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A Uniqueness Theorem for Stability Problems of Functional Equations

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Abstract: In this paper, we present a uniqueness theorem obtained by using direct calculation. This theorem is applicable to stability problems of functional equations whose solutions are monomial or generalized polynomial mappings of degree n. The advantage of this uniqueness theorem is that it simplifies the proof by eliminating the need to repeatedly and cumbersomely prove uniqueness in stability studies.

Keywords: uniqueness; stability; generalized stability; monomial mapping; generalized polynomial mapping of degree n

MSC: 39B82; 39B52

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Citation: Jung, S.-M.; Lee, Y.-H.; Roh, J. A Uniqueness Theorem for Stability Problems of Functional Equations. *Symmetry* **2024**, *16*, 1298. https://doi.org/10.3390/sym16101298

Academic Editor: Serkan Araci

Received: 16 July 2024
Revised: 9 September 2024
Accepted: 18 September 2024
Published: 2 October 2024

1. Introduction

By convention, let $\mathbb{N}$ and $\mathbb{Q}$ be the set of all positive integers and the set of all rational numbers, respectively. Assume that V and W are real vector spaces, and Y is a real normed space. A mapping $A : V \to W$ is said to be additive if it satisfies $A(x+y) = A(x) + A(y)$ for all $x, y \in V$. If A is an additive mapping, we can easily show that $A(rx) = rA(x)$ for all $x \in V$ and all $r \in \mathbb{Q}$.

A mapping $A_n : V^n \to W$ is called n-additive if it is additive in each of its variables. A mapping $A_n : V^n \to W$ is said to be symmetric if $A_n(x_1, x_2, \ldots, x_n) = A_n(y_1, y_2, \ldots, y_n)$, whenever $(y_1, y_2, \ldots, y_n)$ is a permutation of $(x_1, x_2, \ldots, x_n)$. For every n-additive symmetric mapping $A_n : V^n \to W$, we set $A^n(x) = A_n(x, x, \ldots, x)$ for all $x \in V$. Then, we obtain $A^n(rx) = r^n A^n(x)$ whenever $x \in V$ and $r \in \mathbb{Q}$. Such a mapping $A^n(x)$ where $A^n \not\equiv 0$ is called a *monomial mapping of degree n*, or an *n-monomial mapping*. Any mapping $p : V \to W$ is said to be a *generalized polynomial mapping of degree n*, provided that there are a constant mapping $A^0(x) = A^0 \in W$ and i-monomial mappings $A^i : V^i \to W$, $i \in \{1, 2, \ldots, n\}$, such that $p(x) = \sum_{i=0}^{n} A^i(x)$ for all $x \in V$, where $A^n \not\equiv 0$. For details on the terminologies and definitions used above, one may refer to [1].

The purpose of this paper is to prove a theorem that solves the uniqueness problem that arises when studying the (generalized) stability of some functional equations, whose solutions are monomial mappings or generalized polynomial mappings of degree n.

The concept of stability of a functional equation occurs when we replace a functional equation with an inequality that acts as a perturbation of the equation. In 1940 (refer to [2]), the stability problem of the functional equation was raised by Ulam. This problem has attracted the attention of many researchers. In 1941 (refer to [3]), the affirmative answer to this question was given by Hyers. In 1950 (refer to [4]), Aoki generalized Hyers' theorem for additive mappings. Also, in [5], Hyers' result was generalized by Th. M. Rassias for linear mappings by an unbounded Cauchy difference. Moreover, in 1994, a further generalization of Th. M. Rassias' theorem was obtained by Găvruţa (see [6]). After then, the stability problem of various functional equations has been extensively investigated by many mathematicians. For works of the stability problem of a functional equation whose solution is a monomial mapping, one can refer to [7–12]. For recent works of the stability problem of functional equations whose solutions are generalized polynomial mappings of degree 4 or 5 or 6 or 7 or 8 or 9 or 10, one can refer to [13–23].

Our results in this paper can be applicable to generalized stability problems of functional equations whose solutions are monomial or generalized polynomial mappings of degree n. Our main theorem (Theorem 5 in Section 3) states the following:

For any fixed $n \in \mathbb{N}$, let $a, \alpha_1, \alpha_2, \ldots, \alpha_n$ be nonzero real constants such that $a > 1$ and $\alpha_1 < \alpha_2 < \cdots < \alpha_n$. For a given mapping $f : V \to Y$, if there exist mappings $F_1, F_2, \ldots, F_n : V \to Y$ and a function $\phi : V \setminus \{0\} \to [0, \infty)$ that satisfy

$$\left\| f(x) - \sum_{k=1}^{n} F_k(x) \right\| \le \sum_{i=0}^{\infty} \frac{1}{a^{\alpha_1 i}} \phi(a^i x) < \infty \ \text{ or}$$
$$\left\| f(x) - \sum_{k=1}^{n} F_k(x) \right\| \le \sum_{i=0}^{\infty} \frac{1}{a^{\alpha_{\ell+1} i}} \phi(a^i x) + \sum_{i=0}^{\infty} a^{\alpha_\ell i} \phi\left(\frac{1}{a^i} x\right) < \infty \ \text{ or}$$
$$\left\| f(x) - \sum_{k=1}^{n} F_k(x) \right\| \le \sum_{i=0}^{\infty} a^{\alpha_n i} \phi\left(\frac{1}{a^i} x\right) < \infty$$

for some integer $1 \le \ell < n$, where every F_k has the property

$$F_k(ax) = a^{\alpha_k} F_k(x) \quad (\text{for all } x \in V),$$

then the mappings $F_1, F_2, \ldots, F_n$ are uniquely determined.

The above main theorem of this paper is considered to be a further extension and generalization of existing uniqueness theorems. For previous uniqueness theorems related to the stability of functional equations, one can refer to [24,25].

2. Preliminaries

Throughout the paper, unless otherwise stated, we assume that V is a real vector space, Y is a real Banach space, and $f : V \to Y$ is a given arbitrary mapping.

In the following theorem, let Φ be a function that satisfies similar conditions with Găvruţa condition (refer to [6]). Then, we prove that, for any given mapping f, if there is a mapping F (close to f) with some additional properties, then the mapping F is uniquely determined.

