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Solvable Two-Dimensional Dirac Equation with Matrix Potential: Graphene in External Electromagnetic Field

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Abstract: It is known that the excitations in graphene-like materials in external electromagnetic field are described by solutions of a massless two-dimensional Dirac equation which includes both Hermitian off-diagonal matrix and scalar potentials. Up to now, such two-component wave functions were calculated for different forms of external potentials, though as a rule depending on only one spatial variable. Here, we shall find analytically the solutions for a wide class of combinations of matrix and scalar external potentials which physically correspond to applied mutually orthogonal magnetic and longitudinal electrostatic fields, both depending really on two spatial variables. The main tool for this progress is provided by supersymmetrical (SUSY) intertwining relations, specifically, by their most general—asymmetrical—form proposed recently by the authors. This SUSY-like method is applied in two steps, similar to the second order factorizable (reducible) SUSY transformations in ordinary quantum mechanics.

Keywords: two-dimensional Dirac equation; SUSY intertwining relations; matrix potential; graphene in electromagnetic field

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1. Introduction

The extensive study of two-dimensional massless Dirac equations in the presence of external electromagnetic fields [1–7] is due to its connection with the properties of electron carriers in graphene and graphene-like materials [8–11]. The actual task is to find analytically the normalizable solutions of such a Dirac equation where two components of the "spinor" wave function $\Psi(\vec{x}) \equiv (\Psi_A(\vec{x}),\, \Psi_B(\vec{x}))$ correspond to two sublattices of graphene. The potentials in the equation have different origins: the off-diagonal matrix term is provided by the electromagnetic vector-potential $\vec{A} = (A_1(x_1,x_2), A_2(x_1,x_2), 0)$ in the "long derivatives" and leads to the magnetic field $\vec{B} = \vec{\nabla} \times \vec{A}$ along the z-axis, while the scalar potential $A_0(x_1,x_2)$ describes the interaction with electrostatic or some other scalar field:

$$[\sigma_1(-i\partial_1 - A_1(\vec{x})) + \sigma_2(-i\partial_2 - A_2(\vec{x})) + A_0(\vec{x})]\Psi(\vec{x}) = 0, \tag{1}$$

where two-dimensional $\vec{x} \equiv (x_1,\, x_2)$, the derivatives $\partial_i \equiv \frac{\partial}{\partial x_i}$, and the charge is taken as $e = 1$.

It is clear that the case of a pure magnetic field in a massless two-dimensional Dirac equation is explicitly solvable. Indeed, when multiplying (1) without the term A_0 by σ_1, we obtain a pair of decoupled first order equations which can be easily solved. The presence in (1) of a term proportional to the unity matrix σ_0 prevents this decoupling (an analogous problem appears in the presence of the mass term proportional to σ_3). Different methods have been used [12–23] to study such two-dimensional Dirac equations, mainly with strong restrictions on the conditions of the problem. There are problems with only electrostatic or only magnetic fields, as well as the problems with different specific one-dimensional ansatzes for external fields depending on a variable x_1 or radial variable r.

The supersymmetrical method inherent in Schrödinger Quantum Mechanics [24–26] has become one of the most effective tools in the discussed problems [27–33]. As a rule, the main ingredient of SUSY Quantum Mechanics, so-called SUSY intertwining relations, has been explored in different forms. In the present paper, a class of external electromagnetic fields is chosen with both magnetic (matrix) and electrostatic (scalar) terms depending effectively on both spatial coordinates. Such progress is possible due to using a particular form of this approach called asymmetric intertwining relations [34–36]. To date, this technique has been explored to study the massless two-dimensional Dirac equation with scalar potential as well as the Fokker–Planck equation [37]. More specifically, the procedure includes two steps (see Section 2). First, asymmetric intertwining relations provide SUSY diagonalization of the potential in Equation (1) with both the electromagnetic and electrostatic two-dimensional terms (Section 3). In the second stage, an additional asymmetric intertwining connects the Dirac operator with diagonal potential to its partner, for which the potential is similarly diagonal but with constant elements (Section 4). The solutions of such a Dirac equation can be found analytically; solutions to the initial problem are built by applying intertwining operators of both steps (Section 5). Actually, the whole procedure realizes the asymmetric form of factorizable SUSY intertwining of the second order. Such SUSY intertwinings are known in their standard (symmetric) form [26] in the context of SUSY Quantum Mechanics with Schrödinger operator.

2. Asymmetric Intertwining for Two-Dimensional Dirac Equation in an Electromagnetic Field

We start with the asymmetric intertwining relations

$$D_1 N_1 = N_2 D_2 \tag{2}$$

for a pair of two-dimensional massless Dirac operators of the form (1) rewritten as general operators with Hermitian matrix potentials

$$D_{1,2} \equiv -i\sigma_k \partial_k + V_{1,2}(\vec{x}); \quad k = 1,2; \quad V_i(\vec{x}) = \begin{pmatrix} v_{11}^{(i)}(\vec{x}) & v_{12}^{(i)}(\vec{x}) \\ v_{21}^{(i)}(\vec{x}) & v_{22}^{(i)}(\vec{x}) \end{pmatrix}; \quad i = 1,2. \tag{3}$$

Two different intertwining operators have the general matrix form

$$N_1 = A_k \partial_k + A(\vec{x}); \quad N_2 = B_k \partial_k + B(\vec{x}) \tag{4}$$

with constant matrices A_k, B_k and two $\vec{x}-$dependent matrices $A(\vec{x}), B(\vec{x})$:

$$A(\vec{x}) = \begin{pmatrix} a_{11}(\vec{x}) & a_{12}(\vec{x}) \\ a_{21}(\vec{x}) & a_{22}(\vec{x}) \end{pmatrix}; \quad B(\vec{x}) = \begin{pmatrix} b_{11}(\vec{x}) & b_{12}(\vec{x}) \\ b_{21}(\vec{x}) & b_{22}(\vec{x}) \end{pmatrix}. \tag{5}$$

Expanding Equation (2) in powers of derivatives, we obtain

$$\sigma_k A_n - B_k \sigma_n + \sigma_n A_k - B_n \sigma_k = 0, \quad k,n = 1,2, \tag{6}$$
$$i\sigma_k A(\vec{x}) - V_1(\vec{x}) A_k = -B_k V_2(\vec{x}) + iB(\vec{x})\sigma_k, \quad k = 1,2, \tag{7}$$
$$i\sigma_k(\partial_k A(\vec{x})) - V_1(\vec{x}) A(\vec{x}) = -B_k(\partial_k V_2(\vec{x})) - B(\vec{x}) V_2(\vec{x}), \quad k = 1,2. \tag{8}$$

