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Exact Solutions of Population Balance Equation with Aggregation, Nucleation, Growth and Breakage Processes, Using Scaling Group Analysis

Fubiao Lin *, Yang Yang and Xinxia Yang

School of Mathematics and Statistics, Guizhou University of Finance and Economics, Guiyang 550025, China; 15348508488@163.com (Y.Y.); yxx6151130@163.com (X.Y.)
* Correspondence: linfubiao0851@163.com

Abstract: Population balance equations may be employed to handle a wide variety of particle processes has certainly received unprecedented attention, but the absence of explicit exact solutions necessitates the use of numerical approaches. In this paper, a (2 + 1) dimensional population balance equation with aggregation, nucleation, growth and breakage processes is solved analytically by use of the methods of scaling transformation group, observation and trial function. Symmetries, reduced equations, invariant solutions, exact solutions, existence of solutions, evolution analysis of dynamic behavior for solutions are presented. The exact solutions obtained can be compared with the numerical scheme. The obtained results also show that the method of scaling transformation group can be applied to study integro-partial differential equations.

Keywords: integro-partial differential equation; population balance equation; scaling group; exact solution

MSC: 45K05; 22E70; 76M60

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1. Introduction

The areas of applications of population balance equations (PBEs) [1–6] and the references therein are more and more extensive, including gene regulatory processes, cell growth, division, differentiation, death processes, biochemistry and molecular biology, agriculture engineering, astrophysics and astronomy and so on. A general (2 + 1)-dimensional PBE [4] is given by

$$\frac{\partial f(x,t)}{\partial t} + \frac{\partial}{\partial x}[G(x,t)f(x,t)] = \psi(f,x,t), \tag{1}$$

where x is the internal coordinate, it denotes the size of particles. t represents the time. $f(x,t)$ is an average number density. $G(x,t)$ is the growth rate of particle size x. The source $\psi(f,x,t)$ denotes the contribution to $f(x,t)$ of the change in the number of particles, owing to particle aggregation, nucleation, growth and breakage [1–4]. A (2 + 1)-dimensional homogeneous PBE is presented by (1) if the source term $\psi(f,x,t) = 0$.

In the following paragraph, it is of interest to consider the particulate processes for the particle population distributed according to their mass are frequently encountered in applications [1–8] and the references therein. A (2 + 1)-dimensional PBE [4] with particle aggregation, nucleation, growth and breakage processes is written

$$\underbrace{\frac{\partial f(x,t)}{\partial t}}_{\text{average number density}} + \underbrace{\frac{\partial}{\partial x}[G(x,t)f(x,t)]}_{\text{growth due to coating(layering)}} = \underbrace{\frac{1}{2}\int_0^x K(x-y,y)f(x-y,t)f(y,t)\,dy}_{\text{birth due to aggregation}} -$$

$$\underbrace{f(x,t)\int_0^\infty K(x,y)f(y,t)\,dy}_{\text{death due to aggregation}} + \underbrace{\int_x^\infty v(y,t)b(y,t)P(x|y)f(y,t)\,dy}_{\text{birth due to breakage}} - \underbrace{b(x,t)f(x,t)}_{\text{death due to breakage}}, \tag{2}$$

where $v(x,t)$ denotes an average number of particles on breakage of a particle of size x. $b(x,t)$ is the breakage rate or breakage frequency of particles at time t. In general, the breakage rate coefficient function $b(x,t)$ is increasing with respect to the size of the fragmenting particle. $P(x|y)$ is the probability of the particles of size y breaking into the particles of size x, which satisfies the normalization conditions

$$\int_0^y P(x|y)\,dx = 1,\ P(x|y) = 0,\ x > y. \tag{3}$$

All of which are assumed to be time independent, but size dependent.

$K(x,y)$ is the aggregation frequency for particle pairs of mass x and y [4]. The choice of kernel can dramatically affect the rate of coalescence and thereby the shape of the predicted granule size distribution. $K(x,y)$ has a nonnegative symmetry property, that is,

$$K(x,y) = K(y,x) \geq 0.$$

Various intriguing and significant aggregation kernels $K(x,y)$ originating from in industrial applications are homogeneous [4–10], that is, one can find an exponent γ satisfies

$$K(ax,ay) = a^\gamma K(x,y), \tag{4}$$

where every $a,x,y > 0$, γ denotes the degree of homogeneity. For instance:

$$K(x,y) = k_0,\ k_1(x+y),\ k_2xy,\ k_3\Big(\frac{1}{x}+\frac{1}{y}\Big),\ \frac{k_4}{xy}, \tag{5}$$

where the kinetic coefficients $k_i (i = 0, \cdots, 4)$ are positive real constants. Using the property (4), one has

$$xK_x(x,y) + yK_y(x,y) = \gamma K(x,y). \tag{6}$$

Hence, the general solution to Equation (6) is presented by

$$K(x,y) = y^\gamma \bar{K}\Big(\frac{x}{y}\Big), \tag{7}$$

where $\bar{K}$ is an arbitrary function of one variable.

Suppose that the average number of particle breakage $v(x,t)$ is an arbitrary positive constant. The growth rate $G(x,t)$ and breakage rate $b(x,t)$ are both homogeneous with respect to particle size x, that is, which satisfy

$$G(\lambda x,t) = \lambda^\delta G(x,t),\ b(\lambda x,t) = \lambda^\kappa b(x,t),$$

where δ, λ and κ are constants. In particular, the following kinetic functions

$$G(x,t) = gx^n,\ v(x,t) = \nu,\ P(x|y) = \frac{1}{y},\ b(x,t) = kx^{n-1} \tag{8}$$

are considered in this work, where g, ν, k and n are positive constants, the probability density function $P(x|y) = \frac{1}{y}$ satisfies the normalization conditions (3). In addition, assuming that $y = xs$, and using (4) or (7), Equation (2) under the constrain (8) can be simplified to the following form

$$\frac{\partial f(x,t)}{\partial t} + gx^n \frac{\partial f(x,t)}{\partial x} = \frac{1}{2} x^{\gamma+1} \int_0^1 K(1-s,s) f(x(1-s),t) f(xs,t)\, ds -$$
$$f(x,t) \int_0^\infty K(x,y) f(y,t)\, dy + \nu k \int_x^\infty y^{n-2} f(y,t)\, dy - (k+gn) x^{n-1} f(x,t). \quad (9)$$

If $f = f(x,t)$ is any solution of Equation (9), then it has a property that population density vanishes for infinite-sized particles, which means the values of $f(x,t)$ approach 0 as x approaches ∞, that is, $f(\infty,t) = 0$. The regularity condition can be defined as

$$G(\infty,t) f(\infty,t) = 0. \quad (10)$$

Equation (10) does not insist that the number density function $f(x,t)$ itself vanishes at infinite mass if the growth rate function $G(x,t)$ vanishes for large particles. The boundary conditions and initial condition for Equation (9) are

$$f(0,t) = f(x,t)\big|_{x=0},\ f(\infty,t) = 0,\ f(x,0) = f(x,t)\big|_{t=0}.$$

Moments [11] are mathematical formulations that allow us to calculate various properties of the particle size distribution function $f(x,t)$. The jth moment $M_j(t) (j = 0,1)$ of the particle size distribution $f(x,t)$ is defined as

$$M_j(t) = \int_0^\infty x^j f(x,t)\, dx,\ j = 0,1, \quad (11)$$

where $M_0(t)$ is the average total number of particles per unit volume of physical space in the system. $M_1(t)$ is the total volume fraction of all particles.

