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Variable Lebesgue Space over Weighted Homogeneous Tree

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Abstract: An infinite homogeneous tree is a special type of graph that has a completely symmetrical structure in all directions. For an infinite homogeneous tree $T = (\mathcal{V}, \mathcal{E})$ with the natural distance d defined on graphs and a weighted measure μ of exponential growth, the authors introduce the variable Lebesgue space $L^{p(\cdot)}(\mu)$ over $(\mathcal{V}, d, \mu)$ and investigate it under the global Hölder continuity condition for $p(\cdot)$. As an application, the strong and weak boundedness of the maximal operator relevant to admissible trapezoids on $L^{p(\cdot)}(\mu)$ is obtained, and an unbounded example is presented.

Keywords: homogeneous tree; exponential growth; variable Lebesgue space; maximal operator

MSC: 05C05; 42B30; 42B35

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1. Introduction

The generalization of Lebesgue space—variable Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$, which was first introduced by Orlicz [1] in 1931—has been studied profoundly in recent years for its applications to partial differential equations with non-standard growth conditions [2]. Many scholars have paid attention to the boundedness of the Hardy–Littlewood maximal operator M on $L^{p(\cdot)}(\mathbb{R}^n)$, and various conditions have been proposed to ensure this; see [3–5].

In the past few decades, many achievements in function space and operator theory over $\mathbb{R}^n$ have been extended to some more general metric measure spaces, such as the spaces of homogeneous type $(\mathcal{X}, d, \mu)$ in the sense of Coifman and Weiss [6]. In 2018, Cruz-Uribe and Shukla [7] considered the variable Lebesgue space $L^{p(\cdot)}(\mathcal{X})$ over $(\mathcal{X}, d, \mu)$ and pointed out that, if a measurable function $p : [1, +\infty)$ satisfies that, for all $x, y \in \mathcal{X}$ with $0 < d(x, y) < 1/2$,

$$\left| \frac{1}{p(x)} - \frac{1}{p(y)} \right| \le \frac{C}{-\log(d(x,y))}, \tag{1}$$

and for some $p_0 \ge 1$, $x_0 \in \mathcal{X}$, and all $x \in \mathcal{X}$,

$$\left| \frac{1}{p(x)} - \frac{1}{p_0} \right| \le \frac{C}{\log(e + d(x, x_0))}, \tag{2}$$

then the weak boundedness of M on $L^{p(\cdot)}(\mathcal{X})$ holds. Furthermore, an additional condition

$$p_- := \operatorname*{ess\,inf}_{x \in \mathcal{X}} p(x) > 1$$

implies the strong boundedness.

Inequalities (1) and (2) are respectively called the local log-Hölder continuity condition and the global log-Hölder continuity condition. Specifically, (1) requires that, when x is sufficiently close to y, the difference between $p(x)$ and $p(y)$ is limited in the logarithmic form, and (2) claims that $p(x)$ tends towards a constant at a logarithmic rate when x tends towards infinity.

As another type of metric measure space with properties very different from homogeneous spaces (see Remark 1), an infinite homogeneous tree $\mathcal{V}$ equipped with the natural distance d and a weighted measure μ (see Definition 1 through Definition 3) has been considered by many researchers. In 2003, Hebisch and Steger [8] established an abstract Calderón–Zygmund theory, which is suitable for $(\mathcal{V}, d, \mu)$. As applications of this theory, they obtained the weak $(1, 1)$ boundedness of the maximal operator related to admissible trapezoids, $M_{\mathscr{R}}$, and further studied the properties of several other operators. In 2020, Arditti, Tabacco, and Vallarino [9] introduced and investigated the atomic Hardy space $H^1(\mu)$ over $(\mathcal{V}, d, \mu)$ and obtained the boundedness of singular integrals on $H^1(\mu)$. It is worth noting that the weak $(1, 1)$ boundedness of $M_{\mathscr{R}}$ was utilized again. In 2021, Arditti, Tabacco, and Vallarino [10] introduced the space $\mathrm{BMO}(\mu)$ over $(\mathcal{V}, d, \mu)$ as the dual space of $H^1(\mu)$ and studied the interpolation theory involving $L^p(\mu)$, $H^1(\mu)$, and $\mathrm{BMO}(\mu)$.

Throughout these works, the boundedness of $M_{\mathscr{R}}$ on Lebesgue space also has a fundamental importance for studying the properties of function spaces and other operators. Can conditions (1) and (2) ensure the boundedness of $M_{\mathscr{R}}$ on variable Lebesgue space $L^{p(\cdot)}(\mu)$ over $(\mathcal{V}, d, \mu)$? Unfortunately, both (1) and (2) are not applicable to our setting. On the one hand, the metric in $(\mathcal{V}, d, \mu)$ only takes integral values, so the analogue of (1) is trivial. On the other hand, the condition (2) with $p_0 = \infty$ implies the constant function $\mathbf{1} \in L^{p(\cdot)}(\mathcal{X})$, which plays a crucial role in [7]. However, since the measure in $(\mathcal{V}, d, \mu)$ does not satisfy the doubling condition (see Remark 1), from a simple calculation, the analogue of (2) with $p_0 = \infty$ can not imply $\mathbf{1} \in L^{p(\cdot)}(\mu)$. There is another reason forcing us to find a new condition for $p(\cdot)$. From (1) and the doubling condition in $(\mathcal{X}, d, \mu)$, for a ball B in $\mathcal{X}$ with small enough measure, the oscillation of $p(\cdot)$ on B is also small. However, this property does not hold for $(\mathcal{V}, d, \mu)$; thus, another important lemma also fails, although (1) holds unconditionally.

The main purpose of this article is to search for new conditions that can replace conditions (1) and (2) in order to ensure the boundedness of $M_{\mathscr{R}}$ on variable Lebesgue space $L^{p(\cdot)}(\mu)$. From a full utilization of the exponential growth property in $(\mathcal{V}, d, \mu)$, we put forward the global Hölder condition $1/p(\cdot) \in \mathcal{H}$; see Definition 8. This condition solves the above two difficulties together and further leads to the strong and weak boundedness of $M_{\mathscr{R}}$ on $L^{p(\cdot)}(\mu)$.

This paper is organized as follows. In Section 2, we recall some classical concepts in graph theory and define $L^{p(\cdot)}(\mu)$ over $(\mathcal{V}, d, \mu)$. In Section 3, some properties of $L^{p(\cdot)}(\mu)$ are proven. In Section 4, the strong and weak boundedness of $M_{\mathscr{R}}$ on $L^{p(\cdot)}(\mu)$ is obtained, and a counterexample is presented to show the failure of strong boundedness for $p_- = 1$.

Throughout this paper, we use C to denote a positive constant independent of the main parameters, which may vary in different places. Additionally, $\mathbb{Z}$ denotes the set of all integers, $\mathbb{Z}_+ := \mathbb{Z} \cap [0, \infty)$, and $\mathbb{N} := \mathbb{Z} \cap [1, \infty)$. For a locally integrable function f and $E \subset \mathcal{V}$,

$$\fint_E f(v)d\mu(v) := \frac{1}{\mu(E)} \int_E f(v)d\mu(v).$$

2. Propositions

In this section, we review some basic definitions for weighted homogeneous trees, admissible trapezoids, and variable Lebesgue spaces, and we cite or prove some lemmas. These definitions and lemmas are the basis for our subsequent discussions.