Equation (6) provide the following form for the constant coefficient matrices A_k, B_k:

$$A_1 = \begin{pmatrix} a & b \\ d & c \end{pmatrix}; \quad A_2 = \begin{pmatrix} n & -ib \\ id & p \end{pmatrix}; \quad B_1 = \begin{pmatrix} c & d \\ b & a \end{pmatrix}; \quad B_2 = \begin{pmatrix} p & -id \\ ib & n \end{pmatrix} \tag{9}$$

with constant values a, b, c, d, n, p. Equation (7) provides the system of eight linear equations for the matrix elements of $V_{1,2}(\vec{x}), A(\vec{x}), B(\vec{x})$:

$$(p+ic)(v_{12}^{(1)} - v_{12}^{(2)}) = 0; \tag{10}$$
$$(n-ia)(v_{21}^{(1)} - v_{21}^{(2)}) = 0; \tag{11}$$
$$(n+ia)v_{11}^{(1)} + 2idv_{12}^{(1)} - (p+ic)v_{11}^{(2)} = 2b_{12}; \tag{12}$$
$$(n+ia)(v_{21}^{(1)} - v_{21}^{(2)}) + 2idv_{22}^{(1)} - 2ibv_{11}^{(2)} = -2a_{11} + 2b_{22}; \tag{13}$$
$$(p+ic)v_{22}^{(1)} - 2ibv_{12}^{(2)} - (n+ia)v_{22}^{(2)} = -2a_{12}; \tag{14}$$
$$(n-ia)v_{11}^{(1)} + 2idv_{21}^{(2)} - (p-ic)v_{11}^{(2)} = 2a_{21}; \tag{15}$$
$$(p-ic)(v_{12}^{(1)} - v_{12}^{(2)}) + 2idv_{22}^{(2)} - 2ibv_{11}^{(1)} = -2b_{11} + 2a_{22}; \tag{16}$$
$$(p-ic)v_{22}^{(1)} - (n-ia)v_{22}^{(2)} - 2ibv_{21}^{(1)} = -2b_{21}. \tag{17}$$

The first two equations of the system indicate the need to highlight four different possibilities for solutions of (10)–(17), bearing the Hermiticity of both potentials in mind:

$$\begin{array}{rl}
\text{(I)} & p+ic = n-ia = 0; \\
\text{(II)} & v_{12}^{(1)} = v_{12}^{(2)}; \quad v_{21}^{(1)} = v_{21}^{(2)}; \quad (p+ic)(n-ia) \neq 0 \\
\text{(III)} & v_{12}^{(1)} = v_{12}^{(2)}; \quad v_{21}^{(1)} = v_{21}^{(2)}; \quad p+ic = 0; \\
\text{(IV)} & v_{12}^{(1)} = v_{12}^{(2)}; \quad v_{21}^{(1)} = v_{21}^{(2)}; \quad n-ia = 0.
\end{array}$$

Thus, Equations (6) and (7) are reduced to the system of Equations (12) and (17) in the four possible variants above. We have not yet considered the matrix differential Equation (8). It is convenient to solve it in an indirect form using its combination with the derivative ∂_k of Equation (7):

$$V_1(\vec{x})A(\vec{x}) - B(\vec{x})V_2(\vec{x}) = (\partial_k V_1(\vec{x}))A_k + i(\partial_k B(\vec{x}))\sigma_k. \tag{18}$$

Below, all four variants are used to obtain the general solution of the initial intertwining relations (2).

3. SUSY Diagonalization by Means of Intertwining

Substitution of $p+ic = n-ia = 0$, i.e., of Variant I above, into other Equations (12)–(17) together with Equation (18) is quite long, but straightforward. These calculations lead to a system of four nonlinear equations, two of them being differential:

$$v_{11}^{(1)}(ia_{11} + bv_{11}^{(2)} + av_{21}^{(2)}) = v_{11}^{(2)}(ia_{22} + cv_{12}^{(2)} + dv_{22}^{(2)}); \tag{19}$$
$$v_{22}^{(1)}(ia_{22} + cv_{12}^{(2)} + dv_{22}^{(2)}) = v_{22}^{(2)}(ia_{11} + bv_{11}^{(2)} + av_{21}^{(2)}); \tag{20}$$
$$i(v_{12}^{(2)} - v_{12}^{(1)})(ia_{22} + cv_{12}^{(2)} + dv_{22}^{(2)}) = 2\partial_z(ia_{22} + cv_{12}^{(2)} + dv_{22}^{(2)}); \tag{21}$$
$$i(v_{21}^{(2)} - v_{21}^{(1)})(ia_{11} + bv_{11}^{(2)} + av_{21}^{(2)}) = 2\partial_{\bar{z}}(ia_{11} + bv_{11}^{(2)} + av_{21}^{(2)}). \tag{22}$$

Here and below, it is convenient to use the space arguments $\vec{x}$ of the functions in the form of complex variables $z = x_1 + ix_2; \quad \bar{z} = x_1 - ix_2$ and corresponding derivatives $\partial \equiv \frac{1}{2}(\partial_1 - i\partial_2); \ \bar{\partial} \equiv \frac{1}{2}(\partial_1 + i\partial_2)$. In particular, the system (19)–(22) takes the compact form:

$$
\begin{aligned}
v_{11}^{(1)}(z,\bar{z})f_1(z,\bar{z}) &= v_{11}^{(2)}(z,\bar{z})f_2(z,\bar{z}); && (23)\\
v_{22}^{(1)}(z,\bar{z})f_2(z,\bar{z}) &= v_{22}^{(2)}(z,\bar{z})f_1(z,\bar{z}); && (24)\\
v_{12}^{(1)}(z,\bar{z}) &= v_{12}^{(2)}(z,\bar{z}) + 2i\partial\ln(f_2(z,\bar{z})); && (25)\\
v_{21}^{(1)}(z,\bar{z}) &= v_{21}^{(2)}(z,\bar{z}) + 2i\bar{\partial}\ln(f_1(z,\bar{z})), && (26)
\end{aligned}
$$

where two combinations are introduced:

$$
\begin{aligned}
f_1(z,\bar{z}) &\equiv ia_{11}(z,\bar{z}) + bv_{11}^{(2)}(z,\bar{z}) + av_{21}^{(2)}(z,\bar{z}); && (27)\\
f_2(z,\bar{z}) &\equiv ia_{22}(z,\bar{z}) + cv_{12}^{(2)}(z,\bar{z}) + dv_{22}^{(2)}(z,\bar{z}). && (28)
\end{aligned}
$$

One may notice the apparent paradox contained in the system (19)–(22). All these equations are identically fulfilled if

$$ia_{11} + bv_{11}^{(2)} + av_{21}^{(2)} = ia_{22} + cv_{12}^{(2)} + dv_{22}^{(2)} = 0$$

for arbitrary potentials V_1, V_2. The explanation is rather simple; in this case, the Dirac operators D_1, D_2 are proportional to the intertwining operators $D_2 = CN_1, D_1 = N_2C$ up to some constant matrix C. Therefore, the intertwining relation (2) becomes a trivial identity in such a case.

