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An Existence Result for Second-Order Boundary-Value Problems via New Fixed-Point Theorems on Quasi-Metric Space

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Abstract: We introduce the new idea of $(\alpha - \theta_\sigma)$-contraction in quasi-metric spaces in this paper. For these kinds of mappings, we then prove new fixed-point theorems on left K, left M, and left *Smyth*-complete quasi-metric spaces. We also apply our results to infer the existence of a solution to a second-order boundary-value problem.

Keywords: quasi-metric space; left K-Cauchy sequence; α-admissible mapping; fixed point

MSC: Primary 54H25; Secondary 47H10

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Citation: Güngör, G.D.; Altun, I. An Existence Result for Second-Order Boundary-Value Problems via New Fixed-Point Theorems on Quasi-Metric Space. *Symmetry* **2024**, *16*, 99. https://doi.org/10.3390/sym16010099

Academic Editors: Hongkun Xu and Calogero Vetro

Received: 17 November 2023
Revised: 22 December 2023
Accepted: 11 January 2024
Published: 13 January 2024

1. Introduction and Preliminaries

The Banach Contraction Principle is a fundamental result in the field of functional analysis and topology. It provides conditions under which a mapping on a complete metric space has a unique fixed point. The Banach Contraction Principle has significant applications in various fields, including mathematical analysis, numerical methods, and optimization. It provides a powerful tool for establishing the existence and uniqueness of solutions to many kinds of equations. Additionally, it has implications in the study of dynamical systems and stability analysis. However, owing to the strict conditions of the metric space and the specific properties imposed, the need to work with topological structures that have more flexible conditions than the metric space has emerged. Therefore, many generalizations of the Banach Contraction Principle have been obtained in this space by defining the quasi-metric space. Furthermore, quasi-metric spaces are useful in numerous topics of mathematics, like optimization, functional analysis, and computer science. They provide a more general framework for studying approaches related to distances and convergence, allowing for more flexible and adaptable notions of proximity (see [1–6]). We now go over the terms and symbols associated with quasi-metric space:

Definition 1 ([6,7])**.** *Let us consider the following properties for the function $\sigma : \Omega \times \Omega \to \mathbb{R}$, where Ω be a nonempty set: for each $\xi, \zeta, \varsigma \in \Omega$*

(i) $\sigma(\xi, \xi) = 0$.
(ii) $\sigma(\xi, \zeta) \leq \sigma(\xi, \varsigma) + \sigma(\varsigma, \zeta)$ *(triangle inequality).*
(iii) $\sigma(\xi, \zeta) = \sigma(\zeta, \xi) = 0$ *implies* $\xi = \zeta$.
(iv) $\sigma(\xi, \zeta) = 0$ *implies* $\xi = \zeta$.

When (i) and (ii) are met, σ is referred to as a quasi-pseudo metric or simply qpm. When requirements (i), (ii), and (iii) are met, It is clear that every T_1-qm is a qm, every qm is a qpm, and every ordinary metric is a T_1-qm. If (Ω, σ) is a quasi-pseudo metric space (abbreviated qpms), then σ produces a topology τ_σ on Ω, with the following family of open balls serving as its base:

$$\{B_\sigma(\xi, \varepsilon) : \xi \in \Omega \text{ and } \varepsilon > 0\},$$

where $B_\sigma(\xi_0, \varepsilon) = \{\zeta \in \Omega : \sigma(\xi_0, \zeta) < \varepsilon\}$. τ_σ is a T_0 topology on

On the other hand, the mappings, $\sigma^{-1}, \sigma^s, \sigma_+ : \Omega \times \Omega \to [0, \infty)$ defined as

$$\begin{aligned} \sigma^{-1}(\xi, \zeta) &= \sigma(\zeta, \xi) \\ \sigma^s(\xi, \zeta) &= \max\{\sigma(\xi, \zeta), \sigma^{-1}(\xi, \zeta)\} \\ \sigma_+(\xi, \zeta) &= \sigma(\xi, \zeta) + \sigma^{-1}(\xi, \zeta) \end{aligned}$$

are also qpms on Ω, whenever σ is a qpm. If σ is a qm, then σ^s and σ_+ are (equivalent) metrics on Ω.

Let (Ω, σ) be a qm, $\{\xi_n\}$ be a sequence in Ω and $\xi \in \Omega$. If $\{\xi_n\}$ converges to ξ with respect to τ_σ, this is denoted as $\xi_n \stackrel{\sigma}{\to} \xi$ and called σ-convergence. In this case, by the definition of τ_σ, $\xi_n \stackrel{\sigma}{\to} \xi$ if and only if $\sigma(\xi, \xi_n) \to 0$. Similarly, if $\{\xi_n\}$ converges to ξ with respect to $\tau_{\sigma^{-1}}$, this is denoted as $\xi_n \stackrel{\sigma^{-1}}{\to} \xi$ and called σ^{-1}-convergence. In this case, by the definition of $\tau_{\sigma^{-1}}$, $\sigma(\xi_n, \xi) \to 0$ if and only if $\xi_n \stackrel{\sigma^{-1}}{\to} \xi$. Finally, if $\{\xi_n\}$ converges to ξ with respect to τ_{σ^s}, this is denoted as $\xi_n \stackrel{\sigma^s}{\to} \xi$ and called σ^s-convergence. If for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for every $n, k \in \mathbb{N}$ with $n \geq k \geq n_0$ ($k \geq n \geq n_0$), $\sigma(\xi_k, \xi_n) < \varepsilon$, then the sequence $\{\xi_n\}$ in Ω is called left (right) K-Cauchy. Under σ, the right K-Cauchy property under σ^{-1} is implied by the left K-Cauchy property. It is clear that, if

$$\sum_{n=1}^{\infty} \sigma(\xi_n, \xi_{n+1})$$

is convergent, then the sequence $\{\xi_n\}$ is left K-Cauchy.

