sup\{\| x(t) - x(s) \|_E \colon \; t, s \in [0,T], \; |t-s| \leqslant \varepsilon\}.$$

However, we will not use the modulus $\omega^T(x, \varepsilon)$ in this paper (cf. Banaś and Chlebowicz [10]).

Next, let us define:

$$\begin{aligned} \omega^\infty(X,\varepsilon) &= \sup\{\omega^\infty(x,\varepsilon) : \; x \in X\}, \\ \omega_0^\infty(X) &= \lim_{\varepsilon \to 0} \omega^\infty(X,\varepsilon). \end{aligned} \tag{2}$$

It is easily seen that $\omega_0^\infty(X) = 0$ if and only if functions from the set X are equicontinuous on the interval $\mathbb{R}_+$ or equivalently, functions from X are equiuniformly continuous on $\mathbb{R}_+$.

Next, let us consider the function $\overline{\mu}_\infty$ defined on the family $\mathfrak{M}_{BC(\mathbb{R}_+,E)}$ by the formula

$$\overline{\mu}_\infty(X) = \lim_{T \to \infty} \overline{\mu}_T(X), \tag{3}$$

where

$$\overline{\mu}_T(X) = \sup\{\mu(X(t)) : \; t \in [0,T]\}$$

and μ is an MNC given in the Banach space E.

Notice that the existence of the limit in (3) follows from the fact that the function $T \to \overline{\mu}_T(X)$ is nondecreasing and bounded on $\mathbb{R}_+$ (cf. Banaś et al. [21]).

Further, for arbitrarily fixed $t \in \mathbb{R}_+$ we define

$$\operatorname{diam} X(t) = \sup\{\| x(t) - y(t) \|_E \colon \; x, y \in X\}$$

and

$$c(X) = \limsup_{t \to \infty} \; \operatorname{diam} X(t). \tag{4}$$

Finally, taking into account (2)–(4), we define the following quantity

$$\mu_c(X) = \omega_0^\infty(X) + \overline{\mu}_\infty(X) + c(X) \tag{5}$$

(cf. Banaś et al. [21]).

It can be shown that the function μ_c defined by (5) is the MNC in the Banach space $BC(\mathbb{R}_+, E)$ (cf. Banaś et al. [21]). The kernel $ker\mu_c$, of this measure contains all nonempty and bounded subsets X of the space $BC(\mathbb{R}_+, E)$ which are equiuniformly continuous on $\mathbb{R}_+$ and such that all cross-sections $X(t)$ of X are elements of the kernel $ker\mu$ in the Banach space E. Moreover, the thickness of the bundle formed by graphs of functions from X tends to zero at infinity.

Let us also pay attention to the fact that the MNC μ_c is not full and does not have the maximum property. If we take the MNC μ to be sublinear in E then the MNC μ_c is also sublinear in the Banach space $BC(\mathbb{R}_+, E)$ (cf. Banaś et al. [21] for details).

In what follows, keeping in mind further applications of the MNC μ_c defined by (5), we will consider as the Banach space E the sequence space l_∞ consisting of all real sequences (x_n) being bounded. Obviously, we consider the space l_∞ with the classical supremum norm

$$\| x \| = \| (x_n) \| = \sup\{|x_n| : n = 1, 2, \dots\},$$

where $x = (x_n) \in l_\infty$.

It is worthwhile mentioning (Banaś and Goebel [12]) that we do not know formulas expressing the Hausdorff MNC χ in the space l_∞. Therefore, we are forced to consider the MNC defined by (5), where in the component defined by (3) we use one of MNCs constructed in the space l_∞ (cf. Banaś and Geobel [12], Banaś and Mursaleen [22], Akhmerov et al. [16]).

Now, we present the formula for the MNC used in the space $BC(\mathbb{R}_+, l_\infty)$ in our further considerations. For convenience, we will denote the space $BC(\mathbb{R}_+, l_\infty)$ by the symbol BC_∞.

Summing up, we consider the space BC_∞ consisting of functions $x : \mathbb{R}_+ \to l_\infty$ which are continuous and bounded on $\mathbb{R}_+$. Any such a function can be written in the form

$$x(t) = (x_n(t)) = (x_1(t), x_2(t), \dots)$$

for $t \in \mathbb{R}_+$, where the sequence $(x_n(t))$ is an element of the space l_∞ for any fixed t. The norm of the function $x = x(t) = (x_n(t))$ is defined with help of the equality

$$\| x \| = \sup\{\| x(t) \|_{l_\infty} : t \in \mathbb{R}_+\} = \sup_{t \in \mathbb{R}_+} \{\sup\{|x_n(t)| : n = 1, 2, \dots\}\}.$$

In what follows, we present the formula expressing the MNC μ_c in connection with an MNC in the space l_∞, which seems to be the most natural in our setting.

Thus, let us fix a set $X \in \mathfrak{M}_{BC_\infty}$. For $\varepsilon > 0$ and for an arbitrary function $x(t) = (x_n(t))$ belonging to the set X let us consider the modulus $\omega^\infty(x, \varepsilon)$ which now has the form

$$\begin{aligned} \omega^\infty(x, \varepsilon) &= \sup\{\| x(t) - x(s) \|_{l_\infty} : t, s \in \mathbb{R}_+, |t - s| \leqslant \varepsilon\} \\ &= \sup\{\sup\{|x_n(t) - x_n(s)| : n = 1, 2, \dots\} : t, s \in \mathbb{R}_+, |t - s| \leqslant \varepsilon\}. \end{aligned}$$

Then we obtain

$$\omega^\infty(X, \varepsilon) = \sup_{x \in X} \left\{ \sup \left\{ \sup_{n \in \mathbb{N}} |x_n(t) - x_n(s)| : t, s \in \mathbb{R}_+, |t - s| \leqslant \varepsilon \right\} \right\}.$$

Finally, in view of (2) we have

$$\begin{aligned} \omega_0^\infty(X) &= \lim_{\varepsilon \to 0} \omega^\infty(X, \varepsilon) \\ &= \lim_{\varepsilon \to 0} \left\{ \sup_{x \in X} \left\{ \sup \left\{ \sup_{n \in \mathbb{N}} |x_n(t) - x_n(s)| : t, s \in \mathbb{R}_+, |t - s| \leqslant \varepsilon \right\} \right\} \right\}. \end{aligned} \tag{6}$$

Now, to define the second term $\overline{\mu}_\infty$ of the MNC μ_c given by Formula (5), we will assume (as we indicated it above) that in the space l_∞ we take the MNC μ_3 defined on the family $\mathfrak{M}_{l_\infty}$ in the following way (see Banaś and Goebel [12], Banaś and Mursaleen [22]):

$$\mu_3(X) = \limsup_{n \to \infty} \operatorname{diam} X_n,$$

where

$$X_n = \{x_n : x = (x_i) \in X\}$$

and

$$\operatorname{diam} X_n = \sup\{|x_n - y_n| : x = (x_i),\ y = (y_i) \in X\}.$$

Now, keeping in mind the above formula and (3), for $X \in \mathfrak{M}_{BC_\infty}$ and for arbitrarily fixed $T > 0$ we obtain

$$\overline{\mu}^3_T(X) = \sup\left\{\mu^3(X(t)) :\ t \in [0,T]\right\}$$
$$= \sup_{t\in[0,T]}\left\{\limsup_{n\to\infty}\{\sup\{|x_n(t) - y_n(t)| :\ x = x(t),\ y = y(t) \in X\}\}\right\}.$$

