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Spontaneous and Explicit Spacetime Symmetry Breaking in Einstein–Cartan Theory with Background Fields

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Abstract: Explicit and spontaneous breaking of spacetime symmetry under diffeomorphisms, local translations, and local Lorentz transformations due to the presence of fixed background fields is examined in Einstein–Cartan theory. In particular, the roles of torsion and violation of local translation invariance are highlighted. The nature of the types of background fields that can arise and how they cause spacetime symmetry breaking is discussed. With explicit breaking, potential no-go results are known to exist, which if not evaded lead to inconsistencies between the Bianchi identities, Noether identities, and the equations of motion. These are examined in detail, and the effects of nondynamical backgrounds and explicit breaking on the energy–momentum tensor when torsion is present are discussed as well. Examples illustrating various features of both explicit and spontaneous breaking of local translations are presented and compared to the case of diffeomorphism breaking.

Keywords: gravity; torsion; Einstein–Cartan theory; spacetime symmetry breaking

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1. Introduction

Local spacetime symmetries are fundamental features of current theories of gravity, including Einstein's General Relativity (GR). In GR, the underlying geometry is Riemann, which is characterized by the Riemann curvature tensor. In the Einstein equations, the energy–momentum density of the matter fields acts as the source of the curvature.

An extension of GR, which is useful in describing fields with spin, is Einstein–Cartan (EC) theory [1–11]. In this case, the underlying geometry is Riemann–Cartan, which is characterized by both a curvature and a torsion tensor. Using a vierbein formalism, the independent fields are the vierbein and the spin connection. The pure-gravity term in the action has an Einstein–Hilbert form, but in this case a spin density due to the presence of spin fields acts as the source of torsion while the Einstein equations couple energy–momentum and curvature. The torsion in EC theory is fixed by the spin density and is zero in regions of spacetime where the spin density vanishes. Since torsion couples to spin density only very weakly, no experiments have detected it [12,13]. Despite the lack of evidence for torsion, EC theory can be viewed as a viable alternative to GR that has the advantage of incorporating spin in a straightforward manner.

In one approach to EC theory, it is common to consider invariance under diffeomorphisms (Diffs) in a spacetime frame as well as invariance under local Lorentz transformations (LLTs) in a local Lorentz basis as fundamental spacetime symmetries. In GR, the spin connection is completely determined by the vierbein, while in EC theory, the vierbein and spin connection are independent of each other when the torsion is nonzero. However, the spin connection does not propagate as independent degrees of freedom in EC theory. There are also generalizations that go beyond EC theory, which contain additional terms in the pure-gravity action that allow the spin connection to propagate. However, in such extensions, questions concerning unphysical ghost modes, negative energies, or discrepancies with observations must be resolved, and the differences with GR are greater.

In many respects, local spacetime symmetries are similar to local gauge symmetries, where the latter are fundamental in the Standard Model (SM) in particle physics. This has led to considerable interest in EC theory formulated as a gauge theory, where Poincare symmetry is treated as a local gauge symmetry [3–11]. In this context, the fundamental spacetime symmetries consist of LLTs and local translations (LTs), and the vierbein becomes the gauge field for the LTs while the spin connection is the gauge field for the LLTs. At the same time, a theory of gravity must be covariant, which implies invariance under Diffs, as long as the theory contains only physical dynamical fields. In the end, it is largely a matter of choice whether to consider Diffs and LLTs as fundamental versus considering LTs and LLTs in this way.

In parallel with constructing EC theory and its generalizations as gauge theories with local Poincare invariance, much effort has also been devoted to understanding how gravity can be quantized and how the effects of this might be discovered. One idea that has been investigated widely is that in a quantum theory of gravity, small violations of spacetime symmetry can occur. Mechanisms for how this might occur can be found, for example, in string theory [14–16]. A phenomenological framework known as the Standard-Model Extension (SME) has been developed, which is useful in exploring the possibility of spacetime symmetry breaking [17–24]. It is based on the general idea that no matter how such breakings might occur, the effects of these violations should be describable in the context of an effective field theory that contains both the SM and EC theories at low energies.

The SME is constructed by adding to the action any possible interaction terms that involve couplings with SM or gravitational fields that break spacetime symmetry while maintaining observer independence. Such interactions introduce fixed background fields, usually referred to as SME coefficients, which couple with the SM and gravitational fields. Using the SME framework, many experimental searches for spacetime symmetry breaking have been conducted with extremely high sensitivities over the past several decades [25].

When the gravitational sector of the SME was first investigated, it was found that there are fundamental differences depending on whether spacetime symmetry breaking occurs spontaneously or explicitly [20]. Due to the nondynamical nature of the backgrounds with explicit breaking, inconsistencies between the Bianchi identities and equations of motion can occur, which in a Riemann geometry create conflicts with covariant energy–momentum conservation. However, these results, known as no-go results, do not occur with spontaneous breaking. For this reason, it was typically assumed when using the gravity sector of the SME that spacetime symmetry breaking occurs spontaneously [26–28]. Specifically, it was spontaneous breaking of Diffs and LLTs that was most widely investigated, and questions concerning Nambu–Goldstone (NG) bosons, massive Higgs-like fields, and the possibility of a gravitational Higgs mechanism were examined [29,30]. In addition, a linkage between spontaneous breaking of Diffs and LLTs was found, in that when vacuum values exist that spontaneously break Diffs, vacuum values also exist that spontaneously break LLTs, and vice versa. Vector models known as Bumblebee models were studied as examples that illustrate these and other results of spontaneous breaking of Diffs and LLTs [16,20,29–31].

In some subsequent works, explicit breaking of Diffs and LLTs, due to the presence of fixed backgrounds, was examined in more detail, and it was found that in some cases the no-go results can be evaded [32–35]. Noether identities that must hold as a result of observer independence were shown to provide a useful tool for determining whether a particular model must be ruled out or not. It was also found that in some cases a hybrid form of spacetime symmetry breaking can occur, involving both explicit and spontaneous breaking. The question of whether the SME can accommodate explicit spacetime symmetry breaking was reexamined, and it was shown that in some cases the potential no-go results can be evaded and the SME can be used with explicit breaking [36,37]. For simplicity, effects of torsion were largely ignored in much of these works, and LTs were not considered.

However, in [38–42], the roles of torsion and LTs were considered in more detail in gravity theories with explicit-breaking SME coefficients, and some interesting properties were found. For example, it was shown that when a nondynamical background field is present in EC theory, the torsion can be nonzero in regions where there is no spin density associated with matter. Linkages between explicit breaking of LTs and explicit breaking of Diffs and LLTs were explored as well.

All of this led to a complete generalization of the SME being developed, which includes nondynamical backgrounds that can explicitly break Diffs and LLTs [43]. However, for simplicity, many of the effects of torsion were again largely ignored and LTs were not directly considered. Nonetheless, possible linkages between explicit breaking of Diffs and LLTs, spontaneous breaking of these symmetries, or hybrid combinations of both types of breakings were catalogued and investigated. Applications where this new approach can be used and some examples of tests and their sensitivities are described in [44].

Ultimately, any theory with explicit breaking that does not evade the no-go results must be ruled out in Riemann geometry or in Riemann–Cartan geometry if torsion is included. An idea that has been widely explored is that these theories might instead be consistent in a Finsler geometry or some other beyond-Riemann geometry [45–55]. Based on this, the interpretation when working with the generalization of the SME that includes explicit breaking is that any detection of spacetime symmetry breaking involving interactions that do not evade the no-go results would indicate the existence of a beyond-Riemann geometry, such as Finsler geometry [43,44].

The primary goals of this paper are to revisit EC theory when background fields that explicitly or spontaneously break spacetime symmetries are present and to elaborate on and fill in certain features or possibilities that have largely or partially been ignored. A general formalism containing a variety of different types of backgrounds is used. In particular, breaking of all three spacetime symmetries, Diffs, LTs, and LLTs, and the linkages between them, are examined for both explicit and spontaneous breaking with torsion included. In each case, the question of whether no-go results can appear is addressed, and implications concerning covariant conservation of the energy–momentum tensor with torsion present are examined. Specific examples of how LTs are broken either explicitly or spontaneously when torsion is present are provided.

The organization of this paper is as follows: Section 2 gives background on EC Theory for the usual case of when Diffs, LTs, and LLTs are not broken. Readers already familiar with EC theory may want to skip ahead and start with Section 3, which then looks at how these symmetries are broken either spontaneously or explicitly when background fields are present. Section 4 looks at explicit breaking in detail, including the no-go results and Noether identities that follow from observer independence. Spontaneous breaking is examined in Section 5, and examples of Bumblebee theories with spontaneous breaking of LTs with torsion included are presented. Section 6 provides some discussion and conclusions. The notation and conventions used here follow those in [20].

2. EC Theory

In EC theory, there is both curvature and torsion, and it is assumed that the nonmetricity vanishes so that $D_\lambda g_{\mu\nu} = 0$. A vierbein formalism can be used, where ${e_\mu}^a$ is the vierbein and ${\omega_\mu}^{ab}$ is the spin connection. In this notation, Greek letters denote components with respect to the spacetime frame, while Latin letters denote components with respect to the local Lorentz basis. The metric is given in terms of the vierbein as $g_{\mu\nu} = {e_\mu}^a {e_\nu}^b \eta_{ab}$, where η_{ab} is the local Minkowski metric. The connection is the Cartan connection $\Gamma^\lambda{}_{\mu\nu}$, which has an antisymmetric part that defines the torsion tensor:

$$T^\lambda{}_{\mu\nu} = \Gamma^\lambda{}_{\mu\nu} - \Gamma^\lambda{}_{\nu\mu} \, . \quad (1)$$

Using a tilde to denote the symmetric Levi-Civita connection in GR, which has components given by the Christoffel symbol,

$$\tilde{\Gamma}^\lambda{}_{\nu\mu} = \left\{ \begin{array}{c} \lambda \\ \mu\nu \end{array} \right\} , \quad (2)$$

and defining the contorsion tensor as

$$K^{\lambda\mu\nu} = \frac{1}{2}(T^{\lambda\mu\nu} - T^{\mu\nu\lambda} - T^{\nu\mu\lambda}) \,, \tag{3}$$

the Cartan connection can be written as

$$\Gamma^{\lambda}{}_{\mu\nu} = \tilde{\Gamma}^{\lambda}{}_{\nu\mu} + K^{\lambda}{}_{\nu\mu} \,. \tag{4}$$

Assuming that the covariant derivative of the vierbein vanishes gives a relation involving the spin connection $\omega_{\mu}{}^{ab}$ as

$$D_\mu e_\nu{}^a = \partial_\mu e_\nu{}^a - \Gamma^{\lambda}{}_{\mu\nu} e_\lambda{}^a + \omega_\mu{}^a{}_b e_\nu{}^b = 0 \,. \tag{5}$$

With this, the connection and torsion can be found in terms of the vierbein and spin connection:

$$\Gamma^{\lambda}{}_{\mu\nu} = e^{\lambda a}(\partial_\mu e_{\nu a} - \omega_\mu{}^b{}_a e_{\nu b}) \,, \tag{6}$$

$$T_{\lambda\mu\nu} = e_\lambda{}^a [(\partial_\mu e_{\nu a} + \omega_{\mu ab}\, e_\nu{}^b) - (\mu \leftrightarrow \nu)] \,. \tag{7}$$

The curvature tensor is defined as

$$R^{\kappa}{}_{\lambda\mu\nu} = \partial_\mu \Gamma^{\kappa}{}_{\nu\lambda} - \Gamma^{\kappa}{}_{\mu\sigma}\Gamma^{\sigma}{}_{\nu\lambda} - (\mu \leftrightarrow \nu) \,, \tag{8}$$

and its contractions,

$$R_{\mu\nu} = R^{\kappa}{}_{\mu\kappa\nu} \,, \quad R = g^{\mu\nu} R_{\mu\nu} \,, \tag{9}$$

give, respectively, the Ricci tensor and the curvature scalar. Note that in EC theory, the Ricci tensor is not symmetric and neither is the Einstein tensor, $G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R$. The curvature can also be given in terms of the vierbein and spin connection as

$$R^{\kappa}{}_{\lambda\mu\nu} = e^{\kappa}{}_a e_\lambda{}^b [(\partial_\mu \omega_\nu{}^a{}_b + \omega_\mu{}^a{}_c \omega_\nu{}^c{}_b) - (\mu \leftrightarrow \nu)] \,. \tag{10}$$

Bianchi identities for the curvature and torsion in EC theory are off-shell geometric identities, which are given as:

$$\sum_{(\lambda\mu\nu)} [D_\nu R^{\alpha}{}_{\beta\lambda\mu} + T^{\sigma}{}_{\lambda\mu} R^{\alpha}{}_{\beta\sigma\nu}] = 0 \,, \tag{11}$$

$$\sum_{(\lambda\mu\nu)} [D_\nu T^{\alpha}{}_{\lambda\mu} + T^{\sigma}{}_{\lambda\mu} T^{\alpha}{}_{\sigma\nu} - R^{\alpha}{}_{\nu\lambda\mu}] = 0 \,, \tag{12}$$

where the sum in each case is over the cyclic permutations of $(\lambda\mu\nu)$. Two useful off-shell identities can be derived from these by contracting and manipulating terms [20]. The results are

$$(D_\mu - T^{\lambda}{}_{\lambda\mu}) G^{\mu\nu} + T_{\lambda\mu}{}^{\nu} G^{\mu\lambda} - \tfrac{1}{2} R^{\alpha\beta\mu\nu} \hat{T}_{\mu\alpha\beta} = 0 \,, \tag{13}$$

$$G^{\mu\nu} - G^{\nu\mu} = -(D_\sigma - T^{\lambda}{}_{\lambda\sigma}) \hat{T}^{\sigma\mu\nu} \,. \tag{14}$$

2.1. EC Action

The generic form of the action can be written in terms of the vierbein, spin connection, and matter fields as

$$S = S_g + S_{g,m} = \frac{1}{2\kappa} \int d^4x e\, R(e_\mu{}^a, \omega_\mu{}^{ab}) + \int d^4x e \mathcal{L}_m(e_\mu{}^a, \omega_\mu{}^{ab}, f^\psi) \,. \tag{15}$$

Here, S_g is the Einstein–Hilbert term, with the curvature scalar expressed as a function of the vierbein and the spin connection. The matter term $S_{g,m}$ depends on the vierbein, spin connection, and matter fields, where the latter are denoted generically as f^ψ. The specific forms and suitable indices for f^ψ depend on the types of fields that are included, which can consist of tensor and spin fields. The coupling $\kappa = 8\pi G$ (with $c = 1$), and e is the determinant of the vierbein.

Note that a cosmological constant term could also be added to the action, and as an alternative to EC theory, kinetic terms for the torsion would be added as well. This would permit inclusion of teleparallel gravity when the curvature vanishes [56,57]. However, these considerations go beyond the scope of this work, which for simplicity considers only flat spacetime background in vacuum.

Variation of the action term $S_{g,m}$ with respect to $e_\mu{}^a$, $\omega_\mu{}^{ab}$, and f^ψ has the form:

$$\delta S_{g,m} = \int d^4x e \left[T_e^{\mu\nu} e_{\nu a}\, \delta e_\mu{}^a + \tfrac{1}{2} S_\omega{}^\mu{}_{ab}\, \delta\omega_\mu{}^{ab} + \frac{\delta S_{g,m}}{\delta f^\psi} \delta f^\psi \right], \tag{16}$$

which defines $T_e^{\mu\nu}$ as the energy–momentum tensor and $S_\omega{}^\mu{}_{ab}$ as the spin density. Varying the matter fields gives the Euler–Lagrange expression for f^ψ.