In the present context, the typical approach [24–26,34,36] to using SUSY intertwining relations can be formulated as follows. Let us choose one of the Dirac operators, D_2, such that the corresponding Dirac equation is rather simple and the problem of its analytical solution is more easy. Then, the solutions of the partner Dirac equation with operator D_1 are found through the action of the intertwining operator $\Psi^{(1)} = N_1\Psi^{(2)}$ (in turn, the operator $N_2^{\dagger}$ transforms the spinor $\Psi^{(1)}$ into $\Psi^{(2)}$). As the first step in this approach, we choose the potential $V_2(\vec{x})$ as a diagonal matrix:

$$V_2(\vec{x}) = diag(v_{11}^{(2)}(z,\bar{z}),\ v_{22}^{(2)}(z,\bar{z})) \equiv diag(v_1(z,\bar{z}),\ v_2(z,\bar{z})) \tag{29}$$

with real diagonal matrix elements. Then, due to Equations (23) and (24), the fraction $f_2(z,\bar{z})/f_1(z,\bar{z})$ is a real function. Futhermore, Equations (25) and (26) and the Hermiticity of $V_1(\vec{x})$, i.e., $v_{12}^{(1)}(z,\bar{z}) = v_{21}^{(1)\star}(z,\bar{z})$, lead to the following restriction:

$$f_1(z,\bar{z})f_2^{\star}(z,\bar{z}) = c \tag{30}$$

with an arbitrary real constant c. Summarizing these results, the potential V_1 is expressed in terms of the components (29), function $f_2(z,\bar{z})$, and real constant c:

$$V_1(\vec{x}) = \begin{pmatrix} \frac{|f_2|^2}{c}v_1 & 2i\partial\ln(f_2) \\ -2i\bar{\partial}\ln(f_2^{\star}) & \frac{c}{|f_2|^2}v_2 \end{pmatrix} = \begin{pmatrix} \pm f^2 v_1 & 2i(\partial\ln(f) + i\partial\varphi) \\ -2i(\bar{\partial}\ln(f) - i\bar{\partial}\varphi) & \pm f^{-2}v_2 \end{pmatrix}, \tag{31}$$

where the function f_2 is parameterized as

$$f_2(z,\bar{z}) \equiv \sqrt{\pm c}\, f(z,\bar{z})e^{i\varphi(z,\bar{z})}. \tag{32}$$

In the above, the sign $\pm$ corresponds to cases $c > 0$, $c < 0$, respectively, and $f(z,\bar{z})$ is a positive function connecting elements of the initial diagonal potential V_2 as

$$v_2(z,\bar{z}) = f^4(z,\bar{z})v_1(z,\bar{z}). \tag{33}$$

For the physical system with matrix potential V_1 describing a "spin 1/2 particle" in the external electromagnetic field according to Equation (1), the diagonal elements of V_1 define the electrostatic potential

$$A_0(\vec{x}) = v_{11}^{(1)} = v_{22}^{(1)} = \pm(v_1 v_2)^{1/2} = \pm f^2(\vec{x}) v_1(\vec{x}), \tag{34}$$

while the off-diagonal terms define the magnetic field that is orthogonal to the plane x_1, x_2:

$$B_3(\vec{x}) = \vec{\nabla} \times \vec{A}(\vec{x}) = \triangle \ln(f(\vec{x})). \tag{35}$$

The solutions of Dirac equations with such external fields can be obtained from the two-component solutions $\Psi^{(2)}(\vec{x})$ of Dirac equations with diagonal potential V_2 by action of the intertwining operator N; however, the diagonal form of the potential V_2 does not yet provide solvability of the corresponding Dirac problem analytically. Thus far, supersymmetric intertwining relations (2) have connected an initial Dirac problem with operator D_1 to a Dirac problem with a potential V_2 which is diagonal, and as such has a chance of being solvable. By analogy with SUSY separation of variables, this part of the procedure can be called SUSY-diagonalization of the two-dimensional Dirac problem with matrix potential [38].

4. From Diagonal Potential to Constant Potential by Means of Intertwining

Let us now consider variant II with $(p + ic)(n - ia) \neq 0$ for the solution of the system (10)–(17) in the context of the second step of our procedure. Specifically, we consider intertwining relations between two Dirac operators, both with diagonal potential:

$$U_1(\vec{x}) = \begin{pmatrix} v_1(\vec{x}) & 0 \\ 0 & v_2(\vec{x}) \end{pmatrix}; \quad U_2(\vec{x}) = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}, \tag{36}$$

with constant elements m_1, m_2. Here, the potential $U_1(\vec{x})$ is identified with the potential $V_2(\vec{x})$ from the previous step, while the constant partner potential U_2 provides solvability of the problem. The intertwining operators N_1, N_2 have the same general form (4), and the explicit expressions for matrices A_k, B_k, $A(\vec{x})$, $B(\vec{x})$ are found below, obtained by analytical solutions of the system of equations in Section 2.