Hence, we derive the following formula

$$\overline{\mu}^3_\infty(X) = \lim_{T\to\infty}\overline{\mu}^3_T(X)$$
$$= \lim_{T\to\infty}\left\{\sup_{t\in[0,T]}\left\{\limsup_{n\to\infty}\{\sup\{|x_n(t) - y_n(t)| :\ x = x(t),\ y = y(t) \in X\}\}\right\}\right\}. \tag{7}$$

Further, we define the third term of the MNC μ_c in the space BC_∞ given by Formula (4). Indeed, we obtain

$$c(X) = \limsup_{t\to\infty} \operatorname{diam} X(t)$$
$$= \limsup_{t\to\infty}\left\{\sup\left\{\sup_{n\in\mathbb{N}}|x_n(t) - y_n(t)| :\ x = x(t),\ y = y(t) \in X\right\}\right\}. \tag{8}$$

Now, based on Formulas (6)–(8) and taking into account Formula (5) expressing the MNC μ_c in the Banach space $BC(\mathbb{R}_+, E)$, we obtain the formula for the MNC in the space BC_∞ being counterpart of the MNC μ_3 mentioned above. In fact, this MNC has the form

$$\mu^3_c(X) = \omega^\infty_0(X) + \overline{\mu}^3_\infty(X) + c(X) \tag{9}$$

(cf. Banaś et al. [21]).

Let us point out that the function μ^3_c is the MNC in the Banach space BC_∞, which is sublinear but does not have the maximum property. Moreover, μ_c is not full.

The kernel $ker\mu^3_c$ is the family consisting of all nonempty and bounded subsets X of the space BC_∞ such that functions belonging to X are equiuniformly continuous on $\mathbb{R}_+$ and all cross-sections $X(t)$ of X are sets in l_∞ such that the thickness $X(t)$ tends to zero as $n \to \infty$, uniformly with respect to $t \in \mathbb{R}_+$. Moreover, the thickness of the bundle formed by graphs of functions from X tends to zero at infinity.

Let us mention that the MNC μ^3_c defined by (9) will be used in our considerations of the next sections of the paper.

3. Main Result

This section is devoted to investigating the solvability of the IS of the quadratic Hammerstein IEs (1).

Let us recall that the mentioned IS has the form

$$x_n(t) = a_n(t) + f_n(t, x_1(t), x_2(t), \dots)\int_0^\infty g_n(t,\tau)h_n(\tau, x_1(\tau), x_2(\tau), \dots)d\tau,$$

where $t \in \mathbb{R}_+$ and $n = 1, 2, \dots$.

As we indicated earlier, our considerations are located in the Banach space $BC_\infty = BC(\mathbb{R}_+, l_\infty)$. The main tool used in our study is the MNC μ^3_c defined by Formula (9).

Now, we present assumptions under which the IS if IEs (1) will be investigated.

(i) The sequence $(a_n(t))$ is an element of the space BC_∞. Apart from this, the functions $a_n = a_n(t)$ are equicontinuous on $\mathbb{R}_+$.

For further purposes we denote by A the norm of the function $(a_n(t))$ in the space BC_∞ i.e.,

$$A = \sup\{\sup\{|a_n(t)| : n = 1, 2, \dots\} : t \in \mathbb{R}_+\}.$$

(ii) The functions $g_n(t, \tau) = g_n : \mathbb{R}_+^2 \to \mathbb{R}$ are continuous on the set $\mathbb{R}_+^2$ $(n = 1, 2, \dots)$. Moreover, the functions $t \to g_n(t, \tau)$ are equicontinuous on the set $\mathbb{R}_+$ uniformly with respect to $\tau \in \mathbb{R}_+$ i.e., the following condition is satisfied

$$\underset{\varepsilon>0}{\forall}\ \underset{\delta>0}{\exists}\ \underset{n\in\mathbb{N}}{\forall}\ \underset{\tau\in\mathbb{R}_+}{\forall}\ \underset{t_1,t_2\in\mathbb{R}_+}{\forall} [|t_2 - t_1| \leqslant \delta \Rightarrow |g_n(t_2, \tau) - g_n(t_1, \tau)| \leqslant \varepsilon].$$

(iii) For any $n \in \mathbb{N}$ and for each $t \in \mathbb{R}_+$ the improper integral

$$\int_0^\infty |g_n(t, \tau)| d\tau$$

is convergent. Moreover, the integrals $\int_0^\infty |g_n(t, \tau)| d\tau$ are equibounded for any $n = 1, 2, \dots$ and for each $t \in \mathbb{R}_+$.

In what follows we denote by G_1 the finite constant defined by the equality

$$G_1 = \sup\left\{\int_0^\infty |g_n(t, \tau)| d\tau : n = 1, 2, \dots, t \in \mathbb{R}_+\right\}.$$

(iv) The sequence $(g_n(t, \tau))$ is equibounded on $\mathbb{R}_+^2$ i.e., there exists a constant $G_2 > 0$ such that $|g_n(t, \tau)| \leqslant G_2$ for $t, \tau \in \mathbb{R}_+$ and $n = 1, 2, \dots$.

(v) The functions f_n are defined on the set $\mathbb{R}_+ \times \mathbb{R}^\infty$ and take real values for $n = 1, 2, \dots$. Apart from this, the functions $t \to f_n(t, x_1, x_2, \dots)$ are equicontinuous on $\mathbb{R}_+$ uniformly with respect to $x = (x_n) \in l_\infty$ i.e., the following condition is satisfied

$$\underset{\varepsilon>0}{\forall}\ \underset{\delta>0}{\exists}\ \underset{(x_i)\in l_\infty}{\forall}\ \underset{n\in\mathbb{N}}{\forall}\ \underset{t_1,t_2\in\mathbb{R}_+}{\forall} [|t_2 - t_1| \leqslant \delta \Rightarrow |f_n(t_2, x_1, x_2, \dots) - f_n(t_1, x_1, x_2, \dots)| \leqslant \varepsilon].$$

(vi) There exists a function $k : \mathbb{R}_+ \to \mathbb{R}_+$ which is nondecreasing on $\mathbb{R}_+$, $k(0) = 0$ and continuous at 0. Moreover, the following condition is satisfied

$$|f_n(t, x_1, x_2, \dots) - f_n(t, y_1, y_2, \dots)| \leqslant k(r) \sup\{|x_i - y_i| : i \geqslant n\}$$

for any $r > 0$, for $x = (x_i)$, $y = (y_i) \in l_\infty$ such that $\| x \|_{l_\infty} \leqslant r$, $\| y \|_{l_\infty} \leqslant r$ and for all $t \in \mathbb{R}_+$ and $n = 1, 2, \dots$.

(vii) The sequence of functions $\left(\overline{f}_n\right)$, where $\overline{f}_n(t) = |f_n(t, 0, 0, \dots)|$ is an element of the space BC_∞.