Every convergent sequence in a metric space is, in fact, a Cauchy sequence; in qms, this may not be the case. Completeness is one of the indispensable concepts in metric fixed-point theory. However, while completeness is defined in one way in metric spaces, this concept is diversified in quasi-metric spaces since quasi-metric does not have the symmetry property. The literature contains numerous definitions of completeness in these domains (see [8,9]). Let (Ω, σ) be a qms. If every left (right) K-Cauchy sequence is σ (resp. σ^{-1}, σ^s)-convergent, then (Ω, σ) is considered left (right) K (resp. M, Smyth)-complete. You may obtain a more thorough discussion of a few key metric features in [8].

Let us now recall the notion of α-admissibility defined by Samet et al. [10], which has recently become important in metric fixed-point theory. This notion has the effect of weakening the hypotheses in the theorems since it restricts the set of points that are required to satisfy the contraction inequality in metric fixed-point theory. Let $\Omega \neq \emptyset$, $\Gamma : \Omega \to \Omega$ be a mapping and

$$\alpha(\xi, \zeta) \geq 1 \text{ implies } \alpha(\Gamma\xi, \Gamma\zeta) \geq 1.$$

Samet et al. [10] established several universal fixed-point results encompassing several well-known theorems regarding complete metric space by introducing the α-admissibility technique. These discoveries on fixed points offer a framework for investigating the existence and characteristics of fixed points for self-mappings on a complete metric space, employing the α-admissibility method (see [11–17]).

On the other hand, [18] saw the introduction of a novel kind of contractive mapping called a θ-contraction by Jleli and Samet. Within the field of fixed-point theory, this θ-contraction is an appealing generalization. Let us go over a few concepts and associated findings on θ-contraction to gain a better understanding of this method. Let Θ represent the set of all functions $\theta : (0, \infty) \to (1, \infty)$ that meet the specified criteria:

- (θ_1) θ is non-decreasing,
- (θ_2) For each sequence $\{\varrho_n\} \subset (0,\infty)$, $\lim_{n\to\infty} \varrho_n = 0^+$ if only if $\lim_{n\to\infty} \theta(\varrho_n) = 1$,
- (θ_3) There exist $0 < p < 1$ and $\mu \in (0,\infty]$ such that $\lim_{\varrho\to 0^+} \frac{\theta(\varrho)-1}{\varrho^p} = \mu$.

For example, the θ, defined by $\theta(\varrho) = e^{\sqrt{\varrho}}$ for $\varrho \leq 1$ and $\theta(\varrho) = 9$ for $\varrho > 1$, belongs to Θ.

Let (Ω, σ) be a metric space and $\theta \in \Theta$. Then, a self-mapping Γ of Ω is said to be a θ-contraction if there exists $0 < \delta < 1$ satisfying

$$\theta(\sigma(\Gamma\xi, \Gamma\zeta)) \leq [\theta(\sigma(\xi, \zeta))]^{\delta} \tag{1}$$

for each $\xi, \zeta \in \Omega$ with $\sigma(\Gamma\xi, \Gamma\zeta) > 0$.

Various contractions can be obtained by selecting suitable functions for θ in (1), e.g., $\theta_1(\varrho) = e^{\sqrt{\varrho}}$ and $\theta_2(\varrho) = e^{\sqrt{\varrho e^{\varrho}}}$. It has been demonstrated by Jleli and Samet [18] that each θ-contraction on a complete metric space has a unique fixed point. This outcome offers a useful perspective on the presence and uniqueness of fixed points for a large class of contractive mappings. There are various articles accessible if additional papers in the literature about θ-contractions are required (see [19–21]).

In our previous paper [22], by combining the concept of ζ-contraction, which was created with the simulation function used by Khojasteh et al. [23] for the first time in fixed-point theory, and Berinde's almostness idea [24], the concept of almost-ζ-contraction in quasi-metric space was defined, and then the related fixed-point theorem was established. Then, an application was made to a fractional order boundary-value problem.

In this study, we establish the notion of $(\alpha - \theta_\sigma)$-contraction mappings on quasi-metric spaces, taking into account the preceding arguments, and then present some fixed-point results for such mappings quasi-metric spaces. Finally, the obtained theoretical result was applied to the existence of a solution to a second-order boundary-value problem.

2. The Results

In this section, we present our theoretical results.

Let Γ be a self-mapping on qms (Ω, σ), $\alpha : \Omega \times \Omega \to [0,\infty)$ be a function and Γ_α be a set defined by

$$\Gamma_\alpha = \{(\xi, \zeta) \in \Omega \times \Omega : \alpha(\xi, \zeta) \geq 1 \text{ and } \sigma(\Gamma\xi, \Gamma\zeta) > 0\}. \tag{2}$$

Definition 2. *Let Γ be a self-mapping on qms (Ω, σ) satisfying*

$$\sigma(\xi, \zeta) = 0 \Longrightarrow \sigma(\Gamma\xi, \Gamma\zeta) = 0. \tag{3}$$

$\alpha : \Omega \times \Omega \to [0,\infty)$ and $\theta \in \Theta$ be two functions. Then, Γ is called $(\alpha - \theta_\sigma)$-contraction if there exists $0 < \delta < 1$ such that

$$\theta(\sigma(\Gamma\xi, \Gamma\zeta)) \leq [\theta(\sigma(\xi, \zeta))]^{\delta}, \tag{4}$$

for each $(\xi, \zeta) \in \Gamma_\alpha$.