It is relatively easy to establish the model (2), but it is typically difficult to search for exact solutions, except for using numerical methods, for instance, see the literature [2–4,7], whereas we would prefer to have explicit exact solutions that can describe phenomena in chemical engineering and other fields of nonlinear science. Analytical solutions of PBE (2) with zero growth rate (that is, $G(x,t) = 0$) and simultaneous breakage and coalescence for a special case were presented in [12–15]. Exact solutions of PBE (2) with aggregation, nucleation, growth and breakup for the particular cases were considered by using the method of adomian decomposition in [16]. However, at present, the explicit exact solution of Equation (9) has not been reported in the modern literature.

The developed Lie group theory [17,18] presents an approach for computing operators of integro-partial differential equations. In recent years, the developed Lie group analysis was applied to search for explicit exact solutions of PBEs [19–22] and solve integro-partial differential equations, stochastic equations and delay equations [23–32]. The essential obstacle of this approach is in searching for the general solutions of the determining equations, the approaches of solving determining equations of integro-partial differential equations depend on the studied equations, there is no general approach for solving determining equations of integro-partial differential equations [19–31].

The purpose of this work is to present an analytical technique for PBE (9) and to search for explicit exact solutions, in particular, physical explicit exact solutions. The methods of Lie group analysis [17,18] have already been developed to solve PBEs [19–21], and the references therein. However, it seems that none of the literature makes use of the method of scaling transformation group to find explicit exact solutions of PBE (9), except that which has been used in [19] for the simple homogeneous PBEs. Therefore, in the current work, explicit exact solutions of PBE (9) are investigated analytically by the method of scaling transformation group. Firstly, the admitted scaling group of PBE (9) will be obtained. Finally, explicit exact solutions, invariant solutions, and reduced equations of PBE (9) will be constructed.

The paper is structured as follows. In Section 2, a search for symmetries of PBE (9) is investigated using a scaling transformation group. In Section 3, explicit exact solutions, invariant solutions and reduced equations of PBE (9) are considered. At the same time, explicit exact unphysical solutions are presented. Dynamic behavior evolution analysis of particle size distribution for solutions is also given. In the Section 4, some conclusions are made.

2. Admitted Scaling Group

The complete symmetries of Equation (9) are typically laborious to be found with the approach of developed Lie group theory [17,18]. Conversely, the admitted groups of Equation (9) are considered with scaling group [17–19] in this section. Let us consider the following scaling group

$$\bar{t} = ta^{\lambda_1},\ \bar{x} = xa^{\lambda_2},\ \bar{f} = fa^{\mu}, \tag{12}$$

and equation

$$\frac{\partial \bar{f}(\bar{x},\bar{t})}{\partial \bar{t}} + g\bar{x}^n \frac{\partial \bar{f}(\bar{x},\bar{t})}{\partial \bar{x}} = \frac{1}{2}\bar{x}^{\gamma+1}\int_0^1 K(1-s,s)\bar{f}(\bar{x}(1-s),\bar{t})\bar{f}(\bar{x}s,\bar{t})\,ds$$
$$-\bar{f}(\bar{x},\bar{t})\int_0^\infty K(\bar{x},\bar{y})\bar{f}(\bar{y},\bar{t})\,d\bar{y} + \nu k\int_{\bar{x}}^\infty \bar{y}^{n-2}\bar{f}(\bar{y},\bar{t})\,d\bar{y} - (k+gn)\bar{x}^{n-1}\bar{f}(\bar{x},\bar{t}), \tag{13}$$

where a is an arbitrary real group parameter, λ_1, λ_2 and μ are constants. If the transformation group (12) is admitted by Equation (9), then the admitted operator of Equation (9) is written

$$X = \lambda_1 t\frac{\partial}{\partial t} + \lambda_2 x\frac{\partial}{\partial x} + \mu f\frac{\partial}{\partial f}. \tag{14}$$

Using the transformations (12), one has

$$\bar{f}(\bar{x},\bar{t}) = a^{\mu} f\left(\bar{x}a^{-\lambda_2}, \bar{t}a^{-\lambda_1}\right). \tag{15}$$

Substituting (12) and (15) into Equation (13), using the property of kernel (4), one obtains

$$a^{\mu-\lambda_1}\frac{\partial f(x,t)}{\partial t} + a^{\mu+(n-1)\lambda_2} g x^n \frac{\partial f(x,t)}{\partial x}$$
$$= a^{2\mu+(\gamma+1)\lambda_2}\left[\frac{1}{2}x^{\gamma+1}\int_0^1 K(1-s,s)f(x(1-s),t)f(xs,t)\,ds\right. \tag{16}$$
$$\left.-f(x,t)\int_0^\infty K(x,y)f(y,t)\,dy\right] + a^{\mu+(n-1)\lambda_2}\left[\nu k\int_x^\infty y^{n-2}f(y,t)\,dy - (k+gn)x^{n-1}f(x,t)\right].$$

Equation (16) gives

$$\lambda_1 = (1-n)\lambda_2,\ \mu = (n-\gamma-2)\lambda_2. \tag{17}$$

Similar to the previous case, it is not difficult to demonstrate that Equation (9) admits the following translation group T_{τ_0} with a real parameter τ_0

$$T_{\tau_0}:\ \ \bar{x} = x,\ \bar{t} = t+\tau_0,\ \bar{f} = f. \tag{18}$$

Furthermore, the translation group T_{τ_0} corresponding generator $X_1 = \frac{\partial}{\partial t}$ is admitted by Equation (9). In addition, after substituting the invariance conditions (17) into (14), it contains an arbitrary constant λ_2. Therefore, it follows from (14), (17) and (18) that the incomplete admitted operators of Equation (9) with (4) are provided by

$$X_1 = \frac{\partial}{\partial t},\ X_2 = (1-n)t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + (n-\gamma-2)f\frac{\partial}{\partial f}. \tag{19}$$

