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Short Remark on (p_1, p_2, p_3)-Complex Numbers

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Abstract: Movements on surfaces of centered Euclidean spheres and changes between those with different radii mean complex multiplication in R^3. Here, the Euclidean norm, which generates the spheres, is replaced with an inhomogeneous functional and a product is introduced in a geometric analogy. Because a change in the radius now leads to a change in the shape of the sphere, a three-dimensional dynamic complex structure is created. Statements about invariant probability densities, generalized uniform distributions on generalized spheres, geometric measure representations, and dynamic ball numbers associated with this structure are also presented.

Keywords: vector-valued product; vector exponential function; (p_1, p_2, p_3)-complex algebraic structure; dynamic complex structure; Euler type formula; (p_1, p_2, p_3)-spherical coordinates; matrix homogeneous function; Lie group; dynamic ball numbers; invariant densities; generalized uniform distributions

1. Introduction

Ordinary complex multiplication has geometric visualization as a superposition of a movement on a circle and the alternation between two concentric circles. In the analogous three-dimensional complex structure, circles are replaced with Euclidean spheres. The resulting complex structure is called static in the context of this work because a change in the radius variable does not lead to a change in the shape of the sphere. In this sense, concentric spheres can be viewed as parallel. If, however, the Euclidean norm that generates the spherical surfaces is replaced by an inhomogeneous functional, a dynamic three-dimensional complex structure will be achieved.

Multiplication in three-dimensional generalized complex structures, as studied in [1,2], can be interpreted as changing two radius variables and one angle variable. But, in these papers, the product is not primarily defined in a geometric way by distinguishing generalized spheres in the entire space, but by determining the value of the product for the so-called basis elements. In contrast, here, in the spirit of [3], we used generalized spheres to define multiplication by changes in one radius variable and two angle variables.

To become more specific, let $p = (p_1, p_2, p_3)$ denote three positive real numbers and

$$||\mathfrak{x}|| = \frac{|x|^{p_1}}{p_1} + \frac{|y|^{p_2}}{p_2} + \frac{|z|^{p_3}}{p_3},\ \mathfrak{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in R^3$$

is a functional that plays a fundamental role in defining the density of the three-dimensional p-generalized Gaussian probability law. With regard to the large variety of multivariate probability distributions, which could justify the introduction of numerous other functionals and in turn other complex number systems, we refer to [4–19]. We call

$$B(r) = \{\mathfrak{x} \in R^3 : ||\mathfrak{x}|| \le r\}$$

the p-ball, which is a star-shaped set with respect to the origin $\mathfrak{o} = (0,0,0)^T$, its boundary

$$S(r) = \{\mathfrak{x} \in R^3 : ||\mathfrak{x}|| = r\}$$

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the p-sphere of p-radius $r > 0$, and $B(1) = B$ and $S(1) = S$ the unit p-ball and unit p-sphere, respectively.

If $p_1 = p_2 = p_3$, then the functional $\mathfrak{x} \to ||\mathfrak{x}||$ is positively homogeneous of degree p_1, that is $||\lambda\mathfrak{x}|| = |\lambda|^{p_1} \cdot ||\mathfrak{x}||$, $\lambda \in R$. In other words: S and $r \cdot S$ are "parallel", meaning that a change in p-radius r does not change the "shape" of the sphere. This situation is a static structure. If, in particular, $p_1 \geq 1$, then the p-ball B is convex and its generating functional $||.||$ is a norm, but if $0 < p_1 \leq 1$, then B is radially concave in every sector of a suitably defined fan and $||.||$ is an antinorm according to [20]. Three complex numbers in case $p_1 > 0$ are dealt with in [3].

Throughout this paper, however, we always consider the case

$$p_l \neq p_m, l \neq m. \tag{1}$$

This situation is a dynamic structure because $S(r_1)$ and $S(r_2)$ have different "shapes" if $r_1 \neq r_2$. The functional $||.||$ is not homogeneous in any degree with respect to scalars, but it is matrix homogeneous in the sense

$$S(r) = D(r)S$$

where $D(r) = diag(r^{\frac{1}{p_1}}, r^{\frac{1}{p_2}}, r^{\frac{1}{p_3}})$ is a diagonal matrix. A two-dimensional dynamic structure was studied in [21].

From the author's perspective of probability theory, a natural application of the number system can be constructed here for the analysis of certain invariant probability densities where generalized uniform distributions on generalized spheres, geometric measure representations and dynamic ball numbers play a role. For the convenience of the reader, several of these statements are provided from various sources to complete the picture. The reader may have their own application of dynamic structures in mind.

The rest of the paper is organized as follows. The new (p_1, p_2, p_3)-complex structure including the corresponding trigonometric Euler-type formulae is introduced in Section 2, invariant probability densities are considered in Section 3, Section 4 deals with generalized uniform distributions on generalized spheres and dynamic measure disintegration before Section 5 looks at dynamic ball numbers, and a discussion in Section 6 finishes the paper. In the Appendix A, we quote some functions from the literature that could be used as a starting point for the construction of alternative generalized complex number systems.

2. The (p_1, p_2, p_3)-Complex Structure

2.1. Geometric Approach

In this section, we first introduce coordinates that will allow us to describe a group G of movements on the manifold S. The Lie group (S, G) forms the basis of the complex structure constructed here. Different geometric approaches to complex numbers and various coordinate systems were considered in [19,22–29].

Let $\mathfrak{p} = p_1 p_2 p_3$ be a positive real parameter. We recall that $\mathfrak{p}$-generalized trigonometric functions are defined in [28] as

$$\cos_{\mathfrak{p}} \phi = \frac{\cos\phi}{N(\phi)} \text{ and } \sin_{\mathfrak{p}} \phi = \frac{\sin\phi}{N(\phi)} \text{ where } N(\phi) = (|\cos\phi|^{\mathfrak{p}} + |\sin\phi|^{\mathfrak{p}})^{\frac{1}{\mathfrak{p}}}.$$

Definition 1. *Let $M = (0,\infty) \times [0,\pi] \times [0,2\pi)$. The (p_1, p_2, p_3)-spherical coordinate transformation $Pol : M \to R^3 \setminus \{(0,0,0)^T\}$ is defined*

$$\mathfrak{x} = Pol(r,\varphi,\vartheta) = \begin{pmatrix} (p_1 r)^{\frac{1}{p_1}} sign(\cos_{\mathfrak{p}} \varphi)|\cos_{\mathfrak{p}} \varphi|^{\frac{\mathfrak{p}}{p_1}} \\ (p_2 r)^{\frac{1}{p_2}} sign(\cos_{\mathfrak{p}} \vartheta)(\sin_{\mathfrak{p}} \varphi|\cos_{\mathfrak{p}} \vartheta|)^{\frac{\mathfrak{p}}{p_2}} \\ (p_3 r)^{\frac{1}{p_3}} sign(\sin_{\mathfrak{p}} \vartheta)(\sin_{\mathfrak{p}} \varphi|\sin_{\mathfrak{p}} \vartheta|)^{\frac{\mathfrak{p}}{p_3}} \end{pmatrix}.$$

These coordinates allow the manifold S to be described in the very simple form $S = \{Pol(1, \varphi, \vartheta), (\varphi, \vartheta) \in [0, \pi] \times [0, 2\pi)\}$, or simply by equation $r = 1$.