When the full action is varied with respect to the dynamical fields, the result is:

$$\delta S = \int d^4x e \left[(-\frac{1}{\kappa} G^{\mu\nu} + T_e^{\mu\nu}) e_{\nu a}\, \delta e_\mu{}^a + (\frac{1}{2\kappa} \hat{T}^{\lambda\mu\nu} e_{\mu a} e_{\nu b} + \tfrac{1}{2} S_\omega{}^\mu{}_{ab})\, \delta\omega_\mu{}^{ab} + \frac{\delta S_{g,m}}{\delta f^\psi} \delta f^\psi \right], \tag{17}$$

where

$$\hat{T}^{\lambda\mu\nu} = T^{\lambda\mu\nu} + T^\alpha{}_\alpha{}^\mu g^{\lambda\nu} - T^\alpha{}_\alpha{}^\nu g^{\lambda\mu} \tag{18}$$

is the trace-corrected torsion tensor. Setting $\delta S = 0$ gives the equations of motion for the vierbein, spin connection, and matter fields, respectively, as

$$G^{\mu\nu} = \kappa T_e^{\mu\nu}, \tag{19}$$

$$\hat{T}^{\lambda\mu\nu} = -\kappa S_\omega{}^{\lambda\mu\nu}, \tag{20}$$

$$\frac{\delta S_{g,m}}{\delta f^\psi} = 0. \tag{21}$$

2.2. Spacetime Symmetries in EC Theory

The local spacetime symmetries, diffeomorphisms (Diffs), local Lorentz transformations (LLTs), and local translations (LTs), act on $e_\mu{}^a$, $\omega_\mu{}^{ab}$, and f^ψ leaving the action invariant. Under Diffs, the transformation rules are given as Lie derivatives, which can be rewritten using the full covariant derivative D_μ that corrects with both the Cartan connection $\Gamma^\lambda{}_{\mu\nu}$ and the spin connection, depending on what types of fields it acts on. However, the Poincare algebra that includes LLTs and LTs uses a Lorentz covariant derivative, $D_a^{(\omega)} = e^\mu{}_a D_\mu^{(\omega)}$, where $D_\mu^{(\omega)}$ corrects with only the spin connection [5]. For example,

$$D_\mu^{(\omega)} e_\nu{}^a = \partial_\mu e_\nu{}^a + \omega_\mu{}^a{}_b e_\nu{}^b, \tag{22}$$

which does not vanish, since the term with the Cartan connection in (5) is not included. Notice that when acting on a tensor with only local indices, for example, A^b, it can also be written as

$$D_\mu^{(\omega)} A^b = e_\nu{}^b D_\mu A^\nu, \tag{23}$$

where $A^b = e_\nu{}^b A^\nu$.

The transformation rules for the the vierbein and spin connection are then defined as follows: Under Diffs with parameters ξ^μ,

$$\begin{aligned} \delta_{\text{Diff}}\, e_\mu{}^a &= \mathcal{L}_\xi e_\mu{}^a = (\partial_\mu \xi^\alpha) e_\alpha{}^a + \xi^\alpha \partial_\alpha e_\mu{}^a \\ &= (D_\mu \xi^\alpha) e_\alpha{}^a - T^\alpha{}_{\mu\beta} \xi^\beta e_\alpha{}^a - \xi^\alpha \omega_\alpha{}^{ab} e_{\mu b}, \end{aligned} \tag{24}$$

$$\delta_{\text{Diff}}\,\omega_\mu{}^{ab} \;=\; \mathcal{L}_\xi \omega_\mu{}^{ab} = (\partial_\mu \xi^\alpha)\omega_\alpha{}^{ab} + \xi^\alpha \partial_\alpha \omega_\mu{}^{ab}\,, \tag{25}$$

where $\mathcal{L}_\xi$ denotes a Lie derivative along ξ^μ. Under LTs with parameters ϵ^a:

$$\delta_{\text{LT}}\,e_\mu{}^a \;=\; D_\mu^{(\omega)}\epsilon^a + \epsilon^b T^a{}_{b\mu}\,, \tag{26}$$

$$\delta_{\text{LT}}\,\omega_\mu{}^{ab} \;=\; \epsilon^c R^{ab}{}_{c\mu}\,. \tag{27}$$

Under LLTs with parameters $\epsilon_a{}^b = -\epsilon^b{}_a$:

$$\delta_{\text{LLT}}\,e_\mu{}^a = -\epsilon^a{}_b e_\mu{}^b\,, \tag{28}$$

$$\begin{aligned}\delta_{\text{LLT}}\,\omega_\mu{}^{ab} &= D_\mu^{(\omega)}\epsilon^{ab} \\ &= \partial_\mu \epsilon^{ab} + \omega_\mu{}^a{}_c\,\epsilon^{cb} + \omega_\mu{}^b{}_c\,\epsilon^{ac}\,.\end{aligned} \tag{29}$$

Note that these transformations are not independent, since it can be shown with $\xi^\mu = \epsilon^\mu = e^\mu{}_a \epsilon^a$ and $\epsilon^{ab} = \epsilon^\mu \omega_\mu{}^{ab}$ that they are related by [5,7,38]:

$$\delta_{\text{Diff}}(\xi^\mu) = \delta_{\text{LT}}(\epsilon^a) + \delta_{\text{LLT}}(\epsilon^{ab})\,. \tag{30}$$

As a result, symmetry under Diffs and LLTs implies symmetry under LTs and LLTs, and vice versa.

Dynamical tensors in EC theory can have components given with respect to either the spacetime frame or a local basis frame. For example, a vector field can have spacetime components A_μ or local components A_b, where these are related by the vierbein: $A_\mu = e_\mu{}^b A_b$. Under Diffs, LLTs, and LTs, the local frame components, A_b, transform, respectively, as

$$\begin{aligned}&\delta_{\text{Diff}}\,A_b = \xi^\nu \partial_\nu A_b\,, \\ &\delta_{\text{LLT}}\,A_b = -\epsilon_b{}^c A_c\,, \\ &\delta_{\text{LT}}\,A_b = \epsilon^c e^\mu{}_c D_\mu^{(\omega)} A_b\,.\end{aligned} \tag{31}$$

Making the substitutions $\xi^\mu = \epsilon^\mu = e^\mu{}_c \epsilon^c$ and $\epsilon^{bc} = \epsilon^\mu \omega_\mu{}^{bc}$, it follows that Equation (30) holds for these transformations. These symmetry transformations can also be performed on the components A_μ defined with respect to the spacetime frame. The fact that A_b and the vierbein transform results in transformations of A_μ as well, which are given under Diffs, LLTs, and LTs, respectively, as

$$\begin{aligned}&\delta_{\text{Diff}}\,A_\mu = (\partial_\mu \xi^\nu) A_\nu + \xi^\nu \partial_\nu A_\mu\,, \\ &\delta_{\text{LLT}}\,A_\mu = 0\,, \\ &\delta_{\text{LT}}\,A_\mu = (D_\mu^{(\omega)}\epsilon^b) e^\nu{}_b A_\nu + \epsilon^b e^\nu{}_b D_\nu A_\mu + \epsilon^b T^\nu{}_{b\mu} A_\nu\,.\end{aligned} \tag{32}$$

Notice that in this case with $\xi^\nu = \epsilon^\nu = e^\nu{}_c \epsilon^c$, it follows that $\delta_{\text{Diff}}\,A_\mu = \delta_{\text{LT}}\,A_\mu$, as expected from Equation (30), given that $\delta_{\text{LLT}}\,A_\mu = 0$.

2.3. Noether Identities in EC Theory

There are two Noether theorems that are important in theoretical physics [58–60]. The first states that global symmetries give rise to conserved currents, while the second shows that local symmetries give rise to off-shell identities that must hold. It is the second theorem that is used here, and it can be applied to the whole action or individually to any term in the action that is invariant under a local symmetry.

For example, the Einstein–Hilbert term S_g is invariant under Diffs, LLTs, and LTs. With $\delta S_g = 0$ under LTs as given in Equations (26) and (27), the Noether identity that follows from this directly matches the contracted form of the first Bianchi identity in (13). The Noether identity that follows from LLTs using (28) and (29) directly matches the contracted form of the second Bianchi identity in (14). Under Diffs, using (24) and (25), the identity that follows has the form:

$$(D_\mu - T^\lambda{}_{\lambda\mu})G^{\mu\nu} + T_{\lambda\mu}{}^\nu G^{\mu\lambda} - \tfrac{1}{2}R^{\alpha\beta\mu\nu}\hat{T}_{\mu\alpha\beta} \\ -\tfrac{1}{2}\omega^{\nu ab}e_{\alpha a}\,e_{\beta b}\left[G^{\alpha\beta} - G^{\beta\alpha} + (D_\sigma - T^\lambda{}_{\lambda\sigma})\hat{T}^{\sigma\alpha\beta}\right] = 0\,, \tag{33}$$

where this is identically satisfied off-shell by virtue of the identities from LTs and LLTs, or equivalently by virtue of the two Bianchi identities. This gives an illustration of how the three identities under LTs, LLTs, and Diffs are linked as indicated by Equation (30).

Similarly, the matter term $S_{g,m}$ is invariant under LTs, LLTs, and Diffs when the matter fields f^ψ are dynamical and the Lagrangian $\mathcal{L}_m(e_\mu{}^a, \omega_\mu{}^{ab}, f^\psi)$ is a scalar under each transformation. For simplicity, the matter fields can be put on-shell so that Equation (21) holds. In this case, the three identities that follow, respectively, from LTs, LLTs, and Diffs are:

$$(D_\mu - T^\lambda{}_{\lambda\mu})T_e{}^{\mu\nu} + T_{\lambda\mu}{}^\nu T_e{}^{\mu\lambda} + \tfrac{1}{2}R^{\alpha\beta\mu\nu}S_{\omega\,\mu\alpha\beta} = 0\,, \tag{34}$$

$$T_e{}^{\mu\nu} - T_e{}^{\nu\mu} = (D_\sigma - T^\lambda{}_{\lambda\sigma})S_\omega{}^{\sigma\mu\nu}\,, \tag{35}$$

$$(D_\mu - T^\lambda{}_{\lambda\mu})T_e{}^{\mu\nu} + T_{\lambda\mu}{}^\nu T_e{}^{\mu\lambda} + \tfrac{1}{2}R^{\alpha\beta\mu\nu}S_{\omega\,\mu\alpha\beta} \\ -\tfrac{1}{2}\omega^{\nu ab}e_{\alpha a}\,e_{\beta b}\left[T_e{}^{\alpha\beta} - T_e{}^{\beta\alpha} - (D_\sigma - T^\lambda{}_{\lambda\sigma})S_\omega{}^{\sigma\alpha\beta}\right] = 0\,. \tag{36}$$

In this case, there are no geometric Bianchi identities that enforce these identities as was the case with the Einstein–Hilbert term. Instead, these identities are the result of the local symmetries, with the matter fields put on-shell. Note that the gravitational fields are still off-shell in these identities, and only the matter fields have been put on-shell. The fact that the spin connection appears in the identity stemming from Diffs in Equation (36) appears problematic, since it is not a covariant tensor. However, the identity for Diffs is a combination of the identities for LTs and LLTs, as expected from Equation (30). Thus, the identity for Diffs is automatically satisfied as a result of the identities for LTs and LLTs.

2.4. Theoretical Consistency and Energy–Momentum Conservation

In GR, in the absence of spin and torsion, the Einstein equations reduce to $\tilde{G}^{\mu\nu} = \kappa T_e{}^{\mu\nu}$, where the tilde denotes that the connection used to define the curvature and Einstein tensor is the Levi-Civita connection $\tilde{\Gamma}^\lambda{}_{\mu\nu}$. The Einstein tensor in Riemann space is symmetric, obeying $\tilde{G}^{\mu\nu} = \tilde{G}^{\nu\mu}$, and the relevant contracted Bianchi identity is $\tilde{D}_\mu \tilde{G}^{\mu\nu} = 0$, where the covariant derivative, $\tilde{D}_\mu$, uses $\tilde{\Gamma}^\lambda{}_{\mu\nu}$. The identities due to LTs and LLTs in Equations (34) and (35) with zero torsion and putting the matter fields on-shell reduce, respectively, to $\tilde{D}_\mu T_e{}^{\mu\nu} = 0$ and $T_e{}^{\mu\nu} = T_e{}^{\nu\mu}$. These two results automatically satisfy the identity for Diffs in Equation (36) with the torsion set to zero. Thus, in GR, there is complete consistency between the Bianchi identities, the Einstein equations, and the equations of motion for the matter fields. As a result of these, the energy–momentum tensor is both symmetric and covariantly conserved.

In EC theory, there is also full consistency between the contracted Bianchi identities, the Einstein and spin connection equations, and the equations of motion for the matter fields. However, in this case, in the presence of spin density and torsion, $G^{\mu\nu}$ is not symmetric and $D_\mu G^{\mu\nu} \neq 0$, and therefore, $T_e{}^{\mu\nu} \neq T_e{}^{\mu\nu}$ and $D_\mu T_e{}^{\mu\nu} \neq 0$. These properties of energy–momentum in EC theory have been examined and discussed previously [3,61–63]. (See also [5,8,10]). In particular, with spin density and torsion present and on-shell, it has been shown that an effective Riemann geometry with a conserved and symmetric energy–momentum tensor can be identified.

To see this, first use that the curvature tensor $R^\kappa{}_{\lambda\mu\nu}$ can be separated into a Riemann part $\tilde{R}^\kappa{}_{\lambda\mu\nu}$ and additional terms involving the contorsion:

$$R^\kappa{}_{\lambda\mu\nu} = \tilde{R}^\kappa{}_{\lambda\mu\nu} + [\tilde{D}_\mu K^\kappa{}_{\nu\lambda} + K^\kappa{}_{\mu\sigma}K^\sigma{}_{\nu\lambda} - (\mu \leftrightarrow \nu)]\,. \tag{37}$$

The torsion can then be eliminated on-shell in terms of the spin density using Equation (20), which allows the on-shell contorsion tensor to be written as

$$K^{\lambda\mu\nu} = -\frac{\kappa}{2}(S_\omega{}^{\lambda\mu\nu} - S_\omega{}^{\mu\nu\lambda} - S_\omega{}^{\nu\mu\lambda} + g^{\mu\nu}S_\omega{}^{\lambda} - g^{\lambda\mu}S_\omega{}^{\nu})\,, \tag{38}$$

where $S_\omega{}^{\mu} = S_\omega{}^{\sigma}{}_{\sigma}{}^{\mu}$. With this substituted into Equation (37), an effective theory can be found that uses the Riemann curvature. However, there is still the caveat that minimal couplings in the full theory depend on the Cartan connection $\Gamma^\lambda{}_{\mu\nu}$, not $\tilde{\Gamma}^\lambda{}_{\mu\nu}$, so the full theory remains non-Riemann. Nonetheless, with the torsion eliminated in this way on-shell, the Einstein equation in (19) becomes

$$\begin{aligned}\tilde{G}^{\mu\nu} &= \kappa T^{\mu\nu}_{\text{eff}} + \tfrac{\kappa^2}{4}(2S_\omega{}^{\mu\alpha\beta}S_{\omega\,\alpha\beta}{}^{\nu} + 2S_\omega{}^{\mu\nu\sigma}S_{\omega\,\sigma} - S_\omega{}^{\mu\alpha\beta}S_\omega{}^{\nu}{}_{\alpha\beta}) \\ &\quad - \tfrac{\kappa^2}{8}g^{\mu\nu}(S_\omega{}^{\alpha\beta\gamma}S_{\omega\,\gamma\alpha\beta} - S_\omega{}^{\alpha\beta\gamma}S_{\omega\,\alpha\beta\gamma} - S_{\omega\,\sigma}S_\omega{}^{\sigma})\,,\end{aligned} \tag{39}$$

where an effective energy–momentum tensor is defined as

$$T^{\mu\nu}_{\text{eff}} = T^{\mu\nu}_e - \tfrac{1}{2}\tilde{D}_\sigma(S_\omega{}^{\sigma\mu\nu} + S_\omega{}^{\mu\nu\sigma} + S_\omega{}^{\nu\mu\sigma})\,. \tag{40}$$