2.1. Weighted Homogeneous Tree

Let us first review the following concepts about weighted homogeneous trees.

Definition 1 ([8]). *(Weighted homogeneous tree and the level of points.)*

- *An infinite homogeneous tree of order $m + 1$ is a graph $T = (\mathcal{V}, \mathcal{E})$ satisfying the following conditions, where $\mathcal{V}$ is the set of vertices, and $\mathcal{E}$ is the set of edges:*
  - *(i) T is connected and acyclic.*
  - *(ii) Each vertex in $\mathcal{V}$ has exactly $m + 1$ neighbors.*

The natural distance $d(x,y)$ of $x,y \in \mathcal{V}$ is the length of the shortest path between x and y.

- *Let $T = (\mathcal{V}, \mathcal{E})$ be an infinite homogeneous tree; a doubly-infinite geodesic g in T is a connected subset of $\mathcal{V}$ such that:*
  - *(i) For each vertex $v \in g$, there are exactly two neighbors of v in g.*
  - *(ii) For each two vertices $u, v \in g$, the shortest path between u and v is contained in g.*
- *Let $T = (\mathcal{V}, \mathcal{E})$ be an infinite homogeneous tree with a doubly-infinite geodesic g. Choose a mapping $N : g \to \mathbb{Z}$ such that, for all $u, v \in g$,*

$$|N(u) - N(v)| = d(u,v).$$

Then, for any $v \in \mathcal{V}$, define its level $l(v)$ as

$$l(v) = N(v') - d(v, v'),$$

where v' is the unique vertex that minimizes $d(u,v)$ for $u \in g$.

Actually, the function $l : \mathcal{V} \to \mathbb{Z}$ depends on the choice of g, the unique vertex $o \in g$ satisfying $N(o) = 0$ (called the origin of $\mathcal{V}$), and the orientation of g. In what follows, for a given T, we assume that they have been determined, and then l is determined.

Definition 2 ([8]). *Let $T = (\mathcal{V}, \mathcal{E})$ be an infinite homogeneous tree. For $u, v \in \mathcal{V}$, v lies above u, or u lies below v, if*

$$l(v) - l(u) = d(u,v).$$

Starting from the definition of each point's level, we can sort all points from top to bottom based on their levels. Under this sorting, each point $x \in \mathcal{V}$ has one neighbor lying above it and m neighbors lying below it, and the latter neighbors generate m symmetric branches.

For example, the weighted homogeneous tree T for $m = 3$ is actually an upside down "tree", with the bottom endpoint of each branch growing two new branches downwards. We refer the reader to [9], Figure 1.

Definition 3 ([8]). *Let $T = (\mathcal{V}, \mathcal{E})$ be an infinite homogeneous tree of order $m + 1$; the measure μ on $\mathcal{V}$ is defined as*

$$\mu(v) = m^{l(v)} \text{ for } v \in \mathcal{V}.$$

To simplify writing, in this article, we always use $T, \mathcal{V}, \mathcal{E}, m, d, l$, and μ to denote the corresponding concepts in Definition 1 through Definition 3.

Obviously, any function $f : \mathcal{V} \to \mathbb{R}$ is measurable, and if f is non-negative,

$$\int_{\mathcal{V}} f(v) d\mu(v) = \sum_{v \in \mathcal{V}} f(v) m^{l(v)}.$$

The measure of a ball in $\mathcal{V}$ was accurately calculated.

Lemma 1 ([9]). *For $r \in \mathbb{N}$ and a ball $B = B(v_0, r) \subset \mathcal{V}$,*

$$\mu(B) = \frac{m^{l(v_0)}(m^{r+1} + m^r - 2)}{m - 1}.$$

Remark 1. *From Lemma 1, the measure μ is of exponential growth; thus, $(\mathcal{V}, d, \mu)$ does not satisfy the doubling condition on homogeneous space [6] or the upper doubling condition on non-homogeneous space [11].*

2.2. Admissible Trapezoid

Definition 4 ([9])**.** *An admissible trapezoid R is a subset of $\mathcal{V}$ satisfying at least one of the following conditions:*

(i) R consists of a single point v_R.
(ii) There exist $v_R \in \mathcal{V}$ and $h(R) \in \mathbb{N}$ such that

$$R = \{v \in \mathcal{V} : v \text{ lies below } v_R,\ h(R) \le l(v_R) - l(v) < 2h(R)\}.$$

We agree that $h(R) = 1$ in the first case. Then, in both cases, $h(R)$ is called the height of R and is the number of different levels of vertices in R. Meanwhile, the quantity $w(R) = m^{l(v_R)}$ is called the width of R. It is easy to calculate that

$$\mu(R) = h(R)w(R).$$

Denote $\mathscr{R}$ as the set of all admissible trapezoids.

Definition 5 ([9])**.** *Let $R \in \mathscr{R}$ contain more than one vertex; the envelope of R is defined as*

$$R^* = \left\{v \in \mathcal{V} : v \text{ lies below } v_R,\ \frac{1}{2}h(R) \le l(v_R) - l(v) < 4h(R)\right\}.$$

In fact, an admissible trapezoid is an array contained in the tree, where all vertices are divided into $h(R)$ layers from top to bottom, and the number of points in the next layer is q times that of the previous layer, which is why it is called a trapezoid. The envelope of R is another trapezoid with more layers than R.

There are two lemmas that characterize the geometric structure of admissible trapezoids and their envelopes.

Lemma 2 ([9])**.** *Let $R \in \mathscr{R}$; then, $\mu(R^*) \le 4\mu(R)$.*

Lemma 3 ([9])**.** *Let $R, R' \in \mathscr{R}$; if $R \cap R' \neq \Phi$ and $w(R') \le w(R)$, then $R' \subset R^*$.*

Lemma 4. *Let $\{R_\lambda\}_{\lambda\in\Lambda} \subset \mathscr{R}$ satisfy $\sup_{\lambda\in\Lambda} \mu(R_\lambda) < \infty$; then, there exists a pairwise disjoint subcollection $\mathscr{R}' \subset \{R_\lambda\}_{\lambda\in\Lambda}$ such that, for any $\lambda \in \Lambda$, there exists $R \in \mathscr{R}'$ with $R_\lambda \subset R^*$.*

Proof. For any $\lambda \in \Lambda$, $h(R_\lambda) \ge 1$, the condition $\sup_{\lambda\in\Lambda} \mu(R_\lambda) < \infty$ implies that

$$W := \sup_{\lambda\in\Lambda} w(R_\lambda) < \infty.$$

Denote

$$\mathscr{R}_j = \left\{R \in \{R_\lambda\}_{\lambda\in\Lambda} : w(R) = \frac{W}{m^j}\right\},\ \ j \in \mathbb{Z}_+,$$

and use the following method to choose $\mathscr{R}'$:

(a) Let $\mathscr{R}'_0$ be any maximal pairwise disadjoint subcollection of $\mathscr{R}_0$.