Notice that on the basis of assumption (vii), we infer that we can define the finite constant
$\overline{F} = \sup\left\{\overline{f}_n(t) : t \in \mathbb{R}_+, n = 1, 2, \dots\right\}$.

Now, we formulate other assumptions concerning IS (1).

(viii) The functions h_n are defined on the set $\mathbb{R}_+ \times \mathbb{R}^\infty$ and take real values for $n = 1, 2, \dots$. Moreover, there exists a function $m : \mathbb{R}_+ \to \mathbb{R}_+$, which is nondecreasing on $\mathbb{R}_+$, continuous at $r = 0$, $m(0) = 0$ and such that the following condition is satisfied

$$|h_n(t, x_1, x_2, \dots) - h_n(t, y_1, y_2, \dots)| \leqslant m(r) \sup\{|x_i - y_i| : i \geqslant n\}$$

for any $r > 0$, for $x = (x_i)$, $y = (y_i) \in l_\infty$ such that $\| x \|_{l_\infty} \leqslant r$, $\| y \|_{l_\infty} \leqslant r$ and for all $t \in \mathbb{R}_+$ and $n = 1, 2, \dots$.

(ix) The operator h defined on the set $\mathbb{R}_+ \times l_\infty$ by the formula

$$(hx)(t) = (h_n(t, x)) = (h_1(t, x), h_2(t, x), \dots)$$

is bounded, i.e., there exists a positive constant $\overline{h}$ such that $\| (hx)(t) \|_{l_\infty} \leqslant \overline{h}$ for any $x \in l_\infty$ and for each $t \in \mathbb{R}_+$.

(x) For any $n \in \mathbb{N}$ and for each function $x = x(t) = (x_i(t)) \in BC_\infty$ the improper integral

$$\int_0^\infty |h_n(s, x(s))|ds = \int_0^\infty |h_n(s, x_1(s), x_2(s), \dots)|ds$$

is convergent. Moreover, the integrals $\int_0^\infty |h_n(s, x(s))|ds$ are equibounded for $n \in \mathbb{N}$ and for each $x = x(t) \in BC_\infty$.

In view of the above assumption, we can define the finite constant $\overline{H}$ by putting

$$\overline{H} = \sup\left\{ \int_0^\infty |h_n(s, x(s))|ds : \ x \in BC_\infty, \ n = 1, 2, \dots \right\}.$$

(xi) There exists a positive number r_0 which satisfies the inequality

$$\overline{A} + \overline{F}G_1\overline{h} + G_1\overline{h}rk(r) \leqslant r$$

and such that

$$G_1\overline{h}k(r_0) + (r_0k(r_0) + \overline{F})G_1m(r_0) < 1,$$

where the constants $\overline{F}, \ G_1, \ \overline{h}$ were defined above and the constant A was defined in assumption (i).

Remark 1. *Let us notice that on the basis of assumption (vi) we conclude that for $x = (x_i), \ y = (y_i) \in l_\infty$ such that $\| x \|_{l_\infty} \leqslant r, \ \| y \|_{l_\infty} \leqslant r$ and for $t \in \mathbb{R}_+, \ n \in \mathbb{N}$, the following inequality holds*

$$|f_n(t, x_1, x_2, \dots) - f_n(t, y_1, y_2, \dots)| \leqslant k(r) \| x - y \|_{l_\infty},$$

where $k = k(r)$ is the function appearing in assumption (vi).

In the same way, from assumption (viii) we deduce that

$$|h_n(t, x_1, x_2, \dots) - h_n(t, y_1, y_2, \dots)| \leqslant m(r) \| x - y \|_{l_\infty},$$

for $t \in \mathbb{R}_+, \ n \in \mathbb{N}$ and for $r > 0$, where $x = (x_i), \ y = (y_i) \in l_\infty$ are such that $\| x \|_{l_\infty} \leqslant r$, $\| y \|_{l_\infty} \leqslant r$. Moreover, the function $m = m(r)$ is involved in assumption (viii).

Let us observe that the above remark allows us to infer that assumptions (vi) and (viii) are essentially stronger than the assumption requiring that the functions f_n and h_n satisfy the classical Lipschitz condition with the functions $k(r)$ and $m(r)$.

Now, we are in a position to formulate our main existence result concerning IS of IEs (1).

Theorem 2. *Under assumptions (i)-(xi) the IS of IEs (1) has at least one solution $x = (x_n(t))$ in the space $BC_\infty = BC(\mathbb{R}_+, l_\infty)$. Moreover, solutions of IS of IEs (1) are such that the thickness of the bundle formed by graphs of functions belonging to those solutions tends to zero at infinity.*

Proof. At the beginning we define three operators $F, \ H, \ Q$ on the space BC_∞ in the following way:

$$(Fx)(t) = ((F_nx)(t)) = (f_n(t, x(t))) = (f_n(t, x_1(t), x_2(t), \dots)),$$

$$(Hx)(t) = ((H_nx)(t)) = \left(\int_0^\infty g_n(t, \tau)h_n(\tau, x_1(\tau), x_2(\tau), \dots)d\tau \right),$$

$$(Qx)(t) = ((Q_nx)(t)) = (a_n(t) + (F_nx)(t)(H_nx)(t)).$$

At first, we show that the operator F acts from the space BC_∞ into itself.

To prove this fact let us choose a function $x = x(t) = (x_n(t)) \in BC_\infty$.

Next, let us fix a natural number n and $t \in \mathbb{R}_+$. Then, keeping in mind the imposed assumptions and Remark 1, we obtain

$$\begin{aligned} |(F_n x)(t)| &\leqslant |f_n(t, x_1(t), x_2(t), \dots) - f_n(t, 0, 0, \dots)| + |f_n(t, 0, 0, \dots)| \\ &\leqslant k(||x(t)||_{l_\infty}) \sup\{|x_i(t)| : i \geqslant n\} + |\overline{f}_n(t)| \leqslant k(||x||_{BC_\infty})||x||_{BC_\infty} + \overline{F}. \end{aligned} \tag{10}$$

Further, we show that the function Fx is continuous on $\mathbb{R}_+$.

To this end, we will utilize the continuity of the function $x = x(t) = (x_n(t)) \in BC_\infty$ on the interval $\mathbb{R}_+$. Indeed, this means that the following condition is satisfied:

$$\underset{t_0 \in \mathbb{R}_+}{\forall} \ \underset{\varepsilon > 0}{\forall} \ \underset{\delta > 0}{\exists} \ \underset{t \in \mathbb{R}_+}{\forall} [|t - t_0| \leqslant \delta \Rightarrow ||x(t) - x(t_0)||_{l_\infty} \leqslant \varepsilon].$$

Thus, let us take $\varepsilon > 0$ and $t_0 \in \mathbb{R}_+$. Next, choose $\delta > 0$ according to the above condition. Then, for $t \in \mathbb{R}_+$ such that $|t - t_0| \leqslant \delta$, in view of Remark 1 we have:

$$\begin{aligned} |(F_n x)(t) - (F_n x)(t_0)| &\leqslant |f_n(t, x_1(t), x_2(t), \dots) - f_n(t_0, x_1(t), x_2(t), \dots)| \\ &\quad + k(||x(t)||_{l_\infty})||x(t) - x(t_0)||_{l_\infty} \\ &\leqslant |f_n(t, x_1(t), x_2(t), \dots) - f_n(t_0, x_1(t), x_2(t), \dots)| + k(||x||_{BC_\infty})\varepsilon. \end{aligned} \tag{11}$$

Now, taking into account assumption (v), we can find a number $\delta > 0$ such that

$$|f_n(t, x_1(t), x_2(t), \dots) - f_n(t_0, x_1(t), x_2(t), \dots)| \leqslant \varepsilon$$

for $|t - t_0| \leqslant \delta$ and for $n = 1, 2, \dots$. Joining this fact with (11) we obtain the following estimate

$$|(F_n x)(t) - (F_n x)(t_0)| \leqslant (1 + k(||x||_{BC_\infty}))\varepsilon,$$

for $n = 1, 2, \dots$ and for $t \in \mathbb{R}_+$ such that $|t - t_0| \leqslant \delta$.