Lemma 1. *Except in a set of measure zero, which contains $x = 0$ and $y = 0$, the inverse of map Pol is given by $r = r(\mathfrak{x}) = \frac{|x|^{p_1}}{p_1} + \frac{|y|^{p_2}}{p_2} + \frac{|z|^{p_3}}{p_3}$,*

$$\varphi = \varphi(\mathfrak{x}) = \arctan(\delta(\mathfrak{x})) \textit{ where } \delta(\mathfrak{x}) = \left(sign(x) \left(\frac{\frac{|y|^{p_2}}{p_2} + \frac{|z|^{p_3}}{p_3}}{\frac{|x|^{p_1}}{p_1}} \right)^{1/\mathfrak{p}} \right)$$

and

$$\vartheta = \vartheta(\mathfrak{x}) = \arctan(\Theta(\mathfrak{x})) \textit{ where } \Theta(\mathfrak{x}) = sign(yz) \left(\frac{p_2}{p_3} \right)^{\frac{1}{\mathfrak{p}}} \left(\frac{|z|^{p_3}}{|y|^{p_2}} \right)^{\frac{1}{\mathfrak{p}}}.$$

Proof. The first equation follows on using $|\sin_{\mathfrak{p}} t|^{\mathfrak{p}} + |\cos_{\mathfrak{p}} t|^{\mathfrak{p}} = 1$. Notice that $sign(y) = sign(\cos \vartheta)$ and $sign(z) = sign(\sin \vartheta)$. The third equation now follows by

$$\frac{|z|^{\frac{p_3}{\mathfrak{p}}}}{|y|^{\frac{p_2}{\mathfrak{p}}}} \left(\frac{p_2}{p_3} \right)^{\frac{1}{\mathfrak{p}}} sign(\tan \vartheta) = \tan \vartheta$$

and the second equation is proven similarly. For more details and a slight modification, we refer to [19]. □

A general definition of what we mean by a vector-valued product was given in [30]. Here, we first used it in the language of (p_1, p_2, p_3)-spherical coordinates and only later in the language of Cartesian coordinates. The following definition is analogous to one given in [21], where a two-dimensional case was considered.

Definition 2. *The three-complex (p_1, p_2, p_3)-spherical coordinate product of the vectors $\mathfrak{x}_l = Pol(r_l, \varphi_l, \vartheta_l), l = 1, 2$ is defined*

$$\mathfrak{x}_1 \otimes \mathfrak{x}_2 = Pol(r_1 r_2, \varphi_1 \diamond \varphi_2, \vartheta_1 \rhd \vartheta_2)$$

where

$$\begin{aligned} \varphi_1 \diamond \varphi_2 = (\varphi_1 + \varphi_2) I_{[0,\pi]}(\varphi_1 + \varphi_2) + (\varphi_1 + \varphi_2 - \pi) I_{(\pi,2\pi)}(\varphi_1 + \varphi_2) \\ + (\varphi_1 + \varphi_2 - 2\pi) I_{\{2\pi\}}(\varphi_1 + \varphi_2) \end{aligned}$$

and

$$\vartheta_1 \rhd \vartheta_2 = (\vartheta_1 + \vartheta_2) I_{[0,2\pi)}(\vartheta_1 + \vartheta_2) + (\vartheta_1 + \vartheta_2 - 2\pi) I_{[2\pi,4\pi)}(\vartheta_1 + \vartheta_2).$$

Remark 1. ***(a)*** *We first mention that the vector $\mathfrak{e} = Pol(1, 0, 0)$ satisfies the equation*

$$\mathfrak{e} \otimes \mathfrak{x} = Pol(1, 0, 0) \otimes Pol(r, \varphi, \vartheta) = Pol(r, \varphi, \vartheta) = \mathfrak{x}$$

and is therefore called a multiplicative neutral element.

(b) *Moreover, if $\mathfrak{x} \in S(r)$ and $\mathfrak{y} \in S$, then $\mathfrak{x} \otimes \mathfrak{y} \in S(r)$. Thus multiplications by elements of S build a group G and (S, G) is a Lie group.*

Definition 3. *Vector-valued vector (p_1, p_2, p_3)-powers are defined as*

$$\mathfrak{x}^{\otimes 0} = \mathfrak{e}, \mathfrak{x}^{\otimes k} = \mathfrak{x}^{\otimes (k-1)} \otimes \mathfrak{x}, k = 1, 2, \ldots$$

Example 1. *The following basic elements of R^3 are frequently used:*

$$Pol(1,0,0) = \begin{pmatrix} p_1^{\frac{1}{p_1}} \\ 0 \\ 0 \end{pmatrix} = \mathfrak{e}, Pol(1,\frac{\pi}{2},0) = \begin{pmatrix} 0 \\ p_2^{\frac{1}{p_2}} \\ 0 \end{pmatrix} = \mathfrak{i},$$
$$Pol(1,\frac{\pi}{2},\frac{\pi}{2}) = \begin{pmatrix} 0 \\ 0 \\ p_3^{\frac{1}{p_3}} \end{pmatrix} = \mathfrak{j},\ Pol(1,\pi,0) = -\mathfrak{e}.$$

Remark 2. *The multiple ambiguities that occur in the present coordinate system remain here without further comment.*

Example 2. *Let*

$$\mathfrak{x} = \begin{pmatrix} 0 \\ y \\ z \end{pmatrix} = Pol(r,\frac{\pi}{2},\vartheta) \text{ with } r = \frac{|y|^{p_2}}{p_2} + \frac{|z|^{p_3}}{p_3} \text{ and suitably chosen } \vartheta$$

then

$$\mathfrak{x}^{\otimes 2} = Pol(r^2,\pi,2\vartheta(mod(2\pi))) = -r^{\frac{2}{p_1}}\mathfrak{e}.$$

Example 3. *Because of the following equations and for historical reasons, $\mathfrak{i}$ and $\mathfrak{j}$ are called the imaginary units of the complex structure considered here:*

$$\mathfrak{i}\otimes\mathfrak{i} = Pol(1,\pi,0) = -\mathfrak{e}, \mathfrak{j}\otimes\mathfrak{j} = Pol(1,\pi,\pi) = -\mathfrak{e},$$
$$\mathfrak{i}\otimes\mathfrak{j} = Pol(1,\pi,\frac{\pi}{2}) = -\mathfrak{e}.$$

The next example serves as a preparation for Euler's formula.