(b) Assume that $\mathscr{R}'_1, \mathscr{R}'_2, \cdots, \mathscr{R}'_{k-1}$ has been selected and let $\mathscr{R}'_k$ be the any maximal pairwise disadjoint subcollection of

$$\left\{R \in \mathscr{R}_k : R \cap R' = \Phi \text{ for all } R' \in \bigcup_{j=0}^{k-1} \mathscr{R}'_j\right\}.$$

(c) Fix $\mathscr{R}' = \bigcup_{j=0}^{\infty} \mathscr{R}'_j$.

In fact, for any $\lambda \in \Lambda$, there exists a unique $k \in \mathbb{Z}_+$ such that $R_\lambda \in \mathscr{R}_k$. If $R_\lambda \in \mathscr{R}'_k$, the proof is finished. Otherwise, there exists $j \le k$ and $R \in \mathscr{R}'_j \subset \mathscr{R}'$ with $R_\lambda \cap R \neq \Phi$. Since $j \le k$, $w(R_\lambda) \le w(R)$, then, by Lemma 3, $R_\lambda \subset R^*$. □

2.3. Exponent Function and Variable Lebesgue Space

Definition 6. *For $r : \mathcal{V} \to [0, \infty]$, $E \subset \mathcal{V}$, define*

$$r_-(E) = \inf_{v \in E} r(v), \quad r_+(E) = \sup_{v \in E} r(v),$$

and simply write $r_- := r_-(\mathcal{V})$, $r_+ := r_+(\mathcal{V})$.

Denote $\mathcal{P}$ as the set of all functions $p : \mathcal{V} \to [1, \infty)$.

Definition 7. *Let $p \in \mathcal{P}$; the modular of $f : \mathcal{V} \to \mathbb{R}$ associated with p is defined as*

$$\rho_{p(\cdot)}(f) = \int_{\mathcal{V}} |f(v)|^{p(v)} d\mu(v).$$

Then, the variable Lebesgue space $L^{p(\cdot)}(\mu)$ is defined as the set of all functions $f : \mathcal{V} \to \mathbb{R}$ such that

$$\|f\|_{p(\cdot)} := \inf\left\{\lambda > 0 : \rho_{p(\cdot)}\left(\frac{f}{\lambda}\right) \leq 1\right\} < \infty,$$

where we agree that the infimum of an empty set is ∞.

Remark 2. *If $p(x) = p$ for all $x \in \mathcal{V}$, we have*

$$\rho_{p(\cdot)}(f) = \int_{\mathcal{V}} |f(v)|^p d\mu(v) = \|f\|_p^p,$$

then

$$\rho_{p(\cdot)}\left(\frac{f}{\lambda}\right) \leq 1 \Leftrightarrow \frac{1}{\lambda^p}\|f\|_p^p \leq 1 \Leftrightarrow \lambda \geq \|f\|_p,$$

which implies that $\|f\|_{p(\cdot)} = \|f\|_p$, and thus $L^{p(\cdot)}(\mu) = L^p(\mu)$.

In what follows, we abbreviate $L^{p(\cdot)}(\mu)$ and $L^p(\mu)$ as $L^{p(\cdot)}$ and L^p, respectively. By some similar arguments as in [5], the following lemmas about $\rho_{p(\cdot)}$ and $\|\cdot\|_{p(\cdot)}$ can be obtained. We omit the details here.

Lemma 5. *Let $p \in \mathcal{P}$. Then, $\|\cdot\|_{p(\cdot)}$ is a norm; that is:*

(i) $\|f\|_{p(\cdot)} \geq 0$, *and* $\|f\|_{p(\cdot)} = 0 \Leftrightarrow f(v) \equiv 0$.
(ii) $\|\lambda f\|_{p(\cdot)} = |\lambda|\|f\|_{p(\cdot)}$ *for* $\lambda \in \mathbb{R}$.
(iii) $\|f + g\|_{p(\cdot)} \leq \|f\|_{p(\cdot)} + \|g\|_{p(\cdot)}$.

Lemma 6. *Let $p \in \mathcal{P}$; then, $\rho_{p(\cdot)}$ and $\|\cdot\|_{p(\cdot)}$ have the following properties:*

(i) *If $|f(v)| \leq |g(v)|$ for all $v \in \mathcal{V}$, then $\rho_{p(\cdot)}(f) \leq \rho_{p(\cdot)}(g)$, and $\|f\|_{p(\cdot)} \leq \|g\|_{p(\cdot)}$.*
(ii) *For $\lambda > 0$,*

$$\rho_{p(\cdot)}\left(\frac{f}{\lambda}\right) \leq \frac{\rho_{p(\cdot)}(f)}{\lambda}.$$

(iii) *There holds $\|f\|_{p(\cdot)} \leq C_1 \Leftrightarrow \rho_{p(\cdot)}(f) \leq C_2$. Meanwhile, one of the constants C_1, C_2 equals 1 will make the other equal 1.*
(iv) *If $\|f\|_{p(\cdot)} \leq 1$, then $\rho_{p(\cdot)}(f) \leq \|f\|_{p(\cdot)}$; if $\|f\|_{p(\cdot)} > 1$, then $\rho_{p(\cdot)}(f) \geq \|f\|_{p(\cdot)}$.*

Lemma 7. *Let $p, q, r \in \mathcal{P}$ such that, for any $v \in \mathcal{V}$,*

$$\frac{1}{p(v)} = \frac{1}{q(v)} + \frac{1}{r(v)},$$

$f \in L^{q(\cdot)}$, and $g \in L^{r(\cdot)}$; then, $fg \in L^{p(\cdot)}$ with the norm

$$\|fg\|_{p(\cdot)} \leq 3\|f\|_{q(\cdot)}\|g\|_{r(\cdot)}.$$

As we describe above, the boundedness of the maximal operator on $L^{p(\cdot)}$ relies on the following condition.

Definition 8. *A function $r : \mathcal{V} \to [0, \infty]$ is called global Hölder continuous with respect to $v_0 \in \mathcal{V}$ if there exists $C_0 > 0$ and $r_0 \geq 0$ such that, for any $v \in \mathcal{V}$,*

$$|r(v) - r_0| \leq \frac{C_0}{1 + d(v, v_0)}.$$

Remark 3. *Suppose that r is global Hölder continuous with respect to v_0. For another point $v_1 \in \mathcal{V}$, from the inequality*

$$\frac{1}{1 + d(v, v_0)} = \frac{1}{1 + d(v, v_1)}\left(1 + \frac{d(v, v_1) - d(v, v_0)}{1 + d(v, v_0)}\right) \leq \frac{1 + d(v_0, v_1)}{1 + d(v, v_1)},$$

r is global Hölder continuous with respect to v_1. Therefore, we always assume $v_0 = o$ and denote the set of all such r as $\mathcal{H}$.

3. Properties of $L^{p(\cdot)}$ over $(\mathcal{V}, d, \mu)$

In this section, we present some more profound properties of $L^{p(\cdot)}$, which will be utilized in Section 4.