The above reasoning shows that the function Fx is continuous at the point t_0. Keeping in mind that t_0 was chosen arbitrarily, we deduce that Fx is continuous on $\mathbb{R}_+$. Linking this property with the boundedness of Fx, which was established earlier, we conclude that the operator F acts from the space BC_∞ into itself.

Next, we show that the operator H defined above maps the space BC_∞ into itself. To prove this fact let us take a function $x = x(t) = (x_n(t)) \in BC_\infty$. Then, for a fixed number $t \in \mathbb{R}_+$ and for $n \in \mathbb{N}$, in virtue of assumptions (iii) and (ix), we obtain

$$\begin{aligned} |(Hx)(t)| &\leqslant \int_0^\infty |g_n(t, \tau)||h_n(\tau, x_1(\tau), x_2(\tau), \dots)| d\tau \\ &\leqslant \int_0^\infty |g_n(t, \tau)|\overline{h} d\tau \leqslant \overline{h} \int_0^\infty |g_n(t, \tau)| d\tau \leqslant G_1 \overline{h}. \end{aligned} \tag{12}$$

The obtained estimate shows that the function Hx is bounded on the interval $\mathbb{R}_+$.

Now, let us fix $\varepsilon > 0$ and choose a number $\delta > 0$ according to assumption (ii). Then, for arbitrary numbers $t_1, t_2 \in \mathbb{R}_+$ such that $|t_2 - t_1| \leqslant \delta$, based on assumptions (ii) and (ix) (taking, for example, that $t_1 < t_2$) we obtain

$$\begin{aligned} &|(H_n x)(t_2) - (H_n x)(t_1)| \\ &\leqslant \left| \int_0^\infty g_n(t_2, \tau) h_n(\tau, x_1(\tau), x_2(\tau), \dots) d\tau - \int_0^\infty g_n(t_1, \tau) h_n(\tau, x_1(\tau), x_2(\tau), \dots) d\tau \right| \\ &\leqslant \int_0^\infty |g_n(t_2, \tau) - g_n(t_1, \tau)||h_n(\tau, x_1(\tau), x_2(\tau), \dots)| d\tau \\ &\leqslant \int_0^\infty \omega_g(\delta)|h_n(\tau, x_1(\tau), x_2(\tau), \dots)| d\tau, \end{aligned}$$

where $\omega_g(\delta)$ denotes the common modulus of equicontinuity of the sequence of functions $t \to g_n(t,\tau)$ (according to assumption (ii)). Obviously we have that $\omega_g(\delta) \to 0$ as $\delta \to 0$.

Further, let us notice that applying assumption (x), from the above derived estimate we obtain

$$|(H_n x)(t_2) - (H_n x)(t_1)| \leqslant \overline{H}\omega_g(\delta). \tag{13}$$

Hence, we obtain

$$||(Hx)(t_2) - (Hx)(t_1)||_{l_\infty} \leqslant \overline{H}\omega_g(\delta).$$

The above estimate shows that the function Hx is continuous on the interval $\mathbb{R}_+$. Linking this fact with the boundedness of the function Hx on $\mathbb{R}_+$, we deduce that the operator H maps the space BC_∞ into itself.

Now, in view of the fact that the space BC_∞ is a Banach algebra with respect to the coordinatewise multiplication of function sequences and taking into account the definition of the operator Q and assumption (i) we conclude that for arbitrarily fixed function $x = x(t) \in BC_\infty$ the function $(Qx)(t) = ((Q_n x)(t)) = (a_n(t) + (F_n x)(t)(H_n x)(t))$ transforms the interval $\mathbb{R}_+$ into the space l_∞. Indeed, in virtue of the fact that $((F_n x)(t)) \in l_\infty$ for any $t \in \mathbb{R}_+$ and in view of the estimate (12) we obtain

$$|(Q_n x)(t)| \leqslant |a_n(t)| + G_1\overline{h}|(F_n x)(t)|$$

for any $n \in \mathbb{N}$. Hence, on the base of (10) we infer that $((Qx)(t)) = ((Q_n x)(t)) \in l_\infty$ for any $t \in \mathbb{R}_+$.

Next, let us observe that the continuity of the function Qx in the interval $\mathbb{R}_+$ is a consequence of the continuity of the functions Fx and Hx on $\mathbb{R}_+$. In a similar way, we derive the boundedness of the function Qx on $\mathbb{R}_+$, provided we pay attention to the assumption (i).

Summing up, gathering all the above-established properties of the function Qx, we deduce that the operator Q transforms the space BC_∞ into itself.

In what follows, let us notice that keeping in mind estimates (10) and (12), for arbitrarily fixed $n \in \mathbb{N}$ and $t \in \mathbb{R}_+$, we obtain

$$\begin{aligned}
|(Q_n x)(t)| &\leqslant |a_n(t)| + |(F_n x)(t)||(H_n x)(t)| \\
&\leqslant A + \left(k(||x(t)||_{l_\infty})||x(t)||_{l_\infty} + \overline{F}\right)G_1\overline{h} \\
&\leqslant A + \overline{F}G_1\overline{h} + G_1\overline{h}k(||x||_{BC_\infty})||x||_{BC_\infty}.
\end{aligned}$$

Hence, we derive the following estimate

$$||Qx||_{BC_\infty} \leqslant A + \overline{F}G_1\overline{h} + G_1\overline{h}k(||x||_{BC_\infty})||x||_{BC_\infty}.$$

The above estimate and the first inequality from assumption (xi) yields that there exists a number $r_0 > 0$ such that the operator Q transforms the ball B_{r_0} ($B_{r_0} \subset BC_\infty$) into itself.

Further on, we are going to show that the operator Q is continuous on the ball B_{r_0}. To this end, let us notice that taking into account the representation of the operator Q given at the beginning of the proof, it is sufficient to show the continuity of the operators F and H separately.

Thus, let us take an arbitrary number $\varepsilon > 0$ and choose $x \in B_{r_0}$. Further, for an arbitrary point $y \in B_{r_0}$ such that $||x - y||_{BC_\infty} \leqslant \varepsilon$ and for $n \in \mathbb{N}$, $t \in \mathbb{R}_+$, in view of assumption (vi) and Remark 1, we have

$$\begin{aligned}
|(F_n x)(t) - (F_n y)(t)| &\leqslant |f_n(t, x_1(t), x_2(t), \dots) - f_n(t, y_1(t), y_2(t), \dots)| \\
&\leqslant k(r_0)||x - y||_{BC_\infty} \leqslant k(r_0)\varepsilon.
\end{aligned}$$

Hence, we obtain

$$||Fx - Fy||_{BC_\infty} \leqslant k(r_0)\varepsilon.$$

On the basis of the above estimate, we derive the desired continuity of the operator F on the ball B_{r_0}.