Prior to outlining our primary findings, let us put on two crucial points:

- Every self-mapping Γ meets the requirement (3) if (Ω, σ) is a T_1-qms.
- It is clear from (2)–(4) that if Γ is an $(\alpha - \theta_\sigma)$-contraction on a qms (Ω, σ), then

  $$\sigma(\Gamma\xi, \Gamma\zeta) \leq \sigma(\xi, \zeta),$$

  for each $\xi, \zeta \in \Omega$ with $\alpha(\xi, \zeta) \geq 1$.

The next theorem will be discussed using the $(\alpha - \theta_\sigma)$-contraction technique.

Theorem 1. *Let (Ω, σ) be a left K-complete T_1-qms such that (Ω, τ_σ) is Hausdorff topological space, and let $\Gamma : \Omega \to \Omega$ be a $(\alpha - \theta_\sigma)$-contraction. Assume that Γ is τ_σ-continuous and α-admissible. If there exists $\xi_0 \in \Omega$ such that $\alpha(\xi_0, \Gamma\xi_0) \geq 1$, then Γ has a fixed point in Ω.*

Proof. Let $\xi_0 \in \Omega$ be a such that $\alpha(\xi_0, \Gamma\xi_0) \geq 1$. Define a sequence $\{\xi_n\}$ in Ω by $\xi_{n+1} = \Gamma\xi_n$. Since Γ is α-admissible, we have $\alpha(\xi_n, \xi_{n+1}) \geq 1$ for $n \in \mathbb{N}$. If there exist $k \in \mathbb{N}$ such that $\sigma(\xi_k, \Gamma\xi_k) = 0$ then by T_1 property of σ, we have $\Gamma\xi_k = \xi_k$, i.e., ξ_k is a fixed point of Γ. Assume $\sigma(\xi_n, \Gamma\xi_n) > 0$ for $n \in \mathbb{N}$. Hence, the pair $(\xi_n, \xi_{n+1}) \in \Gamma_\alpha$ for $n \in \mathbb{N}$. Since Γ is $(\alpha - \theta_\sigma)$-contraction, then by (θ_1), we obtain

$$\begin{aligned} \theta(\sigma(\xi_n, \xi_{n+1})) &= \theta(\sigma(\Gamma\xi_{n-1}, \Gamma\xi_n)) \\ &\leq [\theta(\sigma(\xi_{n-1}, \xi_n))]^\delta. \end{aligned} \tag{5}$$

Let $\sigma_n = \sigma(\xi_n, \xi_{n+1})$ for $n \in \mathbb{N}$. Then $\sigma_n > 0$ for $n \in \mathbb{N}$ and so, from (5) , we obtain

$$\theta(\sigma_n) \leq [\theta(\sigma_0)]^{\delta^n},$$

i.e.,

$$1 < \theta(\sigma_n) \leq [\theta(\sigma_0)]^{\delta^n} \tag{6}$$

for $n \in \mathbb{N}$. Letting $n \to \infty$ in (6), we obtain

$$\lim_{n\to\infty} \theta(\sigma_n) = 1. \tag{7}$$

From (θ_2), we obtain that $\lim_{n\to\infty} \sigma_n = 0^+$, so from (θ_3), there exist $p \in (0, 1)$ and $\mu \in (0, \infty]$ such that

$$\lim_{n\to\infty} \frac{\theta(\sigma_n) - 1}{(\sigma_n)^p} = \mu.$$

Now, assume that $\mu < \infty$ and let $M = \frac{\mu}{2} > 0$. According to the limit's definition, there exists $n_0 \in \mathbb{N}$ such that, for each $n_0 \leq n$,

$$\left| \frac{\theta(\sigma_n) - 1}{(\sigma_n)^p} - \mu \right| \leq M.$$

Hence, for each $n_0 \leq n$, we have

$$\frac{\theta(\sigma_n) - 1}{(\sigma_n)^p} \geq \mu - M = M.$$

Then, for each $n_0 \leq n$,

$$n(\sigma_n)^p \leq Bn[\theta(\sigma_n) - 1],$$

where $B = 1/M$.

For the second case, assume that $\mu = \infty$ and let $F > 0$ be an arbitrary positive number. According to the limit's definition, there exists $n_0 \in \mathbb{N}$ such that, for each $n_0 \leq n$,

$$\frac{\theta(\sigma_n) - 1}{(\sigma_n)^p} \geq M.$$

Hence, for each $n_0 \leq n$, we have

$$n[\sigma_n]^p \leq Bn[\theta(\sigma_n) - 1],$$

where $B = 1/M$.