Lemma 8. *Let $r \in \mathcal{H}$; then, there exists $C > 0$ such that, for any set $S \subset \mathcal{V}$ and $v \in S$,*

$$\mu(S)^{r(v) - r_+(S)} \leq C, \ \ \mu(S)^{r_-(S) - r(v)} \leq C.$$

Proof. If $\mu(S) \geq 1$, these inequalities hold obviously. Otherwise, let

$$d_0 = \min_{v \in S} d(v, o);$$

then, $\mu(S) \geq m^{-d_0}$, and, for all $v \in S$,

$$|r(v) - r_0| \leq \frac{C_0}{1 + d(v, o)} \leq \frac{C_0}{1 + d_0};$$

thus,

$$r(v) - r_+(S) = -|r_+(S) - r(v)| \geq -(|r_+(S) - r_0| + |r(v) - r_0|) \geq -\frac{2C_0}{1 + d_0},$$

$$r_-(S) - r(v) = -|r(v) - r_-(S)| \geq -(|r(v) - r_0| + |r_-(S) - r_0|) \geq -\frac{2C_0}{1 + d_0}.$$

Therefore,

$$\mu(S)^{r(v) - r_+(S)} \leq (m^{-d_0})^{-\frac{2C_0}{1+d_0}} < m^{2C_0},$$

$$\mu(S)^{r_-(S) - r(v)} \leq (m^{-d_0})^{-\frac{2C_0}{1+d_0}} < m^{2C_0},$$

which completes the proof. □

Lemma 9. *Define*

$$R(v) = \frac{1}{m^{2d(v,o)}}$$

for $v \in \mathcal{V}$; then

$$\int_{\mathcal{V}} R(v) d\mu(v) < \infty.$$

Proof. Denote $B_r := B(o,r)$ for any $r \in \mathbb{N}$; then

$$\int_{\mathcal{V}} R(v) d\mu(v) = \lim_{r\to\infty} \int_{B_r} R(v) d\mu(v) = R(o) + \sum_{k=1}^{\infty} \int_{B_k \setminus B_{k-1}} R(v) d\mu(v).$$

For any $v \in B_k \setminus B_{k-1}$,

$$R(v) = \frac{1}{m^{2k}},$$

and by Lemma 1,

$$\mu(B_k \setminus B_{k-1}) = \mu(B_k) - \mu(B_{k-1}) = m^k + m^{k-1} < 2m^k.$$

Therefore,

$$\int_{\mathcal{V}} R(v) d\mu(v) < 1 + 2 \sum_{k=1}^{\infty} \frac{m^k}{m^{2k}} < \infty,$$

which completes the proof. □

Lemma 10. *Let $p \in \mathcal{P}$, $1/p(\cdot) \in \mathcal{H}$ with $r_0 = 0$; then, $\mathbf{1} \in L^{p(\cdot)}$.*

Proof. For any $v \in \mathcal{V}$, there holds

$$\frac{1}{p(v)} \leq \frac{C_0}{1 + d(v,o)};$$

thus, for $\lambda > 1$,

$$\rho_{p(\cdot)}\left(\frac{\mathbf{1}}{\lambda}\right) = \int_{\mathcal{V}} \lambda^{-p(v)} d\mu(v) \leq \int_{\mathcal{V}} \lambda^{-\frac{1+d(v,o)}{C_0}} d\mu(v).$$

Fix $\lambda_0 = m^{2C_0}$; then, $\lambda_0^{-\frac{1+d(v,o)}{C_0}} = m^{-2-2d(v,o)} < R(v)$ by Lemma 9, and

$$\rho_{p(\cdot)}\left(\frac{\mathbf{1}}{\lambda_0}\right) < \int_{\mathcal{V}} R(v) d\mu(v) < \infty.$$

By Lemma 6, $\mathbf{1}/\lambda_0 \in L^{p(\cdot)}$; then, by Lemma 5, $\mathbf{1} \in L^{p(\cdot)}$. □

Lemma 11. *Let $p_1, p_2 \in \mathcal{P}$ such that $p_1(v) \leq p_2(v)$ for all $v \in \mathcal{V}$, $f \in L^{p_1(\cdot)}$ and $|f(v)| \leq 1$ for all $v \in \mathcal{V}$; then, $f \in L^{p_2(\cdot)}$ with the norm*

$$\|f\|_{p_2(\cdot)} \leq \|f\|_{p_1(\cdot)}.$$

Proof. If $\|f\|_{p_1(\cdot)} \leq 1$, by Lemma 6, $\rho_{p_1(\cdot)}(f) \leq \|f\|_{p_1(\cdot)}$; then

$$\rho_{p_2(\cdot)}(f) = \int_{\mathcal{V}} |f(v)|^{p_2(v)} d\mu(v) \leq \int_{\mathcal{V}} |f(v)|^{p_1(v)} d\mu(v) = \rho_{p_1(\cdot)}(f) \leq \|f\|_{p_1(\cdot)}.$$

By Lemma 6,

$$\rho_{p_2(\cdot)}\left(\frac{f}{\|f\|_{p_1(\cdot)}}\right) \leq 1,$$

thus $\|f\|_{p_2(\cdot)} \le \|f\|_{p_1(\cdot)}$.
If $\|f\|_{p_1(\cdot)} > 1$, let

$$g(v) = \frac{f(v)}{\|f\|_{p_1(\cdot)}};$$

then, $|g(v)| \le 1$ for all $v \in \mathcal{V}$, and $\|g\|_{p_1(\cdot)} = 1$. By the known result, $\|g\|_{p_2(\cdot)} \le \|g\|_{p_1(\cdot)} = 1$, that is, $\|f\|_{p_2(\cdot)} \le \|f\|_{p_1(\cdot)}$. □

4. The Maximal Operator Relevant to Admissible Trapezoids

In this section, we focus on the maximal operator $M_{\mathscr{R}}$. Another maximal operator $M_{\mathscr{R}^*}$ is also needed.

Definition 9 ([9])**.** *The maximal operator $M_{\mathscr{R}}$ is defined as*

$$M_{\mathscr{R}} f(x) = \sup_{R \in \mathscr{R}, R \ni x} \fint_R |f(v)| d\mu(v).$$

Definition 10. *Denote $\mathscr{R}^* = \{R^* : R \in \mathscr{R}\}$; the maximal operator $M_{\mathscr{R}^*}$ is defined as*

$$M_{\mathscr{R}^*} f(x) = \sup_{R^* \in \mathscr{R}^*, R^* \ni x} \fint_{R^*} |f(v)| d\mu(v).$$

Remark 4. *The maximal operators $M_{\mathscr{R}}$ and $M_{\mathscr{R}^*}$ are the variants of the classical Hardy–Littlewood maximal operator in harmonic analysis, where the balls $B \ni x$ are replaced with the admissible trapezoids $R \ni x$ or their envelopes $R^* \ni x$. The new maximal operators clearly retain some properties of the Hardy–Littlewood maximal operator, such as not changing the infinity norm of functions.*

The first inequality in the following lemma is from [8], Theorem 3.1. By using the further expansion of the envelopes, one can obtain the second inequality in the same way.

Lemma 12. *For any $f \in L^1$ and $t > 0$,*

$$\|t\chi_{\{v \in \mathcal{V} : M_{\mathscr{R}} f(v) > t\}}\|_1 \le C\|f\|_1,$$

$$\|t\chi_{\{v \in \mathcal{V} : M_{\mathscr{R}^*} f(v) > t\}}\|_1 \le C\|f\|_1.$$

By Remark 4, Lemma 12, and the Marcinkiewicz interpolation theorem on measure space ([12], Theorem 1.3.2), the following corollary, which shows the strong (p, p) boundedness of $M_{\mathscr{R}}$ and $M_{\mathscr{R}^*}$ for $1 < p < \infty$, is directly obtained.