We shall consider the system of Equations (10)–(17) sequentially. Equations (10) and (11) are fulfilled automatically. Equations (12), (14), (15) and (17) allow off-diagonal elements of $A(\vec{x})$, $B(\vec{x})$ at (5) to be expressed in terms of $v_1(\vec{x})$, $v_2(\vec{x})$:

$$a_{12}(\vec{x}) = -\frac{1}{2}(p + ic)v_2(\vec{x}) + \frac{1}{2}m_2(n + ia); \tag{37}$$

$$a_{21}(\vec{x}) = \frac{1}{2}(n - ia)v_2(\vec{x}) - \frac{1}{2}m_1(p - ic); \tag{38}$$

$$b_{12}(\vec{x}) = \frac{1}{2}(n + ia)v_1(\vec{x}) - \frac{1}{2}m_1(p + ic); \tag{39}$$

$$a_{12}(\vec{x}) = -\frac{1}{2}(p - ic)v_2(\vec{x}) + \frac{1}{2}m_2(n - ia). \tag{40}$$

Equations (13) and (16) for diagonal elements of $A(\vec{x})$, $B(\vec{x})$ can be written as

$$b_{11}(\vec{x}) = a_{22}(\vec{x}) + ibv_1(\vec{x}) - idm_2; \tag{41}$$

$$b_{22}(\vec{x}) = a_{11}(\vec{x}) + idv_2(\vec{x}) - ibm_1. \tag{42}$$

Equation (8) in its initial form is convenient to write now in complex coordinates $z\,\bar{z}$:

$$2i\left[\begin{pmatrix} 0 & \partial \\ \bar{\partial} & 0 \end{pmatrix} - \begin{pmatrix} v_1(z,\bar{z}) & 0 \\ 0 & v_2(z,\bar{z}) \end{pmatrix}\right]\begin{pmatrix} a_{11}(z,\bar{z}) & a_{12}(z,\bar{z}) \\ a_{21}(z,\bar{z}) & a_{22}(z,\bar{z}) \end{pmatrix} =$$
$$-\begin{pmatrix} b_{11}(z,\bar{z}) & b_{12}(z,\bar{z}) \\ b_{21}(z,\bar{z}) & b_{22}(z,\bar{z}) \end{pmatrix}\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}, \tag{43}$$

where $\partial \equiv \partial_z = \frac{1}{2}(\partial_1 - i\partial_2)$, $\bar{\partial} \equiv \partial_{\bar{z}} = \frac{1}{2}(\partial_1 + i\partial_2)$. In components, the matrix Equation (43) is equivalent to the system of linear first-order differential equations:

$$2i(\partial a_{21}(z,\bar{z})) - v_1(z,\bar{z})a_{11}(z,\bar{z}) = -m_1 b_{11}(z,\bar{z}); \tag{44}$$
$$2i(\bar{\partial} a_{12}(z,\bar{z})) - v_2(z,\bar{z})a_{22}(z,\bar{z}) = -m_2 b_{22}(z,\bar{z}); \tag{45}$$
$$2i(\partial a_{22}(z,\bar{z})) - v_1(z,\bar{z})a_{12}(z,\bar{z}) = -m_2 b_{12}(z,\bar{z}); \tag{46}$$
$$2i(\bar{\partial} a_{11}(z,\bar{z})) - v_2(z,\bar{z})a_{21}(z,\bar{z}) = -m_1 b_{21}(z,\bar{z}). \tag{47}$$

Let us define for convenience

$$a_{11}(z,\bar{z}) - im_1 b \equiv g_1(z,\bar{z}); \quad a_{22}(z,\bar{z}) - im_2 d \equiv g_2(z,\bar{z}). \tag{48}$$

Then, after substitution of Equations (37)–(40) and Equations (41) and (42), the system (44)–(47) takes the form

$$i(n-ia)(\partial v_1(z,\bar{z})) = v_1(z,\bar{z})g_1(z,\bar{z}) - m_1 g_2(z,\bar{z}); \tag{49}$$
$$i(p+ic)(\bar{\partial} v_2(z,\bar{z})) = -v_2(z,\bar{z})g_2(z,\bar{z}) + m_2 g_1(z,\bar{z}); \tag{50}$$
$$4i(\bar{\partial} g_1(z,\bar{z})) = (n-ia)(v_1(z,\bar{z})v_2(z,\bar{z}) - m_1 m_2); \tag{51}$$
$$4i(\partial g_2(z,\bar{z})) = -(p+ic)(v_1(z,\bar{z})v_2(z,\bar{z}) - m_1 m_2), \tag{52}$$

where the form of last two equations mean that functions $g_1(z,\bar{z})$, $g_2(z,\bar{z})$ can be expressed in terms of one complex function g:

$$g_1(z,\bar{z}) = -\frac{1}{p+ic}(\partial g(z,\bar{z})); \quad g_2(z,\bar{z}) = \frac{1}{n-ia}(\bar{\partial} g(z,\bar{z})), \tag{53}$$

which satisfies the second order equation:

$$4i(\partial\bar{\partial} g(z,\bar{z})) = -(n-ia)(p+ic)(v_1(z,\bar{z})v_2(z,\bar{z}) - m_1 m_2) \tag{54}$$

and the first two Equations (49) and (50) become

$$i(n-ia)(\partial v_1(z,\bar{z})) = -\frac{1}{p+ic}v_1(z,\bar{z})(\partial g(z,\bar{z})) - \frac{1}{n-ia}m_1(\bar{\partial} g(z,\bar{z})), \tag{55}$$
$$i(p+ic)(\bar{\partial} v_2(z,\bar{z})) = -\frac{1}{n-ia}v_2(z,\bar{z})(\bar{\partial} g(z,\bar{z})) - \frac{1}{p+ic}m_2(\partial g(z,\bar{z})). \tag{56}$$

Thus,

$$\Omega(\partial v_1(z,\bar{z})) + v_1(z,\bar{z})(\partial g(z,\bar{z})) = -L_1(\bar{\partial} g(z,\bar{z})); \tag{57}$$
$$\Omega(\bar{\partial} v_2(z,\bar{z})) + v_2(z,\bar{z})(\bar{\partial} g(z,\bar{z})) = -L_2(\partial g(z,\bar{z})); \tag{58}$$
$$\Omega \equiv i(n-ia)(p+ic); \quad L_1 \equiv \frac{m_1(p+ic)}{n-ia}; \quad L_2 \equiv \frac{m_2(n-ia)}{p+ic}.$$

Below, we shall solve this system of equations by separately considering different options for the choice of constant parameters.