Theorem 1. *For any fixed integer $n > 0$, let $a, \alpha_1, \alpha_2, \ldots, \alpha_n$ be nonzero real constants, such that $a > 1$ and $\alpha_1 < \alpha_2 < \cdots < \alpha_n$, and let $\Phi : V \setminus \{0\} \to [0, \infty)$ be a function satisfying one of the following conditions:*

$$\begin{cases} \lim_{i\to\infty} \frac{1}{a^{\alpha_1 i}} \Phi(a^i x) = 0 & (\text{for all } x \in V \setminus \{0\}), \\ \lim_{i\to\infty} a^{\alpha_1 i} \Phi\left(\frac{1}{a^i} x\right) = \lim_{i\to\infty} \frac{1}{a^{\alpha_2 i}} \Phi(a^i x) = 0 & (\text{for all } x \in V \setminus \{0\}), \\ \qquad \vdots & \qquad \vdots \\ \lim_{i\to\infty} a^{\alpha_{n-1} i} \Phi\left(\frac{1}{a^i} x\right) = \lim_{i\to\infty} \frac{1}{a^{\alpha_n i}} \Phi(a^i x) = 0 & (\text{for all } x \in V \setminus \{0\}), \\ \lim_{i\to\infty} a^{\alpha_n i} \Phi\left(\frac{1}{a^i} x\right) = 0 & (\text{for all } x \in V \setminus \{0\}). \end{cases} \tag{1}$$

For a given mapping $f : V \to Y$, if there exist mappings $F_1, F_2, \ldots, F_n : V \to Y$ such that

$$\left\| f(x) - \sum_{k=1}^{n} F_k(x) \right\| \leq \Phi(x) \tag{2}$$

for all $x \in V \setminus \{0\}$, where every F_k satisfies

$$F_k(ax) = a^{\alpha_k} F_k(x) \quad (\textit{for all } x \in V), \tag{3}$$

then the mappings $F_1, F_2, \ldots, F_n$ are uniquely determined.

Proof. We will prove this theorem by applying mathematical induction. First, we will prove our claim for $n = 1$. In this case, only the first and last conditions of (1) are valid. Let $\Phi : V \setminus \{0\} \to [0, \infty)$ be a function that satisfies one of the following conditions:

$$\begin{cases} \lim_{i\to\infty} \dfrac{1}{a^{\alpha_1 i}} \Phi\big(a^i x\big) = 0 & (\text{for all } x \in V \setminus \{0\}), \\ \lim_{i\to\infty} a^{\alpha_1 i} \Phi\left(\dfrac{1}{a^i} x\right) = 0 & (\text{for all } x \in V \setminus \{0\}), \end{cases} \tag{4}$$

and let $f : V \to Y$ be an arbitrarily given mapping. Assume that F_1, F_1' are mappings such that $\|f(x) - F_1(x)\| \leq \Phi(x)$, $\|f(x) - F_1'(x)\| \leq \Phi(x)$, $F_1(ax) = a^{\alpha_1} F_1(x)$, and $F_1'(ax) = a^{\alpha_1} F_1'(x)$ for all $x \in V$.

If Φ satisfies the first condition in (4), then we have

$$\begin{aligned} \|F_1(x) - F_1'(x)\| &= \lim_{i\to\infty} \left\| \frac{1}{a^{\alpha_1 i}} F_1\big(a^i x\big) - \frac{1}{a^{\alpha_1 i}} F_1'\big(a^i x\big) \right\| \\ &\leq \lim_{i\to\infty} \left\| \frac{1}{a^{\alpha_1 i}} F_1\big(a^i x\big) - \frac{1}{a^{\alpha_1 i}} f\big(a^i x\big) \right\| \\ &\quad + \lim_{i\to\infty} \left\| \frac{1}{a^{\alpha_1 i}} f\big(a^i x\big) - \frac{1}{a^{\alpha_1 i}} F_1'\big(a^i x\big) \right\| \\ &\leq 2 \lim_{i\to\infty} \frac{1}{a^{\alpha_1 i}} \Phi\big(a^i x\big) \\ &= 0 \end{aligned}$$

for all $x \in V \setminus \{0\}$.

We now assume that Φ satisfies the second condition in (4). Then, we obtain

$$\begin{aligned} \|F_1(x) - F_1'(x)\| &= \lim_{i\to\infty} \left\| a^{\alpha_1 i} F_1\left(\frac{1}{a^i} x\right) - a^{\alpha_1 i} F_1'\left(\frac{1}{a^i} x\right) \right\| \\ &\leq \lim_{i\to\infty} \left\| a^{\alpha_1 i} F_1\left(\frac{1}{a^i} x\right) - a^{\alpha_1 i} f\left(\frac{1}{a^i} x\right) \right\| \\ &\quad + \lim_{i\to\infty} \left\| a^{\alpha_1 i} f\left(\frac{1}{a^i} x\right) - a^{\alpha_1 i} F_1'\left(\frac{1}{a^i} x\right) \right\| \\ &\leq 2 \lim_{i\to\infty} a^{\alpha_1 i} \Phi\left(\frac{1}{a^i} x\right) \\ &= 0 \end{aligned}$$

for all $x \in V \setminus \{0\}$. Since $F_1(0) = F_1'(0) = 0$, it holds that $F_1(x) = F_1'(x)$ for all $x \in V$ (for both cases).

Assume that our assertion holds for $n = m - 1$, where $m > 1$ is some integer. Let $\Phi : V \setminus \{0\} \to [0, \infty)$ be a function satisfying one of conditions in (1) for $n = m$, and let $f : V \to Y$ be an arbitrary mapping. Assume that $f, F_1, F_2, \ldots, F_m : V \to Y$ are mappings satisfying (2) and (3) for $n = m$.