Example 4. *For $\mathfrak{x}$ as in Example 2,*

$$\mathfrak{x}^{\otimes(2k)} = (-1)^k r^{\frac{2k}{p_1}}\mathfrak{e}, \quad \mathfrak{x}^{\otimes(2k+1)} = (-1)^k r^{\frac{2k}{p_1}}\mathfrak{x}, \quad k = 0,1,2,\ldots$$

Definition 4. *Let $\oplus$ denote usual component-wise vector addition, $\mathfrak{o}$ the additive neutral element and $\cdot$ multiplication by a scalar. We call $(R^3,\oplus,\otimes,\cdot,\mathfrak{o},\mathfrak{e},\mathfrak{i},\mathfrak{j})$ the algebraic structure of (p_1,p_2,p_3)-complex numbers or (p_1,p_2,p_3)-complex vectors and $||.||$ its generating functional.*

Remark 3. *Complex numbers are often considered synonymously as points in the plane or elements of the two-dimensional vector space R^2. In the same sense, we use the terms (p_1,p_2,p_3)-complex number and (p_1,p_2,p_3)-complex vector synonymously in this paper.*

2.2. Analytical Reformulation

Let I_A denote the indicator of set A.

Definition 5. *The three-complex Cartesian coordinate (p_1,p_2,p_3)-product of vectors $\mathfrak{x}_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ and $\mathfrak{x}_2 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$ is defined by*

$$\mathfrak{x}_1 \odot \mathfrak{x}_2 = \begin{pmatrix} (p_1 r_1 r_2)^{\frac{1}{p_1}} sign(\delta_1 - \delta_2) I(\mathfrak{x}_1, \mathfrak{x}_2) \frac{|1-\delta_1\delta_2|^{\frac{\mathfrak{p}}{p_1}}}{(|\delta^1+\delta_2|^{\mathfrak{p}}+|1-\delta_1\delta_2|^{\mathfrak{p}})^{\frac{1}{p_1}}} \\ (p_2 r_1 r_2)^{\frac{1}{p_2}} sign(1-\Theta_1\Theta_2) \frac{|(\delta_1+\delta_2)(1-\Theta_1\Theta_2)|^{\frac{\mathfrak{p}}{p_2}}}{(|\delta_1+\delta_2|^{\mathfrak{p}}+|1-\delta_1\delta_2|^{\mathfrak{p}})^{\frac{1}{p_2}} (|\Theta_1+\Theta_2|^{\mathfrak{p}}+|1-\Theta_1\Theta_2|^{\mathfrak{p}})^{\frac{1}{p_2}}} \\ (p_3 r_1 r_2)^{\frac{1}{p_3}} sign(\Theta_1+\Theta_2) \frac{|(\delta_1+\delta_2)(\Theta_1+\Theta_2)|^{\frac{\mathfrak{p}}{p_3}}}{(|\delta_1+\delta_2|^{\mathfrak{p}}+|1-\delta_1\delta_2|^{\mathfrak{p}})^{\frac{1}{p_3}} (|\Theta_1+\Theta_2|^{\mathfrak{p}}+|1-\Theta_1\Theta_2|^{\mathfrak{p}})^{\frac{1}{p_3}}} \end{pmatrix}$$

where the indicator function $I(\mathfrak{x}_1, \mathfrak{x}_2)$ is

$$I(\mathfrak{x}_1, \mathfrak{x}_2) = I_{[0,\pi]}(\arctan\delta_1 + \arctan\delta_2) - I_{(\pi,2\pi)}(\arctan\delta_1 + \arctan\delta_2)$$

and, for $l = 1, 2$,

$$\delta_l = sign(x_l) \left(\frac{\frac{|y_l|^{p_2}}{p_2} + \frac{|z_l|^{p_3}}{p_3}}{\frac{|x_l|^{p_1}}{p_1}} \right)^{\frac{1}{\mathfrak{p}}}, \ \Theta_l = sign(y_l z_l) \left(\frac{p_2}{p_3} \right)^{\frac{1}{\mathfrak{p}}} \frac{|z_l|^{\frac{p_3}{\mathfrak{p}}}}{|y_l|^{\frac{p_2}{\mathfrak{p}}}} \ \text{as well as} \ r_l = ||\mathfrak{x}_l||.$$

Theorem 1. *The three-complex Cartesian coordinate (p_1, p_2, p_3)-product of the three-dimensional (p_1, p_2, p_3)-complex vectors $\mathfrak{x}_l = Pol(r_l, \varphi_l, \vartheta_l)$, $l = 1, 2$ coincides with their three-complex (p_1, p_2, p_3)-spherical coordinate product.*

Proof. We first remark that

$$sign(\cos\varphi_1 \diamond \varphi_2) = sign(\cos(\varphi_1 + \varphi_2))(I_1 - I_2)$$

where

$$I_1 = I_{[0,\pi]}(\varphi_1 + \varphi_2), I_2 = I_{(\pi,2\pi)}(\varphi_1 + \varphi_2).$$

It follows, therefore, from the definition of the generalized trigonometric functions and the 2π-periodicity of the sine and cosine functions that

$$\begin{aligned} \mathfrak{x}_1 \otimes \mathfrak{x}_2 &= Pol(r_1 r_2, \varphi_1 \diamond \varphi_2, \vartheta_1 \rhd \vartheta_2) \\ &= \begin{pmatrix} (p_1 r_1 r_2)^{\frac{1}{p_1}} (I_1 - I_2)(sign\cos(\varphi_1+\varphi_2))|\cos_{\mathfrak{p}}(\varphi_1+\varphi_2)|^{\frac{\mathfrak{p}}{p_1}} \\ (p_2 r_1 r_2)^{\frac{1}{p_2}} sign(\cos(\vartheta_1+\vartheta_2))(\sin_{\mathfrak{p}}\varphi_1 \diamond \varphi_2|\cos_{\mathfrak{p}}(\vartheta_1+\vartheta_2)|)^{\frac{\mathfrak{p}}{p_2}} \\ (p_3 r_1 r_2)^{\frac{1}{p_3}} sign(\sin(\vartheta_1+\vartheta_2))(\sin_{\mathfrak{p}}\varphi_1 \diamond \varphi_2|\sin_{\mathfrak{p}}(\vartheta_1+\vartheta_2)|)^{\frac{\mathfrak{p}}{p_3}} \end{pmatrix}. \end{aligned}$$