Corollary 1. *For any $p > 1$ and $f \in L^p$,*

$$\|M_{\mathscr{R}} f\|_p \le C\|f\|_p,$$

$$\|M_{\mathscr{R}^*} f\|_p \le C\|f\|_p.$$

Remark 5. *For $p > 1$, the weak (p, p) boundedness of $M_{\mathscr{R}}$ and $M_{\mathscr{R}^*}$ still holds. Specifically, for any $f \in L^p$ and $t > 0$,*

$$\begin{aligned} \|t\chi_{\{v \in \mathcal{V} : M_{\mathscr{R}} f(v) > t\}}\|_p &= \left(\int_{\{v \in \mathcal{V} : M_{\mathscr{R}} f(v) > t\}} t^p d\mu(v) \right)^{\frac{1}{p}} \\ &\le \left(\int_{\{v \in \mathcal{V} : M_{\mathscr{R}} f(v) > t\}} (M_{\mathscr{R}} f(v))^p d\mu(v) \right)^{\frac{1}{p}} \end{aligned}$$

$$\le \left(\int_{\mathcal{V}} (M_{\mathscr{R}} f(v))^p d\mu(v)\right)^{\frac{1}{p}} = \|M_{\mathscr{R}} f\|_p \le C\|f\|_p,$$

and similarly,

$$\|t\chi_{\{v\in\mathcal{V}:M_{\mathscr{R}^*}f(v)>t\}}\|_p \le C\|f\|_p.$$

We first prove the strong boundedness of $M_{\mathscr{R}}$ on $L^{p(\cdot)}$ for $p_- > 1$.

Theorem 1. *Let $p \in \mathcal{P}$ with $1/p(\cdot) \in \mathcal{H}$ and $p_- > 1$, $f \in L^{p(\cdot)}$; then,*

$$\|M_{\mathscr{R}} f\|_{p(\cdot)} \le C\|f\|_{p(\cdot)}. \tag{3}$$

Proof. Since $M_{\mathscr{R}}|f| = M_{\mathscr{R}} f$ and $f \equiv 0 \Rightarrow M_{\mathscr{R}} f \equiv 0$, (3) suffices for f that are non-negative and not identical to 0. By Lemma 5 and Lemma 6, without the loss of generalization, we assume $\|f\|_{p(\cdot)} = 1$; thus, $\rho_{p(\cdot)}(f) \le 1$, and we can prove $\|M_{\mathscr{R}} f\|_{p(\cdot)} < C$.
Decompose

$$f = f_1 + f_2 := f\chi_{\{v:f(v)>1\}} + f\chi_{\{v:f(v)\le 1\}},$$

and then it suffices to show that, for $i = 1, 2$, $\|M_{\mathscr{R}} f_i\|_{p(\cdot)} < C$.
To estimate $\|M_{\mathscr{R}} f_1\|_{p(\cdot)}$, let $\lambda_1, \lambda_2, \lambda_3$ be constants that will be determined later. Fix $A > 1$ and define

$$\Omega_k = \{v \in \mathcal{V} : M_{\mathscr{R}} f_1(v) > A^k\};$$

then,

$$\mathcal{V} = \bigcup_{k\in\mathbb{Z}} \Omega_k \backslash \Omega_{k+1}.$$

For given $k \in \mathbb{N}$ and $v \in \Omega_k$, there exists $R_v \in \mathscr{R}$ containing v, such that

$$⨍_{R_v} f_1(v) d\mu(v) > A^k.$$

From $f_1(v) > 1$ or $f_1(v) = 0$, and $\|f_1\|_{p(\cdot)} \le 1$, there holds

$$\int_{\mathcal{V}} |f_1(v)| d\mu(v) \le \int_{\mathcal{V}} |f_1(v)|^{p(v)} d\mu(v) \le 1,$$

so

$$\lim_{\mu(R)\to\infty} ⨍_R f_1(v) d\mu(v) = 0,$$

and thus

$$\sup_{v\in\Omega_k} \mu(R_v) < \infty.$$

By Lemma 4, there exists a pairwise disjoint set family $\{R_j^k\}_{j\in\mathbb{N}} \subset \{R_v\}_{v\in\Omega_k}$ (we agree that $\mathbb{N}$ can also represent the finite set $\{1, 2, \cdots, n_0\}$ here), such that, for any $v \in \Omega_k$, there exists $j \in \mathbb{N}$ with $R_v \subset (R_j^k)^*$. By Lemma 2, for any $j \in \mathbb{N}$,

$$⨍_{(R_j^k)^*} f_1(v) d\mu(v) \ge \frac{\mu(R_j^k)}{\mu((R_j^k)^*)} ⨍_{R_j^k} f_1(v) d\mu(v) > \frac{A^k}{4}.$$

Define $S_1^k = (\Omega_k \backslash \Omega_{k+1}) \cap (R_1^k)^*$, $S_2^k = ((\Omega_k \backslash \Omega_{k+1}) \cap (R_2^k)^*) \backslash S_1^k$, $S_3^k = ((\Omega_k \backslash \Omega_{k+1}) \cap (R_3^k)^*) \backslash (S_1^k \cup S_2^k)$, and so on. Therefore, $\{S_j^k\}$ is a pairwise disjoint family for all $k \in \mathbb{Z}$ and $j \in \mathbb{N}$, and

$$\Omega_k \backslash \Omega_{k+1} = \bigcup_{j\in\mathbb{N}} S_j^k$$

for all $k \in \mathbb{Z}$. Let $\lambda_1 = (4A)^{-1}$, $p_{jk} = p_-((R_j^k)^*)$, and then $p_{jk} \geq p_-$. By the Hölder inequality,

$$\begin{aligned}
\rho_{p(\cdot)}(\lambda_1\lambda_2\lambda_3 M_{\mathscr{R}} f_1) &= \sum_{k\in\mathbb{Z}} \int_{\Omega_k \setminus \Omega_{k+1}} (\lambda_1\lambda_2\lambda_3 M_{\mathscr{R}} f_1(v))^{p(v)} d\mu(v) \\
&\leq \sum_{k\in\mathbb{Z}, j\in\mathbb{N}} \int_{S_j^k} \left(\lambda_2\lambda_3 \frac{A^k}{4}\right)^{p(v)} d\mu(v) \\
&\leq \sum_{k\in\mathbb{Z}, j\in\mathbb{N}} \int_{S_j^k} \left(\lambda_2\lambda_3 \fint_{(R_j^k)^*} f_1(w) d\mu(w)\right)^{p(v)} d\mu(v) \\
&\leq \sum_{k\in\mathbb{Z}, j\in\mathbb{N}} \int_{S_j^k} \left(\lambda_2\lambda_3 \left(\fint_{(R_j^k)^*} f_1(w)^{\frac{p_{jk}}{p_-}} d\mu(w)\right)^{\frac{p_-}{p_{jk}}}\right)^{p(v)} d\mu(v).
\end{aligned}$$