Therefore, in two cases, there exist $B > 0$ and $n_0 \in \mathbb{N}$ such that

$$n[\sigma_n]^p \leq Bn[\theta(\sigma_n) - 1],$$

for each $n_0 \leq n$. Using (6), we obtain

$$n[\sigma_n]^p \leq Bn\Big[[\theta(\sigma_0)]^{\delta^n} - 1\Big],$$

for each $n_0 \leq n$. Taking $n \to \infty$ from both sides the last inequality, we have

$$\lim_{n\to\infty} n[\sigma_n]^p = 0.$$

Thus, there exists $n_1 \in \mathbb{N}$ such that $n[\sigma_n]^p \leq 1$ for each $n \geq n_1$, so, we have, for each $n \geq n_1$,

$$\sigma_n \leq \frac{1}{n^{1/p}}. \tag{8}$$

Therefore, for each $n \geq n_1$ from (8) we have

$$\sigma(\xi_n, \xi_{n+1}) \leq \frac{1}{n^{1/p}}$$

and so

$$\sum_{i=n_1}^{\infty} \sigma(\xi_n, \xi_{n+1}) \leq \sum_{i=n_1}^{\infty} \frac{1}{n^{1/p}}.$$

Since the series $\sum\limits_{i=1}^{\infty} \frac{1}{i^{1/p}}$ is convergent we have $\sum\limits_{i=1}^{\infty} \sigma(\xi_n, \xi_{n+1})$ is convergent. This show that $\{\xi_n\}$ is a left K -Cauchy sequence. By the left K-completeness of (Ω, σ), there exists $\varsigma \in \Omega$ such that $\sigma(\varsigma, \xi_n) \to 0$ as $n \to \infty$. Since Γ is τ_σ-continuous then we have $\sigma(\Gamma\varsigma, \Gamma\xi_n) = \sigma(\Gamma\varsigma, \xi_{n+1}) \to 0$ as $n \to \infty$. Since Ω is Hausdorff, we obtain $\Gamma\varsigma = \varsigma$. □

Now, we present an illustrative example.

Example 1. *Let* $\Omega = \{0, 1, 2, \cdots\}$ *and*

$$\sigma(\xi, \zeta) = \begin{cases} 0 & , \quad \xi = \zeta \\ \xi + \zeta & , \quad \xi \neq \zeta \end{cases}.$$

Then (Ω, σ) *is a left K-complete* T_1*-qms. Moreover,* (Ω, τ_σ) *is Hausdorff topological space. Define two mappings* $\alpha : \Omega \times \Omega \to [0, +\infty)$ *by*

$$\alpha(\xi, \zeta) = \begin{cases} 1 & , \quad \xi \geq \zeta > 0 \\ 0 & , \quad \textit{otherwise} \end{cases}$$

and $\Gamma : \Omega \to \Omega$ *by*

$$\Gamma\xi = \begin{cases} \xi & , \quad \xi \in \{0, 1\} \\ \xi - 1 & , \quad \xi \geq 2 \end{cases}.$$

It is easy to see that Γ *is* τ_σ*-continuous and* α*-admissible. Also, for* $\xi_0 = 1$ *we have* $\alpha(\xi_0, \Gamma\xi_0) \geq 1$*. Now, we claim that* Γ *is* $(\alpha - \theta_\sigma)$*-c with* $\theta(\varrho) = e^{\sqrt{\varrho e^{\varrho}}}$ *and* $\delta = e^{-\frac{1}{2}}$*. To see this, we must show that*

$$\frac{\sigma(\Gamma\xi, \Gamma\zeta)}{\sigma(\xi, \zeta)} e^{\sigma(\Gamma\xi, \Gamma\zeta) - \sigma(\xi, \zeta)} \leq e^{-1}. \tag{9}$$

for all $(\xi, \zeta) \in \Gamma_\alpha$*. First, observe that*

$$\begin{aligned} \Gamma_\alpha &= \{(\xi, \zeta) \in \Omega \times \Omega : \alpha(\xi, \zeta) \geq 1 \text{ and } \sigma(\Gamma\xi, \Gamma\zeta) > 0\} \\ &= \{(\xi, \zeta) \in \Omega \times \Omega : \xi > \zeta > 0\} \backslash \{(2, 1)\}. \end{aligned}$$

Let $(\xi, \zeta) \in \Gamma_\alpha$. *Then*

$$\frac{\sigma(\Gamma\xi, \Gamma\zeta)}{\sigma(\xi, \zeta)} e^{\sigma(\Gamma\xi, \Gamma\zeta) - \sigma(\xi, \zeta)} \leq \frac{\xi + \zeta - 1}{\xi + \zeta} e^{-1} \leq e^{-1}.$$

This shows that (9) is true. Hence, all conditions of Theorem 1 are satisfied and so Γ *has a fixed point in* Ω . *Here* 0 *and* 1 *are fixed points of* Γ. *On the other hand, since* $\sigma(\Gamma 0, \Gamma 1) = 1 = \sigma(0, 1)$, *then we have*

$$\theta(\sigma(\Gamma 0, \Gamma 1)) > [\theta(\sigma(0, 1))]^{\delta},$$

for all $\theta \in \Theta$ *and* $\delta \in (0, 1)$. *This situation shows the importance and effect of* α *in Theorem 1 .*

The outcomes that follow are a direct result of the Theorem 1.