Note that $T^{\mu\nu}_{\text{eff}}$ has the form of a Belinfante–Rosenfeld energy–momentum tensor [61–63], where in a theory with spin the extra added spin-density terms lead to a redefined energy–momentum tensor that is symmetric. Although Equation (39) is not yet symmetric, the LLT Noether identity in Equation (35) rewritten in terms of $\tilde{D}_\sigma$ and additional quadratic terms in the spin density can be used to eliminate the antisymmetric part of the effective energy momentum from Equation (39). This leaves an Einstein equation that is effectively Riemann:

$$\begin{aligned}\tilde{G}^{\mu\nu} &= \kappa T^{(\mu\nu)}_{\text{eff}} + \tfrac{\kappa^2}{2}[(S_\omega{}^{\mu\alpha\beta}S_{\omega\,\alpha\beta}{}^{\nu} + S_\omega{}^{\nu\alpha\beta}S_{\omega\,\alpha\beta}{}^{\mu}) + (S_\omega{}^{\mu\nu}{}_{\sigma} + S_\omega{}^{\nu\mu}{}_{\sigma})S_\omega{}^{\sigma}] \\ &\quad - \tfrac{\kappa^2}{8}g^{\mu\nu}[(S_\omega{}^{\alpha\beta\gamma}S_{\omega\,\gamma\alpha\beta} - S_\omega{}^{\alpha\beta\gamma}S_{\omega\,\alpha\beta\gamma} - S_{\omega\,\sigma}S_\omega{}^{\sigma})\,,\end{aligned} \tag{41}$$

where $T^{(\mu\nu)}_{\text{eff}} = \frac{1}{2}(T^{\mu\nu}_{\text{eff}} + T^{\nu\mu}_{\text{eff}})$. In this form, the right-hand side is symmetric and is consistent with $\tilde{G}^{\mu\nu} = \tilde{G}^{\nu\mu}$, and since $\tilde{D}_\mu\tilde{G}^{\mu\nu} = 0$, the combined terms on the right-hand side also have a vanishing covariant divergence with respect to $\tilde{D}_\mu$. Thus, in this context, the spin density acts effectively like additional contributions to the energy–momentum.

Since torsion is extremely weak, and since the coupling κ is small, the quadratic contributions $\sim \kappa^2 S^2_\omega$ can be neglected at leading order in a perturbative treatment. In this approximation,

$$T^{\mu\nu}_{\text{eff}} \simeq T^{(\mu\nu)}_e - \tfrac{1}{2}\tilde{D}_\sigma(S_\omega{}^{\mu\nu\sigma} + S_\omega{}^{\nu\mu\sigma})\,, \tag{42}$$

is symmetric, and the effective Einstein equation reduces to $\tilde{G}^{\mu\nu} \simeq \kappa T^{\mu\nu}_{\text{eff}}$. From this, it follows that $\tilde{D}_\mu T^{\mu\nu}_{\text{eff}} \simeq 0$, and in a flat spacetime limit with negligible torsion, $\partial_\mu T^{\mu\nu}_{\text{eff}} \simeq 0$.

3. Spacetime Symmetry Breaking

Many ideas have been put forward for how spacetime symmetries might be broken, including mechanisms in string theory, above threshold cosmic rays, modified gravity, noncommutative geometry, loop quantum gravity, spacetime foam, Chern Simons gravity, Hořava gravity, and massive gravity. See, for examples, [14–16,64–84].

However, in EC theory working at the level of observer-independent effective field theory, as in the framework of the SME, violation of spacetime symmetry occurs when a fixed background tensor field interacts with a dynamical gravitational or matter field [20,32–37,43,44]. In general, the backgrounds can have components with respect to both spacetime frames and local tangent spaces. The symmetry breaking occurs because the background fields are fixed and do not transform under Diffs, LLTs, or LTs, while fully dynamical fields do transform. The combination of an active transformation for fully dynamical fields with the background fields being held fixed is referred to as a particle transformation. However, at the same time, a physical theory must be observer-independent, which requires invariance under general coordinate transformations and passive changes of basis states in local tangent spaces. These are called observer transformations, and under them the components of fixed backgrounds transform passively, and the action S is mathematically left unchanged. In EC theory with no fixed backgrounds, mathematical observer transformations of Diffs, LLTs, and LTs can be written in the form of inverse transformations to the corresponding active transformations. However, when fixed background fields are present, the particle transformations are broken, while the mathematical observer invariances must still hold.

3.1. EC Theory with Background Fields

The generic form for the action of an EC theory with fixed background fields can be written as

$$\begin{aligned} S &= S_g + S_{\bar{k}_X} + S_{g,\bar{k}_Y} + S_{g,m,\bar{k}_Z} \\ &= \frac{1}{2\kappa}\int d^4x\, e\, R(e_\mu{}^a, \omega_\mu{}^{ab}) - \frac{1}{2\kappa}\int d^4x\, e\, \mathcal{U}(e_\mu{}^a, \bar{k}_X) \\ &\quad + \frac{1}{2\kappa}\int d^4x\, e\, \mathcal{L}_{g,\bar{k}_Y}(e_\mu{}^a, \omega_\mu{}^{ab}, \bar{k}_Y) + \int d^4x\, e\, \mathcal{L}_{g,m,\bar{k}_Z}(e_\mu{}^a, \omega_\mu{}^{ab}, f^\psi, \bar{k}_Z)\,, \end{aligned} \tag{43}$$

where the total action has been divided into four terms. The first term, S_g, has the Einstein–Hilbert form, which depends on the dynamical vierbein and spin connection. The second term couples the vierbein directly to fixed backgrounds denoted generically as $\bar{k}_X$, where the bar indicates that $\bar{k}_X$ is a fixed field, and the indices X generically label the spacetime and local indices carried by it. The remaining two terms contain fixed background fields, which are denoted generically as $\bar{k}_Y$ and $\bar{k}_Z$, where Y and Z generically label the spacetime and local indices carried by each background. In the last term, f^ψ generically denotes all the dynamical matter fields. It is assumed that the three backgrounds $\bar{k}_X$, $\bar{k}_Y$, and $\bar{k}_Z$ are different from each other and that no covariant derivatives act directly on them. However, if symmetry transformations or integrations by parts are performed in the action, this can result in expressions where covariant derivatives act on the backgrounds, in which case it is assumed that $\bar{k}_X$, $\bar{k}_Y$, or $\bar{k}_Z$ are not covariantly constant.

The terms $S_{\bar{k}_X}$, $S_{g,\bar{k}_Y}$, and $S_{g,m,\bar{k}_Z}$ are all symmetry-breaking terms. $S_{\bar{k}_X}$ is a potential term, which includes possible mass terms for the vierbein formed using background fields. There is no dependence on the spin connection in $S_{\bar{k}_X}$, since that would have to originate from covariant derivatives acting on the backgrounds $\bar{k}_X$. $S_{g,\bar{k}_Y}$ is a pure-gravity term, consisting of interactions between the vierbein, spin connection, and the backgrounds $\bar{k}_Y$. To maintain covariance, it is assumed that any terms in $S_{g,\bar{k}_Y}$ consist only of couplings between the backgrounds $\bar{k}_Y$ with the curvature, torsion, or covariant derivatives of the curvature or torsion. Since both $S_{\bar{k}_X}$ and $S_{g,\bar{k}_Y}$ exclude matter contributions, they both carry the dimensional coupling, $1/2\kappa$, as in the Einstein–Hilbert term. The last term, $S_{g,m,\bar{k}_Z}$, is a matter–gravity term, consisting of interactions between the vierbein, spin connection, matter fields, and the backgrounds $\bar{k}_Z$. It also contains conventional matter–gravity couplings that do not couple to the backgrounds $\bar{k}_Z$.

In general, there is some ambiguity between how the backgrounds $\bar{k}_X$, $\bar{k}_Y$ and $\bar{k}_Z$ are determined, since not all of the background coefficients are independent or physical. This is because coordinate changes and field redefinitions can be used to move sensitivity to spacetime symmetry breaking from one sector to another, including between the potential, pure-gravity, and matter–gravity sectors [20,28,36,85,86]. With this in mind, it should be assumed before splitting the full action in (43) into these sectors that any unphysical coefficients have been removed and that field redefinitions and coordinate choices have been made that fix an observable set of backgrounds, $\bar{k}_X$, $\bar{k}_Y$, and $\bar{k}_Z$. Note that in some cases, an observable set of backgrounds might consist of combinations of backgrounds with couplings to different particle species [28].

Under particle Diffs, LLTs, and LTs, all three of the background fields remain fixed, transforming as

$$\bar{k}_X \to \bar{k}_X\,, \qquad \bar{k}_Y \to \bar{k}_Y\,, \qquad \bar{k}_Z \to \bar{k}_Z\,, \tag{44}$$

while under observer Diffs, LLTs, and LTs, they transform passively according to their representations as spacetime or local tensors as indicated by the index labels X, Y, and Z. Since the backgrounds $\bar{k}_X$, $\bar{k}_Y$, and $\bar{k}_Z$ are assumed to be observable, the three symmetry-breaking sectors are distinct and independent. Thus, the potential, $\mathcal{U}(e_\mu{}^a, \bar{k}_X)$, and the Lagrangians, $\mathcal{L}_{g,\bar{k}_Y}(e_\mu{}^a, \omega_\mu{}^{ab}, \bar{k}_Y)$ and $\mathcal{L}_{g,m,\bar{k}_Z}(e_\mu{}^a, \omega_\mu{}^{ab}, f^\psi, \bar{k}_Z)$ must each be a scalar under observer spacetime transformations to maintain observer independence. However, they are not scalars under the broken particle transformations.

3.2. Explicit versus Spontaneous Breaking

To understand the nature of the background fields $\bar{k}_X$, $\bar{k}_Y$, and $\bar{k}_Z$, as well as their effects in the context of effective field theory, a distinction must be made between when the spacetime symmetry breaking is spontaneous versus when it is explicit.

With spontaneous breaking, the symmetry is hidden. It is only the vacuum solution that breaks the symmetry, while the full solutions, including excitations in the form of NG modes and additional massive Higgs-like modes, are fully dynamical and transform appropriately under all spacetime transformations. For example, with spontaneous breaking, the backgrounds $\bar{k}_X$, $\bar{k}_Y$, and $\bar{k}_Z$ all equal vacuum expectation values,

$$\bar{k}_X = \langle K_X \rangle \,, \qquad \bar{k}_Y = \langle K_Y \rangle \,, \qquad \bar{k}_Z = \langle K_Z \rangle \,, \tag{45}$$

where K_X , K_Y, and K_Z are dynamical fields. As dynamical fields, all three sets of components K_X, K_Y, and K_Z undergo field variations and have equations of motion. They are very much like any other dynamical field components except that they have vacuum values $\langle K_X \rangle$, $\langle K_Y \rangle$, and $\langle K_Z \rangle$, which spontaneously break spacetime symmetries. In contrast, with explicit breaking, the backgrounds $\bar{k}_X$, $\bar{k}_Y$, and $\bar{k}_Z$ are fixed nondynamical fields that are inserted into the Lagrangian. They have no dynamical field variations and therefore no equations of motion.

For simplicity, consider the case of a vector background field, which has components $\bar{b}_\mu$ with respect to the spacetime frame and components $\bar{b}_a$ with respect to a local basis. Since the vector is fixed under particle transformations, its components are unchanged in either frame, obeying $\delta \bar{b}_\mu = 0$ and $\delta \bar{b}_a = 0$ under particle Diffs, LLTs, and LTs.

In the case of spontaneous breaking, $\bar{b}_\mu$ and $\bar{b}_a$ are vacuum values of a dynamical field with components B_μ or B_a,

$$\langle B_\mu \rangle = \bar{b}_\mu \,, \quad \langle B_a \rangle = \bar{b}_a \,. \tag{46}$$

The dynamical field components are related to each other by the vierbein as $B_\mu = e_\mu{}^a B_a$. With spontaneous breaking, the vierbein also has a vacuum value, $\langle e_\mu{}^a \rangle$, which in a Minkowski background in Cartesian coordinates is $\langle e_\mu{}^a \rangle = \delta_\mu{}^a$. The vierbein vacuum value relates the two vector vacuum values as

$$\bar{b}_\mu = \langle e_\mu{}^a \rangle \bar{b}_a \,. \tag{47}$$

Since the vacuum solutions, $\bar{b}_\mu$, $\bar{b}_a$, and $\langle e_\mu{}^a \rangle$ carry both spacetime and local indices, all three of the symmetries Diffs, LLTs, and LTs are spontaneously broken. In particular, the existence of a vacuum geometry with $\langle e_\mu{}^a \rangle \neq 0$ requires that both $\bar{b}_\mu$ and $\bar{b}_a$ be nonzero if either of them is.

With spontaneous breaking, any action term, $\mathcal{L} = J^\mu B_\mu$, involving a current J^μ interacting with the dynamical field B_μ, can also be written in terms of local components, since $J^\mu B_\mu = J^a B_a$. Hence, as the vector field separates into a vacuum solution and excitations, any terms in the effective action of the form $J^\mu \bar{b}_\mu$ or $J^a \bar{b}_a$ are physically linked by the vacuum vierbein $\langle e_\mu{}^a \rangle$ and its excitations. Similarly, couplings to $\bar{b}^\mu$, with an upper index, are linked to $\bar{b}_\mu$, since these are related by the vacuum solution for the metric, $\langle g_{\mu\nu} \rangle = \langle e_\mu{}^a \rangle \langle e_\nu{}^b \rangle \eta_{ab}$. In local frames, components $\bar{b}_a$ and $\bar{b}^a = \eta^{ab} \bar{b}_b$ are directly related by the Minkowski metric.

With explicit breaking, there are no vacuum values of physical fields. There are only fixed nondynamical backgrounds. A fixed nonzero background vector must have nonzero components $\bar{b}_\mu$ in the spacetime frame and components $\bar{b}_a$ in the local basis. However, there is no linkage between these given by a physical or vacuum vierbein. Similarly, background components $\bar{b}^\mu$ are not linked to $\bar{b}_\mu$ by the physical metric. However, in local frames η_{ab} is the metric, so components $\bar{b}_a$ and $\bar{b}^a$ are related using it.

Any Lagrangian terms such as $J^\mu \bar{b}_\mu$, $J_\mu \bar{b}^\mu$, and $J^a \bar{b}_a$ are therefore all physically distinct when the symmetry breaking is explicit. Taken separately, each of the backgrounds $\bar{b}_\mu$, $\bar{b}^\mu$, or $\bar{b}_a$ might have couplings that explicitly break one or more of the spacetime symmetries, Diffs, LLTs, or LTs, but not necessarily any two or more of them at the same time. This generalizes as well to tensors with more than one index, possibly including tensors that have both spacetime and local indices. An extensive list of examples of possible distinct tensor backgrounds, including which spacetime symmetries they explicitly break, is given in [43], and examples of their phenomenological implications are explored in [44].