Let $g, G_1, G_2, \ldots, G_m : V \to Y$ be the mappings defined by $g(x) := a^{\alpha_m} f(x) - f(ax)$, $G_1(x) := (a^{\alpha_m} - a^{\alpha_1})F_1(x)$, …, $G_{m-1}(x) := (a^{\alpha_m} - a^{\alpha_{m-1}})F_{m-1}(x)$, and let $\Psi : V \setminus \{0\} \to [0, \infty)$ be the function defined by $\Psi(x) := a^{\alpha_m}\Phi(x) + \Phi(ax)$. Then, Ψ satisfies one of the conditions in (1) for $n = m$. It is easy to show that $G_1, G_2, \ldots, G_{m-1}$ satisfy

$$G_k(ax) = a^{\alpha_k} G_k(x)$$

for all $x \in V$ and $k \in \{1, 2, \ldots, m-1\}$, and

$$\begin{aligned}
\left\| g(x) - \sum_{k=1}^{m-1} G_k(x) \right\| &= \left\| a^{\alpha_m} f(x) - f(ax) - \left(a^{\alpha_m} \sum_{k=1}^{m-1} F_k(x) - \sum_{k=1}^{m-1} F_k(ax) \right) \right\| \\
&= \left\| a^{\alpha_m} \left(f(x) - \sum_{k=1}^{m} F_k(x) \right) - \left(f(ax) - \sum_{k=1}^{m} F_k(ax) \right) \right\| \\
&\le a^{\alpha_m} \left\| f(x) - \sum_{k=1}^{m} F_k(x) \right\| + \left\| f(ax) - \sum_{k=1}^{m} F_k(ax) \right\| \\
&\le a^{\alpha_m}\Phi(x) + \Phi(ax) \\
&= \Psi(x)
\end{aligned}$$

for all $x \in V \setminus \{0\}$.

Since Ψ satisfies one of conditions in (1) for $n = m$ and g, $G_1, \ldots, G_{m-1}$, Ψ satisfy (2) and (3) for $n = m - 1$; by the induction assumption, $G_1, \ldots, G_{m-1}$ are uniquely determined. This implies that if Φ satisfies one of conditions in (1) for $n = m$ and f, and $F_1, F_2, \ldots, F_m$, Φ satisfy (2) and (3) for $n = m$, then $F_1, F_2, \ldots, F_{m-1}$ are uniquely determined, because $F_k(x) = \frac{1}{a^{\alpha_m} - a^{\alpha_k}} G_k(x)$ for each $k \in \{1, 2, \ldots, m-1\}$. In other words, if Φ satisfies one of conditions in (1) for $n = m$ and f, $F_1, F_2, \ldots, F_m$, Φ satisfies (2) and (3) for $n = m$ and simultaneously, if f, $F_1', F_2', \ldots, F_k'$, Φ satisfy (2) and (3) for $n = m$, then $F_k = F_k'$ for every $k \in \{1, 2, \ldots, m-1\}$. Moreover, we have

$$\left\| F_m(x) - F_m'(x) \right\| = \left\| f(x) - \sum_{k=1}^{m} F_k(x) - \left(f(x) - \sum_{k=1}^{m} F_k'(x) \right) \right\| \le 2\Phi(x) \tag{5}$$

for all $x \in V \setminus \{0\}$.

Now, we use (3) and (5) to prove that $F_m = F_m'$. If $\Phi : V \setminus \{0\} \to [0, \infty)$ satisfies one of the conditions other than the last one in (1) for $n = m$, namely the jth condition, then

$$\begin{aligned}
\left\| F_m(x) - F_m'(x) \right\| &= \lim_{i \to \infty} \left\| \frac{1}{a^{\alpha_m i}} \left(F_m(a^i x) - F_m'(a^i x) \right) \right\| \\
&\le 2 \lim_{i \to \infty} \frac{1}{a^{\alpha_m i}} \Phi(a^i x) \\
&\le 2 \lim_{i \to \infty} \frac{1}{a^{\alpha_j i}} \Phi(a^i x) \\
&= 0
\end{aligned}$$

for all $x \in V \setminus \{0\}$ and for some $j \in \{1, 2, \ldots, m\}$, since $\frac{1}{a^{\alpha_m i}} \le \frac{1}{a^{\alpha_j i}}$.

We now assume that Φ satisfies the last condition in (1) for $n = m$. It then follows from (5) that

$$\begin{aligned}\|F_m(x) - F'_m(x)\| &= \lim_{i\to\infty} a^{\alpha_m i}\left\|F_m\left(\frac{1}{a^i}x\right) - F'_m\left(\frac{1}{a^i}x\right)\right\| \\ &\leq 2\lim_{i\to\infty} a^{\alpha_m i}\Phi\left(\frac{1}{a^i}x\right) \\ &= 0\end{aligned}$$

for all $x \in V \setminus \{0\}$.

Finally, since $F_m(0) = F'_m(0) = 0$, it holds that $F_m(x) = F'_m(x)$ for all $x \in V$. With the inductive conclusion, we complete the proof of our assertion. □

In the following two corollaries, we assume that V is a real vector space and Y is a real normed space.

Corollary 1. *Let $a, \alpha_1, \alpha_2, \ldots, \alpha_n$ be nonzero real constants, such that $a > 1$ and $\alpha_1 < \alpha_2 < \cdots < \alpha_n$, and let $\Phi : V \setminus \{0\} \to [0, \infty)$ be a function that satisfies one of the conditions in (1). For a given mapping $f : V \to Y$, if there are mappings $F, F_1, F_2, \ldots, F_n : V \to Y$ such that*

$$\|f(x) - F(x)\| \leq \Phi(x) \quad (\text{for all } x \in V \setminus \{0\}), \tag{6}$$

where $F(x) = \sum_{k=1}^{n} F_k(x)$ and $F_k(ax) = a^{\alpha_k}F_k(x)$ for all $x \in V$, then the mappings $F, F_1, F_2, \ldots, F_n$ are uniquely determined.

Corollary 2. *Let a, p, $\alpha_1, \alpha_2, \ldots, \alpha_n$, and p be nonzero real constants, such that $a > 1$, $p \notin \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ and $\alpha_1 < \alpha_2 < \cdots < \alpha_n$. For a given mapping $f : V \to Y$, if there are mappings $F, F_1, F_2, \ldots, F_n : V \to Y$ and a constant $K > 0$ such that*

$$\|f(x) - F(x)\| \leq K\|x\|^p \quad (\text{for all } x \in V \setminus \{0\}), \tag{7}$$

where $F(x) = \sum_{k=1}^{n} F_k(x)$ and $F_k(ax) = a^{\alpha_k}F_k(x)$ for all $x \in V$, then the mappings $F, F_1, F_2, \ldots, F_n$ are uniquely determined.