Corollary 1. *Let* (Ω, σ) *be a left K-complete* T_1*-qms such that* (Ω, τ_σ) *is Hausdorff topological space, and let* $\Gamma : \Omega \to \Omega$ *be a* τ_σ*-continuous and* α*-admissible such that*

$$\frac{\sigma(\Gamma\xi, \Gamma\zeta)}{\sigma(\xi, \zeta)} e^{\sigma(\Gamma\xi, \Gamma\zeta) - \sigma(\xi, \zeta)} \leq \delta$$

for all $(\xi, \zeta) \in \Gamma_\alpha$. *Finally, if there exists* $\xi_0 \in \Omega$ *such that* $\alpha(\xi_0, \Gamma\xi_0) \geq 1$, *then* Γ *has a fixed point in* Ω.

Proof. If we take $\theta(\varrho) = e^{\sqrt{\varrho e^{\varrho}}}$ in Theorem 1, we obtain the desired result. □

Corollary 2. *Let* (Ω, σ) *be a left K-complete* T_1*-qms such that* (Ω, τ_σ) *is Hausdorff topological space, and let* $\Gamma : \Omega \to \Omega$ *be a* τ_σ*-continuous and* α*-admissible such that*

$$\sigma(\Gamma\xi, \Gamma\zeta) \leq \delta\sigma(\xi, \zeta)$$

for all $(\xi, \zeta) \in \Gamma_\alpha$. *Finally, if there exists* $\xi_0 \in \Omega$ *such that* $\alpha(\xi_0, \Gamma\xi_0) \geq 1$, *then* Γ *has a fixed point in* Ω.

Proof. If we take $\theta(\varrho) = e^{\sqrt{\varrho}}$ in Theorem 1, we obtain the desired result. □

Corollary 3. *Let* (Ω, σ) *be a left K-complete* T_1*-qms such that* (Ω, τ_σ) *is Hausdorff topological space, and let* $\Gamma : \Omega \to \Omega$ *be a* τ_σ*-continuous mapping such that*

$$\theta(\sigma(\Gamma\xi, \Gamma\zeta)) \leq \theta(\sigma(\Gamma\xi, \Gamma\zeta))^{\delta}$$

for all $\xi, \zeta \in \Omega$ *with* $\sigma(\Gamma\xi, \Gamma\zeta) > 0$. *Then* Γ *has a fixed point in* Ω.

Proof. If we take $\sigma(\xi, \zeta) = 1$ in Theorem 1, we obtain the desired result. □

Within Theorem 1, considering the $\tau_{\sigma^{-1}}$-continuity technique, the following theorem can be obtained:

Theorem 2. *Let* (Ω, σ) *be a left M-complete* T_1*-qms such that* $(\Omega, \tau_{\sigma^{-1}})$ *is Hausdorff topological space, and* $\Gamma : \Omega \to \Omega$ *be a* $(\alpha - \theta_\sigma$ *)-contraction. Assume that* Γ *is* $\tau_{\sigma^{-1}}$*-continuous and* α*-admissible. Then,* Γ *has a fixed point in* Ω, *provided that there exists* $\xi_0 \in \Omega$ *such that* $\alpha(\xi_0, \Gamma\xi_0) \geq 1$.

Proof. By the similar proof of Theorem 1, the constructed sequence $\{\xi_n\}$ is left K-Cauchy. Hence from the left M-completeness of the space (Ω, σ), there exists $\varsigma \in \Omega$ such that $\sigma(\xi_n, \varsigma) \to 0$ as $n \to \infty$. Using $\tau_{\sigma^{-1}}$-continuity of Γ, we obtain $\sigma(\Gamma\xi_n, \Gamma\varsigma) = \sigma(\xi_{n+1}, \Gamma\varsigma) \to 0$ as $n \to \infty$. Since $(\Omega, \tau_{\sigma^{-1}})$ is Hausdorff, we obtain $\varsigma = \Gamma\varsigma$. □

The outcomes that follow are a direct result of the Theorem 2.

Corollary 4. *Let (Ω, σ) be a left M-complete T_1-qms such that $(\Omega, \tau_{\sigma^{-1}})$ is Hausdorff topological space, and let $\Gamma : \Omega \to \Omega$ be a $\tau_{\sigma^{-1}}$-continuous and α-admissible such that*

$$\frac{\sigma(\Gamma\xi, \Gamma\zeta)}{\sigma(\xi, \zeta)} e^{\sigma(\Gamma\xi,\Gamma\zeta)-\sigma(\xi,\zeta)} \leq \delta$$

for all $(\xi, \zeta) \in \Gamma_\alpha$. Then, Γ has a fixed point in Ω, provided that there exists $\xi_0 \in \Omega$ such that $\alpha(\xi_0, \Gamma\xi_0) \geq 1$.

Proof. If we take $\theta(\varrho) = e^{\sqrt{\varrho e^{\varrho}}}$ in Theorem 2, we obtain the desired result. □

Corollary 5. *Let (Ω, σ) be a left M-complete T_1-qms such that $(\Omega, \tau_{\sigma^{-1}})$ is Hausdorff topological space, and let $\Gamma : \Omega \to \Omega$ be a $\tau_{\sigma^{-1}}$-continuous and α-admissible such that*

$$\sigma(\Gamma\xi, \Gamma\zeta) \leq \delta\sigma(\xi, \zeta)$$

for all $(\xi, \zeta) \in \Gamma_\alpha$. Then, Γ has a fixed point in Ω, provided that there exists $\xi_0 \in \Omega$ such that $\alpha(\xi_0, \Gamma\xi_0) \geq 1$.