Regardless of whether the symmetry breaking is spontaneous or explicit, the action must be a scalar under observer Diffs, LLTs, and LTs. In this case, background vector components, $\bar{b}_\mu$, $\bar{b}^\mu$, or $\bar{b}_a$, have conventional observer transformations appropriate for the type of index they carry. For example, the transformations of $\bar{b}_a$ under observer Diffs, LLTs, and LTs, are given, respectively, as

$$\begin{aligned} \delta_{\text{Diff}}\, \bar{b}_a &= \xi^\nu \partial_\nu \bar{b}_a \,, \\ \delta_{\text{LLT}}\, \bar{b}_a &= -\epsilon_a{}^c \bar{b}_c \,, \\ \delta_{\text{LT}}\, \bar{b}_a &= \epsilon^c e^\mu{}_c D^{(\omega)}_\mu \bar{b}_a \,, \end{aligned} \tag{48}$$

while $\bar{b}_\mu$ transforms under observer Diffs, LLTs, and LTs as

$$\begin{aligned} \delta_{\text{Diff}}\, \bar{b}_\mu &= (\partial_\mu \xi^\nu) \bar{b}_\nu + \xi^\nu \partial_\nu \bar{b}_\mu \,, \\ \delta_{\text{LLT}}\, \bar{b}_\mu &= 0 \,, \\ \delta_{\text{LT}}\, \bar{b}_\mu &= (D^{(\omega)}_\mu \epsilon^b) e^\nu{}_b \bar{b}_\nu + \epsilon^b e^\nu{}_b D_\nu \bar{b}_\mu + \epsilon^b T^\nu{}_{b\mu} \bar{b}_\nu \,. \end{aligned} \tag{49}$$

Note that technically the parameters ϵ^a and $\epsilon_a{}^b$ for observer transformations, as defined here, should have the opposite signs of those used in (31) and (32); however, the parameters in (48) and (49) have been redefined with a minus sign so that they have the same mathematical form as in (31) and (32).

3.3. Equations of Motion

Before writing the variations of the full action S, it is convenient to make some definitions regarding the separate variations of the potential, pure-gravity, and matter–gravity terms.

First, the variation of the potential term $S_{\bar{k}_X}$ with respect to the vierbein can be written as

$$\delta S_{\bar{k}_X} = \int d^4x\, e\, (-\frac{1}{2\kappa} \bar{\mathcal{U}}^{\mu\nu} e_{\nu a})\, \delta e_\mu{}^a \,, \tag{50}$$

where $\bar{\mathcal{U}}^{\mu\nu}$ is written with a bar over it to denote that it includes contributions from the backgrounds $\bar{k}^X$. A coupling $1/2\kappa$ is included in $S_{\bar{k}_X}$, since it does not include matter fields.

Next, define the variations of $S_{g,\bar{k}_Y}$ with respect to the vierbein and spin connection as:

$$\delta S_{g,\bar{k}_Y} = \int d^4x\, e \left[-\frac{1}{\kappa} \bar{\mathcal{G}}^{\mu\nu} e_{\nu a}\, \delta e_\mu{}^a + \frac{1}{2\kappa} \bar{\mathcal{T}}^{\mu\alpha\beta} e_{\alpha a} e_{\beta b} \delta\omega_\mu{}^{ab} \right]. \tag{51}$$

Here, $\bar{\mathcal{G}}^{\mu\nu}$ and $\bar{\mathcal{T}}^{\mu\alpha\beta}$ are written with bars over them to indicate that they include contributions coming from the background fields $\bar{k}_Y$. Since $S_{g,\bar{k}_Y}$ contains a factor of $1/2\kappa$, defining $\bar{\mathcal{G}}^{\mu\nu}$ and $\bar{\mathcal{T}}^{\mu\alpha\beta}$ in this way indicates that these terms have similar mass dimensions as the curvature and torsion terms arising from the Einstein–Hilbert term.

Lastly, the variations of $S_{g,m,\bar{k}_Z}$ with respect to the vierbein, spin connection, and matter fields are written as:

$$\delta S_{g,m,\bar{k}_Z} = \int d^4x\, e \left[\bar{T}_e^{\mu\nu} e_{\nu a}\, \delta e_\mu{}^a + \tfrac{1}{2} \bar{S}_\omega{}^{\mu\alpha\beta} e_{\alpha a} e_{\beta b} \delta\omega_\mu{}^{ab} + \frac{\delta S_{g,m,\bar{k}_Z}}{\delta f^\Psi} \delta f^\Psi \right]. \tag{52}$$

Here, the energy–momentum tensor, $\bar{T}_e^{\mu\nu}$, and spin density tensor, $\bar{S}_\omega{}^\mu{}_{ab}$, are written with bars over them to indicate they include contributions coming from both matter and the background fields $\bar{k}_Z$. If the backgrounds $\bar{k}_Z$ vanish, $\bar{T}_e^{\mu\nu}$ and $\bar{S}_\omega{}^\mu{}_{ab}$ reduce to the energy–momentum and spin density for the matter fields alone, which can then be written without using bars.

With these definitions, the variation of the full action with respect to the dynamical fields, $e_\mu{}^a$, $\omega_\mu{}^{ab}$, and f^Ψ, has the form

$$\begin{aligned} \delta S = \int d^4x\, e\, &[[-\tfrac{1}{\kappa}(G^{\mu\nu} + \bar{\mathcal{U}}^{\mu\nu} + \bar{\mathcal{G}}^{\mu\nu}) + \bar{T}_e^{\mu\nu}] e_{\nu a} \delta e_\mu{}^a \\ &+ [\tfrac{1}{2\kappa}(\hat{T}^{\mu\alpha\beta} + \bar{\mathcal{T}}^{\mu\alpha\beta}) + \tfrac{1}{2}\bar{S}_\omega{}^{\mu\alpha\beta}] e_{\alpha a} e_{\beta b}\, \delta\omega_\mu{}^{ab} + \tfrac{\delta S_{g,m,\bar{k}_Z}}{\delta f^\Psi} \delta f^\Psi]\,. \end{aligned} \tag{53}$$

Setting $\delta S = 0$ for dynamical variations $\delta e_\mu{}^a$, $\delta\omega_\mu{}^{ab}$, and δf^Ψ gives the equations of motion for the vierbein, spin connection, and matter fields, respectively, as

$$G^{\mu\nu} + \bar{\mathcal{U}}^{\mu\nu} + \bar{\mathcal{G}}^{\mu\nu} = \kappa \bar{T}_e^{\mu\nu}\,, \tag{54}$$

$$\hat{T}^{\lambda\mu\nu} + \bar{\mathcal{T}}^{\lambda\mu\nu} = -\kappa \bar{S}_\omega{}^{\lambda\mu\nu}\,, \tag{55}$$

$$\frac{\delta S_{g,m,\bar{k}_Z}}{\delta f^\Psi} = 0\,. \tag{56}$$

In these equations, the quantities $\bar{\mathcal{U}}^{\mu\nu}$ and $\bar{\mathcal{G}}^{\mu\nu}$ can be interpreted in two different ways. In the first, they act, respectively, as corrections to the curvature, which depend on the backgrounds $\bar{k}_X$ and $\bar{k}_Y$. Alternatively, they can be interpreted as belonging on the right-hand side of (54), where in that case they contribute to the energy–momentum. Similarly, the quantity $\bar{\mathcal{T}}^{\lambda\mu\nu}$ can be interpreted as corrections to the torsion, which depend on the backgrounds $\bar{k}_Y$, or they can go on the right-hand side of (55) and act as contributing to the spin density. However, with these quantities on the right-hand sides, the coupling κ does not appear when $\bar{k}_X$ and $\bar{k}_Y$ interact with the vierbein and spin connection as it does when matter fields and $\bar{k}_Z$ couple to them. For this reason, it is more natural to keep the quantities $\bar{\mathcal{U}}^{\mu\nu}$, $\bar{\mathcal{G}}^{\mu\nu}$ and $\bar{\mathcal{T}}^{\lambda\mu\nu}$ on the left-hand sides of the equations of motion.

3.4. No-Go Results and Noether Identities

The issue of whether no-go conditions apply when local spacetime symmetry breaking occurs can be examined using Noether identities that hold as a result of observer independence. While the background fields break particle spacetime symmetries, the mathematical observer symmetries in the action must still hold so that observer independence is maintained. Thus, the observer transformations can be used to find Noether identities even when background fields are present. Since observer Diffs, LLTs, and LTs are related, it suffices to consider the Noether identities resulting from only LTs and LLTs. The action terms S_g, $S_{\bar{k}X}$, $S_{g,\bar{k}_Y}$, and $S_{g,m,\bar{k}_Z}$ are each separately unchanged under observer LTs and LLTs, and Noether identities can be found from each one and for each symmetry.

For the Einstein–Hilbert term, setting $\delta S_g = 0$ gives

$$\delta S_g = \int d^4x\, e \left[-\frac{1}{\kappa} G^{\mu\nu} e_{\nu a} \delta e_\mu{}^a + \frac{1}{2\kappa} \hat{T}^{\mu\alpha\beta} e_{\alpha a} e_{\beta b}\, \delta\omega_\mu{}^{ab} \right] = 0\,, \tag{57}$$

where $\delta e_\mu{}^a$ and $\delta\omega_\mu{}^{ab}$ are variations given in (26) and (27) for LTs and (28) and (29) for LLTs. The Noether identities that follow from these are the same as the contracted forms of the Bianchi identities in (13) and (14).

The action terms $S_{\bar{k}X}$, $S_{g,\bar{k}_Y}$, and $S_{g,m,\bar{k}_Z}$ are each unchanged as well under observer spacetime transformations. The variations are given as

$$\delta S_{\bar{k}X} = \int d^4x\, e \left[-\frac{1}{\kappa}\bar{\mathcal{U}}^{\mu\nu}\delta e_\mu{}^a + \frac{\delta S_{\bar{k}_X}}{\delta \bar{k}_X}\delta\bar{k}_X \right] = 0\,, \tag{58}$$

$$\delta S_{g,\bar{k}_Y} = \int d^4x\, e \left[-\frac{1}{\kappa}\bar{\mathcal{G}}^{\mu\nu}\delta e_\mu{}^a + \frac{1}{2\kappa}\bar{\mathcal{T}}^{\mu\alpha\beta}\,\delta\omega_\mu{}^{ab} + \frac{\delta S_{g,\bar{k}_Y}}{\delta \bar{k}_Y}\delta\bar{k}_Y \right] = 0\,, \tag{59}$$

$$\delta S_{g,m,\bar{k}_Z} = \int d^4x\, e \left[\bar{T}_e^{\mu\nu} e_{\nu a}\delta e_\mu{}^a + \tfrac{1}{2}\bar{S}_\omega{}^{\mu\alpha\beta} e_{\alpha a} e_{\beta b}\,\delta\omega_\mu{}^{ab} + \frac{\delta S_{g,m,\bar{k}_Z}}{\delta \bar{k}_Z}\delta\bar{k}_Z + \frac{\delta S_{g,m,\bar{k}_Z}}{\delta f^\psi}\delta f^\psi \right] = 0\,, \tag{60}$$

where $\delta e_\mu{}^a$, $\delta\omega_\mu{}^{ab}$, $\delta\bar{k}_X$, $\delta\bar{k}_Y$, $\delta\bar{k}_Z$, and δf^ψ are variations of these fields under observer LTs or observer LLTs.

Notice that variations of the background fields $\bar{k}_X$, $\bar{k}_Y$, and $\bar{k}_Z$ are included in these expressions because these fields transform under observer transformations. However, they do not appear in the dynamical variations of the full action, δS, in (53). Thus, when the Bianchi identities are combined with the results that $\delta S_{\bar{k}X} = \delta S_{g,\bar{k}_Y} = \delta S_{g,m,\bar{k}_Z} = 0$ under observer LTs and LLTs, consistency with the dynamical equations of motion only holds if

$$\int d^4x\, e \frac{\delta S_{\bar{k}_X}}{\delta \bar{k}_X}\delta\bar{k}_X = 0\,, \quad \int d^4x\, e \frac{\delta S_{g,\bar{k}_Y}}{\delta \bar{k}_Y}\delta\bar{k}_Y = 0\,, \quad \int d^4x\, e \frac{\delta S_{g,m,\bar{k}_Z}}{\delta \bar{k}_Z}\delta\bar{k}_Z = 0\,. \tag{61}$$

When the integrals in (61) all vanish under observer LTs, and when the matter fields, f^ψ, are on-shell, obeying (56), the Noether identities that follow from LTs are

$$(D_\mu - T^\lambda{}_{\lambda\mu})\bar{\mathcal{U}}^{\mu\nu} + T_{\lambda\mu}{}^\nu \bar{\mathcal{U}}^{\mu\lambda} = 0\,, \tag{62}$$

$$(D_\mu - T^\lambda{}_{\lambda\mu})\bar{\mathcal{G}}^{\mu\nu} + T_{\lambda\mu}{}^\nu \bar{\mathcal{G}}^{\mu\lambda} + \tfrac{1}{2}R^{\alpha\beta\mu\nu}\bar{\mathcal{T}}_{\mu\alpha\beta} = 0\,, \tag{63}$$

$$(D_\mu - T^\lambda{}_{\lambda\mu})\bar{T}_e{}^{\mu\nu} + T_{\lambda\mu}{}^\nu \bar{T}_e{}^{\mu\lambda} + \tfrac{1}{2}R^{\alpha\beta\mu\nu}\bar{S}_{\omega\,\mu\alpha\beta} = 0\,. \tag{64}$$

Similarly, the identities that follow when observer LLTs are made, and the conditions in (61) hold with f^ψ on-shell, are

$$\bar{\mathcal{U}}^{\mu\nu} - \bar{\mathcal{U}}^{\nu\mu} = 0\,. \tag{65}$$

$$\bar{\mathcal{G}}^{\mu\nu} - \bar{\mathcal{G}}^{\nu\mu} = (D_\sigma - T^\lambda{}_{\lambda\sigma})\bar{\mathcal{T}}^{\sigma\mu\nu} = 0\,. \tag{66}$$

$$\bar{T}_e{}^{\mu\nu} - \bar{T}_e{}^{\nu\mu} = (D_\sigma - T^\lambda{}_{\lambda\sigma})\bar{S}_\omega{}^{\sigma\mu\nu} = 0\,. \tag{67}$$

Comparing these with the contracted Bianchi identities in (13) and (14) and using the equations of motion in (54) and (55) confirm that these are all compatible as long as the conditions in (61) hold. However, if the integrals in (61) do not vanish, a no-go result follows, and the theory is inconsistent [20].

4. Explicit Breaking

With explicit breaking, the backgrounds $\bar{k}_X$, $\bar{k}_Y$, and $\bar{k}_Z$ are nondynamical and do not satisfy Euler–Lagrange equations,

$$\frac{\delta S_{\bar{k}_X}}{\delta \bar{k}_X} \neq 0\,, \qquad \frac{\delta S_{g,\bar{k}_Y}}{\delta \bar{k}_Y} \neq 0\,, \qquad \frac{\delta S_{g,m,\bar{k}_Z}}{\delta \bar{k}_Z} \neq 0\,, \tag{68}$$

which makes satisfying the conditions in (61) problematic. To examine this, it is useful to rewrite the expression in (61) in terms of currents, defining

$$J^X = \frac{\delta S_{g,\bar{k}_X}}{\delta \bar{k}_X} \,, \qquad J^Y = \frac{\delta S_{g,m,\bar{k}_Y}}{\delta \bar{k}_Y} \,, \qquad J^Z = \frac{\delta S_{g,\bar{k}_Z}}{\delta \bar{k}_Z} \,, \tag{69}$$

for the potential, pure-gravity, and matter–gravity sectors, respectively. The conditions in (61) can then be written as

$$\int d^4x\, e\, J^X \delta\bar{k}_X = 0 \,, \quad \int d^4x\, e\, J^Y \delta\bar{k}_Y = 0 \,, \quad \int d^4x\, e\, J^Z \delta\bar{k}_Z = 0 \,. \tag{70}$$

Under LTs, the variations $\delta\bar{k}_X$, $\delta\bar{k}_Y$, and $\delta\bar{k}_Z$ each consist of four local translations with parameters ϵ^a, while for LLTs, they each consist of six local Lorentz transformations with parameters ϵ^{ab}. Thus, using integrations by parts and the fact that the integrands must vanish for all ϵ^a and ϵ^{ab}, up to ten conditions can be extracted from each of the three conditions in (70). Thus, there are a total of up to 30 conditions that must hold if explicit breaking occurs in all three sectors. However, this exceeds the number of available degrees of freedom that are available.