For Lemma 1,

$$\cos\varphi(\mathfrak{x}_l) = \frac{1}{\sqrt{1+\delta_l^2}}, \ \sin\varphi(\mathfrak{x}_l) = \frac{\delta_l}{\sqrt{1+\delta_l^2}}, l = 1, 2.$$

Thus,

$$\sin(\varphi_1+\varphi_2) = \frac{\delta_1+\delta_2}{\sqrt{(1+\delta_1^2)(1+\delta_2^2)}}, \cos(\varphi_1+\varphi_2) = \frac{1-\delta_1\delta_2}{\sqrt{(1+\delta_1^2)(1+\delta_2^2)}}$$

and

$$N(\varphi_1+\varphi_2) = \frac{(|\delta_1+\delta_2|^{\mathfrak{p}} + |1-\delta_1\delta_2|^{\mathfrak{p}})^{\frac{1}{\mathfrak{p}}}}{\sqrt{(1+\delta_1^2)(1+\delta_2^2)}}.$$

With similar representations for $\sin(\vartheta_1+\vartheta_2), \cos(\vartheta_1+\vartheta_2)$ and $N_{\mathfrak{p}}(\theta_1+\theta_2)$, the proof will be finished. □

Remark 4. *The summary of the results of this section shows that* $(R^3, \oplus, \odot, \cdot, \mathfrak{o}, \mathfrak{e}, \mathfrak{i}, \mathfrak{j})$ *is an analytical reformulation of the algebraic structure of* (p_1, p_2, p_3)*-complex numbers. This is why we simply write* $(p_1, p_2 p_3)$*-powers as* $\mathfrak{x}^k$ *instead of* $\mathfrak{x}^{\otimes k}$*.*

2.3. Euler Type Formulae

Definition 6. *The vector-valued, or three-complex,* (p_1, p_2, p_3)*-exponential function is defined*

$$\exp(\mathfrak{x}) = \sum_{k=0}^{\infty} \frac{\mathfrak{x}^k}{k!}, \ \mathfrak{x} \in R^3.$$

The formulas in the next theorems are based on Euler's [31] (1748) famous representation of trigonometric functions using an imaginary unit.

Theorem 2. *The following vector equations are true:*

$$\begin{aligned} \exp(y\mathfrak{i}) &= (\cos y)\mathfrak{e} + (\sin y)\mathfrak{i}, \\ \exp(z\mathfrak{j}) &= (\cos z)\mathfrak{e} + (\sin z)\mathfrak{j}. \end{aligned}$$

Proof. These two statements are proven by straightforward vector series expansions and appropriate rearrangements of terms. □

It follows immediately from this theorem that two well-known formulas about trigonometric functions and the values of the exponential function in imaginary arguments, see (9) below, can be extended to the three-dimensional case as follows:

$$\frac{1}{2}(\exp(y\mathfrak{i}) + \exp(-y\mathfrak{i})) = (\cos y)\mathfrak{e},$$

$$\frac{1}{2}(\exp(y\mathfrak{i}) - \exp(-y\mathfrak{i})) = (\sin y)\mathfrak{i}$$

and

$$\frac{1}{2}(\exp(z\mathfrak{j}) + \exp(-z\mathfrak{j})) = (\cos z)\mathfrak{e},$$

$$\frac{1}{2}(\exp(z\mathfrak{j}) - \exp(-z\mathfrak{j})) = (\sin z)\mathfrak{j}.$$

The following formula is closely related to the Euler type formula (19) in [3].

Theorem 3. *For* $\mathfrak{x}$ *and* r*, as in Example 2,*

$$\exp\begin{pmatrix} 0 \\ y \\ z \end{pmatrix} = \cos(r^{\frac{1}{p_1}})\begin{pmatrix} p_1^{\frac{1}{p_1}} \\ 0 \\ 0 \end{pmatrix} + \frac{\sin(r^{\frac{1}{p_1}})}{r^{\frac{1}{p_1}}}\begin{pmatrix} 0 \\ y \\ z \end{pmatrix}.$$

Proof. It follows from Example 4 that

$$\exp(\mathfrak{x}) = \mathfrak{e} + \mathfrak{x} - \frac{1}{2!}r^{\frac{2}{p_1}}\mathfrak{e} - \frac{r^{\frac{2}{p_1}}}{3!}\mathfrak{x} + \frac{r^{\frac{4}{p_1}}}{4!}\mathfrak{e} + \frac{r^{\frac{4}{p_1}}}{5!}\mathfrak{x} - \frac{r^{\frac{6}{p_1}}}{6!}\mathfrak{e} + - \ldots.$$

Rearranging the terms provides

$$\exp(\mathfrak{x}) = (1 - \frac{r^{\frac{2}{p_1}}}{2!} + \frac{r^{\frac{4}{p_1}}}{4!} - \frac{r^{\frac{6}{p_1}}}{6!} + -)\mathfrak{e} + (1 - \frac{r^{\frac{2}{p_1}}}{3!} + \frac{r^{\frac{4}{p_1}}}{5!} - \frac{r^{\frac{6}{p_1}}}{7!} + -)\mathfrak{x}$$

□

It follows immediately from this theorem that

$$\frac{1}{2}\left(\exp\begin{pmatrix}0\\y\\z\end{pmatrix}+\exp\begin{pmatrix}0\\-y\\-z\end{pmatrix}\right)=\cos(r^{\frac{1}{p_1}})\begin{pmatrix}p_1^{\frac{1}{p_1}}\\0\\0\end{pmatrix},$$

$$\frac{1}{2}\left(\exp\begin{pmatrix}0\\y\\z\end{pmatrix}-\exp\begin{pmatrix}0\\-y\\-z\end{pmatrix}\right)=\frac{\sin(r^{\frac{1}{p_1}})}{r^{\frac{1}{p_1}}}\begin{pmatrix}0\\y\\z\end{pmatrix}$$

and

$$\frac{1}{2}\left(\exp\begin{pmatrix}0\\y\\z\end{pmatrix}+\exp\begin{pmatrix}0\\y\\-z\end{pmatrix}\right)=\cos(r^{\frac{1}{p_1}})\begin{pmatrix}p_1^{\frac{1}{p_1}}\\0\\0\end{pmatrix}+\frac{\sin(r^{\frac{1}{p_1}})}{r^{\frac{1}{p_1}}}\begin{pmatrix}0\\y\\0\end{pmatrix},$$