Proof. If we take $\theta(\varrho) = e^{\sqrt{\varrho}}$ in Theorem 2, we obtain the desired result. □

Corollary 6. *Let (Ω, σ) be a left M-complete T_1-qms such that $(\Omega, \tau_{\sigma^{-1}})$ is Hausdorff topological space, and let $\Gamma : \Omega \to \Omega$ be a $\tau_{\sigma^{-1}}$-continuous mapping such that*

$$\theta(\sigma(\Gamma\xi, \Gamma\zeta)) \leq \theta(\sigma(\Gamma\xi, \Gamma\zeta))^{\delta}$$

for all $\xi, \zeta \in \Omega$ with $\sigma(\Gamma\xi, \Gamma\zeta) > 0$. Then Γ has a fixed point in Ω.

Proof. If we take $\sigma(\xi, \zeta) = 1$ in Theorem 2, we obtain the desired result. □

The Hausdorffness constraint can be dropped if we take the space Ω's left *Smyth* completeness into account. But in this instance, the σ needs to remain a T_1-qm.

Theorem 3. *Let (Ω, σ) be a left Smyth complete T_1-qms and $\Gamma : \Omega \to \Omega$ be a $(\alpha - \theta_\sigma)$-contraction. Assume that Γ is τ_σ or $\tau_{\sigma^{-1}}$-continuous, and α-admissible. Then, Γ has a fixed point in Ω, provided that there exists $\xi_0 \in \Omega$ such that $\alpha(\xi_0, \Gamma\xi_0) \geq 1$.*

Proof. By the similar proof of Theorem 1, the constructed sequence $\{\xi_n\}$ is left K-Cauchy. By the left *Smyth* completeness of the space (Ω, σ), there exists $\varsigma \in \Omega$ such that $\{\xi_n\}$ is σ^s-converges to $\varsigma \in \Omega$, i.e., $\sigma^s(\xi_n, \varsigma) \to 0$ as $n \to \infty$. If Γ is τ_σ-continuous, then

$$\sigma(\Gamma\varsigma, \Gamma\xi_n) = \sigma(\Gamma\varsigma, \xi_{n+1}) \to 0 \text{ as } n \to \infty.$$

Therefore we obtain

$$\sigma(\Gamma\varsigma, \varsigma) \leq \sigma(\Gamma\varsigma, \xi_{n+1}) + \sigma(\xi_{n+1}, \varsigma) \to 0 \text{ as } n \to \infty.$$

If Γ is $\tau_{\sigma^{-1}}$-continuous, then

$$\sigma(\Gamma\xi_n, \Gamma\varsigma) = \sigma(\xi_{n+1}, \Gamma\varsigma) \to 0 \text{ as } n \to \infty.$$

Therefore we have,

$$\sigma(\varsigma, \Gamma\varsigma) \leq \sigma(\varsigma, \xi_{n+1}) + \sigma(\xi_{n+1}, \Gamma\varsigma) \to 0 \text{ as } n \to \infty.$$

Since Γ is T_1-qms, we obtain $\Gamma\varsigma = \varsigma$. □

The outcomes that follow are a direct result of the Theorem 3.

Corollary 7. *Let (Ω,σ) be a left Smyth complete T_1-qms, and let $\Gamma:\Omega\to\Omega$ be a α-admissible and τ_σ(or $\tau_{\sigma^{-1}}$)-continuous such that*

$$\frac{\sigma(\Gamma\xi,\Gamma\zeta)}{\sigma(\xi,\zeta)}e^{\sigma(\Gamma\xi,\Gamma\zeta)-\sigma(\xi,\zeta)}\leq\delta$$

for all $(\xi,\zeta)\in\Gamma_\alpha$. Then, Γ has a fixed point in Ω, provided that there exists $\xi_0\in\Omega$ such that $\alpha(\xi_0,\Gamma\xi_0)\geq 1$.

Proof. If we take $\theta(\varrho)=e^{\sqrt{\varrho e^{\varrho}}}$ in Theorem 3, we obtain the desired result. □

Corollary 8. *Let (Ω,σ) be a left Smyth complete T_1-qms, and let $\Gamma:\Omega\to\Omega$ be a α-admissible and τ_σ(or $\tau_{\sigma^{-1}}$)-continuous such that*

$$\sigma(\Gamma\xi,\Gamma\zeta)\leq\delta\sigma(\xi,\zeta)$$

for all $(\xi,\zeta)\in\Gamma_\alpha$. Then, Γ has a fixed point in Ω, provided that there exists $\xi_0\in\Omega$ such that $\alpha(\xi_0,\Gamma\xi_0)\geq 1$.

Proof. If we take $\theta(\varrho)=e^{\sqrt{\varrho}}$ in Theorem 3, we obtain the desired result. □

Corollary 9. *Let (Ω,σ) be a left Smyth complete T_1-qms, and let $\Gamma:\Omega\to\Omega$ be a τ_σ(or $\tau_{\sigma^{-1}}$)–continuous mapping such that*

$$\theta(\sigma(\Gamma\xi,\Gamma\zeta))\leq\theta(\sigma(\Gamma\xi,\Gamma\zeta))^{\delta}$$

for all $\xi,\zeta\in\Omega$ with $\sigma(\Gamma\xi,\Gamma\zeta)>0$. Then Γ has a fixed point in Ω.