With explicit breaking, there are at most ten additional degrees of freedom in the vierbein and spin connection due to the loss of gauge invariance under LTs and LLTs. These consist of four degrees of freedom that can normally be gauged away using LTs plus another six degrees of freedom that can normally be gauged away using LLTs. When the symmetries are explicitly broken, these degrees of freedom can no longer be gauged away, and hence all ten can potentially become available to satisfy some of the conditions in (70).

However, with at most ten extra degrees of freedom, if explicit breaking of LTs and LLTs occurs in all three sectors, then the three sets of conditions in (70) cannot hold, and the theory will be inconsistent. In the case that two or more sectors with special values of $\bar{k}_X$, $\bar{k}_Y$, and $\bar{k}_Z$, break a combined total of ten or fewer of the symmetries under LTs and LLTs, then enough degrees of freedom might be available. However, with generic values of $\bar{k}_X$, $\bar{k}_Y$, and $\bar{k}_Z$, as considered here, it is not possible to evade a no-go result.

Therefore, it is assumed in the remainder of this paper that only one of the three types of backgrounds $\bar{k}_X$, $\bar{k}_Y$, and $\bar{k}_Z$ can be nonzero in a theory with explicit breaking. In this case, there are up to ten identities that must hold as well as ten additional degrees of freedom in the vierbein, making it possible in principle to evade the no-go results.

With explicit breaking occurring in only one sector, each sector can be examined separately, in which case only one relevant condition in (70) must hold in each case. There are, however, a number of ways in which these conditions might still not hold even if there are enough degrees of freedom. For example, if the extra degrees of freedom resulting from explicit breaking do not appear in the relevant identity or equations of motion, then a no-go result follows. Even if the extra degrees of freedom do appear in the relevant identity, they still need to provide solutions that exist. Furthermore, if it turns out that solutions only exist when background tensors satisfy certain constraints, then the theory is inconsistent unless the backgrounds are defined from the start as obeying these constraints. Thus, the only way to tell for sure if a theory is consistent is to examine whether the no-go conditions for it can be evaded for all possible values of the background as defined by the theory.

4.1. Potential Terms

To examine theories with symmetry breaking in a potential term, the background $\bar{k}^X$ is assumed not to vanish while both $\bar{k}^Y$ and $\bar{k}^Z$ are set to zero. The equations of motion for the vierbein and spin connection in this case are

$$G^{\mu\nu} + \bar{\mathcal{U}}^{\mu\nu} = \kappa T_e^{\mu\nu} \,, \tag{71}$$

$$\hat{T}^{\lambda\mu\nu} = -\kappa S_{\omega}{}^{\lambda\mu\nu} \,. \tag{72}$$

The torsion equation is unchanged, since $\mathcal{U}$ is assumed not to depend on the spin connection. If the no-go results are evaded, the Noether identities that hold for the potential term under observer LTs and LLTs are given in (62) and (65), which have a form that is consistent with the Bianchi identities, the identities (34) and (35), and the equations of motion in (71) and (72).

The relevant condition that must hold to evade the no-go result is

$$\int d^4x\, e\, J^X \delta\bar{k}_X = 0\,, \tag{73}$$

where $J^X = -\frac{\delta\bar{\mathcal{U}}}{\delta\bar{k}^X}$, and $\delta\bar{k}_X$ are the variations of $\bar{k}_X$ under observer LTs and LLTs, To evade the no-go result, enough of the extra degrees of freedom in the vierbein must be present in the theory so that solutions of (73) exist. These conditions have been examined in a number of examples with specific types of backgrounds [34,36,37,43,87], and solutions have been found that evade the no-go results. At the same time, however, other examples are known that do not evade them.

For example, backgrounds having the form of a symmetric two-tensor, such as a background metric, or as a background vierbein, have been widely investigated in theories of massive gravity [83,84]. These are theories that couple the metric or vierbein to a background and construct a potential $\mathcal{U}$ containing a mass term in such a way that the ghost mode that typically appears in massive gravity theories is absent. Models with a spacetime background having the form of a fixed Minkowski tensor $\eta_{\mu\nu}$ or with a fixed vierbein have been explored, which evade the no-go results. Additional ansatz solutions in cosmological or Schwarzschild spacetimes that are consistent have been found as well. In certain cases, backgrounds with spacetime dependence can have consistent solutions provided the metric has a specified form. However, it is not the case that the condition in (73) can be satisfied for generic backgrounds $\bar{k}_X$, and hence no-go results hold in most cases.

4.2. Pure-Gravity Sector

This section examines explicit breaking in the pure-gravity sector due to the appearance of backgrounds $\bar{k}_Y$ (with $\bar{k}_X$ and $\bar{k}_Z$ set to zero). The equations of motion in this case are

$$G^{\mu\nu} + \bar{\mathcal{G}}^{\mu\nu} = \kappa \bar{T}_e^{\mu\nu}\,, \tag{74}$$

$$\hat{T}^{\lambda\mu\nu} + \bar{\mathcal{T}}^{\lambda\mu\nu} = -\kappa \bar{S}_{\omega}{}^{\lambda\mu\nu}\,, \tag{75}$$

and if the no-go results are evaded, the Noether identities for observer LTs and LLTs are in (63) and (66), which have a form that is consistent with the Bianchi identities, the identities (34) and (35), and the equations of motion in (74) and (75).

Note that in an approximately flat and torsionless limit, and assuming the no-go conditions are evaded, the identities in (63) and (66) reduce, respectively, to

$$\partial_\mu \bar{\mathcal{G}}^{\mu\nu} \simeq 0\,, \tag{76}$$

$$\bar{\mathcal{G}}^{\mu\nu} \simeq \bar{\mathcal{G}}^{\nu\mu}\,. \tag{77}$$

However, if $\bar{k}_Y$ has spacetime dependence, it becomes problematic for $\partial_\mu \bar{\mathcal{G}}^{\mu\nu} \simeq 0$ to hold. One way to avoid such problems is to assume that the backgrounds $\bar{k}_Y$ are constant. This is the assumption made for backgrounds in the gravity sector of the SME when they result from spontaneous breaking. However, with explicit breaking it is harder to justify such an assumption, since $\bar{k}_Y$ is a pre-determined quantity. Nonetheless, it is assumed here that $\bar{k}_Y$ is constant. Presumably, if the no-go results cannot be evaded in this case, they will be even more problematic in the case where $\bar{k}_Y$ has spacetime dependence.

The relevant condition that must hold to evade the no-go result in this case is

$$\int d^4x\, e\, J^Y \delta\bar{k}_Y = 0\,, \tag{78}$$

where $\delta\bar{k}_Y$ are variations under observer LTs and LLTs. In covariant form, the pure-gravity term $S_{g,\bar{k}_Y}$ is assumed to consist of expressions that couple the backgrounds $\bar{k}_Y$ with products of the curvature and torsion, as well as with covariant derivatives of the curvature or torsion. Generic examples of possible terms are given in [20]. Since $\bar{k}_Y$ is presumably small, the action is assumed to be linear in $\bar{k}_Y$.

In [38], a specific example of a pure-gravity term with explicit breaking is given with Lagrangian $\mathcal{L}_{g,\bar{k}_Y} = e^\mu{}_a e^\nu{}_b \bar{k}_{cd}{}^{ab} R^{cd}{}_{\mu\nu}$, where the background $\bar{k}_Y$ in this case is a nondynamical field $\bar{k}_{cd}{}^{ab}$ that matches a term in the SME. With $\bar{k}_{cd}{}^{ab}$ included in the action, the equations of motion for the vierbein and spin connection both include contributions from the background. Interestingly, it is observed in [38] that the nondynamical background $\bar{k}_{cd}{}^{ab}$ can potentially act as a source of torsion even in vacuum.

Another example of a pure-gravity term is $\mathcal{L}_{g,\bar{k}_Y} = \bar{k}^{\lambda\mu\nu} K_{\lambda\mu\nu}$, where in this case $\bar{k}_Y$ is a background $\bar{k}^{\lambda\mu\nu}$, and $K_{\lambda\mu\nu}$ is the contorsion tensor. Varying the action $S_{g,\bar{k}_Y}$ with respect to $\omega_{\lambda\mu\nu}$ gives $\bar{\mathcal{T}}^{\lambda\mu\nu} = -\bar{k}^{\mu\lambda\nu}$ in the equation of motion in (55). Thus, in regions of space where the matter fields and spin density equal zero, Equation (55) reduces to

$$\hat{T}^{\lambda\mu\nu} = -\bar{\mathcal{T}}^{\lambda\mu\nu} = \bar{k}^{mu\lambda\nu}\,, \tag{79}$$

showing that here too the background can act as a source of torsion even in vacuum.

Perturbation Theory

In examples such as these, with explicit breaking in the pure-gravity sector, their theoretical consistency depends on whether the identities that follow from (78) under observer LTs and LLTs can hold or not. At the same time, in the context of an effective field theory that is used to analyze tests of gravity on Earth and in the solar system, it typically suffices to use first-order perturbation theory, since gravity is weak and torsion has never been detected. In this case, quadratic or higher-order terms in the curvature and torsion can be neglected, and a post-Newtonian framework can be developed. Such higher-order terms would have the added effect of potentially modifying the number of propagating degrees of freedom in the theory, which would be a significant departure from EC theory. For these reasons, only first-order terms in the curvature or torsion are considered here in the pure-gravity sector.

With a perturbative approach, both the consistency and the usefulness of the theory must be examined in the case of explicit breaking. Such an investigation was carried out in Riemann space, with zero torsion, in [32], where it was shown that while it may be possible to evade no-go results nonperturbatively in the pure-gravity sector, in a leading-order perturbative treatment the no-go constraints can nonetheless render a post-Newtonian framework useless. This is because with explicit breaking the extra degrees of freedom in the metric or vierbein, which are normally gauge degrees of freedom, do not appear in a way that allows the no-go results to be evaded. This is due to the fact that the linearized curvature is gauge invariant. Hence, the needed extra degrees of freedom disappear from it. As a result, conditions must be imposed either on the curvature tensor, which limits the geometry, or on the background fields themselves, which invalidates the premise that they are prescribed quantities.

A similar general argument concerning the consistency and usefulness of a perturbative approach can be made in Riemann–Cartan space as well. This is because the linearized curvature and torsion are both gauge invariant under infinitesimal Diffs, LTS, and LLTs, when excitations $(e_\mu{}^a - \delta^a_\mu)$ and $\omega_\mu{}^{ab}$ are small. This causes the extra modes in the vierbein and spin connection due to the gauge breaking to disappear at the linearized level.

To see this, consider a zeroth-order flat background with zero torsion, where $e_\mu{}^a = \delta^a_\mu$ and $\omega_\mu{}^{ab} = 0$. With such a background, when the vierbein acts on a field it converts local indices to spacetime indices, and hence the distinction between them can be dropped. Linearizing the curvature in (10) and the torsion in (7) in this way gives the first-order expressions:

$$R_{\kappa\lambda\mu\nu} \simeq \partial_\mu \omega_{\nu\kappa\lambda} - \partial_\nu \omega_{\mu\kappa\lambda} \, , \tag{80}$$

$$T_{\lambda\mu\nu} \simeq \partial_\mu e_{\nu\lambda} - \partial_\nu e_{\mu\lambda} + \omega_{\mu\lambda\nu} - \omega_{\nu\lambda\mu} \, , \tag{81}$$

The linearized infinitesimal transformations under Diffs, LTs, and LLTs are:

$$\delta_{\rm Diff}\, e_{\mu\nu} \simeq \partial_\mu \epsilon_\nu \, , \qquad \delta_{\rm Diff}\, \omega_{\lambda\mu\nu} \simeq 0 \, , \tag{82}$$

$$\delta_{\rm LT}\, e_{\mu\nu} \simeq \partial_\mu \epsilon_\nu \, , \qquad \delta_{\rm LT}\, \omega_{\lambda\mu\nu} \simeq 0 \, , \tag{83}$$

$$\delta_{\rm LLT}\, e_{\mu\nu} \simeq \epsilon_{\mu\nu} \, , \qquad \delta_{\rm LLT}\, \omega_{\lambda\mu\nu} \simeq \partial_\lambda \epsilon_{\mu\nu} \, , \tag{84}$$

where the Diffs use $\xi^\mu = \epsilon^\mu$. When excitations of this form are inserted into (80) and (81), $R_{\kappa\lambda\mu\nu}$ and $T_{\lambda\mu\nu}$ both vanish at first order. Since the gauge modes have the form of the ten excitations ϵ_μ and $\epsilon_{\mu\nu}$, the fact that they disappear at the linearized level means that the consistency conditions stemming from (78) generally cannot be satisfied unless constraints are imposed on the torsion and curvature or on the backgrounds themselves, either of which renders the framework useless.

This argument extends as well to higher-dimensional operators in the pure-gravity action, which are coupled to background fields. In a first-order treatment, such terms consist of operators with one or more covariant derivatives acting on the curvature or torsion. However, when the action is linearized, the covariant derivatives reduce to partial derivatives at first order, for example,

$$D_\alpha D_\beta R_{\kappa\lambda\mu\nu} \simeq \partial_\alpha \partial_\beta R_{\kappa\lambda\mu\nu} \, , \tag{85}$$

and likewise with the linearized torsion. As a result, all the potential gauge degrees of freedom drop out of these higher-dimensional operators as well.

The end result is that a perturbative pure-gravity approach with explicit breaking in EC theory is generally not useful or is inconsistent. See [43] as well for additional arguments and examples that reach the same conclusion.

Despite the breakdown of theoretical consistency at the perturbative level with explicit breaking, experiments can still conduct tests of gravity using the version of the SME that includes nondynamical backgrounds. Any detection of a signal due to explicit breaking in a model that does not evade the no-go results would then have to be interpreted as giving evidence of a geometry that goes beyond Riemann or Riemann–Cartan geometry, such as Finsler geometry. This is the approach taken in [43], while [44] examines a number of experimental tests involving both the pure-gravity and matter–gravity sectors.

The pure-gravity sector also includes interactions between gravity waves and fixed background fields. In general, the effects of spacetime symmetry breaking on gravity waves can be investigated using a variety of approaches. See, for example [88–96]. However, in the context of effective field theory, as is being considered here, a perturbative approach using linearized gravity coupled to backgrounds of the form $\bar{k}_Y$ can be used [91–93]. In this framework, completely generalized operators consisting of partial derivatives acting on metric excitations $h_{\mu\nu}$ in a Minkowski background coupled directly to $\bar{k}_Y$ are included, as opposed to restricting to couplings with only the linearized curvature tensor. To avoid conflicts with translation invariance, it is assumed that the backgrounds $\bar{k}_Y$ are constant or approximately constant. At the linearized level, breaking of Diffs becomes breaking of a gauge symmetry, while breaking of Lorentz symmetry becomes global breaking in the Minkowski background. In this context, the linearized metric includes additional degrees of freedom associated with gauge breaking under Diffs. However, in general, they may not appear in such a way that evades the no-go results. Thus, any detection of spacetime breaking from gravity waves, which breaks Diffs but does not evade the no-go conditions, would be indicative as well of a geometry that goes beyond Riemann or Riemann–Cartan.