Proof. If we put $\Phi(x) := K\|x\|^p$ for all $x \in V \setminus \{0\}$, then Φ satisfies one of the conditions of (1) and $f, F_1, F_2, \ldots, F_n$ satisfy conditions (2) and (3), given in Theorem 1. Therefore, $F_1, F_2, \ldots, F_n$ are the unique mappings that satisfy conditions (3) and (7). □

3. Main Theorem

In this section, we assume that V is a real vector space and Y is a real normed space.

In the following three lemmas, we will introduce special conditions that satisfy the conditions of (1).

Lemma 1. *Let a and α be nonzero real constants with $a > 1$. If a function $\phi : V \setminus \{0\} \to [0, \infty)$ satisfies the condition*

$$\Phi(x) := \sum_{i=0}^{\infty} \frac{1}{a^{\alpha i}}\phi(a^i x) < \infty$$

for all $x \in V \setminus \{0\}$, then the function $\Phi : V \setminus \{0\} \to [0, \infty)$ satisfies

$$\lim_{m\to\infty} \frac{1}{a^{\alpha m}}\Phi(a^m x) = 0$$

for all $x \in V \setminus \{0\}$.

Proof. If ϕ satisfies $\Phi(x) = \sum_{i=0}^{\infty} \frac{1}{a^{\alpha i}} \phi(a^i x) < \infty$ for all $x \in V \setminus \{0\}$, then we have

$$\lim_{m\to\infty} \frac{1}{a^{\alpha m}} \Phi(a^m x) = \lim_{m\to\infty} \sum_{i=0}^{\infty} \frac{1}{a^{\alpha(m+i)}} \phi\big(a^{m+i} x\big) = \lim_{m\to\infty} \sum_{i=m}^{\infty} \frac{1}{a^{\alpha i}} \phi\big(a^i x\big) = 0$$

for all $x \in V \setminus \{0\}$. □

In the following lemma, we introduce some special conditions that satisfy one of the second to nth conditions of (1).

Lemma 2. *For any fixed integer $n > 0$, let ℓ be an integer with $1 \le \ell < n$. Assume that a, α_ℓ, and $\alpha_{\ell+1}$ are nonzero real constants, such that $a > 1$ and $\alpha_\ell < \alpha_{\ell+1}$. If a function $\phi : V \setminus \{0\} \to [0, \infty)$ satisfies the following conditions*

$$\Phi_\ell(x) := \sum_{i=0}^{\infty} \frac{1}{a^{\alpha_{\ell+1} i}} \phi\big(a^i x\big) < \infty \quad \textit{and} \quad \Phi'_\ell(x) := \sum_{i=0}^{\infty} a^{\alpha_\ell i} \phi\left(\frac{1}{a^i} x\right) < \infty$$

for all $x \in V \setminus \{0\}$, then the functions $\Phi_\ell, \Phi'_\ell : V \setminus \{0\} \to [0, \infty)$ have the following properties:

$$\lim_{m\to\infty} \frac{1}{a^{\alpha_{\ell+1} m}} \Phi_\ell(a^m x) = \lim_{m\to\infty} a^{\alpha_\ell m} \Phi_\ell\left(\frac{1}{a^m} x\right) = 0$$

and

$$\lim_{m\to\infty} \frac{1}{a^{\alpha_{\ell+1} m}} \Phi'_\ell(a^m x) = \lim_{m\to\infty} a^{\alpha_\ell m} \Phi'_\ell\left(\frac{1}{a^m} x\right) = 0$$

for all $x \in V \setminus \{0\}$.

Proof. If ϕ satisfies $\Phi_\ell(x) = \sum_{i=0}^{\infty} \frac{1}{a^{\alpha_{\ell+1} i}} \phi(a^i x) < \infty$ and $\Phi'_\ell(x) = \sum_{i=0}^{\infty} a^{\alpha_\ell i} \phi(\frac{1}{a^i} x) < \infty$ for all $x \in V \setminus \{0\}$, then we have

$$\begin{aligned}
&\lim_{m\to\infty} \frac{1}{a^{2\alpha_{\ell+1} m}} \Phi'_\ell\big(a^{2m} x\big) \\
&\quad = \lim_{m\to\infty} \frac{1}{a^{2\alpha_{\ell+1} m - 2\alpha_\ell m}} \sum_{i=-2m}^{\infty} a^{\alpha_\ell i} \phi\left(\frac{1}{a^i} x\right) \\
&\quad = \lim_{m\to\infty} \frac{1}{a^{2(\alpha_{\ell+1}-\alpha_\ell)m}} \sum_{i=1}^{2m} \frac{1}{a^{\alpha_\ell i}} \phi\big(a^i x\big) + \lim_{m\to\infty} \frac{1}{a^{2(\alpha_{\ell+1}-\alpha_\ell)m}} \sum_{i=0}^{\infty} a^{\alpha_\ell i} \phi\left(\frac{1}{a^i} x\right) \\
&\quad = \lim_{m\to\infty} \frac{1}{a^{2(\alpha_{\ell+1}-\alpha_\ell)m}} \sum_{i=1}^{2m} \frac{1}{a^{\alpha_\ell i}} \phi\big(a^i x\big) + \lim_{m\to\infty} \frac{1}{a^{2(\alpha_{\ell+1}-\alpha_\ell)m}} \Phi'_\ell(x) \\
&\quad = \lim_{m\to\infty} \frac{1}{a^{(\alpha_{\ell+1}-\alpha_\ell)m}} \sum_{i=1}^{m-1} \frac{1}{a^{\alpha_\ell i + (\alpha_{\ell+1}-\alpha_\ell)m}} \phi\big(a^i x\big) \\
&\qquad + \lim_{m\to\infty} \sum_{i=m}^{2m} \frac{1}{a^{2\alpha_{\ell+1} m - \alpha_\ell(2m-i)}} \phi\big(a^i x\big) \\
&\quad = \lim_{m\to\infty} \frac{1}{a^{(\alpha_{\ell+1}-\alpha_\ell)m}} \sum_{i=1}^{m-1} \frac{1}{a^{\alpha_{\ell+1} i + (\alpha_{\ell+1}-\alpha_\ell)(m-i)}} \phi\big(a^i x\big) \\
&\qquad + \lim_{m\to\infty} \sum_{i=m}^{2m} \frac{1}{a^{\alpha_{\ell+1} i + (\alpha_{\ell+1}-\alpha_\ell)(2m-i)}} \phi\big(a^i x\big) \\
&\quad \le \lim_{m\to\infty} \frac{1}{a^{(\alpha_{\ell+1}-\alpha_\ell)m}} \sum_{i=0}^{m} \frac{1}{a^{\alpha_{\ell+1} i}} \phi\big(a^i x\big) + \lim_{m\to\infty} \sum_{i=m}^{2m} \frac{1}{a^{\alpha_{\ell+1} i}} \phi\big(a^i x\big) \\
&\quad = 0
\end{aligned}$$