Proof. If we take $\sigma(\xi,\zeta)=1$ in Theorem 3, we obtain the desired result. □

Let (Ω,σ) be a qms and $\alpha:\Omega\times\Omega\to[0,\infty)$ be a function. In this case, (Ω,σ) is said to have the property A_σ (respectively B_σ), if for every sequence $\{\xi_n\}$ in Ω and $\varsigma\in\Omega$ satisfying both $\alpha(\xi_n,\xi_{n+1})\geq 1$ and $\xi_n\overset{\sigma}{\to}\varsigma$, then $\alpha(\xi_n,\varsigma)\geq 1$ (respectively $\alpha(\varsigma,\xi_n)\geq 1$) for every $n\in\mathbb{N}$. In Theorem 3, the property A_σ or B_σ property of the space can be considered instead of the continuity of mapping.

Theorem 4. *Let (Ω,σ) be a left Smyth complete T_1 qms and $\Gamma:\Omega\to\Omega$ be a $(\alpha-\theta_\sigma)$-contraction. Presume that Γ is α-admissible and Ω has one of the properties A_σ, $A_{\sigma^{-1}}$, B_σ, $B_{\sigma^{-1}}$. Then, Γ has a fixed point in Ω, provided that there exists $\xi_0\in\Omega$ such that $\alpha(\xi_0,\Gamma\xi_0)\geq 1$.*

Proof. By the similar proof of Theorem 1, the constructed iterative sequence $\{\xi_n\}$ is left K-Cauchy. By the left *Smyth* completeness of the space (Ω,σ), there exists $\varsigma\in\Omega$ such that $\{\xi_n\}$ is σ^s-converges to $\varsigma\in\Omega$; that is, $\sigma^s(\xi_n,\varsigma)\to 0$ as $n\to\infty$. If Ω has the property A_σ or $A_{\sigma^{-1}}$, then $\alpha(\xi_n,\varsigma)\geq 1$. Therefore we obtain

$$\begin{aligned}\sigma(\varsigma,\Gamma\varsigma)&\leq\sigma(\varsigma,\xi_{n+1})+\sigma(\xi_{n+1},\Gamma\varsigma)\\&\leq\sigma(\varsigma,\xi_{n+1})+\sigma(\Gamma\xi_n,\Gamma\varsigma)\\&\leq\sigma(\varsigma,\xi_{n+1})+\sigma(\xi_n,\varsigma)\to 0\text{ as }n\to\infty.\end{aligned}$$

If Ω has the property B_σ or $B_{\sigma^{-1}}$, then $\alpha(\varsigma,\xi_n)\geq 1$. Therefore, we obtain,

$$\begin{aligned}\sigma(\Gamma\varsigma,\varsigma)&\leq\sigma(\Gamma\varsigma,\Gamma\xi_n)+\sigma(\Gamma\xi_n,\varsigma)\\&\leq\sigma(\varsigma,\xi_n)+\sigma(\xi_{n+1},\varsigma)\to 0\text{ as }n\to\infty.\end{aligned}$$

Since Γ is T_1-qms, we obtain $\Gamma\varsigma=\varsigma$. □

The outcomes that follow are a direct result of the Theorem 4.

Corollary 10. *Let (Ω, σ) be a left Smyth complete T_1-qms, and let $\Gamma : \Omega \to \Omega$ be a α-admissible mapping such that*

$$\frac{\sigma(\Gamma\xi, \Gamma\zeta)}{\sigma(\xi, \zeta)} e^{\sigma(\Gamma\xi,\Gamma\zeta)-\sigma(\xi,\zeta)} \leq \delta$$

for all $(\xi, \zeta) \in \Gamma_\alpha$. Assume that Ω has one of the property A_σ, $A_{\sigma^{-1}}$, B_σ, $B_{\sigma^{-1}}$. Then, Γ has a fixed point in Ω, provided that there exists $\xi_0 \in \Omega$ such that $\alpha(\xi_0, \Gamma\xi_0) \geq 1$.

Proof. If we take $\theta(\varrho) = e^{\sqrt{\varrho e^{\varrho}}}$ in Theorem 4, we obtain the desired result. □

Corollary 11. *Let (Ω, σ) be a left Smyth complete T_1-qms, and let $\Gamma : \Omega \to \Omega$ be a α-admissible mapping such that*

$$\sigma(\Gamma\xi, \Gamma\zeta) \leq \delta\sigma(\xi, \zeta)$$

for all $(\xi, \zeta) \in \Gamma_\alpha$. Assume that Ω has one of the property A_σ, $A_{\sigma^{-1}}$, B_σ, $B_{\sigma^{-1}}$. Then, Γ has a fixed point in Ω, provided that there exists $\xi_0 \in \Omega$ such that $\alpha(\xi_0, \Gamma\xi_0) \geq 1$.

Proof. If we take $\theta(\varrho) = e^{\sqrt{\varrho}}$ in Theorem 4, we obtain the desired result. □

Remark 1. *We can achieve similar theorems in qm spaces by taking into account the concept of right completeness in the sense of K, M, and Smyth.*

3. Application

In this section, we obtain an existing result about the solution of a second-order boundary-value problem (shortly BVP) by applying Theorem 1. We will consider the BVP as follows:

$$\begin{cases} -\frac{d^2\xi}{dt^2} = H(t, \xi(t)),\ t \in [0,1] \\ \xi(0) = \xi(1) = 0 \end{cases}, \tag{10}$$

where $H : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function. In the literature, there have been existence theorems provided for the problem (10) that consider certain requirements on H (see [25–30]). In this instance, we will examine different conditions on H and offer a novel theorem. It is evident that the following integral equation is equal to the problem (10):

$$\xi(t) = \int_0^1 G(t,\tau)H(\tau, \xi(\tau))d\tau,\ t \in [0,1], \tag{11}$$

where $G(t, \tau)$ is associated Green's function defined as

$$G(t,\tau) = \begin{cases} t(1-\tau) & , \quad 0 \leq t \leq \tau \leq 1 \\ \tau(1-t) & , \quad 0 \leq \tau \leq t \leq 1 \end{cases}.$$