For additional applications with explicit breaking, see, for example [97–100].

4.3. Matter–Gravity Sector

Setting aside the potential and pure-gravity terms, the action term for the matter–gravity sector is $S_{g,m,\bar{k}_Z}$, which contains the fixed backgrounds $\bar{k}_Z$. The dynamical equations of motion for the vierbein and spin connection are

$$G^{\mu\nu} = \kappa \bar{T}_e^{\mu\nu} \, , \tag{86}$$
$$\hat{T}^{\lambda\mu\nu} = -\kappa \bar{S}_\omega{}^{\lambda\mu\nu} \, , \tag{87}$$

where for simplicity the matter fields are put on-shell. Comparing these to Equations (19) and (20) shows that they have the same form as in EC theory except that the energy–momentum and spin density tensors now have bars over them, indicating that they depend on the background fields, $\bar{k}_Z$.

The energy–momentum and spin density for ordinary matter are contained in $\bar{T}_e^{\mu\nu}$ and $\bar{S}_\omega{}^{\lambda\mu\nu}$. The background $\bar{k}_Z$ contributes to $\bar{T}_e^{\mu\nu}$ as well when it couples to both matter fields and the vierbein. Similarly, $\bar{k}_Z$ contributes to $\bar{S}_\omega{}^{\lambda\mu\nu}$ when it couples to both matter fields and the spin connection. The latter contributions arise, for example, when covariant derivatives act on the matter fields, such as $D_\mu \psi$ or $D_\mu A_\nu$, and then also couple with $\bar{k}_Z$, Thus, all terms with couplings to $\bar{k}_Z$ in the matter–gravity sector combine with both gravitational and matter fields, which implies that in regions of spacetime where matter is absent, $\bar{T}_e^{\mu\nu}$ and $\bar{S}_\omega{}^{\lambda\mu\nu}$ both vanish. From (87), it follows that the torsion vanishes as well in the absence of matter; therefore, in vacuum the theory is no different from EC theory.

Assuming that the no-go results are evaded, the Noether identities for observer LTs and LLTs in this case are given in (64) and (67), which are consistent with the Bianchi identities and the equations of motion in (86) and (87). Note how the identities in (64) and (67) match the identities (34) and (35) in EC theory when bars are placed over $\bar{T}_e^{\mu\nu}$ and $\bar{S}_\omega{}^{\lambda\mu\nu}$.

The condition in (70) that must hold to evade the no-go results in this case is

$$\int d^4x \, e \, J^Z \, \delta\bar{k}_Z = 0 \, . \tag{88}$$

where $\delta\bar{k}_Z$ can be observer Diffs, LTs, or LLTs of the backgrounds $\bar{k}_Z$.

4.3.1. Energy–Momentum

Because of the similarity with EC theory, and assuming the no-go results can be evaded, the same procedure as described in Section 2.4 can be applied here as well. An effective energy–momentum tensor with Belinfante–Rosenfeld form can be defined as in Section 2.4, but with bars added to the energy–momentum and spin density so that

$$\bar{T}_{\rm eff}^{\mu\nu} = \bar{T}_e^{\mu\nu} - \tfrac{1}{2}\tilde{D}_\sigma(\bar{S}_\omega{}^{\sigma\mu\nu} + \bar{S}_\omega{}^{\mu\nu\sigma} + \bar{S}_\omega{}^{\nu\mu\sigma}) \, . \tag{89}$$

The Einstein tensor can again be divided into a Riemann part and a non-Riemann part, where on-shell the torsion in the non-Riemann part can be written in terms of $\bar{S}_\omega{}^{\lambda\mu\nu}$. In the limit where quadratic contributions $\sim \kappa^2 \bar{S}_\omega^2$ can be neglected, the effective energy–momentum tensor is

$$\bar{T}_{\rm eff}^{\mu\nu} \simeq \bar{T}_e^{(\mu\nu)} - \tfrac{1}{2}\tilde{D}_\sigma(\bar{S}_\omega{}^{\mu\nu\sigma} + \bar{S}_\omega{}^{\nu\mu\sigma}) \, , \tag{90}$$

and the effective Einstein equation reduces to $\tilde{G}^{\mu\nu} \simeq \kappa \bar{T}_{\rm eff}^{\mu\nu}$. From this, it follows that as long as the no-go results are evaded, $\tilde{D}_\mu \bar{T}_{\rm eff}^{\mu\nu} \simeq 0$ and $T_{\rm eff}^{\mu\nu} = T_{\rm eff}^{\nu\mu}$ must hold at leading order in a perturbative treatment. This results in a perturbative theory that is effectively Riemann.

In particular, in a flat spacetime limit with negligible torsion, $\partial_\mu T_{\rm eff}^{\mu\nu} \simeq 0$ would need to hold. If a background $\bar{k}_Z$ has spacetime dependence, this could lead to violations of global translation invariance and energy–momentum conservation. A breakdown of energy–momentum conservation in a flat torsionless limit is problematic as well, because no measurements detect such a violation. For these reasons, it is assumed here that $\bar{k}_Z$ is constant or very close to constant over relevant distance scales, so that $\partial_\mu \bar{k}_Z \simeq 0$ holds. In this case, as long as the no-go results can be evaded, the Noether identities and equations of motion are consistent, and the theory closely parallels EC theory.

4.3.2. No-Go Results

To evade the no-go results, the identities that follow from (88) must be solvable when $\delta\bar{k}_Z$ are given as observer Diffs, LTs, and LLTs. In general, it is possible that these conditions can hold nonperturbatively, since the current J^Y_m is not typically gauge invariant. Thus, the extra degrees of freedom in the vierbein do not necessarily disappear and can take values that satisfy the consistency conditions in (88). However, to fully evade the no-go results, solutions must exist for the extra modes, without placing additional constraints on the background fields. Note that there are no additional degrees of freedom in the torsion, which according to (87) is fully determined by the spin density.

Arguments concerning consistency at the level of perturbation theory are less conclusive in the matter–gravity sector compared to the pure-gravity sector. Indeed, there are known examples of theories with explicit breaking in the matter–gravity sector that evade the no-go results. See, for example, [36,37,43]. It is also the case that the currents J^Y typically contain both gravitational and matter fields coupled together, which can make linearizing in a systematic way ambiguous. For example, if small matter excitations only couple with the zeroth-order vierbein ${e_\mu}^a \simeq \delta^a_\mu$ and spin connection ${\omega_\mu}^{ab} \simeq 0$, then the extra degrees of freedom are generally suppressed. At the same time, even with quadratic couplings retained in a perturbative treatment, the general arguments given in [43] show that the consistency conditions stemming from (88) can run into experimental constraints that conflict with a particular model. For example, the experimental sensitivities that are relevant for J^Z, $\bar{k}^Z$, vierbein excitations, and matter fields might be orders of magnitude apart and therefore incompatible. This can result in a theory being perturbatively inconsistent and not useful to use as a phenomenological framework.

4.3.3. Constant Vector Background in Matter–Gravity Sector

As concrete examples, consider theories where the background $\bar{k}_Z$ is either a constant spacetime vector $\bar{b}_\mu$ or a constant local vector $\bar{b}_a$. To first order in a perturbative approach, with ${e_\mu}^a = \delta^a_\mu$ and ${\omega_\mu}^{ab} = 0$ at zeroth order, the observer transformations under Diffs, LTs, and LLTs, with infinitesimal parameters $\xi^\mu = \epsilon^\mu = \delta^\mu_a \epsilon^a$ and ${\epsilon_a}^b$, are given in this case as:

$$\delta_{\rm Diff}\, \bar{b}_\mu \simeq (\partial_\mu \epsilon^\nu)\bar{b}_\nu \,, \qquad \delta_{\rm LT}\, \bar{b}_\mu \simeq (\partial_\mu \epsilon^\nu)\bar{b}_\nu \,, \qquad \delta_{\rm LLT}\, \bar{b}_\mu \simeq 0 \,, \tag{91}$$

$$\delta_{\rm Diff}\, \bar{b}_a \simeq 0 \,, \qquad \delta_{\rm LT}\, \bar{b}_a \simeq 0 \,, \qquad \delta_{\rm LLT}\, \bar{b}_a \simeq -{\epsilon_a}^b \bar{b}_b \,. \tag{92}$$

Here, the constant spacetime components $\bar{b}_\mu$, obeying $\partial_\nu \bar{b}_\mu \simeq 0$, break particle Diffs and LTs but not particle LLTs when they couple to dynamical quantities. This is consistent with the relation in (30), which shows that Diffs and LTs are equal when LLTs vanish. At the same time, the constant local background $\bar{b}_a$, with $\partial_\nu \bar{b}_a = 0$, breaks particle LLTs, but at leading order it does not break particle Diffs or LTs in contrast to the result in (30). Notice, however, that at second order in small quantities, $\delta_{\rm LT}\, \bar{b}_a = \epsilon^c {e^\mu}_c D^{(\omega)}_\mu \bar{b}_a = \epsilon^c {e^\mu}_c {\omega_\mu}^{ab} \bar{b}_a$, as in (48). Hence, if ${\omega_\mu}^{ab} \neq 0$ at first order, the relation in (30) would be applicable, and it would confirm that LLTs and LTs are directly linked when Diffs vanish.

As this example illustrates, a theory with background $\bar{b}_\mu$ violates different symmetries than a theory with $\bar{b}_a$. It also shows that a constant background can break particle LTs without necessarily breaking global translations in a flat torsionless limit, since $\partial_\nu \bar{b}_\mu = 0$ and $\partial_\nu \bar{b}_a = 0$. The conditions that must hold to evade the no-go results are the identities derived from (88), with $\delta\bar{b}_\mu$ or $\delta\bar{b}_a$ given as infinitesimal Diffs, LTs, and LLTs (as defined in (91) and (92)) must hold.

For the case of constant $\bar{b}_\mu$, LLTs are unbroken to leading order, while Diffs and LTs have the same form of breaking with $\xi^\mu = \epsilon^\mu$. The resulting identity under LTs is

$$[(D_\mu - T^\lambda{}_{\lambda\mu})J^\mu]\bar{b}_\nu = 0\,. \tag{93}$$

Thus, unless $(D_\mu - T^\lambda{}_{\lambda\mu})J^\mu = 0$, the theory is inconsistent and the no-go results hold. In a limit with weak gravity and negligible torsion, the condition in (93) reduces to $\partial_\mu J^\mu_Y \simeq 0$. In models where the current is approximately conserved in this manner, consistency can hold. However, in general, consistency would require cancelations between matter fields, gravitational fields, and the background $\bar{b}_\mu$ to occur, which generally have very different experimental sensitivities as argued in [43].

For constant $\bar{b}_a$, with broken LLTs, the conditions that must hold are

$$J^a\bar{b}^b - J^b\bar{b}^a = 0\,. \tag{94}$$

With LLTs, the extra gauge degrees of freedom are the antisymmetric components in the vierbein, and these are not sufficient at leading order in J^a_Y to make (94) hold for generic values of a constant vector $\bar{b}_a$. Thus, a no-go result holds at the perturbative level.

In summary, in theories with constant or nearly constant explicit-breaking backgrounds $\bar{b}_\mu$ or $\bar{b}_a$ in interaction with a matter–gravity current, the question of whether no-go results can be evaded depends in a case-by-case manner on the type of current and level of perturbation theory that is used. However, at leading order in perturbation theory, the result for most models is either that the no-go conditions hold or that experimental constraints imply that they cannot hold.

4.3.4. Stückelberg Approach

Since a Stückelberg approach is commonly used in gravity theories with explicit breaking of Diffs in Riemann space [101], it is examined here in Riemann–Cartan space. For simplicity, the example of a constant background vector $\bar{b}_\mu$ is considered again. The technique involves introducing a set of Stückelberg scalar fields ϕ^a and using them to replace $\bar{b}_\mu$ as

$$\bar{b}_\mu \to (\partial_\mu\phi^a)\bar{b}_a\,, \tag{95}$$

where $\bar{b}_a$ is constant and where the scalars ϕ^a are dynamical fields. This procedure (often called a trick) restores the broken Diffs. At the same time, since the scalars ϕ^a are dynamical, there are equations of motion for them that must hold.

In the Stückelberg approach, Diffs are spontaneously broken by the vacuum value for ϕ^a, which is given as $\langle\phi^a\rangle = \delta^a_\nu x^\nu$. When ϕ^a takes this value, $(\partial_\mu\phi^a)\bar{b}_a = \bar{b}_\mu$, reproducing the fixed background $\bar{b}_\mu$. The idea then is that any fixed nondynamical field that explicitly breaks Diffs can be reproduced as a gauge-fixed vacuum solution in a theory with spontaneous breaking caused by the Stückelberg fields.

Making the substitution (95) in the matter–gravity action term changes it to a new action in terms of ϕ^a:

$$\begin{aligned} &S_{g,m,\bar{k}_Z} = \int d^4x\, e\, J^\mu\bar{b}_\mu \\ &\to \int d^4x\, e\, J^\mu(\partial_\mu\phi^a)\bar{b}_a\,. \end{aligned} \tag{96}$$

Varying ϕ^a, using integration by parts, and discarding a boundary term, gives the result

$$[(D_\mu - T^\lambda{}_{\lambda\mu})J^\mu]\bar{b}_a = 0\,. \tag{97}$$

Therefore, the condition $(D_\mu - T^\lambda{}_{\lambda\mu})J^\mu = 0$ holds as a result of the equations of motion for ϕ^a, illustrating that the Stückelberg approach still works when there is torsion.

However, regardless of whether the condition $(D_\mu - T^\lambda{}_{\lambda\mu})J^\mu = 0$ holds as the result of invariance under observer Diffs or LTs or as the result of using a Stückelberg approach, the consistency of a theory still depends in both cases on whether this condition can hold for generic constant vectors $\bar{b}_\mu$. If not, the no-go result applies, and the theory is inconsistent. Thus, while the Stückelberg method can be useful in exploring theories with explicit-breaking background fields for which the no-go results are evaded, it does not provide a way to make a theory that is inconsistent into one that is consistent.

5. Spontaneous Breaking

Spontaneous breaking of spacetime symmetry has been widely investigated both theoretically and experimentally [17–25,102,103]. Specific mechanisms for how spontaneous breaking can occur have been identified and explored [14–16], and examples and properties of the NG modes that can arise have been studied [29,30,104–121]. In the SME, both the pure-gravity and matter–gravity sectors have been constructed in EC theory [20]. In addition, perturbative frameworks in Riemann space with spontaneous breaking have been developed. In the pure-gravity sector, these include a post-Newtonian framework [26] and a linearized perturbative framework suitable for gravity waves [91]. In the matter–gravity sector, techniques leading to the construction of a consistent perturbative framework have been found [27,28]. These different frameworks have been used to analyze a wide range of experimental tests of spacetime symmetry in gravity theories [25]. In addition, specific models exhibiting spontaneous spacetime symmetry have been constructed and investigated [15,16,29,30], including Bumblebee models, where numerous applications have been explored (for examples, see [122–137]).

With spontaneous breaking, the generic form of the action is given in (43), and the equations of motion are given in (54)–(56). In general, the energy–momentum tensor does not have to be symmetric or covariantly conserved when the torsion is nonzero. These equations show that the torsion can be sourced by spin density from ordinary matter as well as by contributions that depend on the backgrounds.