and

$$\lim_{m\to\infty} a^{\alpha_\ell m}\Phi'_\ell\left(\frac{1}{a^m}x\right) = \lim_{m\to\infty}\sum_{i=0}^{\infty} a^{\alpha_\ell(i+m)}\phi\left(\frac{1}{a^{i+m}}x\right) = \lim_{m\to\infty}\sum_{i=m}^{\infty} a^{\alpha_\ell i}\phi\left(\frac{1}{a^i}x\right) = 0.$$

Moreover, we also obtain

$$\begin{aligned}\lim_{m\to\infty}\frac{1}{a^{\alpha_{\ell+1}m}}\Phi_\ell(a^m x) &= \lim_{m\to\infty}\sum_{i=0}^{\infty}\frac{1}{a^{\alpha_{\ell+1}(m+i)}}\phi(a^{m+i}x)\\ &= \lim_{m\to\infty}\sum_{i=m}^{\infty}\frac{1}{a^{\alpha_{\ell+1}i}}\phi(a^i x)\\ &= 0\end{aligned}$$

and

$$\begin{aligned}&\lim_{m\to\infty} a^{2\alpha_\ell m}\Phi_\ell\left(\frac{1}{a^{2m}}x\right)\\ &= \lim_{m\to\infty}\frac{1}{a^{2\alpha_{\ell+1}m-2\alpha_\ell m}}\sum_{i=-2m}^{-1}\frac{1}{a^{\alpha_{\ell+1}i}}\phi(a^i x)\\ &\quad+\lim_{m\to\infty}\frac{1}{a^{2\alpha_{\ell+1}m-2\alpha_\ell m}}\sum_{i=0}^{\infty}\frac{1}{a^{\alpha_{\ell+1}i}}\phi(a^i x)\\ &= \lim_{m\to\infty}\frac{1}{a^{2\alpha_{\ell+1}m-2\alpha_\ell m}}\sum_{i=1}^{2m} a^{\alpha_{\ell+1}i}\phi\left(\frac{1}{a^i}x\right)+\lim_{m\to\infty}\frac{1}{a^{2\alpha_{\ell+1}m-2\alpha_\ell m}}\Phi_\ell(x)\\ &= \lim_{m\to\infty}\frac{1}{a^{\alpha_{\ell+1}m-\alpha_\ell m}}\sum_{i=1}^{m-1}\frac{a^{(\alpha_{\ell+1}-\alpha_\ell)i}}{a^{(\alpha_{\ell+1}-\alpha_\ell)m}}a^{\alpha_\ell i}\phi\left(\frac{1}{a^i}x\right)\\ &\quad+\lim_{m\to\infty}\sum_{i=m}^{2m}\frac{a^{(\alpha_{\ell+1}-\alpha_\ell)i}}{a^{2(\alpha_{\ell+1}-\alpha_\ell)m}}a^{\alpha_\ell i}\phi\left(\frac{1}{a^i}x\right)\\ &\le \lim_{m\to\infty}\frac{1}{a^{\alpha_{\ell+1}m-\alpha_\ell m}}\sum_{i=1}^{\infty}a^{\alpha_\ell i}\phi\left(\frac{1}{a^i}x\right)+\lim_{m\to\infty}\sum_{i=m}^{2m}a^{\alpha_\ell i}\phi\left(\frac{1}{a^i}x\right)\\ &= 0\end{aligned}$$

for all $x \in V\setminus\{0\}$.

Since $\lim_{m\to\infty}\frac{1}{a^{2\alpha_{\ell+1}m}}\Phi'_\ell(a^{2m}x) = 0$ and $\lim_{m\to\infty} a^{2\alpha_\ell m}\Phi_\ell(\frac{1}{a^{2m}}x) = 0$ for all $x \in V\setminus\{0\}$, we have

$$\lim_{m\to\infty}\frac{1}{a^{\alpha_{\ell+1}(2m+1)}}\Phi'_\ell(a^{2m+1}x) = \frac{1}{a^{\alpha_{\ell+1}}}\lim_{m\to\infty}\frac{1}{a^{2\alpha_{\ell+1}m}}\Phi'_\ell(a^{2m}ax) = 0$$

and

$$\lim_{m\to\infty} a^{\alpha_\ell(2m+1)}\Phi_\ell\left(\frac{1}{a^{2m+1}}x\right) = a^{\alpha_\ell}\lim_{m\to\infty} a^{2\alpha_\ell m}\Phi_\ell\left(\frac{1}{a^{2m}}\frac{1}{a}x\right) = 0$$

for all $x \in V\setminus\{0\}$. From the above two equalities, we conclude that

$$\lim_{m\to\infty}\frac{1}{a^{\alpha_{\ell+1}m}}\Phi'_\ell(a^m x) = 0 \quad\text{and}\quad \lim_{m\to\infty} a^{\alpha_\ell m}\Phi_\ell\left(\frac{1}{a^m}x\right) = 0$$

for all $x \in V\setminus\{0\}$. □

In the following lemma, we will introduce a special condition that satisfies the last condition of (1).

Lemma 3. *Let a and α be nonzero real constants with $a > 1$. If a function $\phi : V \setminus \{0\} \to [0, \infty)$ satisfies the following condition*

$$\Phi(x) := \sum_{i=0}^{\infty} a^{\alpha i} \phi\left(\frac{1}{a^i} x\right) < \infty$$

for all $x \in V \setminus \{0\}$, then the function $\Phi : V \setminus \{0\} \to [0, \infty)$ satisfies

$$\lim_{m\to\infty} a^{\alpha m} \Phi\left(\frac{1}{a^m} x\right) = 0$$

for all $x \in V \setminus \{0\}$.