4.1. Case A: The Parameters Are Real

Let us study the case with real function $g(z,\bar{z})$ and real values of parameters Ω, L_1, L_2. Taking into account the reality of $v_1(z,\bar{z})$ and $v_2(z,\bar{z})$, it is useful to come back to the Cartesian coordinates x_1, x_2 in Equations (57) and (58). Separately, both the real and imaginary parts of (57) are integrated explicitly with two "constants of integration" $s_1(x_1)$, $s_2(x_2)$, which are arbitrary real functions of their arguments:

$$\exp\left(g(\vec{x})/\Omega\right) = L_1^{-1}(s_2(x_2) - s_1(x_1)); \tag{59}$$

$$v_1(\vec{x}) = L_1 \frac{s_2(x_2) + s_1(x_1)}{s_2(x_2) - s_1(x_1)}. \tag{60}$$

Analogously, Equation (58) can be integrated with a similar result:

$$\exp\left(g(\vec{x})/\Omega\right) = L_2^{-1}(\tilde{s}_2(x_2) - \tilde{s}_1(x_1)); \tag{61}$$

$$v_2(\vec{x}) = L_2 \frac{\tilde{s}_2(x_2) + \tilde{s}_1(x_1)}{\tilde{s}_2(x_2) - \tilde{s}_1(x_1)} \tag{62}$$

and arbitrary real $\tilde{s}_1(x_1)$, $\tilde{s}_2(x_2)$. Together, Equations (59) and (61) allow us to connect $\tilde{s}_1(x_1)$, $\tilde{s}_2(x_2)$ with their analogues:

$$\tilde{s}_1(x_1) = L_1^{-1}L_2(s_1(x_1) + \delta); \quad \tilde{s}_2(x_2) = L_1^{-1}L_2(s_2(x_2) + \delta), \tag{63}$$

with an arbitrary real constant δ. These relations have to be substituted into expression (62).

Using these connections in the second-order differential Equation (54) for the function $g(\vec{x})$ and differentiating it by $\partial_1\partial_2$, we obtain the following simple third-order equation with separable variables:

$$\frac{s_1'''(x_1)}{s_1'(x_1)} + \frac{s_2'''(x_2)}{s_2'(x_2)} = -4L_1L_2 = -4m_1m_2. \tag{64}$$

After separation ofthe variables in (64) and integration of one-dimensional equations, we have

$$s_1''(x_1) = \lambda_1^2 s_1(x_1) + \omega_1\lambda_1^2; \quad s_2''(x_2) = \lambda_2^2 s_2(x_2) + \omega_2\lambda_2^2, \tag{65}$$

where ω_1, ω_2 are integration constants and λ_1, λ_2 are arbitrary constants which satisfy the following relation:

$$\lambda_1^2 + \lambda_2^2 = -4m_1m_2.$$

The solutions of (65) are known:

$$s_1(x_1) = \frac{1}{2}(\sigma_1 e^{\lambda_1 x_1} + \delta_1 e^{-\lambda_1 x_1}) - \omega_1;$$

$$s_2(x_2) = \frac{1}{2}(\sigma_2 e^{\lambda_2 x_2} + \delta_2 e^{-\lambda_2 x_2}) - \omega_2.$$

Because we used derivatives of Equation (54), it is necessary to check the results. Substitution of (65) into (54) gives us two relations between the parameters:

$$\omega_1\lambda_1^2 + \omega_2\lambda_2^2 + 2\delta L_1L_2 = 0;$$
$$\lambda_1^2(\sigma_1\delta_1 - \omega_1^2) + \lambda_2^2(\sigma_2\delta_2 - \omega_2^2) = 0$$

Now, depending on the values of the constants, we can list all possible solutions $s_1(x_1)$, $s_2(x_2)$ within Section 4.1. All are expressed in terms of hyperbolic, trigonometric, and exponential functions, and have to be inserted into (63) to find $u_1(\vec{x})$, $u_2(\vec{x})$ according to (60) and (62):

I. $\lambda_1^2 > 0; \quad \lambda_2^2 > 0; \quad \sigma_1\delta_1 > 0; \quad \sigma_2\delta_2 > 0.$

By additional translation of $x_{1,2}$ functions, $s_1(x_1)$, $s_2(x_2)$ takes the form

$$s_1(x_1) = \sigma_1 \cosh(\lambda_1 x_1) - \omega_1; \quad s_2(x_2) = \sigma_2 \cosh(\lambda_2 x_2) - \omega_2$$

with the restriction

$$\lambda_1^2(\sigma_1^2 - \omega_1^2) + \lambda_2^2(\sigma_2^2 - \omega_2^2) = 0.$$

II. $\lambda_1^2 > 0; \quad \lambda_2^2 > 0; \quad \sigma_1\delta_1 > 0; \quad \sigma_2\delta_2 < 0$

$$s_1(x_1) = \sigma_1 \cosh(\lambda_1 x_1) - \omega_1; \quad s_2(x_2) = \sigma_2 \sinh(\lambda_2 x_2) - \omega_2,$$

with restriction

$$\lambda_1^2(\sigma_1^2 - \omega_1^2) - \lambda_2^2(\sigma_2^2 + \omega_2^2) = 0.$$

III. $\lambda_1^2 > 0; \quad \lambda_2^2 > 0; \quad \sigma_1\delta_1 > 0; \quad \delta_2 = 0$

$$s_1(x_1) = \sigma_1 \cosh(\lambda_1 x_1) - \omega_1; \quad s_2(x_2) = \frac{1}{2}\sigma_2 \exp(\lambda_2 x_2) - \omega_2,$$

with restriction

$$\lambda_1^2(\sigma_1^2 - \omega_1^2) - \lambda_2^2\omega_2^2 = 0.$$

IV. $\lambda_1^2 < 0; \quad \lambda_2^2 < 0; \quad \sigma_1 = \delta_1^\star; \quad \sigma_2 = \delta_2^\star$

$\lambda_1 \equiv i\Lambda_1; \quad \lambda_2 \equiv i\Lambda_2$

$$s_1(x_1) = \sigma_1 \cos(\Lambda_1 x_1) - \omega_1; \quad s_2(x_2) = \sigma_2 \cos(\Lambda_2 x_2) - \omega_2;$$

with restriction

$$\Lambda_1^2(\sigma_1^2 - \omega_1^2) + \Lambda_2^2(\sigma_2^2 - \omega_2^2) = 0.$$

V. $\lambda_1^2 > 0; \quad \lambda_2^2 < 0; \quad \sigma_1\delta_1 > 0; \quad \sigma_2 = \delta_2^\star$

$\lambda_2 \equiv i\Lambda_2$

$$s_1(x_1) = \sigma_1 \cosh(\lambda_1 x_1) - \omega_1; \quad s_2(x_2) = \sigma_2 \cos(\Lambda_2 x_2) - \omega_2;$$

with restriction

$$\lambda_1^2(\sigma_1^2 - \omega_1^2) - \Lambda_2^2(\sigma_2^2 - \omega_2^2) = 0.$$