As originally shown in [20], in an EC theory with spontaneous breaking of Diffs and LLTs, the no-go results are evaded. The same is true using LTs and LLTs as the basic symmetries, since these yield an equivalent set of Noether identities. The no-go results are evaded because the background fields, $\bar{k}_Y$, $\bar{k}_Y$, and $\bar{k}_Z$ are vacuum expectation values of dynamical fields K_Y, K_Y, and K_Z, which obey the equations of motion:

$$\frac{\delta S_{g,\bar{k}_X}}{\delta K_X} = 0\,, \qquad \frac{\delta S_{g,m,\bar{k}_Z}}{\delta K_Z} = 0\,, \qquad \frac{\delta S_{g,m,\bar{k}_Z}}{\delta K_Z} = 0\,. \tag{98}$$

Thus, the backgrounds $\bar{k}_X$, $\bar{k}_Y$, and $\bar{k}_Z$ are the vacuum solutions to these equations. As a result, all three conditions in (61) hold for the vacuum solutions. Since each of the action terms are individually scalars under observer LTs and LLTs, the Noether identities for LTs in (62)–(64) and for LLTs in (65)–(67) all hold. These identities are consistent with each other, with the contracted forms of the Bianchi identities in (13) and (14), and with the equations of motion.

Notice that with spontaneous breaking, there is nothing that prevents the breaking from happening in more than one particle sector. Thus, in principle $\bar{k}_X$, $\bar{k}_Y$, and $\bar{k}_Z$ can all have nonzero values at the same time. However, to avoid potential issues with breakdown of global translation invariance and energy–momentum conservation in a flat limit, the backgrounds $\bar{k}_X$, $\bar{k}_Y$, and $\bar{k}_Z$ can be approximated as constant or nearly constant on relevant experimental distance scales.

When excitations about the vacuum are included, the symmetry becomes hidden, but NG modes and massive Higgs-like excitations combine with the backgrounds and other dynamical excitations to keep the symmetry unbroken in the action, and the Equations in (98) continue to hold. The counting of degrees of freedom in the vierbein and spin connection is unchanged from the case of EC theory, and all ten of the spacetime symmetries, consisting of LLTs and either Diffs or LTs, can be used to gauge away ten degrees of freedom. The torsion does not propagate, and on-shell it is fixed by the spin density.

Just as in EC theory, an effective theory can be found as described in Section 2.4, where the curvature is split into Riemann and non-Riemann parts, and the torsion is eliminated on-shell using the equations in (55). The relevant equations in the effective theory are found by making the replacements

$$\kappa T_e{}^{\mu\nu} \to (-\bar{\mathcal{U}}^{\mu\nu} - \bar{\mathcal{G}}^{\mu\nu} + \kappa \bar{T}_e{}^{\mu\nu}) \, , \tag{99}$$

$$\kappa S_{\omega\,\mu\alpha\beta} \to (\bar{\mathcal{T}}_{\mu\alpha\beta} + \kappa \bar{S}_{\omega\,\mu\alpha\beta}) \, , \tag{100}$$

in Equations (37) through (41). In a limit where the torsion and background fields are weak, so that quadratic terms in $(\bar{\mathcal{T}}_{\mu\alpha\beta} + \kappa \bar{S}_{\omega\,\mu\alpha\beta})$ can be neglected, the curvature becomes Riemann, and the Einstein equations reduce to $\tilde{G}^{\mu\nu} \simeq \kappa \bar{T}^{\mu\nu}_{\rm eff}$. It follows from this that $\tilde{D}_\mu \bar{T}^{\mu\nu}_{\rm eff} \simeq 0$ and $\bar{T}^{\mu\nu}_{\rm eff} \simeq \bar{T}^{\mu\nu}_{\rm eff}$ hold in a weak-torsion limit. As the bar indicates, $\bar{T}^{\mu\nu}_{\rm eff}$ in general depends on the backgrounds $\bar{k}_X$, $\bar{k}_Y$, and $\bar{k}_Z$. Nonetheless, with spontaneous breaking, the energy–momentum is covariantly conserved in an effective theory with Riemann curvature in a weak-torsion limit, just as it is in EC theory with no symmetry breaking.

5.1. Bumblebee Models

Bumblebee models are useful for studying how spontaneous breaking of spacetime symmetry can occur [14–16,20,29,30]. They allow various features and properties of the symmetry breaking to be explored, including how NG modes and Higgs-like modes can appear, and whether a Higgs mechanism might occur.

The Bumblebee field has spacetime components B_μ that are connected to local components B_a by the vierbein, so that $B_\mu = e_\mu{}^a B_a$. Its defining feature is that it has a potential V that has a minimum when B_μ and the vierbein have nonzero vacuum values, which spontaneously break Diffs and LLTs. In addition to having an Einstein–Hilbert term, the action for a Bumblebee model can take a range of forms, with different kinetic terms and with either minimal or nonminimal couplings.

The simplest form to consider is where there are only minimal couplings and where the kinetic term has a Maxwell form, in which case the Bumblebee Lagrangian can be written as

$$\mathcal{L}_B = -\frac{1}{4} B_{\mu\nu} B^{\mu\nu} - V(B_\mu B^\mu + b^2) \, . \tag{101}$$

Here, $B_{\mu\nu}$ is the field strength, which in Riemann space can be defined using covariant derivatives, which simply reduce to partial derivatives. However, in EC theory the torsion enters when covariant derivatives are used:

$$B_{\mu\nu} = D_\mu B_\nu - D_\nu B_\nu = \partial_\mu B_\nu - \partial_\nu B_\mu - T^\lambda{}_{\mu\nu} B_\lambda \, . \tag{102}$$

For an electromagnetic field, including the term with torsion would break local $U(1)$ gauge symmetry, and therefore a definition in terms of only partial derivatives would be preferable. However, in Bumblebee models, the potential V breaks local $U(1)$ invariance, and therefore either definition of $B_{\mu\nu}$ can be considered. In the case where covariant derivatives are used, and the torsion is included, contributions to the spin density can occur.

The potential V is given as a function of the combination $(B_\mu B^\mu + b^2)$, where b is a constant. One possibility is to define V as a smooth quadratic function,

$$V(B_\mu B^\mu + b^2) = \frac{1}{2} \lambda (B_\mu B^\mu + b^2)^2 \, , \tag{103}$$

where λ is a constant. This potential has a minimum when

$$V' = \lambda (B_\mu B^\mu + b^2) = 0 \, . \tag{104}$$

Thus, the components B_μ and B_a as well as the vierbein all have nonzero vacuum values when V is at its minimum,

$$\langle B_\mu \rangle = \bar{b}_\mu \,, \qquad \langle {e_\mu}^a \rangle = \delta^a_\mu \,. \qquad \langle B_a \rangle = \bar{b}_a \,, \tag{105}$$

Here, $\bar{b}_\mu$ and $\bar{b}_a$ are both assumed to be constant, and a Minkowski background is assumed for the vierbein. The vacuum metric is then given as $\langle g_{\mu\nu} \rangle = \eta_{\mu\nu}$, while the local metric is η_{ab}. The vacuum values for the background vector are related by $\bar{b}_\mu = \langle {e_\mu}^a \rangle \bar{b}_a$, which must obey $\bar{b}^\mu \bar{b}_\mu = \bar{b}^a \bar{b}_a = -b^2$ so that $V' = 0$ holds. Each of these vacuum values is fixed under particle spacetime transformations. Thus, when one of them transforms under observer transformations, it spontaneously breaks the symmetry.

5.1.1. Vacuum Solution with $\langle {\omega_\mu}^{ab} \rangle = 0$

In EC theory, the vacuum solution for the spin connection can be chosen as

$$\langle {\omega_\mu}^{ab} \rangle = 0 \,. \tag{106}$$

With this and the vierbein vacuum, $\langle {e_\mu}^a \rangle = \delta^a_\mu$, the vacuum values for the Bumblebee field strength, spin density, torsion, energy–momentum, and curvature all vanish:

$$\langle B_{\mu\nu} \rangle \neq 0 \,, \qquad \langle {\bar{S}_\omega}^{\lambda\mu\nu} \rangle \neq 0 \,, \qquad \langle {T^\lambda}_{\mu\nu} \rangle \neq 0 \,, \qquad \langle \bar{T}_e^{\mu\nu} \rangle \neq 0 \,, \qquad \langle {R^\kappa}_{\lambda\mu\nu} \rangle \neq 0 \,. \tag{107}$$

Together the vacuum values, $\bar{b}_\mu$, $\bar{b}_a$, and $\langle {e_\mu}^a \rangle$ spontaneously break all three spacetime symmetries: Diffs, LTs, and LLTs. However, each individually spontaneously breaks different symmetries.

While the vierbein vacuum value spontaneously breaks all three symmetries, the constant local background $\bar{b}_a$ spontaneously breaks three LLTs, but not Diffs or LTs. The background $\bar{b}_a$ does not break LTs, because $D^{(\omega)}_\mu \bar{b}_a = 0$ to lowest order when $\bar{b}_a$ is constant and $\langle {\omega_\mu}^{ab} \rangle = 0$, and Diffs are not broken when $\bar{b}_a$ is constant. In contrast, the spacetime vector $\bar{b}_\mu$ breaks Diffs and LTs, but not LLTs. It breaks LTs because the vierbein breaks LTs, and similarly for Diffs. Notice, however, that a constant vector $\bar{b}_\mu$ only breaks one Diff, where ξ^μ is in the same direction as the vacuum value $\bar{b}^\mu$. Similarly, at leading order, $\bar{b}_\mu$ only breaks one LT where ϵ^a is in the same direction as $\langle {e_\mu}^a \rangle \bar{b}^\mu$. Finally, since $\bar{b}_a$ and $\langle {e_\mu}^a \rangle$ transform inversely under observer LLTs, $\bar{b}_\mu$ does not transform under LLTs.

The fate of the NG modes in this Bumblebee model was examined in [29,30] for spontaneous breaking of Diffs and LLTs, where it was found that massless NG modes for the broken LLTs can appear and propagate, while the NG modes for Diffs disappear and do not propagate. However, it is also possible to consider symmetry breaking where LTs and LLTs are the independent transformations, and to look at whether the formation of the NG modes is changed in any way from the case where Diffs and LLTs are independent.

As described in [29,30], the excitations around the vacuum value in B_μ consist of a combination of excitations of both the local vector B_a and the vierbein ${e_\mu}^a$, which take the form

$$\langle B_\mu \rangle + \delta B_\mu = (\langle {e_\mu}^a \rangle + \delta {e_\mu}^a)(\langle B_a \rangle + \delta B_a) \,, \tag{108}$$

where δB_μ, $\delta {e_\mu}^a$, and δB_a are the relevant excitations, which are assumed to be small. Thus, to first order,

$$\delta B_\mu \simeq (\delta {e_\mu}^a) \bar{b}_a + \langle {e_\mu}^a \rangle \delta B_a \,. \tag{109}$$

To identify the physical degrees of freedom in the case where the NG modes are associated with Diffs and LLTs, ten gauge degrees of freedom must first be fixed. This is necessary with spontaneous breaking because the symmetries still hold when all the excitations are included. One choice for the gauge fixing is to set the antisymmetric components in $\delta {e_\mu}^a$ to zero, which fixes the six LLTs, and then set $(\delta {e_\mu}^a) \bar{b}_a = 0$, which fixes the four gauge degrees of freedom under Diffs. The result is that

$$\delta B_\mu \simeq \langle {e_\mu}^a \rangle \delta B_a \,, \tag{110}$$

where there are now four degrees of freedom remaining in δB_a. Since three LLTs are spontaneously broken by $\bar{b}_a$, there are three NG modes that can appear. These have the form of excitations generated by the broken LLTs, which stay in the minimum of the potential V. The remaining fourth mode is a massive mode, which does not stay in the minimum of the potential V. The three NG modes for LLTs have the form $\delta B_a \simeq -{\epsilon_a}^b \bar{b}_b$, where the broken generators, ${\epsilon_a}^b$, become the fields for the NG excitations. These obey $(\delta B_a)\bar{b}^a = 0$, while the massive mode in δB_a is an excitation along the direction of $\bar{b}_a$.

In δB_μ, as given in (110), it is only the NG modes for LLTs that appear, while the NG mode for the broken Diff disappears. Under the broken Diff, the NG mode would appear as an excitation $(\partial_\mu \xi^\nu)\langle {e_\nu}^a \rangle \bar{b}_a$, where in this case ξ^μ for the broken transformation would become the field for the NG mode. The reason it disappears is because gauge fixing $(\delta {e_\mu}^a)\bar{b}_a = 0$ imposes a condition on any excitations that might arise as NG modes in the vierbein:

$$(\partial_\mu \xi^\nu)\langle {e_\nu}^a \rangle \bar{b}_a - \epsilon^a{}_b \langle {e_\mu}^b \rangle \bar{b}_a = 0 \,. \tag{111}$$

This condition locks the NG mode for the broken Diff to the NG modes for the broken LLTs, and they cancel in the vierbein excitations. The net result is that only the three NG modes for the broken LLTs appear in δB_μ.

A corresponding analysis can be carried out starting from (109) and finding the NG modes when it is LTs and LLTs that are spontaneously broken. In this case, the gauge freedoms under LTs and LLTs must be fixed, which can be accomplished by again setting the antisymmetric components in $\delta {e_\mu}^a$ to zero, and then using the four LTs to set $(\delta {e_\mu}^a)\bar{b}_a = 0$. The result is again that $\delta B_\mu \simeq \langle {e_\mu}^a \rangle \delta B_a$ consists of three NG modes for spontaneous breaking of LLTs and one massive mode. The gauge fixing in this case locks excitations having the form of LTs and LLTs in $(\delta {e_\mu}^a)\bar{b}_a$, imposing the condition that $(\partial_\mu \epsilon^a)\bar{b}_a - \epsilon^a{}_b \langle {e_\mu}^b \rangle \bar{b}_a = 0$. Here, the first excitation $(\partial_\mu \epsilon^a)\bar{b}_a$ is what would be the NG mode for the broken LT; however, it is locked to the NG modes for the broken LLTs by the gauge fixing, which therefore causes them to cancel in the vierbein excitations. The net result is again that only the three NG modes for the broken LLTs appear in δB_μ.

Thus, regardless of whether Diffs and LLTs are used or LTs and LLTs are used, the only NG modes that propagate as physical modes are the three NG modes stemming from spontaneous breaking of LLTs. No NG modes for either Diffs or LTs appear in the theory.

5.1.2. Vacuum Solution with $\langle {\omega_\mu}^{ab} \rangle \neq 0$

An alternative vacuum structure can be considered for the case of a constant background $\bar{b}_a$, which spontaneously breaks both LTs and LLTs in the local frame. It occurs when the vacuum vierbein has a Minkowski solution, $\langle {e_\mu}^a \rangle = \delta^a_\mu$, and the spin connection also has a nonzero vacuum value,

$$\langle {\omega_\mu}^{ab} \rangle \neq 0 \,. \tag{112}$$

The condition that the Bumblebee vacuum is in the minimum of the potential, obeying $V' = 0$, is still satisfied when $\langle B_a \rangle = \bar{b}_a$ and $\langle B_\mu \rangle = \langle {e_\mu}^a \rangle \bar{b}_a$. Spontaneous breaking of LTs occurs in this case, because under observer transformations, $\delta_{\rm LT}\bar{b}_a = \epsilon^c e^\mu{}_c D^{(\omega)}_\mu \bar{b}_a$, and therefore the vacuum solution obeys

$$\delta_{\rm LT}\bar{b}_a = \epsilon^c \langle e^\mu{}_c \rangle \langle {\omega_{\mu a}}^b \rangle \bar{b}_a \neq 0 \,. \tag{113}$$

In this case, the constant vector $\bar{b}_a$ spontaneously breaks three LTs, where the broken generators ϵ^a are transverse to $\bar{b}^a$. Thus, three NG modes for the broken LTs are expected. At the same time, spontaneous breaking of LLTs occurs, since $\delta_{\rm LLT}\bar{b}_a = -{\epsilon_a}^b \bar{b}_b \neq 0$ under observer LLTs. Thus, there are three broken LLTs and three NG modes for LLTs are expected.