Now, let us take arbitrary points $x = (x_i)$, $y = (y_i) \in B_{r_0}$. Then, keeping in mind assumption (viii), for fixed $t \in \mathbb{R}_+$ and $n \in \mathbb{N}$, we obtain

$$\begin{aligned}
&|(H_n x)(t) - (H_n y)(t)| \\
\leqslant \int_0^\infty |g_n(t,\tau)||h_n(\tau, x_1(\tau), x_2(\tau), \dots) &- h_n(\tau, y_1(\tau), y_2(\tau), \dots)|d\tau \\
&\leqslant \int_0^\infty |g_n(t,\tau)|m(r_0) \sup\{|x_i(\tau) - y_i(\tau)| : i \geqslant n\} d\tau \\
&\leqslant m(r_0) \int_0^\infty |g_n(t,\tau)|(||x(\tau) - y(\tau)||_{l_\infty}) d\tau \\
&\leqslant m(r_0) \sup\{||x(s) - y(s)||_{l_\infty} : s \in \mathbb{R}_+\} \int_0^\infty |g_n(t,\tau)| d\tau.
\end{aligned}$$

Hence, taking into account assumption (iii), we obtain the following inequality

$$|(H_n x)(t) - (H_n y)(t)| \leqslant G_1 m(r_0) ||x - y||_{BC_\infty}.$$

This implies

$$||Hx - Hy||_{BC_\infty} \leqslant G_1 m(r_0) ||x - y||_{BC_\infty}.$$

From the above estimate we infer that the operator H is continuous on the ball B_{r_0}.

In what follows, we will study the behavior of the operators F, H, and Q with respect to the components of the MNC defined by the Formula (5). Let us recall that those components are defined successively by Formulas (2), (3) and (4) (cf. also the extensions of those formulas given by (6), (7) and (8)). To realize our goal, let us fix an arbitrary number $\varepsilon > 0$. Next, choose $t, s \in \mathbb{R}_+$ such that $|t - s| \leqslant \varepsilon$ and take a nonempty subset X of the ball B_{r_0}. Then, for a function $x = x(t) = (x_n(t)) \in X$ and for a fixed natural number n, in the similar way as in (12), we obtain

$$\begin{aligned}
|(F_n x)(t) - (F_n x)(s)| &\leqslant k(r_0) \sup\{|x_i(t) - x_i(s)| : i \geqslant n\} \\
+ \sup\{|f_n(t, x_1, x_2, \dots) - f_n(s, x_1, x_2, \dots)| &: |t - s| \leqslant \varepsilon, ||x||_{l_\infty} = ||(x_n)||_{l_\infty} \leqslant r_0\} \\
&\leqslant k(r_0)\omega^\infty(x, \varepsilon) + \omega^1_\infty(f, \varepsilon),
\end{aligned}$$

where

$$\omega^1_\infty(f, \varepsilon) = \sup_{n \in \mathbb{N}} \{\sup\{|f_n(t, x_1, x_2, \dots) - f_n(s, x_1, x_2, \dots)| : |t - s| \leqslant \varepsilon, ||x||_{l_\infty} = ||(x_n)||_{l_\infty} \leqslant r_0\}\}.$$

Obviously, taking into account assumption (v) we conclude that $\omega^1_\infty(f, \varepsilon) \to 0$ as $\varepsilon \to 0$.

Now, from the last estimate we infer that

$$\omega^\infty(Fx, \varepsilon) \leqslant k(r_0)\omega^\infty(x, \varepsilon) + \omega^1_\infty(f, \varepsilon). \tag{14}$$

Further on, let us notice that utilizing assumptions (ii), (ix), (x) and assuming additionally that $s < t$, in a similar way as in (14), we can obtain the following estimate

$$|(H_n x)(t) - (H_n x)(s)| \leqslant \overline{H}\omega_g(\varepsilon),$$

where the quantity $\omega_g(\varepsilon)$ was defined earlier as the common modulus of equicontinuity of the sequence of functions $t \to g_n(t, \tau)$. Let us recall that $\omega_g(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Obviously, from the above estimate, we obtain the following one

$$\omega^\infty(Hx, \varepsilon) \leqslant \overline{H}\omega_g(\varepsilon). \tag{15}$$

Now, for a fixed function $x \in X$ and for arbitrary numbers $t, s \in \mathbb{R}_+$, in virtue of the representation of the operator Q, we obtain

$$||(Qx)(t) - (Qx)(s)||_{l_\infty} \leqslant ||a(t) - a(s)||_{l_\infty} + ||(Hx)(t)||_{l_\infty}||(Fx)(t) - (Fx)(s)||_{l_\infty} \\ +||(Fx)(s)||_{l_\infty}||(Hx)(t) - (Hx)(s)||_{l_\infty},$$

where we put $a(t) = (a_n(t))$.

Next, let us fix $\varepsilon > 0$ and assume that $|t - s| \leqslant \varepsilon$. Then, keeping in mind (12), (14), (10) and (15), from the above inequality we obtain

$$\omega^\infty(Qx, \varepsilon) \leqslant \omega^\infty(a, \varepsilon) + G_1\overline{h}\Big[k(r_0)\omega^\infty(x, \varepsilon) + \omega^1_\infty(f, \varepsilon)\Big] \\ +\big(r_0k(r_0) + \overline{F}\big)\overline{H}\omega_g(\varepsilon).$$

Hence, taking into account the above established properties of the functions $\varepsilon \to \omega^1_\infty(f, \varepsilon)$, $\varepsilon \to \omega_g(\varepsilon)$ and assumption (i), we derive the following inequality

$$\omega^\infty_0(QX) \leqslant G_1\overline{h}k(r_0)\omega^\infty_0(X). \tag{16}$$

Now, we are going to consider the second component of the MNC μ^3_c defined by Formula (9). Recall that the mentioned term is denoted by $\overline{\mu}^3_\infty$ and is given by Formula (7). To this end let us fix a nonempty set $X \subset B_{r_0}$ and choose arbitrary functions $x = x(t)$, $y = y(t) \in X$. Then, for fixed $t \in \mathbb{R}_+$ and $n \in \mathbb{N}$, we obtain:

$$\begin{aligned} |(Q_nx)(t) - (Q_ny)(t)| &\leqslant |(F_nx)(t)(H_nx)(t) - (F_ny)(t)(H_ny)(t)| \\ &\leqslant |(H_nx)(t)||(F_nx)(t) - (F_ny)(t)| + |(F_ny)(t)||(H_nx)(t) - (H_ny)(t)|. \end{aligned} \tag{17}$$