Remark 1. *The investigation of symmetries of a new integro-partial differential equation is usually started by using the method of scaling transformation group. On one hand, the found symmetries can be applied to verify determining equations when we study complete group analysis of integro-partial differential equations by use of the method of developed Lie group analysis [17,18]. On the other* *hand, the obtained symmetries can also be used to construct exact and self-similar solutions, such as the literature [19–22].*

3. Results: Explicit Exact Solutions

In this section, explicit exact solutions of Equation (9) with the kernel (5) are studied by using generators (19) and group transformations of solutions. By means of translation group (18) with τ_0, explicit exact solutions to Equation (9) can be shifted with respect to time t, that is,

$$f(x,t) = \bar{f}(x, t+\tau_0),\ \tau_0 \in \mathbb{R}.$$

3.1. Case $K(x,y) = k_0$

For the case of constant aggregation kernel $K(x,y) = k_0$, in terms of (4) one derives that $\gamma = 0$. The invariants corresponding to generator X_1 are given by $J_1 = x, J_2 = f$. Thus, an invariant solution of Equation (9) is presented by

$$f(x,t) = \varphi(x),$$

where φ satisfies the equation

$$gx^n\varphi' = \frac{1}{2}k_0 x\int_0^1 \varphi(x(1-s))\varphi(xs)\,ds$$
$$-k_0\varphi\int_0^\infty \varphi(s)\,ds + \nu k\int_x^\infty s^{n-2}\varphi(s)\,ds - (k+gn)x^{n-1}\varphi. \tag{20}$$

Since Equation (20) involving three different type of integrals $\int_0^1, \int_0^\infty$ and $\int_x^\infty$, by use of the methods of trial function and observation [19–22], one can suppose that trial exponential function

$$\varphi(x) = \alpha\exp(-\sigma x), \sigma > 0$$

is a solution to Equation (20) with $n = 2$, where α and σ are constants, calculations of parameters g, ν and α are performed on Matlab, which leads to $g = 0, \nu = 2, \alpha = \frac{2k}{k_0}$. Hence, an explicit exact solution of Equation (9) with $g = 0, n = 2$ is given by

$$f(x,t) = \frac{2k}{k_0}\exp(-\sigma x),\ \sigma > 0,\ g = 0,\ n = 2,\ \nu = 2, \tag{21}$$

where recalling (11), the zeroth moment $M_0(t)$ and the first moment $M_1(t)$ are provided by

$$M_0(t) = \int_0^\infty f(x,t)\,dx = \frac{2k}{k_1\sigma},\ M_1(t) = \int_0^\infty xf(x,t)\,dx = \frac{2k}{k_1\sigma^2},$$
$$M_0(t) = \sigma M_1(t),\ \frac{dM_0}{dt} = 0,\ \frac{dM_1}{dt} = 0. \tag{22}$$

The results of (22) demonstrate the average total number of particles and total volume of particles are conserved. The values of $f(x,t)$ approach 0 as particle size x becomes large, which implies that solution (21) satisfies the property that population density vanishes for infinite-sized particles. Hence, the corresponding boundary conditions and initial condition of the Cauchy problem of solution (21) are, respectively, given by

$$f(0,t) = \frac{2k}{k_0},\ f(\infty,t) = 0,\ f(x,0) = \frac{2k}{k_0}\exp(-\sigma x),\ \sigma > 0.$$

Using the same methods which were used in the previous case, one can obtain explicit exact solutions to Equation (9) with $n = 2$ that are presented by

$$f(x,t) = \Big(-\frac{12g\sigma}{k_0}x + \beta\Big)\exp(-\sigma x),\ \sigma > 0,\ n = 2, \tag{23}$$

where σ is a constant, analytical calculations of parameters β, k and ν are performed on Matlab and the calculated values are, respectively, given by

$$\beta = \frac{36g}{k_0},\ k = 12g,\ \nu = 3;\ \beta = \frac{12g}{k_0},\ k = 2g, \nu = 1;\ \beta = -\frac{48g}{k_0},\ k = -23g,\ \nu = \frac{48}{23}.$$

The zeroth moment $M_0(t)$ and the first moment $M_1(t)$ for solution (23) are presented by

$$M_0(t) = \int_0^\infty f(x,t)\,dx = \frac{k_0\beta - 12g}{k_0\sigma},\ M_1(t) = \int_0^\infty xf(x,t)\,dx = \frac{k_0\beta - 24g}{k_0\sigma^2}.$$

Remark 2. *Explicit exact solutions to Equation (9) with $n = 2$ and constant aggregation kernel $K(x,y) = k_0$ are provided by (23). Whereas, in the field of practical industrial application of PBE, the population density $f(x,t)$ and kinetic parameters g and k are required to satisfy the constraints $f(x,t) > 0, g > 0, k > 0$. For such $\beta = -\frac{48g}{k_0}, k = -23g, \nu = \frac{48}{23}$, the obtained solution (23) contradict the requirements that $f(x,t) > 0, g > 0, k > 0$.*

If $n = 1$, the invariants corresponding to generator X_2 are presented by $J_1 = t, J_2 = xf$. Thus, the invariant solution is presented by $f(x,t) = \frac{1}{x}\varphi(t)$, substituting this expression into Equation (9), which leads to the improper integral $\int_0^\infty \frac{1}{x}\,dx$ is divergent. Thus, the invariant solution for generator X_2 in this case can not be obtained.

If $n = 1$, the invariants corresponding to generator $X_2 + \frac{1}{\alpha}X_1, \alpha \neq 0$ are provided by $J_1 = x\exp(-\alpha t), J_2 = f\exp(\alpha t)$. Therefore, an invariant solution of Equation (9) is given by

$$f(x,t) = \exp(-\alpha(t+\tau_0))\varphi(z),\ z = x\exp(-\alpha(t+\tau_0)),\ n = 1,\ \alpha \neq 0,$$

where φ satisfies the equation

$$(g-\alpha)z\varphi' + (g+k-\alpha)\varphi = \frac{1}{2}k_0 z\int_0^1 \varphi(z(1-s))\varphi(zs)\,ds - k_0\varphi\int_0^\infty \varphi(s)\,ds + \nu k\int_z^\infty s^{-1}\varphi(s)\,ds.$$

If $n \neq 1$, the invariants corresponding to generator X_2 are given by $J_1 = xt^q, J_2 = t^{-m}f$, where $q = \frac{1}{n-1}, m = \frac{n-2}{1-n}$. Thus, an invariant solution to Equation (9) is presented by

$$f(x,t) = (t+\tau_0)^m\varphi(z),\ z = x(t+\tau_0)^q,\ q = \frac{1}{n-1},\ m = \frac{n-2}{1-n}. \tag{24}$$