VI. $\lambda_1^2 > 0; \quad \lambda_2^2 < 0; \quad \sigma_1\delta_1 < 0; \quad \sigma_2 = \delta_2^\star$

$\lambda_2 \equiv i\Lambda_2$

$$s_1(x_1) = \sigma_1 \sinh(\lambda_1 x_1) - \omega_1; \quad s_2(x_2) = \sigma_2 \cos(\Lambda_2 x_2) - \omega_2;$$

with restriction:

$$\lambda_1^2(\sigma_1^2 + \omega_1^2) + \Lambda_2^2(\sigma_2^2 - \omega_2^2) = 0.$$

4.2. Case B: $m_2 = 0$.

The case with $m_2 = L_2 = 0$ and real Ω, L_1, $g(\vec{x})$ is considered below. For such a choice, Equation (58) provides us with

$$v_2(\vec{x}) = \exp(g(\vec{x})/\Omega) = \frac{C}{s_2(x_2) - s_1(x_1)}, \quad v_1(\vec{x}) = L_1 \frac{s_1(x_1) + s_2(x_2)}{s_2(x_2) - s_1(x_1)}, \tag{66}$$

while Equation (54) looks like

$$(s_2''(x_2) - s_1''(x_1))(s_2(x_2) - s_1(x_1)) - ((s_1'(\vec{x}_1))^2 + (s_2'(\vec{x}_2))^2) = CL_1(s_1(x_1) + s_2(x_2)). \tag{67}$$

After differentiation, the latter equation is again amenable to separation of variables, similarly to (64), except with zero on the right-hand side. It has two different solutions depending on the value of separation constant:

$$s_1(x_1) = a_1 x_1^2 + b_1 x_1 + c_1; \quad s_2(x_2) = a_2 x_2^2 + b_2 x_2 + c_2 \tag{68}$$

and

$$s_1''(x_1) = \lambda^2 (s_1(x_1) + \omega_1); \quad s_2''(x_2) = -\lambda^2 (s_2(x_2) + \omega_2), \tag{69}$$

with the latter having three kinds of explicit solutions to insert into (66):

$$\begin{aligned}
s_1(x_1) &= \sigma_1 \cosh(\lambda x_1) - \omega_1; \quad s_2(x_2) = \sigma_2 \cos(\lambda x_2) - \omega_2; \\
s_1(x_1) &= \sigma_1 \sinh(\lambda x_1) - \omega_1; \quad s_2(x_2) = \sigma_2 \cos(\lambda x_2) - \omega_2; \\
s_1(x_1) &= \frac{1}{2}\sigma_1 \exp(\lambda x_1) - \omega_1; \quad s_2(x_2) = \sigma_2 \cos(\lambda x_2) - \omega_2;
\end{aligned}$$

and corresponding restrictions for the constants:

$$\begin{aligned}
&\omega_2^2 - \omega_1^2 + \sigma_1^2 - \sigma_2^2 = 0; \\
&\omega_2^2 - \omega_1^2 - \sigma_1^2 - \sigma_2^2 = 0; \\
&\omega_2^2 - \omega_1^2 - \sigma_2^2 = 0; \\
&\lambda^2(\omega_2 - \omega_1) - CL_1 = 0.
\end{aligned}$$

As for the polynomial solution (68), a few restrictions have to be fulfilled simultaneously:

$$A \equiv 2a_1 + 2a_2 + CL_1 = 0. \tag{70}$$

Finally, solution (68) leads to two different opportunities for the components of $U_1(\vec{x})$. The first is

$$v_1(\vec{x}) = -L_1 \frac{[CL_1(c_2 x_1^2 - c_1 x_2^2) + 2(c_1^2 - c_2^2)]}{CL_1(c_2 x_1^2 + c_1 x_2^2) + 2(c_1 - c_2)^2}; \tag{71}$$

$$v_2(\vec{x}) = -\frac{2(c_1 - c_2)}{CL_1(c_2 x_1^2 + c_1 x_2^2) + 2(c_1 - c_2)^2}; \tag{72}$$

$$a_1 = \frac{Cc_2 L_1}{2(c_1 - c_2)}; \quad a_2 = -\frac{Cc_1 L_1}{2(c_1 - c_2)}; \; c_1 \neq c_2,$$

while the second, with $c_1 = c_2 = 0$ and functions $s_1(x_1) = a_1 x_1^2$, $s_2(x_2) = -(a_1 + \frac{1}{2}CL_1)x_2^2$, is

$$v_1(\vec{x}) = -L_1 \frac{a_1 x_1^2 - (a_1 + \frac{CL_1}{2})x_2^2}{(a_1 + \frac{CL_1}{2})x_2^2 + a_1 x_1^2}; \quad v_2(\vec{x}) = -\frac{C}{(a_1 + \frac{CL_1}{2})x_2^2 + a_1 x_1^2}. \tag{73}$$

4.3. Case C: $p + ic = 0$.

Let us again consider intertwining of two Dirac operators, both with diagonal potential:

$$W_1(\vec{x}) = \begin{pmatrix} w_{11}^{(1)}(\vec{x}) & 0 \\ 0 & w_{22}^{(1)}(\vec{x}) \end{pmatrix} \equiv \begin{pmatrix} v_1(\vec{x}) & 0 \\ 0 & v_2(\vec{x}) \end{pmatrix}; \quad W_2(\vec{x}) = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}, \tag{74}$$

and with constant elements k_1, k_2. Here, the matrix potential $W_1(\vec{x})$ is identified with the potential $V_2(\vec{x})$ from Section 3, with the constant partner potential W_2 providing solvability of the problem. This means that Case C is an alternative option in relation to Cases B and C from the Sections 4.1 and 4.2. The difference is that we now take $p + ic = 0$, i.e., variant III from Section 2, and it is clear that variant IV can be considered analogously. In this case, the system (12)–(17) is expressed as follows:

$$\begin{aligned} b_{11} &= a_{22} + ibv_1 - idk_2; \quad b_{12} = \frac{n+ia}{2}v_1; \quad b_{21} = -pw_2 + \frac{n-ia}{2}k_2; \\ a_{12} &= \frac{n+ia}{2}k_2 = Const; \quad a_{21} = -pk_1 + \frac{n-ia}{2}v_1. \end{aligned}$$