Further, we intend to estimate the components on the right-hand side of inequality (17). To realize this goal, let us fix a natural number n and a number $T > 0$. Then. for $t \in [0, T]$ and for $k \in \mathbb{N}$, $k \geqslant n$, in view of assumptions (viii) and (iii), for arbitrarily fixed functions $x, y \in X$, we obtain

$$\int_0^\infty |g_k(t,\tau)||h_k(\tau, x_1(\tau), x_2(\tau), \ldots) - h_k(\tau, y_1(\tau), y_2(\tau), \ldots)|d\tau \\ \leqslant m(r_0)\int_0^\infty |g_k(t,\tau)|(\sup\{|x_i(\tau) - y_i(\tau)| :\ i \geqslant k\})d\tau \\ \leqslant m(r_0)\int_0^\infty |g_k(t,\tau)|\left\{\sup_{t\in[0,T]}\left\{\sup_{i\geqslant k}|x_i(t) - y_i(t)|\right\}\right\}d\tau \\ \leqslant G_1m(r_0)\left\{\sup_{t\in[0,T]}\left\{\sup_{i\geqslant k}\{\sup\{|x_i(t) - y_i(t)| :\ x = x(t),\ y = y(t) \in X\}\}\right\}\right\}.$$

From the above estimate, we obtain

$$\sup_{t\in[0,T]}\left\{\sup_{k\geqslant n}\{\sup\{|(H_kx)(t) - (H_ky)(t)| :\ x = x(t),\ y = y(t) \in X\}\}\right\} \\ \leqslant G_1m(r_0)\left\{\sup_{t\in[0,T]}\left\{\sup_{i\geqslant n}\{\sup\{|x_i(t) - y_i(t)| :\ x = x(t),\ y = y(t) \in X\}\}\right\}\right\}.$$

Hence, keeping in mind Formula (7), we obtain the following inequality

$$\overline{\mu}^3_\infty(HX) \leqslant G_1m(r_0)\overline{\mu}^3_\infty(X). \tag{18}$$

In the similar way as above, for arbitrarily fixed $n \in \mathbb{N}$, $t \in \mathbb{R}_+$ and for $x = x(t)$, $y = y(t) \in X$, taking into account assumption (vi), we obtain

$$|(F_n x)(t) - (F_n y)(t)| \leqslant k(r_0) \sup\{|x_i(t) - y_i(t)| : i \geqslant n\}.$$

This yields the following estimate

$$\sup_{t\in[0,T]}\left\{\sup_{i\geqslant n}\{\sup\{|(F_i x)(t) - (F_i y)(t)| : x = x(t),\ y = y(t) \in X\}\}\right\}$$
$$\leqslant k(r_0)\left\{\sup_{t\in[0,T]}\left\{\sup_{i\geqslant n}\{\sup\{|x_i(t) - y_i(t)| : x = x(t),\ y = y(t) \in X\}\}\right\}\right\}.$$

Now, taking into account the above estimate and Formula (7), we derive the following inequality

$$\overline{\mu}^3_\infty(FX) \leqslant k(r_0)\overline{\mu}^3_\infty(X). \tag{19}$$

Finally, linking estimates (17), (12), (10), (18) and (19), we obtain

$$\overline{\mu}^3_\infty(QX) \leqslant G_1\overline{h}k(r_0)\overline{\mu}^3_\infty(X) + (r_0 k(r_0) + \overline{F})G_1 m(r_0)\overline{\mu}^3_\infty(X). \tag{20}$$

In what follows, we will investigate the third component of the MNC μ^3_c defined by (9) i.e., the term $c(X)$ expressed by Formula (8). To this end, let us fix a nonempty subset X of B_{r_0}. Let us take functions $x = x(t)$, $y = y(t) \in X$. Further, fix $T > 0$ and take $t \geqslant T$. Then, for an arbitrary number $n \in \mathbb{N}$, based on calculations performed before estimate (18), we obtain

$$|(H_n x)(t) - (H_n y)(t)| \leqslant G_1 m(r_0)\left\{\sup_{t\geqslant T}\left\{\sup_{i\geqslant n}|x_i(t) - y_i(t)|\right\}\right\}.$$

The above estimate leads to the following inequality

$$\sup_{t\geqslant T}\left\{\sup\left\{\sup_{n\in\mathbb{N}}|(H_n x)(t) - (H_n y)(t)| : x = x(t),\ y = y(t) \in X\right\}\right\}$$
$$\leqslant G_1 m(r_0)\left\{\sup_{t\geqslant T}\left\{\sup\left\{\sup_{n\in\mathbb{N}}|x_n(t) - y_n(t)| : x = x(t),\ y = y(t) \in X\right\}\right\}\right\}.$$

Hence, we obtain

$$c(HX) \leqslant G_1 m(r_0) c(X). \tag{21}$$

In the sequel, arguing in the style of calculations preceded estimate (19), we derive the inequality

$$c(FX) \leqslant k(r_0)c(X). \tag{22}$$

Finally, combining estimates (17), (21), (22), (10) and (12), we arrive to the following inequality

$$c(QX) \leqslant G_1\overline{h}k(r_0)c(X) + (r_0 k(r_0) + \overline{F})G_1 m(r_0)c(X). \tag{23}$$

Now, linking estimates (16), (20), (23) and taking into account Formula (9) expressing the MNC μ^3_c, we obtain

$$\mu^3_c(QX) \leqslant G_1\overline{h}k(r_0)\omega^\infty_0(X)$$
$$+G_1\overline{h}k(r_0)\overline{\mu}^3_\infty(X) + \big(r_0 k(r_0) + \overline{F}\big)G_1 m(r_0)\overline{\mu}^3_\infty(X)$$
$$+G_1\overline{h}k(r_0)c(X) + \big(r_0 k(r_0) + \overline{F}\big)G_1 m(r_0)c(X).$$

Hence, we deduce the following estimate

$$\mu_c^3(QX) \leqslant \left[G_1\overline{h}k(r_0) + \left(r_0k(r_0) + \overline{F}\right)G_1m(r_0)\right]\mu_c^3(X).$$

Next, keeping in mind the above estimate, in virtue of the facts derived in the above-conducted proof, and taking into account assumption (xi) as well as Theorem 1, we infer that there exists at least one element $x \in B_{r_0}$ being a fixed point of the operator Q in the ball B_{r_0}. Obviously the function $x = x(t)$ is a solution of IS of IEs (1) in the space BC_∞.

Moreover, in view of the remark made after Theorem 1 and the description of the kernel of the MNC μ_c given after Formula (5), we conclude that the thickness of the bundle formed by graphs of solutions of IS of IEs (1) tends to zero at infinity. The proof is complete. □

The above proved theorem can be treated as the characterization of the set of solutions of IS of IEs (1) in terms of the concept of asymptotic stability. To show this fact, we adopt the definition of asymptotic stability accepted in the paper of Banaś and Rzepka [23] (cf. also Hu and Yan [24]).

Indeed, let us consider a nonempty subset Ω of the space $BC_\infty = BC(\mathbb{R}_+, l_\infty)$. Let Q be an operator defined on Ω with values in the space BC_∞.

Consider the operator equation having the form

$$x(t) = (Qx)(t), \;\; t \in \mathbb{R}_+. \tag{24}$$

Definition 2. *We say that solutions of equation (24) are asymptotically stable if there exists a ball $B(x_0, r)$ in the space BC_∞ with $B(x_0, r) \cap \Omega \neq 0$ such that for every $\varepsilon > 0$ there exists a number $T > 0$ with the property*

$$||x(t) - y(t)|| \leqslant \varepsilon$$

for all solutions x, y of equation (24) such that $x, y \in B(x_0, r) \cap \Omega$ and for $t \geqslant T$.