Substituting (24) into Equation (9), one can obtain that the reduced equation is given by

$$\Big(gz^n + \frac{1}{n-1}z\Big)\varphi' = \frac{1}{2}k_0 z\int_0^1 \varphi(z(1-s))\varphi(zs)\,ds$$
$$-k_0\varphi\int_0^\infty \varphi(s)\,ds + \nu k\int_z^\infty s^{n-2}\varphi(s)\,ds - \Big[(k+ng)z^{n-1} - \frac{n-2}{n-1}\Big]\varphi. \tag{25}$$

Using the methods of observation and trial function [19–22], assuming that trial exponential function

$$\varphi(z) = \alpha\exp(-\sigma z),\ \sigma > 0$$

is a solution to Equation (25) with $n = 2$, one can derive that $g = 0, \alpha = \frac{\nu k}{k_0}, \sigma = \frac{(2-\nu)k}{2}$, $0 < \nu < 2$. Therefore, an explicit exact solution of Equation (9) with $g = 0, n = 2$ is presented by

$$f(x,t) = \frac{\nu k}{k_0}\exp\Big[-\frac{(2-\nu)k}{2}x(t+\tau_0)\Big],\ 0 < \nu < 2,\ g = 0,\ n = 2. \tag{26}$$

The values of $f(x,t)$ are close to 0 as x is close to ∞, which implies solution (26) has a property that population density vanishes for infinite-sized particles. In addition, the values $f(x,t)$ approach 0 as t approaches ∞, which shows solution (26) is asymptotically stable. The boundary conditions and initial condition for solution (26) are, respectively, provided by

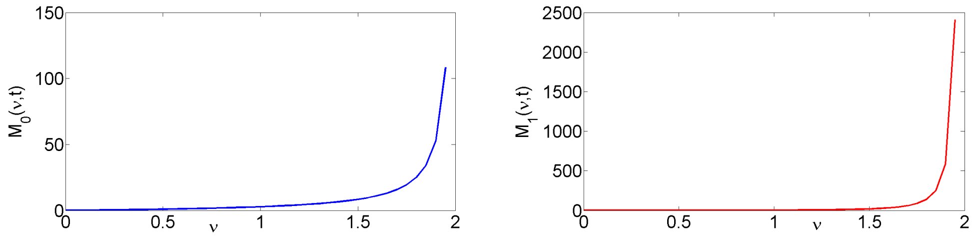
$$f(0,t) = \frac{\nu k}{k_0},\ f(\infty,t) = 0,\ f(x,0) = \frac{\nu k}{k_0}\exp\Big[-\frac{(2-\nu)k}{2}x\tau_0\Big].$$

Recalling that (11) and using (26), by calculations, the zeroth moment $M_0(t)$ and the first moment $M_1(t)$ are presented by

$$M_0(t) = \int_0^\infty f(x,t)\,dx = \frac{2\nu}{(2-\nu)(t+\tau_0)k_0},$$

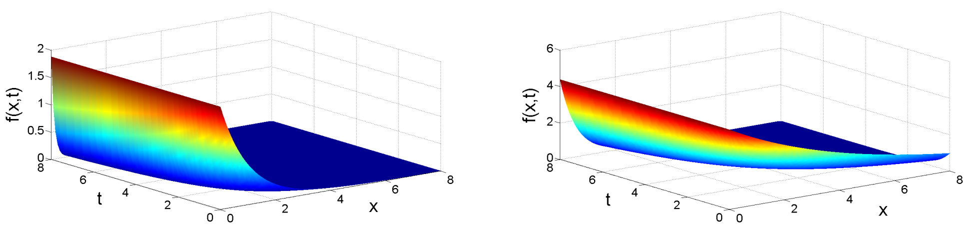
$$M_1(t) = \int_0^\infty x f(x,t)\,dx = \frac{4\nu}{(2-\nu)^2(t+\tau_0)^2 k k_0}.$$

The zeroth moment $M_0(t)$ depends on t, τ_0, ν and k_0, but the first moment $M_1(t)$ depends on t, τ_0, ν, k_0 and k. The moment functions $M_0(t)$ and $M_1(t)$ are both decreasing as time t increases, moreover the values of $M_j(t)(j = 0,1)$ approach 0 as t approaches ∞, which demonstrates the average total number and total volume of particles are decreasing as t increases. Figure 1 interprets the moments $M_0(\nu,t)$ and $M_1(\nu,t)$ are both increasing for case $\tau_0 = 1, k = 1.5, k_0 = 0.6$, as ν changes from 0 to 2 at time $t = 0.2$. Figure 2 shows the evolution of dynamic behavior for solution (26) with $\nu = 0.75$ and $\nu = 1.75$ for case $\tau_0 = 1, k = 1.5, k_0 = 0.6$. Figure 3 interprets the evolution of dynamic behavior for solution (26) with $t = 0.2$ and $t = 1.8$ for case $\tau_0 = 1, k = 1.5, k_0 = 0.6$, when ν changes from 0 to 2.



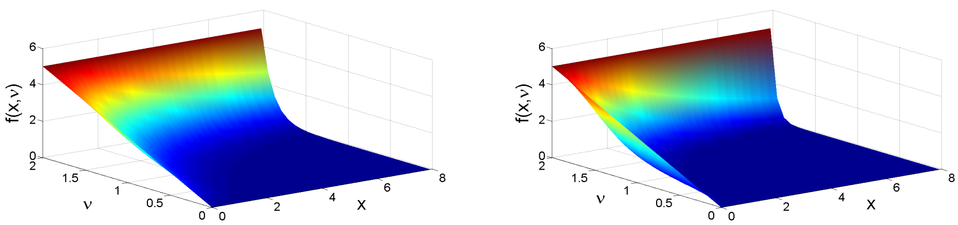
(**a**) $M_0(\nu,t), t = 0.2, \nu \in (0,2)$ (**b**) $M_1(\nu,t), t = 0.2, \nu \in (0,2)$

Figure 1. Evolution of dynamic behavior of moments $M_0(\nu,t)$ and $M_1(\nu,t)$ for solution (26) with $\tau_0 = 1, k = 1.5, k_0 = 0.6$.



(**a**) $f(x,t,\nu), \nu = 0.75, (x,t) \in [0,8] \times [0,8]$ (**b**) $f(x,t,\nu), \nu = 1.75, (x,t) \in [0,8] \times [0,8]$

Figure 2. Evolution of dynamic behavior for solution (26) with $\tau_0 = 1, k = 1.5, k_0 = 0.6$.



(**a**) $f(x,t,\nu), t = 0.2, (x,\nu) \in [0,8] \times [0,2]$ (**b**) $f(x,t,\nu), t = 1.8, (x,\nu) \in [0,8] \times [0,2]$

Figure 3. Evolution of dynamic behavior for solution (26) with $\tau_0 = 1, k = 1.5, k_0 = 0.6$.