Proof. If ϕ satisfies $\Phi(x) = \sum\limits_{i=0}^{\infty} a^{\alpha i} \phi(\frac{1}{a^i} x) < \infty$ for all $x \in V \setminus \{0\}$, then we have

$$\begin{aligned} \lim_{m\to\infty} a^{\alpha m} \Phi\left(\frac{1}{a^m} x\right) &= \lim_{m\to\infty} \sum_{i=0}^{\infty} a^{\alpha m + \alpha i} \phi\left(\frac{1}{a^{m+i}} x\right) \\ &= \lim_{m\to\infty} \sum_{i=m}^{\infty} a^{\alpha i} \phi\left(\frac{1}{a^i} x\right) \\ &= 0 \end{aligned}$$

for all $x \in V \setminus \{0\}$. □

In the following theorem, we present practical ways to use Theorem 1 together with the three lemmas mentioned above. First, we combine Theorem 1 and Lemma 1 to prove the following theorem.

Theorem 2. *Assume that V is a real vector space and Y is a real normed space. For every fixed $n \in \mathbb{N}$, let $a, \alpha_1, \alpha_2, \ldots, \alpha_n$ be nonzero real constants, such that $a > 1$ and $\alpha_1 < \alpha_2 < \cdots < \alpha_n$. Assume that a function $\phi : V \setminus \{0\} \to [0, \infty)$ satisfies the following condition*

$$\Phi(x) := \sum_{i=0}^{\infty} \frac{1}{a^{\alpha_1 i}} \phi(a^i x) < \infty$$

for all $x \in V \setminus \{0\}$. For any given mapping $f : V \to Y$, if there exist mappings $F_1, F_2, \ldots$, $F_n : V \to Y$ satisfying the inequality

$$\left\| f(x) - \sum_{k=1}^{n} F_k(x) \right\| \le \Phi(x) \tag{8}$$

for all $x \in V \setminus \{0\}$, where each F_i satisfies (3) for all $x \in V$, then the mappings $F_1, F_2, \ldots, F_n$ are uniquely determined.

Proof. Since ϕ satisfies $\Phi(x) = \sum\limits_{i=0}^{\infty} \frac{1}{a^{\alpha_1 i}} \phi(a^i x) < \infty$ for all $x \in V \setminus \{0\}$, it follows from Lemma 1 that

$$\lim_{m\to\infty} \frac{1}{a^{\alpha_1 m}} \Phi(a^m x) = 0$$

for all $x \in V \setminus \{0\}$. In view of Theorem 1 with the first condition of (1), we conclude that $F_1, F_2, \ldots, F_n$ are the unique mappings satisfying (3) and (8). □

Corollary 3. *Let V, Y, n, a, $\alpha_1, \alpha_2, \ldots, \alpha_n$, f, ϕ, and Φ be given under the same conditions as in Theorem 2. Assume that a mapping $F : V \to Y$ satisfies the following inequality*

$$\|f(x) - F(x)\| \leq \Phi(x) \tag{9}$$

for all $x \in V \setminus \{0\}$. If $F : V \to Y$ can be expressed as $F(x) = \sum_{k=1}^{n} F_k(x)$ and every F_k has the property (3), then F is the unique mapping that satisfies (9).

Assume that V is a real vector space and Y is a real normed space. Now, we combine Theorem 1 and Lemma 2 to prove the following theorem.

Theorem 3. *For every fixed integer $n > 0$, let ℓ be an integer with $1 \leq \ell < n$. Assume that $a, \alpha_1, \alpha_2, \ldots, \alpha_n$ are nonzero real constants such that $a > 1$ and $\alpha_1 < \alpha_2 < \cdots < \alpha_n$, and also suppose $f : V \to Y$ is an arbitrary mapping. Assume moreover that a function $\phi : V \setminus \{0\} \to [0, \infty)$ satisfies the conditions*

$$\Phi_\ell(x) := \sum_{i=0}^{\infty} \frac{1}{a^{\alpha_{\ell+1} i}} \phi(a^i x) < \infty \quad \text{and} \quad \Phi'_\ell(x) := \sum_{i=0}^{\infty} a^{\alpha_\ell i} \phi\left(\frac{1}{a^i} x\right) < \infty$$

for all $x \in V \setminus \{0\}$. If mappings $F_1, F_2, \ldots, F_n : V \to Y$ satisfy the inequality

$$\left\| f(x) - \sum_{k=1}^{n} F_k(x) \right\| \leq \Phi_\ell(x) + \Phi'_\ell(x) \tag{10}$$

for all $x \in V \setminus \{0\}$, where each F_k satisfies (3) for all $x \in V$, then the mappings $F_1, F_2, \ldots, F_n$ are uniquely determined.

Proof. If ϕ satisfies the conditions

$$\Phi_\ell(x) = \sum_{i=0}^{\infty} \frac{1}{a^{\alpha_{\ell+1} i}} \phi(a^i x) < \infty \quad \text{and} \quad \Phi'_\ell(x) = \sum_{i=0}^{\infty} a^{\alpha_\ell i} \phi\left(\frac{1}{a^i} x\right) < \infty$$

for all $x \in V \setminus \{0\}$, it then follows from Lemma 2 that

$$\lim_{m \to \infty} \frac{1}{a^{\alpha_{\ell+1} m}} \Phi_\ell(a^m x) = \lim_{m \to \infty} a^{\alpha_\ell m} \Phi_\ell\left(\frac{1}{a^m} x\right) = 0,$$
$$\lim_{m \to \infty} \frac{1}{a^{\alpha_{\ell+1} m}} \Phi'_\ell(a^m x) = \lim_{m \to \infty} a^{\alpha_\ell m} \Phi'_\ell\left(\frac{1}{a^m} x\right) = 0$$

for all $x \in V \setminus \{0\}$. Now, we apply Theorem 1 with $\Phi_\ell(x) + \Phi'_\ell(x)$ in place of $\Phi(x)$ to conclude that $F_1, F_2, \ldots, F_n$ are the unique mappings which satisfy both (3) and (10). □

Corollary 4. *Let V, Y, n, a, $\alpha_1, \alpha_2, \ldots, \alpha_n$, f, ϕ, Φ, and Φ' be given under the same conditions as in Theorem 3. If there is a mapping $F : V \to Y$ that satisfies the following inequality*

$$\|f(x) - F(x)\| \leq \Phi_\ell(x) + \Phi'_\ell(x) \quad (\text{for all } x \in V \setminus \{0\}), \tag{11}$$

where $F(x) = \sum_{k=1}^{n} F_k(x)$ and every F_k has the property (3), then the mappings $F, F_1, F_2, \ldots, F_n$ are uniquely determined.