$$\frac{1}{2}\left(\exp\begin{pmatrix}0\\y\\z\end{pmatrix}-\exp\begin{pmatrix}0\\y\\-z\end{pmatrix}\right)=\frac{\sin(r^{\frac{1}{p_1}})}{r^{\frac{1}{p_1}}}\begin{pmatrix}0\\0\\z\end{pmatrix}$$

as well as

$$\frac{1}{2}\left(\exp\begin{pmatrix}0\\y\\z\end{pmatrix}+\exp\begin{pmatrix}0\\-y\\z\end{pmatrix}\right)=\cos(r^{\frac{1}{p_1}})\begin{pmatrix}p_1^{\frac{1}{p_1}}\\0\\0\end{pmatrix}+\frac{\sin(r^{\frac{1}{p_1}})}{r^{\frac{1}{p_1}}}\begin{pmatrix}0\\0\\z\end{pmatrix},$$

$$\frac{1}{2}\left(\exp\begin{pmatrix}0\\y\\z\end{pmatrix}-\exp\begin{pmatrix}0\\-y\\z\end{pmatrix}\right)=\frac{\sin(r^{\frac{1}{p_1}})}{r^{\frac{1}{p_1}}}\begin{pmatrix}0\\y\\0\end{pmatrix}.$$

Remark 5. *Imagine we apply the results of this section to the vector* $\begin{pmatrix}y\\z\end{pmatrix}=\begin{pmatrix}y(t)\\z(t)\end{pmatrix}$ *and the variable* $r=r(t), t\in T$ *where* T *is a time interval. An application-oriented interpretation of Theorems 2 and 3 could then state that points from the space* R^3 *are transformed into fixed values of oscillating quantities or functions at any time through complex* (p_1, p_2, p_3)*-exponentiation.*

3. Invariant Probability Densities

It is well-known that a function ϕ that is defined in R^3 is said to be invariant with respect to transformation $T: R^3 \to R^3$ if it satisfies the equation

$$\phi(T(x,y,z)) = \phi(x,y,z) \text{ for all } (x,y,z)^T \in \mathbb{R}^3.$$

Definition 7. *A probability density* $\phi = \phi_{g;[p_1,p_2,p_3]}$ *defined in* R^3 *is called* $[p_1, p_2, p_3]$*-spherical if it is of the form*

$$\phi_{g;[p_1,p_2,p_3]}(x,y,z) = C(g;[p_1,p_2,p_3])g(||(x,y,z)^T||), (x,y,z)^T \in \mathbb{R}^3$$

where the function $g: [0,\infty) \to [0,\infty)$ *is a density generating function satisfying*

$$0 < \int_0^\infty r^{\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}-1} g(r)dr < \infty$$

and $C(g;[p_1,p_2,p_3])$ *is a normalizing constant.*

Proposition 1. *If ϕ is a $[p_1, p_2, p_3]$-spherical probability density then it has in accordance with Remark 1 the invariance property which says that, for every $(x_1, y_1, z_1)^T \in S$,*

$$\phi((x_1, y_1, z_1)^T \odot (x_2, y_2, z_2)^T) = \phi((x_2, y_2, z_2)^T) \text{ for all } (x_2, y_2, z_2)^T \in \mathbb{R}^3.$$

Remark 6. *This property of $[p_1, p_2, p_3]$-spherical probability densities may be the basis for testing a statistical hypothesis on the $[p_1, p_2, p_3]$-sphericity of a probability density because it does not refer in any way to whether the tails of this distribution are heavy or light. To this end, let us be provide a disjoint partitioning $\mathfrak{P}$ of size N of the sphere S and the relative frequency distribution over $\mathfrak{P}$, which comes from a sample of values $z/||z||$, where the elements z follow the density function ϕ. Then, a first rough test for the spherical distribution of z consists of visually comparing the relative frequency distributions of the values $z/||z||$ before and after multiplicative transformation of all sample elements z with a fixed element from S. A wide range of mathematical statistic techniques can be applied to refine this test and to equip it with sophisticated mathematical properties.*

Example 5. *The Kotz-type density generating function with parameters of β and γ from $(0, \infty)$ and $M > 1 - \frac{1}{p_1} - \frac{1}{p_2} - \frac{1}{p_3}$ is*

$$g(r) = r^{M-1} e^{-\beta r^\gamma} I_{(0,\infty)}(r)$$

and the corresponding (p_1, p_2, p_3)-spherical probability density is

$$\phi^{(p_1,p_2,p_3)}_{Kt;M,\beta,\gamma}(x, y, z) = C^{(p_1,p_2,p_3)}_{Kt;M,\beta,\gamma}\left(\frac{|x|^{p_1}}{p_1} + \frac{|y|^{p_2}}{p_2} + \frac{|z|^{p_3}}{p_3}\right)^{M-1} e^{-\beta(\frac{|x|^{p_1}}{p_1} + \frac{|y|^{p_2}}{p_2} + \frac{|z|^{p_3}}{p_3})^\gamma}$$

with

$$C^{(p_1,p_2,p_3)}_{Kt;M,\beta,\gamma} = \frac{\gamma \beta^{(M-1+\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3})/\gamma}}{\Gamma((M-1+\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3})/\gamma) 8B(\frac{1}{p_1}, \frac{1}{p_2}, \frac{1}{p_3}) \prod_{i=1}^{3} p_i^{1/p_i - 1}}.$$

Elements from the corresponding class of probability distributions are considered light-tailed distributions and the following ones are considered heavy-tailed distributions. In this and the next example, the calculation of the constant is achieved by integrating the density generating function g.