Remark 3. *An analytical solution for PBE (9) with $g = 0, K(x,y) = k_0, \nu = 2, P(x|y) = \frac{1}{y}$, $n = 2$ was developed in the literatures [12,13]. However, for the case where $0 < \nu < 2$, an exact solution (26) of PBE (9) with the same remaining constraints is presented.*

Applying the methods of observation and trial function [19–22] to Equation (25), in a similar way, one can obtain explicit exact solutions to Equation (9) with $n = 2$, the corresponding boundary conditions and initial condition, the zeroth moment $M_0(t)$ and the first moment $M_1(t)$ are, respectively, presented by

$$f(x,t) = [\alpha x(t+\tau_0) + \beta]\exp[-\sigma x(t+\tau_0)],\ n = 2, \tag{27}$$

$$f(0,t) = \beta,\ f(\infty,t) = 0,\ f(x,0) = (\alpha x\tau_0 + \beta)\exp(-\sigma x\tau_0),$$
$$M_0(t) = \int_0^\infty f(x,t)\,dx = \frac{\alpha+\beta\sigma}{(t+\tau_0)\sigma^2},\ M_1(t) = \int_0^\infty xf(x,t)\,dx = \frac{2\alpha+\beta\sigma}{(t+\tau_0)^2\sigma^3},$$

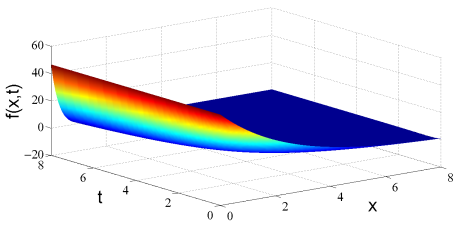
where parameters α, β, σ and ν are given by (28), (29) and (30), respectively.

$$\alpha = \frac{12g(2g-k)}{k_0},\ \beta = \frac{12g}{k_0},\ \sigma = k-2g,\ \nu = 1,\ k > 2g; \tag{28}$$

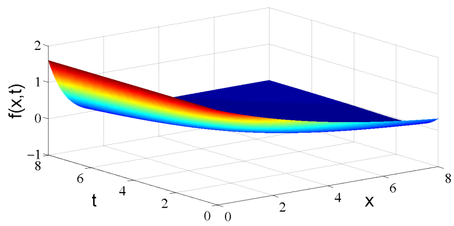
$$\begin{aligned} \alpha &= \frac{12g\big(7g-k-5\sqrt{g(13g-k)}\big)}{k_0},\ \beta = \frac{12\big(2g-\sqrt{g(13g-k)}\big)}{k_0}, \\ \sigma &= k-7g+5\sqrt{g(13g-k)},\ \nu = \frac{12\big(2g-\sqrt{g(13g-k)}\big)}{k},\ 0 < k \le 13g; \end{aligned} \tag{29}$$

$$\begin{aligned} \alpha &= \frac{12g\big(7g-k+5\sqrt{g(13g-k)}\big)}{k_0},\ \beta = \frac{12\big(2g+\sqrt{g(13g-k)}\big)}{k_0}, \\ \sigma &= k-7g-5\sqrt{g(13g-k)},\ \nu = \frac{12\big(2g+\sqrt{g(13g-k)}\big)}{k},\ 12g < k \le 13g. \end{aligned} \tag{30}$$

Evolution of dynamic behavior for explicit exact solutions (27) are considered as follows. Since parameters α and β can be positive or negative, if the values of $f(x,t)$ are positive, which is a physical solution, if the values of $f(x,t)$ are negative, which is an unphysical solution. In terms of $\sigma > 0$, the values of $f(x,t)$ tend to get closer and closer to 0 as t or x get closer and closer to ∞, which shows solutions (27) are asymptotically stable and have a property that population density vanishes for infinite sized particles. If $\alpha + \beta\sigma > 0, 2\alpha + \beta\sigma > 0$, then the moments $M_0(t)$ and $M_1(t)$ decrease as t increases, which demonstrates the average total number and total volume of particles become less and less as t becomes more and more. Finally, they approach 0 as time t sufficiently approaches ∞. Figure 4 shows evolution of dynamic behavior of solution (27) with kinetic parameters (30) and $\tau_0 = 1, k_0 = 0.3$, $(x,t) \in [0,8] \times [0,8]$ for $\nu = 2.847, g = 0.4, k = 4.88$ and $\nu = 1.8462, g = 0.02, k = 0.26$.



(**a**) $f(x,t), \nu = 2.847, g = 0.4, k = 4.88$



(**b**) $f(x,t), \nu = 1.8462, g = 0.02, k = 0.26$

Figure 4. Evolution of dynamic behavior for solution (27) with (30) and $\tau_0 = 1, k_0 = 0.3$.

3.2. Case $K(x,y) = k_1(x+y)$

Applying the property (4) to the kernel $K(x,y) = k_1(x+y)$, one finds that $\gamma = 1$. The invariants for generator X_1 are given by $J_1 = x, J_2 = f$. Hence, an invariant solution to Equation (9) is presented by

$$f(x,t) = \varphi(x),$$

where the reduced equation is

$$gx^n\varphi' = \frac{1}{2}k_1x^2\int_0^1 \varphi(x(1-s))\varphi(xs)\,ds - k_1x\varphi\int_0^\infty \varphi(s)\,ds$$
$$-k_1\varphi\int_0^\infty s\varphi(s)\,ds + \nu k\int_x^\infty s^{n-2}\varphi(s)\,ds - (k+gn)x^{n-1}\varphi. \quad (31)$$

Using the methods of observation and trial function [19–22], by (31) one can derive that a solution of Equation (9) with $g=0, n=3, \nu=2$ is given by

$$f(x,t) = \frac{2k}{k_1}\exp(-\sigma x),\ g=0,\ n=3,\ \nu=2,\ \sigma>0,$$

where the values of population density $f(x,t)$ are close to 0 as particle size x approaches ∞, the boundary conditions and initial condition, the zeroth moment $M_0(t)$ and the first moment $M_1(t)$ are, respectively, presented by

$$f(0,t) = \frac{2k}{k_1},\ f(\infty,t)=0,\ f(x,0) = \frac{2k}{k_1}\exp(-\sigma x),$$
$$M_0(t) = \int_0^\infty f(x,t)\,dx = \frac{2k}{k_1\sigma},\ M_1(t) = \int_0^\infty xf(x,t)\,dx = \frac{2k}{k_1\sigma^2},\ M_0(t) = \sigma M_1(t),$$

$$\frac{dM_0}{dt} = 0,\ \frac{dM_1}{dt} = 0. \quad (32)$$

Equation (32) demonstrates the average total number and total volume of particles are conserved.

If $n=1$, the invariant solutions to Equation (9) corresponding to generator X_2 can not be obtained.