Notice that in the light of Definition 2, we can formulate Theorem 2 exposing the property of the asymptotic stability of solutions of IS of IEs (1).

In fact, we have the following version of the mentioned theorem.

Theorem 3. *Under assumptions (i)–(xi) the IS of IEs (1) has at least one solution $x = (x_n(t))$ in the space BC_∞. Moreover, solutions of the IS of IEs (1) are asymptotically stable.*

4. An Example

This section is dedicated to presenting an example that illustrates the applicability of our main result contained in Theorem 2 (cf. also Theorem 3).

Namely, we will consider the following IS of IEs:

$$\begin{aligned} x_n(t) &= \cos\left(\frac{n^2t+1}{t+n^2}\right) \\ &+ \left(\frac{2x_n(t)}{x_n^2(t)+n} + \frac{x_{n+1}^2(t)+1}{2n+3}\right)\int_0^\infty \frac{1}{\beta n + t^2 + \tau^2}\,\frac{\arctan(n + x_n^2(\tau))}{\gamma + 2n + \tau^2}\,d\tau, \end{aligned} \tag{25}$$

where $t \in \mathbb{R}_+$ and $n = 1, 2, \ldots$. Moreover, β and γ are positive constants which will be specified later.

Observe that IS of IEs (25) is a particular case of IS (1) if we put

$$a_n(t) = \cos\left(\frac{n^2t+1}{t+n^2}\right), \tag{26}$$

$$f_n(t,x_1,x_2,\dots)=\frac{2x_n}{x_n^2+n}+\frac{x_{n+1}^2+1}{2n+3}, \tag{27}$$

$$g_n(t,\tau)=\frac{1}{\beta n+t^2+\tau^2}, \tag{28}$$

$$h_n(t,x_1,x_2,\dots)=\frac{\arctan(n+x_n^2)}{\gamma+2n+\tau^2} \tag{29}$$

for $n=1,2,\dots$ and for $t\in\mathbb{R}_+$.

To show that IS of IEs (25) has a solution in the space BC_∞ we will apply Theorem 2. To this end we show that functions defined by (26)–(29) satisfy assumptions (i)–(xi) of that theorem.

Let us start with the observation that the function $a_n(t)$ given by (26) satisfies the Lipschitz condition with the constant $L=1$ for $n=1,2,\dots$. Hence, we deduce that these functions are equicontinuous on $\mathbb{R}_+$. Apart from this, we obtain

$$A=\sup\{|a_n(t)|:\ n=1,2,\dots,\ t\in\mathbb{R}_+\}=1.$$

Further on, let us observe that the functions $f_n=f_n(t,x_1,x_2,\dots)$ defined by (27) act from the set $\mathbb{R}_+\times\mathbb{R}^\infty$ into $\mathbb{R}$ for $n=1,2,\dots$. Since these functions do not depend explicitly on t, we infer that there is a satisfied assumption (v).

Next, let us fix a number $r>0$ and choose $x=(x_i)$, $y=(y_i)\in l_\infty$ such that $||x||_{l_\infty}\leqslant r$, $||y||_{l_\infty}\leqslant r$. Then, taking into account Formula (27), for an arbitrary number $n\in\mathbb{N}$, we have

$$\begin{aligned}
&|f_n(t,x_1,x_2,\dots)-f_n(t,y_1,y_2,\dots)|\\
&\leqslant 2\left|\frac{x_n}{n+x_n^2}-\frac{y_n}{n+y_n^2}\right|+\frac{1}{2n+3}\left|x_{n+1}^2-y_{n+1}^2\right|\\
&\leqslant \frac{5}{2}|x_n-y_n|+\frac{2}{5}r|x_{n+1}-y_{n+1}|\\
&\leqslant \max\left\{\frac{5}{2},\ \frac{2}{5}r\right\}\{|x_n-y_n|+|x_{n+1}-y_{n+1}|\}\\
&\leqslant 2\max\left\{\frac{5}{2},\ \frac{2}{5}r\right\}\sup\{|x_i-y_i|:\ i\geqslant n\}.
\end{aligned}$$

The above inequality implies that there is a satisfied assumption (vi), where we can put $k(r)=2\max\{\frac{5}{2},\ \frac{2}{5}r\}$.

Further, let us notice that $\overline{f}_n(t)=|f_n(t,0,0,\dots)|=0$. This implies that the functions f_n $(n=1,2,\dots)$ satisfy assumption (vii) with $\overline{F}=0$.

In what follows let us consider the functions $g_n(t,\tau)$ defined by (28). Observe that these functions are continuous on the set $\mathbb{R}_+^2$. Further, taking arbitrarily fixed numbers $t_1,t_2\in\mathbb{R}_+$ (without loss of generality we may assume that $t_1<t_2$) and $n\in\mathbb{N}$, $\tau\in\mathbb{R}_+$, we obtain the following estimates

$$\begin{aligned}
|g_n(t_2,\tau)-g_n(t_1,\tau)|&=\frac{1}{\beta n+\tau^2+t_1^2}-\frac{1}{\beta n+\tau^2+t_2^2}\\
=\frac{t_2^2-t_1^2}{(\beta n+\tau^2+t_1^2)(\beta n+\tau^2+t_2^2)}&\leqslant|t_2-t_1|\frac{t_1+t_2}{(\beta n+\tau^2+t_1^2)(\beta n+\tau^2+t_2^2)}\\
&=|t_2-t_1|\left(\frac{t_1}{\beta n+\tau^2+t_1^2}+\frac{t_2}{\beta n+\tau^2+t_2^2}\right)\\
&\leqslant\frac{1}{\sqrt{\beta n+\tau^2}}|t_2-t_1|\leqslant\frac{1}{\sqrt{\beta n}}|t_2-t_1|\leqslant\frac{1}{\sqrt{\beta}}|t_2-t_1|.
\end{aligned}$$

From the above estimate we conclude that the functions $g_n(t,\tau)$ $(n=1,2,\dots)$ satisfy assumption (ii).

Next, let us notice that the following inequality holds for arbitrary $t, \tau \in \mathbb{R}_+$ and for $n \in \mathbb{N}$:

$$|g_n(t,\tau)| = \frac{1}{\beta n + \tau^2 + t_2} \leqslant \frac{1}{\beta n} \leqslant \frac{1}{\beta}.$$

Thus, the functions $g_n(t,\tau)$ satisfy assumption (iv) with the constant $G_2 = \frac{1}{\beta}$.

Now, keeping in mind that the function $g_n(t,\tau)$ $(n = 1,2,\dots)$ is continuous on $\mathbb{R}_+^2$, for any arbitrarily fixed $n \in \mathbb{N}$ we obtain

$$\int_0^\infty |g_n(t,\tau)|d\tau = \int_0^\infty \frac{1}{\beta n + t^2 + \tau^2} d\tau = \frac{\pi}{2\sqrt{\beta n + t^2}} \leqslant \frac{\pi}{2\sqrt{\beta}}.$$

Hence, we derive that there is a satisfied assumption (iii) with the constant $G_1 = \frac{\pi}{2\sqrt{\beta}}$.