Example 6. *The Pearson Type VII density generating function with parameters of $\nu > 0$ and $M > \max\{1, \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3}\}$ is*

$$g(r) = (1 + \frac{r}{\nu})^{-M} I_{(0,\infty)}(r)$$

and the corresponding (p_1, p_2, p_3)-spherical probability density is

$$\varphi^{(p_1,p_2,p_3)}_{PT7;M,\nu}(x, y, z) = C^{(p_1,p_2,p_k)}_{PT7;M,\nu}\left(1 + \frac{1}{\nu}\left(\frac{|x|^{p_1}}{p_1} + \frac{|y|^{p_2}}{p_2} + \frac{|z|^{p_3}}{p_3}\right)\right)^{-M}. \tag{2}$$

where

$$C^{(p_1,p_2,p_3)}_{PT7;M,\nu} = \frac{\Gamma(M) \prod_{i=1}^{3} p_i^{1-1/p_i}}{8\nu^{\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}} \cdot \Gamma(M - \frac{1}{p_1} - \frac{1}{p_2} - \frac{1}{p_3})\Gamma(\frac{1}{p_1})\Gamma(\frac{1}{p_2})\Gamma(\frac{1}{p_3})}.$$

4. Generalized Uniform Distribution on the Sphere S and Dynamic Geometric Disintegration of the Lebesgue Measure in R^3

Let $\mathfrak{B}$ denote the Borel-σ field in R^3, μ is the Lebesgue measure on $\mathfrak{B}$, $(\Omega, \mathfrak{A}, P)$ is a probability space and $X : \Omega \to B$ is a random vector, that is a $(\mathfrak{A}, \mathfrak{B})$-measurable function. We assume that the random vector X is uniformly distributed on B,

$$P(X \in M) = \frac{\mu(M)}{\mu(B)}, M \in \mathfrak{B} \cap B,$$

and define a non-negative random variable and a random vector taking values in S by

$$R = ||X|| \quad \text{and} \quad U = D(\frac{1}{R})X,$$

respectively. For $A \in \mathfrak{B}(S) = \mathfrak{B} \cap S$, we call

$$CPC(A) = \{D(r)\mathfrak{x} : \mathfrak{x} \in A, r > 0\}$$

a $D(.)$-transformed central projection cone and

$$Se(A, r) = CPC(A) \cap B(r)$$

the corresponding $D(.)$-transformed ball sector generated by A, respectively. We denote the volume of such a sector

$$f_A(r) = \mu(Se(A, r))$$

and define the (p_1, p_2, p_3)-spherical or functional $||.||$-related surface content of $D(r)A$ as

$$O(D(r)A) = f_A'(r), r > 0.$$

In the case $p_1 = p_2 = p_3 = 2$ not being under consideration here, this notion coincides with the Euclidean surface content measure. Note that

$$\mu(Se(A, r)) = \frac{\mathfrak{p}^2 p_1^{\frac{1}{p_1}-1} p_2^{\frac{1}{p_2}-1} p_3^{\frac{1}{p_3}-1}}{\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3}} r^{\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}} \int\limits_{Pol^{*-1}(A)} F_1(\varphi) F_2(\vartheta) \frac{d\varphi d\vartheta}{N_{\mathfrak{p}}^2(\varphi) N_{\mathfrak{p}}^2(\vartheta)}$$

with $N_q(t) = (|\sin t|^q + |\cos t|^q)^{\frac{1}{q}}$, $Pol^*(\varphi, \vartheta) = Pol(1, \varphi, \vartheta)$ and

$$F_1(\varphi) = |\cos_{\mathfrak{p}}(\varphi)|^{\frac{\mathfrak{p}}{p_1}-1} |\sin_{\mathfrak{p}}(\varphi)|^{\frac{\mathfrak{p}}{p_2}+\frac{\mathfrak{p}}{p_3}-1}, F_2(\vartheta) = |\cos_{\mathfrak{p}}(\vartheta)|^{\frac{\mathfrak{p}}{p_2}-1} |\sin_{\mathfrak{p}}(\vartheta)|^{\frac{\mathfrak{p}}{p_3}-1}.$$

Example 7. *The volume of the p-ball $B(r)$ satisfies*

$$\mu(B(r)) = \frac{8}{\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3}} B(\frac{1}{p_1}, \frac{1}{p_2}, \frac{1}{p_3}) p_1^{\frac{1}{p_1}-1} p_2^{\frac{1}{p_2}-1} p_3^{\frac{1}{p_3}-1} r^{\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}}. \tag{3}$$

Example 8. *The dual surface content measure of the generalized sphere S satisfies*

$$O(S) = 8B(\frac{1}{p_1}, \frac{1}{p_2}, \frac{1}{p_3}) p_1^{\frac{1}{p_1}-1} p_2^{\frac{1}{p_2}-1} p_3^{\frac{1}{p_3}-1}$$

where $B(a, b, c) = \Gamma(a)\Gamma(b)\Gamma(c)/\Gamma(a + b + c)$ means the multi Beta function.

Definition 8. *The probability law*

$$\omega(A) = \frac{O(A)}{O(S)}, A \in \mathfrak{B}(S)$$

is called the functional $||.||$*-related or* (p_1, p_2, p_3)*-spherical uniform distribution on* $\mathfrak{B}(S)$.

Note that the random vector U follow this distribution, $U \sim \omega$, is stochastically independent of the random variable R, and the probability density of R is

$$f(r) = (\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3})r^{\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}-1}, r \in (0,1). \tag{4}$$

Moreover, if any random variable $\tilde{R}$ follows density f in (4) and any random vector $\tilde{U}$ satisfies $\tilde{U} \sim \omega$, then $\tilde{X} = \tilde{R} \cdot \tilde{U}$ is uniformly distributed on the unit p-ball B. The following theorem is proven analogously to Theorem 1 in [21] and using (7) in [19].

Theorem 4. *If h is integrable over a Borel set A, then*

$$\int\limits_A h(x)dx = \int\limits_0^\infty (r^{\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}-1} \int\limits_{A^*(r)} h(Pol(r,\varphi,\vartheta)))J^*(\varphi,\vartheta)d\varphi d\vartheta dr$$

where

$$A^*(r) = \{\varphi \in M_n^* : Pol(1,\varphi) \in [D(r^{-1})A] \cap S\}$$

and

$$J^*(\varphi,\vartheta) = \mathfrak{p}^2 p_1^{\frac{1}{p_1}-1} p_2^{\frac{1}{p_2}-1} p_3^{\frac{1}{p_3}-1} \frac{F_1(\varphi)F_2(\vartheta)}{N_{\mathfrak{p}}^2(\varphi)N_{\mathfrak{p}}^2(\vartheta)}.$$

The following result extends formula (7) in [21] to being three-dimensional.