Notice that the spin connection is not forced to have a nonzero vacuum value so that $V' = 0$ holds. Instead, it acquires a vacuum value spontaneously at the same time that $e_\mu{}^a$ and B_a take vacuum values making $V' = 0$. However, as long as $\langle \omega_\mu{}^{ab} \rangle$ is constant, in addition to $\bar{b}_a$ and $\langle e_\mu{}^a \rangle$ being constant, then the Bumblebee field strength, spin density, torsion, energy–momentum, and curvature can all have constant nonvanishing vacuum values:

$$\langle B_{\mu\nu} \rangle \neq 0\,, \qquad \langle \bar{S}_\omega{}^{\lambda\mu\nu} \rangle \neq 0\,, \qquad \langle T^\lambda{}_{\mu\nu} \rangle \neq 0\,, \qquad \langle \bar{T}_e^{\mu\nu} \rangle \neq 0\,, \qquad \langle R^\kappa{}_{\lambda\mu\nu} \rangle \neq 0\,. \tag{114}$$

Thus, the Bumblebee vacuum Lagrangian in this case is a constant, whereas it was zero when the spin connection had a vacuum solution $\langle \omega_\mu{}^{ab} \rangle = 0$.

Since the spin connection has mass dimension equal to one, its vacuum value can be written as $\langle \omega_\mu{}^{ab} \rangle \sim \bar{\omega}$ (dropping indices), where $\bar{\omega}$ has units of mass. Here, $\bar{\omega}$ sets the energy scale for the spin connection vacuum solution. Similarly, $\langle B_a \rangle \sim \langle B_\mu \rangle \sim b$, where b also has mass units. The vacuum Bumblebee field strength is nonzero because of the torsion contribution in (102), which gives

$$\langle B_{\mu\nu} \rangle = -\langle T^\lambda{}_{\mu\nu} \rangle \langle e_\lambda{}^a \rangle \bar{b}_a\,, \tag{115}$$

where the torsion vacuum value is

$$\langle T_{\lambda\mu\nu} \rangle = \langle \omega_{\mu\lambda\nu} \rangle - \langle \omega_{\nu\lambda\mu} \rangle\,. \tag{116}$$

This shows that the torsion has mass dimension one and scales as $\langle T_{\lambda\mu\nu} \rangle \sim \bar{\omega}$, while the field strength has mass dimension two and scales as $\langle B_{\mu\nu} \rangle \sim \bar{\omega} b$. The vacuum spin density $\langle \bar{S}_\omega{}^{\lambda\mu\nu} \rangle$ is found by varying the Lagrangian term $-\frac{1}{4} B_{\mu\nu} B^{\mu\nu}$ with respect to the spin connection and substituting in the vacuum solutions. It has mass dimension three and scales as $\langle \bar{S}_\omega{}^{\lambda\mu\nu} \rangle \sim \bar{\omega} b^2$. Since the torsion equation for the vacuum is $\langle \hat{T}^{\lambda\mu\nu} \rangle = -\kappa \langle \bar{S}_\omega{}^{\lambda\mu\nu} \rangle$, this shows that on-shell the torsion scales as $\langle T_{\lambda\mu\nu} \rangle \sim \kappa \bar{\omega} b^2$, where the dimensional coupling $\kappa \sim M_{\rm Pl}^{-2}$ has mass dimension minus two, with $M_{\rm Pl}$ equal to the Planck mass. The fact that the torsion scales as both $\sim \bar{\omega}$ and $\sim \kappa \bar{\omega} b^2$ on-shell shows that $\kappa b^2 \sim 1$ or that $b \sim M_{\rm Pl}$, and therefore $\langle T_{\lambda\mu\nu} \rangle \sim \bar{\omega}$ and $\kappa \langle \bar{S}_\omega{}^{\lambda\mu\nu} \rangle \sim \bar{\omega}$. Similarly, the energy–momentum is found by varying $-\frac{1}{4} e B_{\mu\nu} B^{\mu\nu}$ with respect to the vierbein. The result is that $\bar{T}_e^{\mu\nu}$ is dimension four and scales as $\sim \bar{\omega}^2 b^2$. However, $\kappa \bar{T}_e^{\mu\nu} \sim \bar{\omega}^2$. Finally, the vacuum curvature, which has mass dimension two, scales as $\langle R^\kappa{}_{\lambda\mu\nu} \rangle \sim \bar{\omega}^2$.

Notice how these equations set the Bumblebee mass scale b to the Planck mass, but the scale for the spin connection vacuum $\sim \bar{\omega}$ is left undetermined. However, since the vacuum torsion scales as $\sim \bar{\omega}$ and the vacuum curvature scales as $\sim \bar{\omega}^2$, on experimental grounds the parameter $\bar{\omega}$ must be very small. This would be consistent with $\langle \omega_\mu{}^{ab} \rangle$ arising spontaneously as a small vacuum value. While the potential term scales as $V \sim b^4 \sim M_{\rm Pl}^4$, it is zero in the vacuum solution. In contrast, the other operator terms in the action all scale as $\sim \bar{\omega}^2 M_{\rm Pl}^2$. Matching the scale of these other nonvanishing vacuum operators on experimental grounds to a cosmological constant term that scales as $\sim \Lambda M_{\rm Pl}^2$ indicates that an appropriate scale for the vacuum spin connection would be $\bar{\omega} \sim \sqrt{\Lambda}$, and hence $\bar{\omega}$ must be extremely small.

To examine the NG modes for spontaneous breaking with $\langle \omega_\mu{}^{ab} \rangle \neq 0$, consider again the excitations in (109). Ten gauge symmetries must be fixed. The six LLTs can be fixed by setting the antisymmetric components of the vierbein to zero, and the four LTs can be chosen so that the excitations $(\delta e_\mu{}^a) \bar{b}_a = 0$. This leaves $\delta B_\mu \simeq \langle e_\mu{}^a \rangle \delta B_a$, where δB_a consists of three NG modes for the broken LTs and three NG modes for the broken LTTs all acting together, which are given as

$$\delta B_a = \epsilon^c \langle e^\mu{}_c \rangle \langle \omega_{\mu a}{}^b \rangle \bar{b}_b - \epsilon_a{}^b \bar{b}_b\,, \tag{117}$$

where ϵ^a and $\epsilon_a{}^b$ become the fields for the NG modes. These combined excitations obey $(\delta B_a) \bar{b}^a = 0$ and therefore remain in the minimum of the potential V. The massive mode is along the direction of $\bar{b}_b$, and is assumed to be highly suppressed due to the mass scale $b \sim M_{\rm Pl}$.

With the gauge freedom fixed by setting $(\delta e_\mu{}^a)\bar{b}_a = 0$, a condition is imposed on any excitations arising as NG modes for LTs and LLTs in the vierbein:

$$(\delta e_\mu{}^a)\bar{b}_a = (\partial_\mu \epsilon^a)\bar{b}_a + \epsilon^b \langle \omega_\mu{}^a{}_b \rangle \bar{b}_a + \epsilon^b \langle T^a{}_{b\mu} \rangle \bar{b}_a - \epsilon^a{}_b \langle e_\mu{}^b \rangle \bar{b}_a = 0 \,. \tag{118}$$

Substituting for the torsion, rearranging, and making insertions of the vacuum vierbein, this condition can be rewritten as:

$$\langle e^\mu{}_a \rangle (\partial_\mu \epsilon^b)\bar{b}_b - [\epsilon^c \langle e^\mu{}_c \rangle \langle \omega_{\mu a}{}^b \rangle \bar{b}_b - \epsilon_a{}^b \bar{b}_b] = 0 \,. \tag{119}$$

This form shows that the excitation $\langle e^\mu{}_a \rangle (\partial_\mu \epsilon^b)\bar{b}_b$, which involves the LT with ϵ^a along the direction of $\bar{b}^a$, is locked to the combination of NG modes for the three LTs and the three LLTs, which appear together in δB_a in (117). Thus, the condition in (119) prevents a fourth NG mode for the LTs, with ϵ^a parallel to $\bar{b}^a$, from appearing in δB_μ.

Notice also that the locked excitation $(\partial_\mu \epsilon^b)\bar{b}_b$ takes a form in the spacetime frame that is the same as a Diff NG mode: $(\partial_\mu \xi^\nu)\bar{b}_\nu$, where $\bar{b}_\nu = \langle e_\nu{}^b \rangle \bar{b}_b$ and the Diff generator is $\xi^\nu = \langle e^\nu{}_a \rangle \epsilon^a$. In fact, if the vacuum value $\langle \omega_{\mu a}{}^b \rangle$ is set equal to zero, so that LTs are no longer broken, then the condition in (119) reduces to the condition that locks the Diff NG mode to the NG modes from LLTs just as in the previous example.

The net result of this example, however, is that with $\langle \omega_\mu{}^{ab} \rangle \neq 0$, there are three excitations in δB_μ, which consist of a combination of three NG modes for LTs and three NG modes for LLTs. The fourth mode is the Bumblebee massive mode, which scales as $\sim b \sim M_{\rm Pl}$, and therefore its excitations are heavily suppressed and are unlikely to matter at lower energies.

The nature of the full solution depends on which components of the spin connection acquire vacuum values. This would determine which components of the vacuum values in (114) are nonvanishing, and how they combine to give an overall constant vacuum Lagrangian. With the gravitational, NG, and massive mode excitations included, the physical viability of such a theory could be evaluated. Additional terms involving nonminimal couplings or additional couplings to the torsion could be considered as well. See, for example, [138]). In addition, adding terms that allow the spin connection to propagate would allow investigation of the possibility of a Higgs mechanism for the spin connection. However, exploring possibilities like these goes beyond the scope of the present work, which is focused on the nature of the spontaneous breaking of spacetime symmetries, the resulting vacuum structure, and the appearance of NG modes.

6. Discussion and Conclusions

In this paper, the processes of explicit and spontaneous spacetime symmetry breaking in EC theory when fixed background fields are present have been reviewed and reexamined with a focus being placed on the roles of torsion and LTs.

The original investigation of spacetime symmetry breaking in gravity [20], published 20 years ago, found that with explicit breaking no-go results consisting of inconsistencies between the Bianchi identities and the equations of motion can arise, but that with spontaneous breaking the no-go results are evaded. The construction of the gravity sector of the SME based on the idea of spontaneous breaking of spacetime symmetries has provided a phenomenological framework that has been used in numerous gravity tests. These include both pure-gravity and matter–gravity tests, where extremely high sensitivities to possible symmetry breaking have been attained. In addition, implications of spontaneous breaking of spacetime symmetries have been explored, including looking at various scenarios and mechanisms for the symmetry breaking as well as addressing questions concerning the appearance of NG modes, Higgs-like modes, and the possibility of a gravitational Higgs mechanism.

However, over the past decade, gravity theories with explicit breaking of spacetime symmetry breaking have been investigated in more detail, and it was found in certain cases that it is possible to evade the no-go results. Seeing how this occurs involves looking at the relationships between the Bianchi identities, Noether identities stemming from observer independence, energy–momentum conservation, and the equations of motion. It was found that the no-go results can most readily be evaded in theories where the extra degrees of freedom in the vierbein due to explicit breaking can appear and have nonperturbative interactions with matter and gravitational fields. It is also the case that highly specific forms of background fields are able to evade the no-go results, while generic forms typically do not. In addition, in perturbative treatments, which are widely used in gravity tests, the extra modes often disappear or are highly restricted so that useful solutions evading the no-go results do not exist. This makes post-Newtonian approaches not useful when the symmetry breaking is explicit. Nonetheless, to investigate the possibility of explicit breaking experimentally, a generalization of the SME with gravity has been developed, where the interpretation is that if any violations are discovered in cases where the no-go results are not evaded, they would give evidence of a geometry that goes beyond Riemann or Riemann–Cartan, such as Finsler geometry.

Much of the progress that has been made in understanding the differences between explicit and spontaneous breaking has relied on investigations in which Diffs and LLTs are broken, while the breaking of LTs has been looked at to a lesser extent. In addition, the effects of torsion have often been ignored, including the question of how to interpret the fact that energy–momentum is not covariantly conserved when torsion is present.

The primary goal of this paper has been to revisit EC theory with background fields included and to look more closely at the effects of torsion and breaking of LTs in theories with either spontaneous or explicit breaking. One immediate effect involving LTs in theories with fixed backgrounds is the possibility of violation of global translation invariance in a flat spacetime limit and breaking energy–momentum conservation. For this reason, the investigation here has been limited to the case where the backgrounds are constant or nearly constant on relevant distance scales, which is an assumption that is often made in the SME. The idea is that if no-go results arise for a constant background, then having a background with spacetime dependence would be even more problematic. Interestingly, it is possible for LTs to be broken by backgrounds that are constant, where presumably breaking of global translations is then not an issue. However, even with restriction to constant backgrounds, it is found that the no-go results still typically occur with explicit breaking. Moreover, even if a theory is not totally ruled out, in a perturbative treatment, the no-go results can still render it useless as a phenomenological framework.

There are noteworthy effects involving torsion that can occur in EC theory when background fields are present. One is that background fields can act as a source of spin density and torsion even in a vacuum. However, for this to work with explicit breaking the no-go results must be evaded, while with spontaneous breaking there are no no-go results. As long as the spin connection has a nonvanishing coupling to a background field, spin density and torsion can result in a vacuum solution. Another effect of torsion is that energy–momentum is not covariantly conserved when it is present. However, in theories with background fields, it is found that the effects of torsion can be accounted for in ways that parallel what happens in EC theory. With no symmetry breaking in EC theory, there are weak limits where the curvature is Riemann and the spin density effectively acts like additional contributions to the energy–momentum. This same interpretation can hold as well when background fields are present. However, with explicit breaking, it can only hold in highly restricted cases, and it cannot hold at all if the no-go results are not evaded. In contrast, with spontaneous breaking, the no-go results are evaded, and thus the same interpretation of energy–momentum conservation when torsion is included can hold as in EC theory.

The effects of torsion and spontaneous breaking of LTs were looked at specifically for the case of a simple Bumblebee model with a Maxwell kinetic term. To consider the effects of torsion, a field strength defined with covariant derivatives that included coupling to the torsion was chosen. Two scenarios were examined, one where the vacuum value of the spin connection vanishes, and the other where it spontaneously takes a constant value. In the latter case, spontaneous breaking of LTs occurs in the local frame, whereas such breaking does occur when the vacuum spin connection is zero. The spin density, curvature, and energy–momentum density can all have constant vacuum values when the spin connection has one. A comparison of the NG modes that can occur with the two types of vacuum structures reveals that the NG modes for LLTs are present as propagating degrees of freedom in both cases. However, in the case where the spin connection has a vacuum value, the Bumblebee modes are due to a combination of both breaking of LLTs and LTs. At the same time, in both scenarios, the NG mode due to spontaneous breaking of the Diff or the LT along the direction of the Bumblebee background vector does not propagate.

Overall, the results found here continue to show that explicit breaking of spacetime symmetry is much more unnatural and problematic compared to spontaneous breaking. The effects of torsion or breaking of LTs do not alter this conclusion. The processes involved in spontaneous breaking of spacetime symmetry are far more elegant and compatible with EC theory than the corresponding processes that occur with explicit breaking.

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Abbreviations

The following abbreviations are used in this manuscript:

GR	General Relativity
EC	Einstein–Cartan Theory
Diff	Diffeomorphism
LLT	Local Lorentz transformation
LT	Local translation
SM	Standard Model
SME	Standard-Model Extension
NG	Nambu–Goldstone

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