Let $r(\cdot) = 1/p(\cdot)$; then, $r \in LH$, and $r_+((R_j^k)^*) = 1/p_{jk}$. By Lemma 8, there exists $\lambda_2 \in (0,1)$ such that, for all $v \in (R_j^k)^*$,

$$\mu((R_j^k)^*)^{r(v) - r_+((R_j^k)^*)} \leq \lambda_2^{-\frac{1}{p_-}} \Leftrightarrow \lambda_2 \mu((R_j^k)^*)^{-\frac{p_-}{p_{jk}}} \leq \mu((R_j^k)^*)^{-\frac{p_-}{p(v)}}.$$

Since $f_1(v) > 1$ or $f_1(v) = 0$,

$$\int_{(R_j^k)^*} f_1(w)^{\frac{p(w)}{p_-}} d\mu(w) \leq \int_{(R_j^k)^*} f_1(w)^{p(w)} d\mu(w) \leq \rho_{p(\cdot)}(f_1) \leq 1;$$

thus, for $\lambda_3 \in (0,1)$,

$$\begin{aligned}
\rho_{p(\cdot)}(\lambda_1\lambda_2\lambda_3 M_{\mathscr{R}} f_1) &\leq \sum_{k\in\mathbb{Z}, j\in\mathbb{N}} \int_{S_j^k} \left(\lambda_2 \mu((R_j^k)^*)^{-\frac{p_-}{p_{jk}}} \left(\lambda_3 \int_{(R_j^k)^*} f_1(w)^{\frac{p_{jk}}{p_-}} d\mu(w)\right)^{\frac{p_-}{p_{jk}}}\right)^{p(v)} d\mu(v) \\
&\leq \lambda_3 \sum_{k\in\mathbb{Z}, j\in\mathbb{N}} \int_{S_j^k} \mu((R_j^k)^*)^{-p_-} \left(\int_{(R_j^k)^*} f_1(w)^{\frac{p(w)}{p_-}} d\mu(w)\right)^{\frac{p_- p(v)}{p_{jk}}} d\mu(v) \\
&\leq \lambda_3 \sum_{k\in\mathbb{Z}, j\in\mathbb{N}} \int_{S_j^k} \mu((R_j^k)^*)^{-p_-} \left(\int_{(R_j^k)^*} f_1(w)^{\frac{p(w)}{p_-}} d\mu(w)\right)^{p_-} d\mu(v) \\
&\leq \lambda_3 \sum_{k\in\mathbb{Z}, j\in\mathbb{N}} \int_{S_j^k} (M_{\mathscr{R}^*}(f_1(\cdot)^{\frac{p(\cdot)}{p_-}})(v))^{p_-} d\mu(v) \\
&\leq \lambda_3 \int_{\mathcal{V}} (M_{\mathscr{R}^*}(f_1(\cdot)^{\frac{p(\cdot)}{p_-}})(v))^{p_-} d\mu(v).
\end{aligned}$$

Since $p_- > 1$, by Corollary 1, there exists $\lambda_3 \in (0,1)$ such that

$$\int_{\mathcal{V}} (M_{\mathscr{R}^*}(f_1(\cdot)^{\frac{p(\cdot)}{p_-}})(v))^{p_-} d\mu(v) \leq \frac{1}{\lambda_3} \int_{\mathcal{V}} f_1(v)^{p(v)} d\mu(v),$$

and thus

$$\rho_{p(\cdot)}(\lambda_1\lambda_2\lambda_3 M_{\mathscr{R}} f_1) \leq \int_{\mathcal{V}} f_1(v)^{p(v)} d\mu(v) \leq 1;$$

that is,

$$\|M_{\mathscr{R}} f_1\|_{p(\cdot)} \leq \frac{1}{\lambda_1\lambda_2\lambda_3}.$$

To estimate $\|M_{\mathscr{R}} f2\|_{p(\cdot)}$, divide $\mathcal{V}$ as the union of sets

$$\mathcal{V}_1 = \{v \in \mathcal{V} : p(v) > p_0\},\ \mathcal{V}_2 = \{v \in \mathcal{V} : p(v) = p_0\},\ \ \mathcal{V}_3 = \{v \in \mathcal{V} : p(v) < p_0\},$$

where $p_0 = \lim_{d(v,o)\to\infty} p(v)$. From $1/p(\cdot) \in \mathcal{H}$ and $p_- > 1$, this limit exists (or equals ∞) and is greater than 1.
We first estimate $\|\chi_{\mathcal{V}_i} f2\|_{p_0}$ for $i = 1, 2, 3$. For $v \in \mathcal{V}_1$, define q as

$$\frac{1}{p_0} = \frac{1}{p(v)} + \frac{1}{q(v)};$$

then, $q \in \mathcal{P}$, $1/q(\cdot) \in \mathcal{H}$, and $\lim_{d(v,o)\to\infty} q(v) = \infty$. By Lemma 7 and Lemma 10,

$$\|\chi_{\mathcal{V}_1} f2\|_{p_0} \leq 3\|\chi_{\mathcal{V}_1} f2\|_{p(\cdot)}\|\mathbf{1}\|_{q(\cdot)} \leq C\|\chi_{\mathcal{V}_1} f2\|_{p(\cdot)} \leq C\|f2\|_{p(\cdot)} \leq C.$$

For $v \in \mathcal{V}_2$, $p(v) = p_0$; thus,

$$\|\chi_{\mathcal{V}_2} f2\|_{p_0} = \|\chi_{\mathcal{V}_2} f2\|_{p(\cdot)} \leq \|f2\|_{p(\cdot)} \leq 1.$$

For $v \in \mathcal{V}_3$, $p(v) < p_0$, and $|f(v)| \leq 1$; thus, by Lemma 11,

$$\|\chi_{\mathcal{V}_3} f2\|_{p_0} \leq \|\chi_{\mathcal{V}_3} f2\|_{p(\cdot)} \leq \|f2\|_{p(\cdot)} \leq 1.$$

Therefore, by the Minkowski inequality,

$$\|f2\|_{p_0} \leq \|\chi_{\mathcal{V}_1} f2\|_{p_0} + \|\chi_{\mathcal{V}_2} f2\|_{p_0} + \|\chi_{\mathcal{V}_3} f2\|_{p_0} \leq C.$$

Finally, we estimate $\|\chi_{\mathcal{V}_i} M_{\mathscr{R}} f2\|_{p(\cdot)}$ for $i = 1, 2, 3$. For $v \in \mathcal{V}_1$, since $p(v) > p_0$ and $|M_{\mathscr{R}} f_2(v)| \leq 1$, by Lemma 11 and Corollary 1,

$$\|\chi_{\mathcal{V}_1} M_{\mathscr{R}} f2\|_{p(\cdot)} \leq \|\chi_{\mathcal{V}_1} M_{\mathscr{R}} f2\|_{p_0} \leq \|M_{\mathscr{R}} f2\|_{p_0} \leq C\|f2\|_{p_0} \leq C.$$

For $v \in \mathcal{V}_2$, $p(v) = p_0$; thus,

$$\|\chi_{\mathcal{V}_2} M_{\mathscr{R}} f2\|_{p(\cdot)} = \|\chi_{\mathcal{V}_2} M_{\mathscr{R}} f2\|_{p_0} \leq C\|M_{\mathscr{R}} f2\|_{p_0} \leq C.$$