Hence, $\xi \in C^2[0,1]$ is a solution of (10) if and only if it is a solution of (11). It is clear that

$$\int_0^1 G(t,\tau)d\tau = \frac{t(1-t)}{2}.$$

Let (Ω, σ) be the T_1-qms, where $\Omega = C[0,1]$ and σ is given by

$$\sigma(\xi, \varsigma) = \max\left\{ \sup_{t\in[0,1]} \{\xi(t) - \varsigma(t)\}, 2 \sup_{t\in[0,1]} \{\varsigma(t) - \xi(t)\} \right\}.$$

It is clear that (Ω, σ) is left K-complete. Also (Ω, τ_σ) is Hausdorff topological space.

Theorem 5. *Under the following assumption, the second-order BVP given by (10) has a solution: for all* $\xi, \varsigma \in \Omega$

$$\max\left\{\sup_{\tau\in[0,1]}\{H(\tau,\xi(\tau)) - H(\tau,\varsigma(\tau))\}, 2\sup_{\tau\in[0,1]}\{H(\tau,\varsigma(\tau)) - H(\tau,\xi(\tau))\}\right\} \leq K\sigma(\xi,\varsigma),$$

where $K < 8$.

Proof. Consider the operator $T : \xi \to \xi$ defined by

$$\Gamma\xi(t) = \int_0^1 G(t,\tau)H(\tau,\xi(\tau))d\tau$$

Then for any $\xi, \varsigma \in \xi$ and $t \in [0,1]$ we have

$$\begin{aligned}
\sigma(\Gamma\xi,\Gamma\varsigma) &= \max\left\{\sup_{t\in[0,1]}\{\Gamma\xi(t) - \Gamma\varsigma(t)\}, 2\sup_{t\in[0,1]}\{\Gamma\varsigma(t) - \Gamma\xi(t)\}\right\} \\
&= \max\left\{\begin{array}{l} \sup_{t\in[0,1]}\left\{\int_0^1 G(t,\tau)H(\tau,\xi(\tau))d\tau - \int_0^1 G(t,\tau)H(\tau,\varsigma(\tau))d\tau\right\}, \\ 2\sup_{t\in[0,1]}\left\{\int_0^1 G(t,\tau)H(\tau,\varsigma(\tau))d\tau - \int_0^1 G(t,\tau)H(\tau,\xi(\tau))d\tau\right\} \end{array}\right\} \\
&= \max\left\{\begin{array}{l} \sup_{t\in[0,1]}\left\{\int_0^1 G(t,\tau)\{H(\tau,\xi(\tau)) - H(\tau,\varsigma(\tau))\}d\tau\right\}, \\ \sup_{t\in[0,1]}\left\{\int_0^1 G(t,\tau)2\{H(\tau,\varsigma(\tau)) - H(\tau,\xi(\tau))\}d\tau\right\} \end{array}\right\} \\
&\leq \max\left\{\begin{array}{l} \sup_{t\in[0,1]}\left\{\int_0^1 G(t,\tau)K\sigma(\xi,\varsigma)d\tau\right\}, \\ \sup_{t\in[0,1]}\left\{\int_0^1 G(t,\tau)K\sigma(\xi,\varsigma)d\tau\right\} \end{array}\right\} \\
&= K\sigma(\xi,\varsigma)\sup_{t\in[0,1]}\left\{\int_0^1 G(t,\tau)d\tau\right\} \\
&= K\sigma(\xi,\varsigma)\sup_{t\in[0,1]}\left\{\frac{t(1-t)}{2}\right\} \\
&= \frac{K}{8}\sigma(\xi,\varsigma)
\end{aligned}$$

Consequently, Γ is a $(\alpha - \theta_\sigma)$-contraction with the functions $\theta(\varrho) = e^{\sqrt{\varrho}}$ and $\alpha(\xi,\varsigma) = 1$. It is evident that the remaining requirements of Theorem 1 are met. Thus, $\zeta \in C[0,1]$ exists, and it is the operator Γ's fixed point. Thus, the solution to Equation (10) is guaranteed in $C[0,1]$. □

4. Conclusions

We introduced the notion of $(\alpha - \theta_\sigma)$-contraction mappings on quasi-metric spaces. Then, we provided some fixed-point results for such mappings. In addition, we presented an illustration to back up our theoretical results. Finally, we provide an existence theorem for the second-order boundary-value problem. The outcomes of this paper are new and contribute to the fixed-point theory and applications. For future research, both new theoretical results can be proved by expanding contraction, and the theoretical results can be used to obtain the existing results for some kinds of equations, such as differential equations and integral equations, including fractional order.

Author Contributions: Conceptualization, I.A.; Formal analysis, I.A.; Funding acquisition, G.D.G.; Investigation, G.D.G.; Methodology, G.D.G.; Validation, G.D.G.; Writing—original draft, I.A.; Writing—review and editing, I.A. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: No new data were created or analyzed in this study.

Conflicts of Interest: The authors declare no conflicts of interest.

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