If $n=1$, the invariants for generator $X_2 + \frac{1}{\alpha}X_1, \alpha \neq 0$ are given by $J_1 = x\exp(-\alpha t)$, $J_2 = f\exp(2\alpha t)$. Therefore, an invariant solution to Equation (9) with $n=1$ is presented by

$$f(x,t) = \exp(-2\alpha(t+\tau_0))\varphi(z),\ z = x\exp(-\alpha(t+\tau_0)),\ n=1,\ \alpha\neq 0,$$

where φ satisfies the equation

$$(g-\alpha)z\varphi' + (g+k-2\alpha)\varphi = \frac{1}{2}k_1z^2\int_0^1 \varphi(z(1-s))\varphi(zs)\,ds$$
$$-k_1z\varphi\int_0^\infty \varphi(s)\,ds - k_1\varphi\int_0^\infty s\varphi(s)\,ds + \nu k\int_z^\infty s^{-1}\varphi(s)\,ds.$$

If $n \neq 1$, the invariants for generator X_2 are presented by $J_1 = xt^q, J_2 = t^{-m}f$, $q = \frac{1}{n-1}, m = \frac{n-3}{1-n}$. Hence, an invariant solution to Equation (9) can be presented by

$$f(x,t) = (t+\tau_0)^m\varphi(z),\ z = x(t+\tau_0)^q,\ q = \frac{1}{n-1},\ m = \frac{n-3}{1-n}, \quad (33)$$

where φ satisfies the equation

$$\left(\frac{1}{n-1}z + gz^n\right)\varphi' = \frac{1}{2}k_1z^2\int_0^1 \varphi(z(1-s))\varphi(zs)\,ds - k_1z\varphi\int_0^\infty \varphi(s)\,ds$$
$$-k_1\varphi\int_0^\infty s\varphi(s)\,ds + \nu k\int_z^\infty s^{n-2}\varphi(s)\,ds - \left[(k+gn)z^{n-1} - \frac{n-3}{n-1}\right]\varphi. \quad (34)$$

Noticing that (33) and (34), with the help of the approaches of observation and trial function [19–22], exact solutions of Equation (9) with $n=3$, the boundary conditions and initial condition, the moments $M_0(t)$ and $M_1(t)$ are, respectively, provided by

$$f(x,t) = \frac{24g}{k_1}(1 - 2\sqrt{g}x\sqrt{t+\tau_0})\exp(-4\sqrt{g}x\sqrt{t+\tau_0}),\ k=6g,\ \nu=\frac{5}{3},\ n=3, \quad (35)$$

$$f(0,t) = \frac{24g}{k_1},\ f(\infty,t) = 0,\ f(x,0) = \frac{24g}{k_1}(1 - 2\sqrt{g}x\sqrt{\tau_0})\exp(-4\sqrt{g}x\sqrt{\tau_0}),$$

$$M_0(t) = \int_0^\infty f(x,t)\,dx = \frac{3\sqrt{g}}{k_1\sqrt{t+\tau_0}},\ M_1(t) = \int_0^\infty xf(x,t)\,dx = \frac{6g\sqrt{g(t+\tau_0)} - 3g}{4k_1(\sqrt{g(t+\tau_0)})^3}.$$

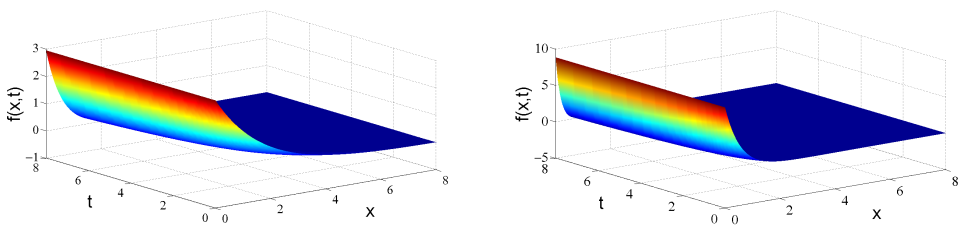
$$f(x,t) = \big(\alpha x\sqrt{t+\tau_0} + \beta\big)\exp\big(-\sigma x\sqrt{t+\tau_0}\big),\ n = 3, \tag{36}$$
$$f(0,t) = \beta,\ f(\infty,t) = 0,\ f(x,0) = (\alpha x\sqrt{\tau_0} + \beta)\exp(-\sigma x\sqrt{\tau_0}),$$
$$M_0(t) = \int_0^\infty f(x,t)\,dx = \frac{\alpha + \beta\sigma}{\sqrt{t+\tau_0}\sigma^2},\ M_1(t) = \int_0^\infty xf(x,t)\,dx = \frac{2\alpha + \beta\sigma}{(t+\tau_0)\sigma^3},$$
$$\sigma = 4\sqrt{\frac{g(5\nu - 12)}{3\nu - 4}},\ k = \frac{48g}{5\nu - 12},\ \alpha = -\frac{48g}{k_1}\sqrt{\frac{g(5\nu - 12)}{3\nu - 4}},\ \beta = \frac{48g\nu}{k_1(5\nu - 12)},$$

$$112\nu^3 - 777\nu^2 + 1752\nu - 1296 = 0. \tag{37}$$

A real root of cubic Equation (37) is given by

$$\nu = \frac{239}{1792\kappa} + \kappa + \frac{37}{16},\ \kappa = \sqrt[3]{\frac{\sqrt{14063}}{2744} + \frac{1867}{28672}},$$

an approximate value of ν is 3.0689825. The values of $f(x,t)$ approach 0 as t or x sufficiently approaches ∞, which shows the obtained solutions (35) and (36) to Equation (9) with $n = 3$ are asymptotically stable, and have a property that population density vanishes for infinite-sized particles. The computed results of the moments $M_0(t)$ and $M_1(t)$ for solutions (35) and (36), demonstrate the average total number and total volume of particles are decreasing as t increases. Figure 5 shows evolution of dynamic behavior of solution (36) with $\nu = 3.0689825$ and $\tau_0 = 1$, $(x,t) \in [0,8] \times [0,8]$ for $g = 0.02, k_1 = 0.3, k = 0.287$, $\sigma = 0.4534$ and $g = 0.2, k_1 = 1, k = 2.87, \sigma = 1.4338$.



(**a**) $g = 0.02, k_1 = 0.3, k = 0.287, \sigma = 0.4534$ (**b**) $g = 0.2, k_1 = 1, k = 2.87, \sigma = 1.4338$

Figure 5. Evolution of dynamic behavior for solution (36) with $\nu = 3.0689825$ and $\tau_0 = 1$.