As we often did before, we set V to be a real vector space and Y to be a real normed space. Finally, we combine Theorem 1 and Lemma 3 to prove the following theorem.

Theorem 4. *For any fixed $n \in \mathbb{N}$, let $a, \alpha_1, \alpha_2, \ldots, \alpha_n$ be nonzero real constants such that $a > 1$ and $\alpha_1 < \alpha_2 < \cdots < \alpha_n$. Assume that a function $\phi : V \setminus \{0\} \to [0, \infty)$ satisfies the following condition*

$$\Phi(x) := \sum_{i=0}^{\infty} a^{\alpha_n i} \phi\left(\frac{1}{a^i} x\right) < \infty$$

for all $x \in V \setminus \{0\}$. For a given mapping $f : V \to Y$, if mappings $F_1, F_2, \ldots, F_n : V \to Y$ satisfy the inequality

$$\left\| f(x) - \sum_{k=1}^{n} F_k(x) \right\| \leq \Phi(x) \quad (\textit{for all } x \in V \setminus \{0\}), \tag{12}$$

where every F_k satisfies (3) for all $x \in V$, then the mappings $F_1, F_2, \ldots, F_n$ are uniquely determined.

Proof. If ϕ satisfies the condition, $\Phi(x) = \sum_{i=0}^{\infty} a^{\alpha_n i} \phi(\frac{1}{a^i} x) < \infty$, then we may use Lemma 3 to show that

$$\lim_{m \to \infty} a^{\alpha_n m} \Phi\left(\frac{1}{a^m} x\right) = 0$$

for all $x \in V \setminus \{0\}$. Finally, we use Theorem 1 with the last condition of (1) to conclude that $F_1, F_2, \ldots, F_n$ are the unique mappings satisfying (3) and (12). □

Corollary 5. *Let V, Y, n, a, $\alpha_1, \alpha_2, \ldots, \alpha_n$, f, ϕ, and Φ be given under the same conditions as in Theorem 4. If there exists a mapping $F : V \to Y$ that satisfies the inequality*

$$\|f(x) - F(x)\| \leq \Phi(x) \quad (\textit{for all } x \in V \setminus \{0\}), \tag{13}$$

where $F(x) = \sum_{k=1}^{n} F_k(x)$ and every F_k has property (3), then the mapping F is uniquely determined.

The following main theorem results from Theorems 2–4.

Theorem 5 (Main Theorem)**.** *Assume that V is a real vector space and Y is a real normed space. For any fixed $n \in \mathbb{N}$, let $a, \alpha_1, \alpha_2, \ldots, \alpha_n$ be nonzero real constants, such that $a > 1$ and $\alpha_1 < \alpha_2 < \cdots < \alpha_n$. For a given mapping $f : V \to Y$, if there exist mappings $F_1, F_2, \ldots, F_n : V \to Y$ and a function $\phi : V \setminus \{0\} \to [0, \infty)$ that satisfy*

$$\left\| f(x) - \sum_{k=1}^{n} F_k(x) \right\| \leq \sum_{i=0}^{\infty} \frac{1}{a^{\alpha_1 i}} \phi(a^i x) < \infty \quad \textit{or} \tag{14}$$

$$\left\| f(x) - \sum_{k=1}^{n} F_k(x) \right\| \leq \sum_{i=0}^{\infty} \frac{1}{a^{\alpha_{\ell+1} i}} \phi(a^i x) + \sum_{i=0}^{\infty} a^{\alpha_\ell i} \phi\left(\frac{1}{a^i} x\right) < \infty \quad \textit{or} \tag{15}$$

$$\left\| f(x) - \sum_{k=1}^{n} F_k(x) \right\| \leq \sum_{i=0}^{\infty} a^{\alpha_n i} \phi\left(\frac{1}{a^i} x\right) < \infty \tag{16}$$

for all $x \in V \setminus \{0\}$ and for some $\ell \in \{1, 2, \ldots, n-1\}$, where every F_k has property (3), then the mappings $F_1, F_2, \ldots, F_n$ are uniquely determined.

Now, we introduce a corollary that further improves the applicability of the above main theorem.

Corollary 6. *Assume that V is a real vector space and Y is a real normed space. For any fixed $n \in \mathbb{N}$, let $a, \alpha_1, \alpha_2, \ldots, \alpha_n$ be nonzero real constants, such that $a > 1$ and $\alpha_1 < \alpha_2 < \cdots < \alpha_n$. For a given mapping $f : V \to Y$, if there exist a mapping $F : V \to Y$ and a function $\phi : V \setminus \{0\} \to [0, \infty)$ that satisfy*

$$\|f(x) - F(x)\| \le \sum_{i=0}^{\infty} \frac{1}{a^{\alpha_1 i}} \phi(a^i x) < \infty \ \ or \tag{17}$$

$$\|f(x) - F(x)\| \le \sum_{i=0}^{\infty} \frac{1}{a^{\alpha_{\ell+1} i}} \phi(a^i x) + \sum_{i=0}^{\infty} a^{\alpha_\ell i} \phi\left(\frac{1}{a^i} x\right) < \infty \ \ or \tag{18}$$

$$\|f(x) - F(x)\| \le \sum_{i=0}^{\infty} a^{\alpha_n i} \phi\left(\frac{1}{a^i} x\right) < \infty \tag{19}$$

for all $x \in V \setminus \{0\}$ and for some $\ell \in \{1, 2, \ldots, n-1\}$, where $F(x) = \sum_{k=1}^{n} F_k(x)$ and every F_k has the property (3), then the mapping F is uniquely determined.

4. Examples

Assume that V is a real vector space and Y is a real normed space.