The matrix differential Equation (18) takes the form

$$\begin{aligned} i(n-ia)\partial v_1(\vec{x}) &= v_1(\vec{x})(a_{11}(\vec{x}) - ibk_1) - k_1(a_{22}(\vec{x}) - idk_2); \\ v_2(\vec{x})(a_{22}(\vec{x}) - idk_2) &= k_2(a_{11}(\vec{x}) - ibk_1); \\ \partial a_{22}(\vec{x}) &= 0; \\ 2i\bar{\partial}a_{11}(\vec{x}) &= \frac{n-ia}{2}(v_1(\vec{x})v_2(\vec{x}) - k_1k_2), \end{aligned}$$

which by convenient definition

$$r_1(\vec{x}) \equiv a_{11}(\vec{x}) - ibk_1; \quad r_2(\vec{x}) \equiv a_{22}(\vec{x}) - idk_2$$

can be reduced to two equations $(r_2(\vec{x}) = r_2(\bar{z}))$

$$ik_2(n-ia)\partial(\frac{v_1(\vec{x})}{r_2(\bar{z})}) = v_1(\vec{x})v_2(\vec{x}) - k_1k_2, \tag{75}$$

$$4i(\bar{\partial}r_1(\vec{x})) = (n-ia)(v_1(\vec{x})v_2(\vec{x}) - k_1k_2), \tag{76}$$

such that both $v_1(\vec{x})$ and $v_2(\vec{x})$ are expressed in terms of one function $\kappa(\vec{x})$:

$$v_1(\vec{x}) = \frac{4r_2(\bar{z})}{k_2(n-ia)}(\bar{\partial}\kappa(\vec{x})); \quad v_2(\vec{x}) = \frac{k_2(n-ia)}{r_2(\bar{z})}(\partial\kappa(\vec{x})); \quad r_1(\vec{x}) = (n-ia)(\partial\kappa(\vec{x})), \tag{77}$$

where function $\kappa(\vec{x})$ satisfies the following second-order differential equation:

$$i(\partial\bar{\partial}\kappa(\vec{x})) = (\partial\kappa(\vec{x}))(\bar{\partial}\kappa(\vec{x})) - k; \quad 4k \equiv k_1k_2. \tag{78}$$

Because the right-hand side here is real, the real part of the function $\kappa(\vec{x})$ is a sum of two mutually conjugated functions:

$$\kappa(\vec{x}) \equiv \alpha(z) + \bar{\alpha}(\bar{z}) + i\xi(\vec{x}).$$

Due to Equation (77), the reality of both diagonal elements $v_1(\vec{x})$, $v_2(\vec{x})$ leads to the reality of $(\partial\kappa(z,\bar{z}))(\bar{\partial}\kappa(z,\bar{z}))$, which, in terms of α and ξ, means that

$$Im((\partial\kappa(\vec{x}))(\bar{\partial}\kappa(\vec{x}))) = \alpha'(z)\bar{\partial}\xi(\vec{x}) + \bar{\alpha}'(\bar{z})\partial\xi(\vec{x}) = 0,$$

i.e., function $\xi(\vec{x})$ is an arbitrary real function of the specific real argument:

$$\xi(\vec{x}) = \Phi(i(\alpha(z) - \bar{\alpha}(\bar{z}))) \equiv \Phi(X); \quad X(z,\bar{z}) \equiv i(\alpha(z) - \bar{\alpha}(\bar{z})). \tag{79}$$

From Equation (78), we obtain a nonlinear differential equation for the function Φ:

$$\Phi''(X) - (\Phi'(X))^2 + 1 = \frac{k}{\alpha'(z)\bar{\alpha}'(\bar{z})}. \tag{80}$$

The left-hand side in (80) depends only on the variable X, which by definition satisfies the following equation:

$$(\frac{1}{\alpha'(z)}\partial + \frac{1}{\bar{\alpha}'(\bar{z})}\bar{\partial})X(z,\bar{z}) = 0;$$

therefore, due to (80), we have

$$(\frac{1}{\alpha'(z)}\partial+\frac{1}{\bar{\alpha}'(\bar{z})}\bar{\partial})\frac{1}{\alpha'(z)\bar{\alpha}'(\bar{z})}=0,$$

allowing us to define possible forms of the function $\alpha(z)$. Indeed, the variables in the latter equation can be separated:

$$\frac{\alpha''(z)}{(\alpha'(z))^2}+\frac{\bar{\alpha}''(\bar{z})}{(\bar{\alpha}'(\bar{z}))^2}=0,$$

informing us that exactly two options exist for the function $\alpha(z)$:

$$a)\ \ \alpha'(z)=\omega;\quad b)\ \ \alpha'(z)=\frac{i\lambda}{z}, \tag{81}$$

with ω being an arbitrary constant and λ an arbitrary real constant.

For the first option, $r_2(\bar{z})$ must be constant:

$$\begin{aligned} v_1(\vec{x}) &= 4e(1+\Phi'(X(z,\bar{z})));\quad v_2(\vec{x})=\frac{|\omega|^2}{e}(1-\Phi'(X(z,\bar{z}))); \\ \Phi''(X) &- (\Phi'(X))^2+1=\frac{k}{|\omega|^2};\quad e\equiv\frac{r_2(\bar{z})\bar{\omega}}{k_2(n-ia)}, \end{aligned} \tag{82}$$

and the parameters must provide that the constant e is real. Depending on the sign of $(\frac{k}{|\omega|^2}-1)$, one of two solutions is realized:

$$\begin{aligned} &\Phi_1'(X)=-\eta\cos(\eta X+\mu);\quad \Phi_2'(X)=-\tilde{\eta}\ln(\cosh(\tilde{\eta}X+\tilde{\mu})); \\ &(\frac{k}{|\omega|^2}-1)\equiv\eta^2>0;\ (\frac{k}{|\omega|^2}-1)\equiv-\tilde{\eta}^2<0. \end{aligned} \tag{83}$$

For the second option, Equation (77) states that $r_2(\bar{z})=\gamma\bar{z}$, constant $\frac{i\gamma}{n-ia}$ must be real, and

$$v_1(\vec{x})=\frac{4\lambda^2}{\beta}(1+\Phi'(X));\quad v_2(\vec{x})=\frac{\beta}{z\bar{z}}(1-\Phi'(X));\quad \beta\equiv\frac{ik_2(n-ia)}{\gamma}. \tag{84}$$

Here, $\alpha(z)=i\lambda\ln(z)$, $\bar{\alpha}(\bar{z})=-i\lambda\ln(\bar{z})$, and the equation for $\Phi(X)$ takes the form

$$\Phi''(X)-(\Phi'(X))^2+1=k\lambda^2e^{\frac{X}{\lambda}}. \tag{85}$$

See [39] for more on solutions of this equation in terms of special functions.