3.3. Case $K(x,y) = k_2xy$

For the case of product aggregation kernel $K(x,y) = k_2xy$, according to (4) one finds that $\gamma = 2$. The invariants for generator X_1 are given by $J_1 = x, J_2 = f$. Hence, an invariant solution to Equation (9) can be written as

$$f(x,t) = \varphi(x),$$

where function φ has to satisfy the equation

$$gx^n\varphi' = \frac{1}{2}k_2x^3\int_0^1 (1-s)s\varphi(x(1-s))\varphi(xs)\,ds$$
$$-k_2x\varphi\int_0^\infty s\varphi(s)\,ds + \nu k\int_x^\infty s^{n-2}\varphi(s)\,ds - (k+gn)x^{n-1}\varphi.$$

If $n = 1$, the invariant solution to Equation (9) corresponding to generator X_2 cannot be obtained. However, the invariants corresponding to generator $X_2 + \frac{1}{\alpha}X_1, n = 1, \alpha \neq 0$ are presented by $J_1 = x\exp(-\alpha t), J_2 = f\exp(3\alpha t)$. Thus, an invariant solution of Equation (9) with $n = 1$ is provided by

$$f(x,t) = \exp(-3\alpha(t+\tau_0))\varphi(z),\ z = x\exp(-\alpha(t+\tau_0)),\ n = 1,\ \alpha \neq 0,$$

where the reduced equation is

$$(g-\alpha)z\varphi' + (g+k-3\alpha)\varphi = \frac{1}{2}k_2z^3\int_0^1 (1-s)s\varphi(z(1-s))\varphi(zs)\,ds$$

$$-k_2z\varphi\int_0^\infty s\varphi(s)\,ds + \nu k\int_z^\infty s^{-1}\varphi(s)\,ds.$$

If $n \neq 1$, the invariants for generator X_2 are presented by $J_1 = xt^q, J_2 = t^{-m}f$, $q = \frac{1}{n-1}, m = \frac{n-4}{1-n}$. Thus, an invariant solution to Equation (9) can be given by

$$f(x,t) = (t+\tau_0)^m\varphi(z),\ z = x(t+\tau_0)^q,\ q = \frac{1}{n-1},\ m = \frac{n-4}{1-n},$$

where φ satisfies the equation

$$\left(\frac{1}{n-1}z + gz^n\right)\varphi' = \frac{1}{2}k_2z^3\int_0^1 (1-s)s\varphi(z(1-s))\varphi(zs)\,ds$$

$$-k_2z\varphi\int_0^\infty s\varphi(s)\,ds + \nu k\int_z^\infty s^{n-2}\varphi(s)\,ds - \left[(k+gn)z^{n-1} - \frac{n-4}{n-1}\right]\varphi.$$

3.4. Case $K(x,y) = k_3\left(\frac{1}{x}+\frac{1}{y}\right)$

For the case of aggregation kernel $K(x,y) = k_3\left(\frac{1}{x}+\frac{1}{y}\right)$, the property of kernel (4) leads to $\gamma = -1$. The invariants for generator X_1 are $J_1 = x, J_2 = f$. An invariant solution to Equation (9) has the representation

$$f(x,t) = \varphi(x),$$

where the reduced equation is

$$gx^{n+1}\varphi' = \frac{1}{2}k_3x\int_0^1 \left(\frac{1}{1-s}+\frac{1}{s}\right)\varphi(x(1-s))\varphi(xs)\,ds - k_3\varphi\int_0^\infty \varphi(s)\,ds$$

$$-k_3x\varphi\int_0^\infty s^{-1}\varphi(s)\,ds + \nu kx\int_x^\infty s^{n-2}\varphi(s)\,ds - (k+gn)x^n\varphi.$$

If $n = 1$, in an analogous way, the invariant solution for generator X_2 cannot be obtained. However, the invariants corresponding to generator $X_2 + \frac{1}{\alpha}X_1$ are $J_1 = x\exp(-\alpha t), J_2 = f$. An invariant solution to Equation (9) is given by

$$f(x,t) = \varphi(z),\ z = x\exp(-\alpha(t+\tau_0)),\ n = 1,\ \alpha \neq 0,$$

where the reduced equation is

$$(g-\alpha)z^2\varphi' = \frac{1}{2}k_3z\int_0^1 \left(\frac{1}{1-s}+\frac{1}{s}\right)\varphi(z(1-s))\varphi(zs)\,ds - k_3\varphi\int_0^\infty \varphi(s)\,ds$$

$$-k_3z\varphi\int_0^\infty s^{-1}\varphi(s)\,ds + \nu kz\int_z^\infty s^{-1}\varphi(s)\,ds - (g+k)z\varphi. \tag{38}$$

Using the methods of observation and trial function [19–22], by solving Equation (38), an explicit unphysical exact solution to Equation (9) is provided by

$$f(x,t) = -kz\exp(-\sigma z),\ z = x\exp(k(t+\tau_0)),\ \nu = -\frac{2(g+k)}{k},\ n = 1,\ \sigma = \frac{kk_3}{2(g+k)}.$$

If $n \neq 1$, the invariants corresponding to generator X_2 are $J_1 = xt^q, J_2 = ft, q = \frac{1}{n-1}$. Thus, an invariant solution of Equation (9) is presented by

$$f(x,t) = (t+\tau_0)^{-1}\varphi(z),\ z = x(t+\tau_0)^q,\ q = \frac{1}{n-1},$$

where the reduced equation is

$$\Big(\frac{1}{n-1}z + gz^n\Big)z\varphi' = \frac{1}{2}k_3 z\int_0^1 \Big(\frac{1}{1-s}+\frac{1}{s}\Big)\varphi(z(1-s))\varphi(zs)\,ds - k_3\varphi\int_0^\infty \varphi(s)\,ds$$

$$-k_3 z\varphi\int_0^\infty s^{-1}\varphi(s)\,ds + \nu kz\int_z^\infty s^{n-2}\varphi(s)\,ds - [(k+gn)z^{n-1}-1]z\varphi.$$