For $v \in \mathcal{V}_3$, define q as

$$\frac{1}{p(v)} = \frac{1}{p_0} + \frac{1}{q(v)},$$

then $q \in \mathcal{P}$, $1/q(\cdot) \in \mathcal{H}$, and $\lim_{d(v,o)\to\infty} q(v) = \infty$. By Lemma 7 and Lemma 10,

$$\|\chi_{\mathcal{V}_3} M_{\mathscr{R}} f2\|_{p(\cdot)} \leq 3\|\chi_{\mathcal{V}_3} M_{\mathscr{R}} f2\|_{p_0}\|\mathbf{1}\|_{q(\cdot)} \leq C\|M_{\mathscr{R}} f2\|_{p_0} \leq C.$$

Therefore, by Lemma 5,

$$\|M_{\mathscr{R}} f2\|_{p(\cdot)} \leq \|\chi_{\mathcal{V}_1} M_{\mathscr{R}} f2\|_{p(\cdot)} + \|\chi_{\mathcal{V}_2} M_{\mathscr{R}} f2\|_{p(\cdot)} + \|\chi_{\mathcal{V}_3} M_{\mathscr{R}} f2\|_{p(\cdot)} \leq C,$$

which, combined with the estimate of $\|M_{\mathscr{R}} f_1\|_{p(\cdot)}$, finishes the proof. □

While $p_- = 1$, the strong boundedness of $M_{\mathscr{R}}$ may fail. In fact, though $p(v) > 1$ for all $v \in \mathcal{V}$, $M_{\mathscr{R}}$ can still be unbounded on $L^{p(\cdot)}$.

Theorem 2. *For $v \in \mathcal{V}$, let*

$$p(v) = 1 + \frac{1}{d(v,o)+1};$$

then, $p \in \mathcal{P}$, $1/p(\cdot) \in \mathcal{H}$, and $M_{\mathscr{R}}$ is unbounded on $L^{p(\cdot)}$.

Proof. The fact that $p \in \mathcal{P}$ is obvious, and the inequality

$$\left|\frac{1}{p(v)} - 1\right| = \frac{1}{d(v,o)+2} < \frac{1}{d(v,o)+1}$$

implies $1/p(\cdot) \in \mathcal{H}$. In order to prove that $M_{\mathscr{R}}$ is unbounded, fix $f(v) = \chi_{\{o\}}(v)$; there holds

$$\|f\|_{p(\cdot)} := \inf\left\{\lambda > 0 : \frac{1}{\lambda^2} \leq 1\right\} = 1,$$

and then we show $M_{\mathscr{R}} f \notin L^{p(\cdot)}$.
Denote v_k as the unique vertex that lies above o with $d(o, v_k) = k$. For integer $k \geq 2$, define

$$S_k = \{v \in \mathcal{V} : v \text{ lies below } v_{2k-1},\ d(v_{2k-1}, v) = k\}.$$

Note that o and all vertices in S_k are contained in the admissible trapezoid

$$R_k = \{v \in \mathcal{V} : v \text{ lies below } v_{2k-1},\ k \leq d(v_{2k-1}, v) < 2k\},$$

and $\mu(R_k) = h(R_k)w(R_k) = km^{2k-1}$; thus, for any $v \in S_k$,

$$M_{\mathscr{R}} f(v) \geq \frac{1}{\mu(R_k)} \int_{R_k} f(v) d\mu(v) = \frac{1}{km^{2k-1}}.$$

Meanwhile, it is easy to calculate that any $v \in S_k$ satisfies $d(v,o) \geq k-1$, the number of vertices in S_k is m^k, and each vertex in S_k has measure m^{k-1}. Therefore,

$$\int_{S_k} (M_{\mathscr{R}} f(v))^{p(v)} d\mu(v) \geq \left(\frac{1}{km^{2k-1}}\right)^{1+\frac{1}{k}} m^{k-1} m^k = \frac{1}{k^{1+\frac{1}{k}} m^{2-\frac{1}{k}}} > \frac{1}{\sqrt[3]{3}m^2 k};$$

thus, for any $\lambda > 0$,

$$\begin{aligned}
\rho_{p(\cdot)}\left(\frac{M_{\mathscr{R}} f}{\lambda}\right) &= \int_{\mathcal{V}} \left(\frac{M_{\mathscr{R}} f(v)}{\lambda}\right)^{p(v)} d\mu(v) \\
&> \min\left\{\frac{1}{\lambda}, \frac{1}{\lambda^2}\right\} \sum_{k=2}^{\infty} \int_{S_k} (M_{\mathscr{R}} f(v))^{p(v)} d\mu(v) \\
&> \frac{1}{\sqrt[3]{3}m^2} \min\left\{\frac{1}{\lambda}, \frac{1}{\lambda^2}\right\} \sum_{k=2}^{\infty} \frac{1}{k} = \infty,
\end{aligned}$$

which completes the proof. □

However, the weak boundedness holds for $p_- = 1$.

Theorem 3. *Let $p \in \mathcal{P}$ with $1/p(\cdot) \in \mathcal{H}$, $f \in L^{p(\cdot)}$; then, for any $t > 0$,*

$$\|t\chi_{\{v \in \mathcal{V} : M_{\mathscr{R}} f(v) > t\}}\|_{p(\cdot)} \leq C\|f\|_{p(\cdot)}.$$

Proof. From the same reason as mentioned in the proof of Theorem 1, we also assume that f is non-negative with the norm $\|f\|_{p(\cdot)} = 1$. Then, we prove $\|t\chi_{\{v \in \mathcal{V} : M_{\mathscr{R}} f(v) > t\}}\|_{p(\cdot)} < C$ for any $t > 0$.
Decompose

$$f = f_1 + f_2 := f\chi_{\{v : f(v) > 1\}} + f\chi_{\{v : f(v) \leq 1\}};$$

then,

$$\{v \in \mathcal{V} : M_{\mathscr{R}} f(v) > t\} \subset \left\{v \in \mathcal{V} : M_{\mathscr{R}} f_1(v) > \frac{t}{2}\right\} \cup \left\{v \in \mathcal{V} : M_{\mathscr{R}} f_2(v) > \frac{t}{2}\right\} =: F_1 \cup F_2,$$

and thus it suffices to show that, for $i = 1, 2$, $\|t\chi_{F_i}\|_{p(\cdot)} < C$.

To estimate $\|t\chi_{F_1}\|_{p(\cdot)}$, let $\lambda_1, \lambda_2, \lambda_3$ be constants that will be determined later. For any $v \in F_1$, there exists $R_v \in \mathscr{R}$ containing v such that

$$\fint_{R_v} f_1(v) d\mu(v) > \frac{t}{2}.$$

From the same reason as in Theorem 1 again, there exists a pairwise disjoint set family $\{R_j\}_{j\in\mathbb{N}} \subset \{R_v\}_{v\in F_1}$ such that, for any $v \in F_1$, there exists $j \in \mathbb{N}$ with $R_v \subset (R_j)^*$. Then, for any $j \in \mathbb{N}$,

$$\fint_{R_j} f_1(v) d\mu(v) > \frac{t}{2}.$$

Define $S_1 = F_1 \cap (R_1)^*$, $S_2 = (F_1 \cap (R_2)^*) \backslash S_1$, $S_3 = (F_1 \cap (R_3)^*) \backslash (S_1 \cup S_2)$, and so on. Therefore, $\{S_j\}$ is a pairwise disjoint family for all $j \in \mathbb{N}$, and