5. Wave Functions and Electromagnetic Fields

In the previous sections, we performed two consecutive transformations of the Dirac operator with the matrix potential using first-order intertwining operators, similar to the SUSY intertwining in ordinary quantum mechanics. In this context, such operations are known as second-order reducible (i.e., factorizable) SUSY transformations [26]. Unlike that case, however, for the present problem with the Dirac operator we have used the asymmetrical form of intertwining [34,36] in both steps.

The resulting Dirac equation, with potential which is a diagonal matrix with constant elements at the diagonal (such as U_2 in (36)), is amenable to a simple analytic solution. Indeed, one of two components of $\Psi^{(2)}(\vec{x})\equiv(\Psi_A^{(2)}(\vec{x}),\Psi_B^{(2)}(\vec{x}))^T$ can be excluded, leading to the second-order equation (Helmholtz equation) for another component:

$$(\Delta+m_1m_2)\Psi_A^{(2)}(\vec{x})=0;\quad \Psi_B^{(2)}(\vec{x})=\frac{2i}{m_2}\bar{\partial}\Psi_A^{(2)}(\vec{x}). \tag{86}$$

After separation of the variables in (86), its solution can be written as a linear combination with arbitrary complex coefficients $\sigma_{k_1k_2}$:

$$\Psi_A^{(2)}(\vec{x}) = \sum_{k_1,k_2} \sigma_{k_1k_2} \exp(k_1x_1)\exp(k_2x_2) \tag{87}$$

where the sum (actually, the integral) is over k_1, k_2, that is, arbitrary complex constants such that $k_1^2 + k_2^2 = -m_1m_2$. The coefficients in the sum have to be determined using the boundary conditions for the wave functions.

According to the intertwining relations of form (2), the solutions $\Psi^{(1)}(\vec{x}) \equiv (\Psi_A^{(1)}(\vec{x}), \Psi_B^{(1)}(\vec{x}))^T$ of the initial Dirac equation with potential $V_1(\vec{x})$ can be constructed using the sequential action of two intertwining operators $N_1\tilde{N}_1$. The first operator N_1 intertwines two Dirac operators: the initial one with potential V_1, and one with diagonal potential (29) (see (2)). The second $\tilde{N}_1$ analogously intertwines the Dirac operator with potential $V_2(\vec{x}) \equiv U_1(\vec{x}) \equiv W_1(\vec{x})$ and the operator with either potential $U_2(\vec{x})$ or $W_2(\vec{x})$, depending on whether we follow the exploration described in Sections 4.1 and 4.2 or Section 4.3 . These intertwining operators have the general form (4), and the corresponding explicit expressions for the coefficients A_k and $A(\vec{x})$ are derived in Sections 3 and 4.

The two-component wave functions $\Psi^{(1)}(\vec{x})$ obtained by the above procedure can describe graphene and similar materials in an external field (i.e., two-dimensional electrostatic plus non-homogeneous orthogonal magnetic). Analytical expressions for the strength of these fields are known from the analytical expression for the initial potential $V_1(\vec{x})$. The strength of electrostatic field is directed along the (x_1, x_2) plane (see (34)):

$$\vec{E}(\vec{x}) = -\vec{\nabla}A_0(\vec{x}) = -\vec{\nabla}(f^2(\vec{x})v_1(\vec{x})), \tag{88}$$

and the strength of B_3 of the magnetic one (see (35)) is

$$B_3(\vec{x}) = \triangle \ln(f(\vec{x})). \tag{89}$$

The functions $f(\vec{x})$ are different for the different cases in Sections 4.1–4.3 and can be calculated from the components $v_1(\vec{x})$, $v_2(\vec{x})$ according to (33). These components are provided by (60) and (62) for Section 4.1, by (66) for Section 4.2, by (82) and (84) for Section 4.3. The explicit expressions for these functions, mainly in terms of trigonometric and hyperbolic functions, lead to corresponding expressions for the electromagnetic strengths in terms of the same elementary functions.

For examples of possible configurations of external fields, we can use the particular case of the polynomial solutions (68) and (70) for $c_1 = c_2 = 0$. The components $v_1(\vec{x})$, $v_2(\vec{x})$ are provided by (73), while the function $f(\vec{x})$ is defined from

$$f^4(\vec{x}) = \frac{v_2(\vec{x})}{v_1(\vec{x})} = \frac{2C}{L_1}[2a_1x_1^2 - (2a_1 + CL_1)x_2^2]^{-1}.$$

According to (88) and (89), the strengths are

$$E_1(\vec{x}) = 4\sqrt{2CL_1}a_1x_1 \frac{2a_1x_1^2 - 3(2a_1 + CL_1)x_2^2}{[2a_1x_1^2 + (2a_1 + CL_1)x_2^2]^3[2a_1x_1^2 - (2a_1 + CL_1)x_2^2]^{1/2}},$$

$$E_2(\vec{x}) = 2\sqrt{2CL_1}(2a_1 + CL_1)x_2 \frac{6a_1x_1^2 - (2a_1 + CL_1)x_2^2}{[2a_1x_1^2 + (2a_1 + CL_1)x_2^2]^3[2a_1x_1^2 - (2a_1 + CL_1)x_2^2]^{1/2}},$$

and

$$B_3(\vec{x}) = 2(4a_1 + CL_1)(2a_1x_1^2 + (2a_1 + CL_1)x_2^2)(2a_1x_1^2 - (2a_1 + CL_1)x_2^2)^{-2},$$

with the possibility of choosing arbitrary suitable values for all constant parameters.

6. Conclusions

In this paper, a modification of the well-known method of asymmetrical intertwining relations from SUSY Quantum Mechanics has been used to build a massless two-dimensional Dirac equation with nontrivial matrix potential with solutions that can be found analytically. It was necessary to use factorizable second-order intertwining, which includes two steps: the first allows the Dirac operator to be diagonalized, while the second connects the latter operator with an explicitly solvable Dirac problem containing the diagonal matrix potential with constant elements.

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