Corollary 1. *If A has finite volume, then the Lebesgue measure of A satisfies the dynamic geometric disintegration formula*

$$\mu(A) = O(S) \int\limits_0^\infty r^{\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}-1} \mathfrak{F}(A,r)dr$$

where

$$\mathfrak{F}(A,r) = \frac{O([D(r^{-1})A] \cap S)}{O(S)}$$

is the (p_1, p_2, p_3)*-spherical dynamical intersection proportion function (ipf) of the set A.*

Note that the shape of $S(r)$ changes if the p-radius r changes (unless p has exclusively equal components). The representation of $\mu(A)$ given in this corollary may be understood as a generalization of Cavalieri's and Torricelli's method of indivisibles, where the indivisibles are the sets

$$[D_p(r^{-1})A] \cap S_p,\ r > 0.$$

For more details on the generalized methods of indivisibles, we refer to [32–34]. In [32], the classical method of Cavalieri and Torricelli has been generalized to a multidimensional situation in which the measure is not the ordinary volume or Gaussian measure. For so-called moderate or large deviation areas whose distance from the origin approaches infinity, it is shown how their Gaussian content essentially depends on their properties in the neighborhood of the point on the surface of the area that is closest to the origin. The surface content of subsets of spheres plays a crucial role in describing these properties and is essentially expressed by the properties of a function that later received the name intersection percentage function or intersection proportion function. The latter function is, in turn, closely linked to another function that was later called the sector function. The application of the classic method of Cavalieri and Toricelli was often very successful, but in certain cases it was also fraught with contradictions. The constructive role of Fubini's theorem in this regard was discussed and the resulting geometric measure representation of the generalized method of indivisibles was subsequently applied to various probabilistic and statistical problems. For example, statements that applied to Gaussian populations were extended to general spherical populations in [33], analogies about exponentially distributed populations were derived, and exact distribution statements in non-linear models were made possible. A geometric-measure theoretic approach to the so-called skew normal distribution in [34] allowed to unify several known representations of this distribution from a geometric point of view and to generalize these results for spherically distributed sample vectors. An extension of such results to general norm contoured two-dimensional populations is possible on the basis of some later results.

5. The (p_1, p_2, p_3)-Ball and Sector Number Functions

If density level sets of probability laws are p-spheres, that is spheres with respect to the functional $||.||$, then a factorial component of normalizing constants is the so-called ball numbers. The general connection between measure theory and geometry behind this statement was developed in several steps. In [15], it is said that the ratios $\mu(B_{a,p}(r))/r^n$ and $O_{a,p,q}(E_{a,p}(r))/(nr^{n-1})$ did not depend on the radius r and their constant values agreed, where $E_{a,p}(r)$ denotes a p-generalized n-dimensional ellipsoid, $B_{a,p}(r)$ is the elliptic ball of the elliptic radius r enclosed by it, and $O_{a,p,q}$ is the suitably defined non-Euclidean surface content. In several papers, it was shown what influence generalized circle numbers had on the normalization constants of general norm contoured distributions in R^2. Because the ball number function agreed with the suitably defined non-Euclidean surface content divided by dimension n, the primary influence of the surface content on the normalizing constant is shown as an alternative. This is the case, for example, in [16,19,35]. The dynamic, matrix-homogeneous situation was treated for the first time in a two-dimensional case in [21].

Remark 7. *It follows from the above results that*

$$\frac{\mu(B(r))}{r^{\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}}} = \pi(S) = \frac{O(S(r))}{(\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3})r^{\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}-1}}$$

where

$$\pi(S) = \frac{8B(\frac{1}{p_1}, \frac{1}{p_2}, \frac{1}{p_3})p_1^{\frac{1}{p_1}-1} p_2^{\frac{1}{p_2}-1} p_3^{\frac{1}{p_3}-1}}{\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}}.$$

Obviously, these equations generalize the well-known properties of the circle number π, both in terms of the dimension of the circle or sphere and its generalized shape.

Definition 9. *The number $\pi(S)$ is called the ball number of the p-sphere S and the function $S \to \pi(S)$ is called the p-ball number function in R^3.*

Remark 8. *Let $A \in \mathfrak{B}(S)$ and $\pi(A) = \frac{\pi^*(A)}{\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}}$ where $\pi^*(A) = \int\limits_{Pol^{*-1}(A)} J^*(\varphi,\vartheta)d\varphi d\vartheta$. Because the equations*

$$\frac{\mu(Se(A,r))}{r^{\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}}} = \pi(A) = \frac{O(D(r)A)}{(\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3})r^{\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}-1}}$$

hold true for every fixed p, the function $A \to \pi(A)$ is called the S-sector number function.

6. Discussion and Conclusions

The geometric method used in the present work is a further development of the geometric method established in [3]. The latter should therefore be additionally illustrated here in order to subsequently deepen our understanding of the present approach. The vector-valued product introduced in [3], formula (12), for the homogeneous Euclidean case can be rewritten in the notation as

$$\mathfrak{x}_1 \odot \mathfrak{x}_2 = S(\mathfrak{x}_1, \mathfrak{x}_2)||\mathfrak{x}_1|| \cdot ||\mathfrak{x}_2|| \begin{pmatrix} c \\ s\varepsilon \end{pmatrix}.$$

The real numbers $c = \frac{x_1 x_2 - \xi_1 \xi_2}{||\mathfrak{x}_1|| \cdot ||\mathfrak{x}_2||}$ and $s = \frac{x_1 \xi_2 + \xi_2 \xi_1}{||\mathfrak{x}_1|| \cdot ||\mathfrak{x}_2||}$ satisfied the equation $c^2 + s^2 = 1$ and could therefore be interpreted as the cosine and sine of an angle φ, respectively. With the rotation matrix $D = \begin{pmatrix} \frac{y_2}{\xi_2} & \frac{-z_2}{\xi_2} \\ \frac{z_2}{\xi_2} & \frac{y_2}{\xi_2} \end{pmatrix}$ and the unit vector $\epsilon^* = \begin{pmatrix} \frac{y_1}{\xi_1} \\ \frac{z_1}{\xi_1} \end{pmatrix}$, the vector $\varepsilon = D\epsilon^*$ also had the structure of a unit vector and could therefore be written with an angle ϑ as $\varepsilon = \begin{pmatrix} \cos\vartheta \\ \sin\vartheta \end{pmatrix}$. With $r = ||\mathfrak{x}_1|| \cdot ||\mathfrak{x}_2||$, this resulted in the representation

$$\mathfrak{x}_1 \odot \mathfrak{x}_2 = S(\mathfrak{x}_1, \mathfrak{x}_2) r \begin{pmatrix} \cos\varphi \\ \sin\varphi\cos\vartheta \\ \sin\varphi\sin\vartheta \end{pmatrix}, \tag{5}$$

which in turn is reminiscent of the use of ordinary spherical coordinates.

Using p-generalized spherical coordinates from [28],

$$SPH(r, \varphi, \vartheta) = r \begin{pmatrix} \cos_p \varphi \\ \sin_p \varphi \cos_p \vartheta \\ \sin \varphi_p \sin_p \vartheta \end{pmatrix},$$

representation (5) was generalized appropriately to introduce p-generalized three-dimensional complex numbers in [3]. While the latter coordinates are particularly suitable for describing points on l_p-spheres, the coordinate system introduced in Definition 1 is aimed at describing points on (p_1, p_2, p_3)-spheres. A dynamic complex structure of the type considered here was introduced for the first time in [36] for the two-dimensional case.