Further on, we intend to verify assumption (viii). To this end let us take $r > 0$ and choose arbitrary $x = (x_i)$, $y = (y_i) \in l_\infty$ such that $||x||_{l_\infty} \leqslant r$ and $||y||_{l_\infty} \leqslant r$. Then, for an arbitrary $t \in \mathbb{R}_+$ and $n \in \mathbb{N}$, in view of Formula (29), we obtain:

$$\begin{aligned} &|h_n(t,x_1,x_2,\dots) - h_n(t,y_1,y_2,\dots)| \\ &= \frac{1}{\gamma + 2n + t^2} |\arctan(n + x_n^2) - \arctan(n + y_n^2)| \\ &\leqslant \frac{1}{\gamma+2} |x_n^2 - y_n^2| \leqslant \frac{1}{\gamma+2} |x_n - y_n|(|x_n| + |y_n|) \\ &\leqslant \frac{1}{\gamma+2} 2r|x_n - y_n| \leqslant \frac{2r}{\gamma+2} \sup\{|x_i - y_i| : i \geqslant n\}. \end{aligned}$$

From the obtained estimate we infer that the functions $h_n(t,x_1,x_2,\dots)$ $(n = 1,2,\dots)$ satisfy assumption (vii) with the function $m(r) = \frac{2r}{\gamma+2}$.

Additionally, we obtain

$$|h_n(t,x_1,x_2,\dots)| \leqslant \frac{\pi}{2} \cdot \frac{1}{\gamma + 2n} \leqslant \frac{\pi}{2(\gamma+2)}$$

for $n \in \mathbb{N}$ and $t \in \mathbb{R}_+$. This yields that there is a satisfied assumption (ix) with the constant $\overline{h} = \frac{\pi}{2(\gamma+2)}$.

In what follows, fixing $n \in \mathbb{N}$ and keeping in mind Formula (29), for an arbitrary function $x = (x_n(t)) \in BC_\infty$ we obtain

$$\begin{aligned} &\int_0^\infty |h_n(s,x_1(s),x_2(s),\dots)|ds = \int_0^\infty \frac{\arctan(n + x_n^2(s))}{\gamma + 2n + s^2} ds \\ &\leqslant \frac{\pi}{2} \int_0^\infty \frac{ds}{\gamma + 2n + s^2} = \frac{\pi}{2} \cdot \frac{\pi}{2} \cdot \frac{1}{\sqrt{\gamma + 2n}} = \frac{\pi^2}{4} \cdot \frac{1}{\sqrt{\gamma + 2n}} \leqslant \frac{\pi^2}{4\sqrt{\gamma+2}}. \end{aligned}$$

This shows that there is satisfied assumption (x) with the constant $\overline{H} = \frac{\pi^2}{4\sqrt{\gamma+2}}$.

Finally, combining the calculated values of the constants $\overline{A}$, $\overline{F}$, G_1, G_2, $\overline{h}$, $\overline{H}$ and keeping in mind the formulas expressing the functions $k(r)$ and $m(r)$, we conclude that the first inequality from assumption (xi) has the form

$$1 + \frac{\pi^2}{2\sqrt{\beta}(\gamma+2)} \cdot r \cdot \max\left\{\frac{5}{2}, \frac{2}{5}r\right\} \leqslant r.$$

We can easily check that taking, for example $\beta = 36$, $\gamma = 34$ we convert the above inequality to the form

$$1 + \frac{\pi^2}{432} \cdot r \cdot \max\left\{\frac{5}{2}, \frac{2}{5}r\right\} \leqslant r. \tag{30}$$

Then, for $r = r_0 = 3$, we see that there is satisfied inequality (30).

On the other hand, the second inequality from assumption (xi) has the form

$$G_1 \overline{h} k(r_0) + r_0 k(r_0) G_1 m(r_0) < 1. \tag{31}$$

Taking into account the above established values of the constants G_1, $\overline{h}$ and the form of the functions $k(r)$ and $m(r)$ we obtain the following form of inequality (31):

$$\frac{\pi^2}{432} \max\left\{\frac{5}{2}, \frac{2}{5} r_0\right\} + \frac{\pi r_0^2}{108} \cdot \max\left\{\frac{5}{2}, \frac{2}{5} r_0\right\} < 1. \tag{32}$$

It is easily seen that the number $r_0 = 3$ satisfies inequality (32).

Now, taking into account Theorem 2, we infer that IS of IEs (25) has at least one solution $x(t) = (x_n(t))$ which belongs to the ball B_3 in the space BC_∞. Moreover, in the light of Theorem 3 all solutions of IS of IEs (25) belonging to the ball B_3 are asymptotically stable.

5. Discussion

We explain the new results of this study.

1°. To the best of our knowledge, taking into account the existing literature, there are no results concerning the existence of asymptotically stable solutions of an IS of IEs of Hammerstein type. In this regard, the results obtained in the paper are new and original.

2°. The basic tool used in the paper is the technique of suitable chosen measures of noncompactness. That technique is applied in the space BC_∞ of functions defined on the real half-axis $\mathbb{R}_+$ with values in the space of bounded real sequences. The space BC_∞ is very convenient in the study of ISs of IEs of various types.

3°. In our opinion, the results obtained in the paper can be generalized for ISs of IEs of Urysohn type. The results in this direction will appear elsewhere in due course.

6. Conclusions

Our study is rather developed since it discusses the subject of IS of IEs. It is significant that the study of an IE generally requires extensive and developed theory as well as the use of advanced tools of functional analysis.

In our investigations, we decided to use the advanced technique of nonlinear analysis depending on measures of noncompactness. Such an approach enables us to simplify the very extensive considerations expected in such a study.

The research presented above indicates that our direction of investigation seems suitable for realizing the goals of the paper.

The investigations conducted in the paper are realized through the broad description of the tools used in our study. Those tools are mainly determined by the concept of an MNC. Therefore, we first present the mentioned concept as well as the fixed point theorem of the Darbo type closely associated with the concept of MNCs. That theorem plays an essential role in our investigations.

The main part of our paper is fulfilled by detailed considerations connected with the solvability of IS of IEs of the Hammerstein type. All details of those considerations are presented step-by-step in the paper. Let us pay attention to the fact that it would be very interesting to provide a numerical simulation of the investigations of the paper associated with the existence of solutions of IS of IEs and the asymptotic stability of those solutions. However, such a task requires us to prepare a new, very extensive paper since we would consider the IS of the equations in question.

Such a paper will be prepared in the future and appear elsewhere.

Author Contributions: Conceptualization, J.B. and J.M.; Data curation, J.B. and J.M.; Formal analysis, J.B. and J.M.; Methodology, J.B. and J.M.; Project administration, J.B. and J.M.; Validation, J.M.; Visualization, J.B. and J.M.; Writing-original draft, J.M. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Data is contained within the article.

Conflicts of Interest: The authors declare no conflicts of interest.

Abbreviations

The following abbreviations are used in this manuscript:

IS	Infinite system
IE	Integral equation
IS of IEs	Infinite system of integral equations
MNC	Measure of noncompactness

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