$$F_1 = \bigcup_{j\in\mathbb{N}} S_j.$$

Let $\lambda_1 = 1/2$, $p_j = p_-(R_j)$; then, $p_j \geq p_-$. By the Hölder inequality,

$$\begin{aligned}
\rho_{p(\cdot)}(\lambda_1\lambda_2\lambda_3 t\chi_{F_1}) &= \int_{F_1} (\lambda_1\lambda_2\lambda_3 t)^{p(v)} d\mu(v) \\
&= \sum_{j\in\mathbb{N}} \int_{S_j} \left(\lambda_2\lambda_3 \frac{t}{2}\right)^{p(v)} d\mu(v) \\
&\leq \sum_{j\in\mathbb{N}} \int_{S_j} \left(\lambda_2\lambda_3 \fint_{R_j} f_1(w) d\mu(w)\right)^{p(v)} d\mu(v) \\
&\leq \sum_{j\in\mathbb{N}} \int_{S_j} \left(\lambda_2\lambda_3 \left(\fint_{R_j} f_1(w)^{\frac{p_j}{p_-}} d\mu(w)\right)^{\frac{p_-}{p_j}}\right)^{p(v)} d\mu(v).
\end{aligned}$$

Let $r(\cdot) = 1/p(\cdot)$; then, $r \in LH$, and $r_+(R_j) = 1/p_j$. By Lemma 8, there exists $\lambda_2 \in (0, 1)$ such that, for all $v \in R_j$,

$$\mu(R_j)^{r(v)-r_+(R_j)} \leq \lambda_2^{-\frac{1}{p_-}} \Leftrightarrow \lambda_2\mu(R_j)^{-\frac{p_-}{p_j}} \leq \mu(R_j)^{-\frac{p_-}{p(v)}}.$$

Since $f_1(v) > 1$ or $f_1(v) = 0$,

$$\int_{R_j} f_1(w)^{\frac{p(w)}{p_-}} d\mu(w) \leq \int_{R_j} f_1(w)^{p(w)} d\mu(w) \leq \rho_{p(\cdot)}(f_1) \leq 1;$$

thus, for $\lambda_3 \in (0, 1)$,

$$\begin{aligned}
\rho_{p(\cdot)}(\lambda_1\lambda_2\lambda_3 t\chi_{F_1}) &\leq \sum_{j\in\mathbb{N}} \int_{S_j} \left(\lambda_2\mu(R_j)^{-\frac{p_-}{p_j}} \left(\lambda_3 \int_{R_j} f_1(w)^{\frac{p_j}{p_-}} d\mu(w)\right)^{\frac{p_-}{p_j}}\right)^{p(v)} d\mu(v) \\
&\leq \lambda_3 \sum_{j\in\mathbb{N}} \int_{S_j} \mu(R_j)^{-p_-} \left(\int_{R_j} f_1(w)^{\frac{p(w)}{p_-}} d\mu(w)\right)^{\frac{p_- p(v)}{p_j}} d\mu(v) \\
&\leq \lambda_3 \sum_{j\in\mathbb{N}} \int_{S_j} \mu(R_j)^{-p_-} \left(\int_{R_j} f_1(w)^{\frac{p(w)}{p_-}} d\mu(w)\right)^{p_-} d\mu(v)
\end{aligned}$$

$$
\begin{aligned}
&\leq \lambda_3 \sum_{j\in\mathbb{N}} \int_{S_j} \mu(R_j)^{-1} \left(\int_{R_j} f_1(w)^{p(w)} d\mu(w) \right) d\mu(v) \\
&\leq \lambda_3 \sum_{j\in\mathbb{N}} \frac{\mu(S_j)}{\mu(R_j)} \int_{R_j} f_1(w)^{p(w)} d\mu(w) \\
&\leq 4\lambda_3 \int_{F_1} f_1(w)^{p(w)} d\mu(w) \leq 4\lambda_3.
\end{aligned}
$$

Fix $\lambda_3 = 1/4$. Then, $\rho_{p(\cdot)}(\lambda_1\lambda_2\lambda_3 t\chi_{F_1}) \leq 1$; that is,

$$
\|t\chi_{F_1}\|_{p(\cdot)} \leq \frac{1}{\lambda_1\lambda_2\lambda_3}.
$$

To estimate $\|t\chi_{F_2}\|_{p(\cdot)}$, note that $F_2 = \Phi$ for $t > 2$, and we only consider the case $0 < t \leq 2$. Divide F_2 as the union of sets

$$
F_{21} = \{v \in F_2 : p(v) > p_0\},\ F_{22} = \{v \in F_2 : p(v) = p_0\},\ \ F_{23} = \{v \in F_2 : p(v) < p_0\},
$$

where $p_0 = \lim_{d(v,o)\to\infty} p(v)$. From $1/p(\cdot) \in \mathcal{H}$ and $p_- \geq 1$, this limit exists (or equals ∞) and is not smaller than 1.
By using the same method as that in Theorem 1,

$$
\|f_2\|_{p_0} \leq C.
$$

For $v \in F_{21}$, since $p(v) > p_0$ and $|t\chi_{F_{21}}(v)/2| \leq 1$, by Lemma 5, Lemma 11, and Lemma 12,

$$
\|t\chi_{F_{21}}\|_{p(\cdot)} = 2\left\|\frac{1}{2}t\chi_{F_{21}}\right\|_{p(\cdot)} \leq 2\left\|\frac{1}{2}t\chi_{F_{21}}\right\|_{p_0} \leq \|t\chi_{F_2}\|_{p_0} \leq C\|f_2\|_{p_0} \leq C.
$$

For $v \in F_{22}$, $p(v) = p_0$; thus,

$$
\|t\chi_{F_{22}}\|_{p(\cdot)} = \|t\chi_{F_{22}}\|_{p_0} \leq \|t\chi_{F_2}\|_{p_0} \leq C.
$$

For $v \in F_{23}$, define q as

$$
\frac{1}{p(v)} = \frac{1}{p_0} + \frac{1}{q(v)},
$$

then $q \in \mathcal{P}$, $1/q(\cdot) \in \mathcal{H}$ and $\lim_{d(v,o)\to\infty} q(v) = \infty$. By Lemma 7 and Lemma 10,

$$
\|t\chi_{F_{23}}\|_{p(\cdot)} \leq 3\|t\chi_{F_{23}}\|_{p_0}\|\mathbf{1}\|_{q(\cdot)} \leq C\|t\chi_{F_2}\|_{p_0} \leq C.
$$

Therefore, by Lemma 5,

$$
\|t\chi_{F_2}\|_{p(\cdot)} \leq \|t\chi_{F_{21}}\|_{p(\cdot)} + \|t\chi_{F_{22}}\|_{p(\cdot)} + \|t\chi_{F_{23}}\|_{p(\cdot)} \leq C,
$$

which, combined with the estimate of $\|t\chi_{F_1}\|_{p(\cdot)}$, finishes the proof. □

5. Conclusions

We study the variable Lebesgue space over the weighted homogeneous tree. Under the global Hölder condition for exponent $p(\cdot)$, some properties are obtained. Furthermore, the weak and strong boundedness of the maximal operator $M_{\mathscr{R}}$ on variable Lebesgue space is proven, and a counterexample for clarifying the range of $p(\cdot)$ is provided.

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