Finally, the importance of the vector representation of complex numbers rather than the pretty unclear representation of $z = x + iy$ should be emphasized against the background of the formulas developed here.

When complex numbers are introduced, one of the things that is usually said is that

$$\text{they are definitely not real numbers,} \tag{6}$$

$$i^2 = -1 \tag{7}$$

and

$$\text{they can be interpreted as points in Gauss's number plane.} \tag{8}$$

It is clear that a maximum satisfactory mathematical rigor is only achieved when an interpretation has been replaced with an axiom and the unsatisfiable Equation (7) has been replaced by a well-defined one, as in [37].

If the Euler number $e = 2.71828\ldots$ and the usual exponential function $x \to e^x, x \in R$ are given, one can ask whether there is an abstract quantity or so called imaginary number i that satisfies (6)–(8) and also the equations

$$\frac{1}{2}(e^{it} + e^{-it}) = \cos t \text{ and } \frac{1}{2}(e^{it} - e^{-it}) = i \sin t, \tag{9}$$

which are stated to be valid in many mathematical sources. For a science like mathematics, which is based, among other things, on the completely exact and pedantically precise derivation of all its statements, it is astonishing that there does not seem to be a derivation of Equation (9) that does not use a so-called artifice, such as equating the number 1 with the vector $(1,0)^T$ or something similar, contrary to all mathematical rules. But, if we interpret the expression on the right side of Euler's well-known formula

$$e^{it} = \cos t + i \sin t, t \in R$$

in terms of points in the plane, or vectors,

$$e^{it} = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} \in R^2,$$

then Equation (9) becomes

$$\frac{1}{2}(e^{it} + e^{-it}) = \begin{pmatrix} \cos t \\ 0 \end{pmatrix} \text{ and } \frac{1}{2}(e^{it} + e^{-it}) = \begin{pmatrix} 0 \\ \sin t \end{pmatrix} \quad (10)$$

through a simple vector calculation. Gauss's interpretation of complex numbers as points on the plane was transformed into the status of an axiom in [37], thereby probably defining complex numbers completely precisely for the first time. Some initial consequences that arise from the vector representation of complex numbers for the characteristic functions of probability distributions were presented elsewhere.

Complex numbers are used in numerous areas of science and technology. Similarly, dynamic models of the type presented here can find wide application. However, due to the variety of practical tasks, the development of numerous other dynamic models generated by a functional other than $||.||$ may also be desirable. This can then be realized following the central themes of the present work. This concerns both the creation of new number structures and the probabilistic treatment of them in the sense of the present work. Some functionals that may be of interest from the perspective of probability distributions are presented in Appendix A. In addition, it can be useful to develop stochastic representations and simulation techniques in the newly created number structures.

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Appendix A

The following explanations show examples of functions that occur, for example, in connection with the definition of probability densities, and for which one could develop one's own complex structures. This is intended to support ideas about the development of problem-oriented or individual complex structures. The functions given below and the associated probability densities have also often been considered in higher dimensions in the original papers mentioned here and elsewhere.

The function

$$s(x,y) = (xy)^{\lambda-1} e^{-\frac{\alpha_x x + \alpha_y y}{1-\rho}}$$

considered in [38] is the density-generating function of the joint probability distribution of two positively correlated random gamma variables and its level lines describe eccentric ellipses.

When examining a mixture of two bivariate normal distributions with different correlation coefficients, the following function was set to zero in [39] in order to consider the contour lines of the resulting two-dimensional density function:

$$s(x,y) = \frac{\exp(-\frac{x^2 - 2\rho_1 xy + y^2}{2(1-\rho_1^2)})}{(1-\rho_1^2)^{1/2}} + \frac{\exp(-\frac{x^2 - 2\rho_2 xy + y^2}{2(1-\rho_2^2)})}{(1-\rho_2^2)^{1/2}}.$$

The corresponding figures are presented in [40].

In [10], multivariate so called g- and h-distributions are introduced. In the two-dimensional case, contour lines of such distributions can be, for example, concentric Euclidean circles or concentric squares, which could be called static models. But dynamic models are also possible in which eccentric rectangles, that is parallel rectangles that propagate at different speeds in different directions when a generalized radius is increased.

While contour lines in the latter work are generated by fitting to the data, the contour lines in [41] are eccentric ellipses that are generated by setting the following function to zero

$$s(x,y) = b^2(\gamma)(x\cos\alpha(\gamma) - y\sin\alpha(\gamma) - x_0(\gamma))^2 + a^2(\gamma)(y\cos\alpha(\gamma) + x\sin\alpha(\gamma) - y_0(\gamma))^2 - a^2(\gamma)b^2(\gamma).$$

The same authors' more general approach in [12] contains, among other things, the following Gumbel type function

$$s(x,y) = \exp(-e^{-x} - \alpha e^{-x-y} - e^{-y} - x - y).$$

The following function, which can generate, in part, boomerang-like contour lines, is presented here as an example from the overview work in [42]:

$$s(x,y) = \exp\,(1, x, x^2)\begin{pmatrix} m_{00} & m_{01} & m_{02} \\ m_{10} & m_{11} & m_{12} \\ m_{20} & m_{21} & m_{22} \end{pmatrix}\begin{pmatrix} 1 \\ y \\ y^2 \end{pmatrix}.$$

A cascade of L_p-norms was used in [43] to construct a function that can be used to describe the density contour sets of the so-called L_p-nested symmetric distributions:

$$s(x,y,z) = \left(|x|^{p_1} + (|y|^{p_2} + |z|^{p_2})^{\frac{p_1}{p_2}}\right)^{\frac{1}{p_1}}.$$

The functional introduced at the beginning of the present article,

$$||\mathfrak{x}|| = \frac{|x|^{p_1}}{p_1} + \frac{|y|^{p_2}}{p_2} + \frac{|z|^{p_3}}{p_3},\ \mathfrak{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in R^3,$$

was used in [17,19] for studying p-spherical distributions.

All of these functions and their contour lines or surfaces can also occur in contexts other than the probabilistic ones suggested here. The more application contexts that arise, the more it becomes advisable to build individual number systems based on them, analogous to the approach here. In any case of such a "system on demand", the necessary basis for it may be the creation of a suitable coordinate system and the Lie group of transformations based on them. It is clear that a 'suitable coordinate system' does not necessarily mean generalized polar or spherical coordinates.

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