3.5. Case $K(x,y) = \frac{k_4}{xy}$

Applying the property (4) to kernel $K(x,y) = \frac{k_4}{xy}$, one can obtain $\gamma = -2$. The invariants for generator X_1 are $J_1 = x, J_2 = f$. An invariant solution to Equation (9) has the representation

$$f(x,t) = \varphi(x),$$

where the reduced equation is

$$gx^{n+1}\varphi' = \frac{1}{2}k_4\int_0^1 (1-s)^{-1}s^{-1}\varphi(x(1-s))\varphi(xs)\,ds - k_4\varphi\int_0^\infty s^{-1}\varphi(s)\,ds$$

$$+\nu kx\int_x^\infty s^{n-2}\varphi(s)\,ds - (k+gn)x^n\varphi.$$

If $n = 1$, similarly the invariant solution for generator X_2 cannot be found. However, the invariants corresponding to generator $X_2 + \frac{1}{\alpha}X_1$ are $J_1 = x\exp(-\alpha t), J_2 = f\exp(-\alpha t)$. So an invariant solution of Equation (9) is presented by

$$f(x,t) = \exp(\alpha(t+\tau_0))\varphi(z),\ z = x\exp(-\alpha(t+\tau_0)),\ n = 1,\ \alpha \neq 0,$$

where the reduced equation is

$$(g-\alpha)z^2\varphi' = \frac{1}{2}k_4\int_0^1 (1-s)^{-1}s^{-1}\varphi(z(1-s))\varphi(zs)\,ds - k_4\varphi\int_0^\infty s^{-1}\varphi(s)\,ds$$

$$+\nu kz\int_z^\infty s^{-1}\varphi(s)\,ds - (g+k)z\varphi.$$

If $n \neq 1$, the invariants corresponding to generator X_2 are $J_1 = xt^q, J_2 = ft^{-m}$, $q = \frac{1}{n-1}, m = \frac{n}{1-n}$. Hence, an invariant solution of Equation (9) is presented by

$$f(x,t) = (t+\tau_0)^m\varphi(z),\ z = x(t+\tau_0)^q,\ q = \frac{1}{n-1},\ m = \frac{n}{1-n},$$

where φ satisfies the equation

$$\Big(\frac{1}{n-1}z + gz^n\Big)z\varphi' = \frac{1}{2}k_4\int_0^1 (1-s)^{-1}s^{-1}\varphi(z(1-s))\varphi(zs)\,ds$$

$$-k_3\varphi\int_0^\infty s^{-1}\varphi(s)\,ds + \nu kz\int_z^\infty s^{n-2}\varphi(s)\,ds - \Big[(k+gn)z^{n-1} - \frac{n}{n-1}\Big]z\varphi.$$

Remark 4. *In the analysis of microbial or bacterial populations property of binary division by cells causes ν to be identically 2. It attains a minimum value of 2 during the uniform binary breakage process, but being an average number is not restricted to being an integer. However, in a multiple-splitting process, detailed modeling of the breakage process is indispensable for obtaining the value of ν. Its determination from experiments also implies a potential alternative.*

Following is a summary of the main explicit exact solutions of this work, explicit exact physical and unphysical solutions of equation (9) with kernel (5) are listed in Table 1.

Table 1. Explicit exact solutions of Equation (9) with homogeneous aggregation kernels (5).

No.	Explicit Exact Physical or Unphysical Solutions
1	$f(x,t) = \frac{2k}{k_0}\exp(-\sigma x),\ \sigma > 0,\ g = 0,\ n = 2,\ \nu = 2$
2	$f(x,t) = \left(-\frac{12g\sigma}{k_0}x + \beta\right)\exp(-\sigma x),\ \sigma > 0,\ n = 2,\ \beta = \frac{36g}{k_0},\ k = 12g,\ \nu = 3;$ $\beta = \frac{12g}{k_0},\ k = 2g,\ \nu = 1;\ \beta = -\frac{48g}{k_0},\ k = -23g,\ \nu = \frac{48}{23}$
3	$f(x,t) = \frac{\nu k}{k_0}\exp\left[-\frac{(2-\nu)k}{2}x(t+\tau_0)\right],\ 0 < \nu < 2,\ g = 0,\ n = 2$
4	$f(x,t) = [\alpha x(t+\tau_0) + \beta]\exp[-\sigma x(t+\tau_0)],\ n = 2,$ $\alpha = \frac{12g(2g-k)}{k_0},\ \beta = \frac{12g}{k_0},\ \sigma = k - 2g,\ \nu = 1,\ k > 2g;$ $\alpha = \frac{12g\left(7g-k-5\sqrt{g(13g-k)}\right)}{k_0},\ \beta = \frac{12\left(2g-\sqrt{g(13g-k)}\right)}{k_0},$ $\sigma = k - 7g + 5\sqrt{g(13g-k)},\ \nu = \frac{12\left(2g-\sqrt{g(13g-k)}\right)}{k},\ 0 < k \le 13g;$ $\alpha = \frac{12g\left(7g-k+5\sqrt{g(13g-k)}\right)}{k_0},\ \beta = \frac{12\left(2g+\sqrt{g(13g-k)}\right)}{k_0},$ $\sigma = k - 7g - 5\sqrt{g(13g-k)},\ \nu = \frac{12\left(2g+\sqrt{g(13g-k)}\right)}{k},\ 12g < k \le 13g$
5	$f(x,t) = \frac{2k}{k_1}\exp(-\sigma x),\ g = 0,\ n = 3,\ \nu = 2,\ \sigma > 0$
6	$f(x,t) = \frac{24g}{k_1}(1 - 2\sqrt{g}x\sqrt{t+\tau_0})\exp(-4\sqrt{g}x\sqrt{t+\tau_0}),\ k = 6g,\ \nu = \frac{5}{3},\ n = 3$
7	$f(x,t) = \left(\alpha x\sqrt{t+\tau_0} + \beta\right)\exp\left(-\sigma x\sqrt{t+\tau_0}\right),\ n = 3,$ $\sigma = 4\sqrt{\frac{g(5\nu-12)}{3\nu-4}},\ k = \frac{48g}{5\nu-12},\ \alpha = -\frac{48g}{k_1}\sqrt{\frac{g(5\nu-12)}{3\nu-4}},\ \beta = \frac{48g\nu}{k_1(5\nu-12)},$ $112\nu^3 - 777\nu^2 + 1752\nu - 1296 = 0,\ \nu = \frac{239}{1792\kappa} + \kappa + \frac{37}{16},\ \kappa = \sqrt[3]{\frac{\sqrt{14{,}063}}{2744} + \frac{1867}{28{,}672}}$
8	$f(x,t) = -kz\exp(-\sigma z),\ z = x\exp(k(t+\tau_0)),\ \nu = -\frac{2(g+k)}{k},\ n = 1,\ \sigma = \frac{kk_3}{2(g+k)}$

4. Conclusions and Discussion

The scale transformation group method is a useful technique for finding symmetries of the PBE (9). By analyzing the scaling properties of the PBE (9), this method discovers the symmetries that keep the equation unchanged. These symmetries are then used to simplify the form of the PBE (9) and reduce the number of independent variables. The simplified equations are analytically solved by using standard techniques, which lead to rich results in this work. More importantly, the admitted scaling group, incomplete symmetries, exact solutions, invariant solutions, unphysical solutions and reduced equations have been derived by scaling transformation group for the PBE (9) with the kernel (5), aggregation, nucleation, breakage and growth processes. The existence of solutions is also demonstrated. The analysis of the dynamic behavior of some solutions for the PBE (9) is provided. The exact solutions can be employed to verify the accuracy of numerical solutions and discretization. In the future, this method would be expected to be an effective tool in various fields including physics, chemistry, biology, engineering, finance, and economics.

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