Example 1. *Let $f : V \to Y$ be an arbitrary mapping and $F : V \to Y$ an additive-quadratic-cubic-quartic ($AQCQ'$) mapping. If there exists a function $\phi : V \setminus \{0\} \to [0, \infty)$ that satisfies*

$$\|f(x) - F(x)\| \le \sum_{i=0}^{\infty} \frac{1}{2^i} \phi(2^i x) < \infty \ \ or \tag{20}$$

$$\|f(x) - F(x)\| \le \sum_{i=0}^{\infty} \frac{1}{2^{2i}} \phi(2^i x) + \sum_{i=0}^{\infty} 2^i \phi\left(\frac{1}{2^i} x\right) < \infty \ \ or \tag{21}$$

$$\|f(x) - F(x)\| \le \sum_{i=0}^{\infty} \frac{1}{2^{3i}} \phi(2^i x) + \sum_{i=0}^{\infty} 2^{2i} \phi\left(\frac{1}{2^i} x\right) < \infty \ \ or \tag{22}$$

$$\|f(x) - F(x)\| \le \sum_{i=0}^{\infty} \frac{1}{2^{4i}} \phi(2^i x) + \sum_{i=0}^{\infty} 2^{3i} \phi\left(\frac{1}{2^i} x\right) < \infty \ \ or \tag{23}$$

$$\|f(x) - F(x)\| \le \sum_{i=0}^{\infty} 2^{4i} \phi\left(\frac{1}{2^i} x\right) < \infty \tag{24}$$

for all $x \in V \setminus \{0\}$, then the mapping F is uniquely determined.

Proof. If we set $a = 2$, $\alpha_1 = 1$, $\alpha_2 = 2$, $\alpha_3 = 3$, $\alpha_4 = 4$, F_1 an additive mapping, F_2 a quadratic mapping, F_3 a cubic mapping, and F_4 a quartic mapping, then $F = F_1 + F_2 + F_3 + F_4$ with $F_1(2x) = 2F_1(x)$, $F_2(2x) = 2^2 F_2(x)$, $F_3(2x) = 2^3 F_3(x)$, and $F_4(2x) = 2^4 F_4(x)$. It then follows from Corollary 6 that F is the only additive-quadratic-cubic-quartic mapping that satisfies either (20), (21), (22), (23), or (24). □

For a given mapping $f : V \to Y$, we use the abbreviations $D_1 f, D_2 f, D_3 f : V^2 \to Y$ defined by

$$\begin{aligned}
D_1 f(x, y) := {} & f(x + 2y) + f(x - 2y) - 4\big(f(x + y) + f(x - y)\big) \\
& - f(4y) + 4f(3y) - 6f(2y) + 4f(y) + 6f(x), \\
D_2 f(x, y) := {} & f(x + 2y) - 4f(x + y) + 6f(x) - 4f(x - y) \\
& + f(x - 2y) - f(2y) - f(-2y) + 4f(y) + 4f(-y), \\
D_3 f(x, y) := {} & f(x + ay) + f(x - ay) - a^2 f(x + y) - a^2 f(x - y) \\
& + 2(a^2 - 1) f(x) - \frac{a^4 - a^2}{12} [f(2y) + f(-2y) - 4f(y) - 4f(-y)], \\
D_4 f(x, y) := {} & f(x + 5y) - 5f(x + 4y) - 10f(x + 3y) + 10f(x + 2y) \\
& + 5f(x + y) - f(x)
\end{aligned}$$

for all $x, y \in V$, where $a \notin \{-1, 0, 1\}$ is a fixed integer. In [26], M. E. Gordji et al. showed that if f satisfies the functional equation $D_1 f = 0$, then f is an AQCQ′ mapping. And in [27], J. R. Lee et al. showed that if f satisfies the functional equation $D_2 f = 0$, then f is an AQCQ′ mapping, while in [28], K. Ravi et al. showed that if f satisfies $D_3 f = 0$, then f is an AQCQ′ mapping. Also, in [29], D. Z. Djoković et al. showed that if f satisfies $D_4 f = 0$, then f is a generalized polynomial mapping of degree 4 (Theorem 3 in [29]). Moreover, in [20], Y. H. Lee et al. obtained stability results of $D_4 f = 0$.

Using Example 1, we can improve the stability results obtained separately by Gordji et al., Lee et al., Ravi, and Y. H. Lee et al. all at once, as shown in the following example.

Example 2. *For each $\ell \in \{0, 1, 2, 3, 4\}$ and any given function $\varphi : V^2 \to [0, \infty)$, we define the conditions φ_ℓ by*

$$\begin{aligned}
&\sum_{i=0}^{\infty} \frac{1}{2^i}\varphi(2^i x, 2^i y) < \infty && (\text{for } \ell = 0),\\
&\sum_{i=0}^{\infty} \frac{1}{4^i}\varphi(2^i x, 2^i y) + 2^i \varphi\left(\frac{1}{2^i}x, \frac{1}{2^i}x\right) < \infty && (\text{for } \ell = 1),\\
&\sum_{i=0}^{\infty} \frac{1}{8^i}\varphi(2^i x, 2^i y) + 4^i \varphi\left(\frac{1}{2^i}x, \frac{1}{2^i}x\right) < \infty && (\text{for } \ell = 2),\\
&\sum_{i=0}^{\infty} \frac{1}{16^i}\varphi(2^i x, 2^i y) + 8^i \varphi\left(\frac{1}{2^i}x, \frac{1}{2^i}x\right) < \infty && (\text{for } \ell = 3),\\
&\sum_{i=0}^{\infty} 16^i \varphi\left(\frac{1}{2^i}x, \frac{1}{2^i}x\right) < \infty && (\text{for } \ell = 4)
\end{aligned} \qquad (\varphi_\ell)$$

for all $x, y \in V$. For a fixed $\ell \in \{0, 1, 2, 3, 4\}$ and $m \in \{1, 2, 3, 4\}$, if a function $\varphi : V^2 \to [0, \infty)$ satisfies the conditions of (φ_ℓ) and a mapping $f : V \to Y$ satisfies $f(0) = 0$ and the following inequality

$$\|D_m f(x, y)\| \le \varphi(x, y)$$

for all $x, y \in V$, then there exists a unique mapping $F : V \to Y$ such that $D_m F(x, y) = 0$ for all $x, y \in V$ and

$$\begin{aligned}
\|f(x) - F(x)\| &\le \frac{1}{6}\sum_{i=0}^{\infty} \frac{1}{2^{i+1}}\rho(2^i x) + \frac{1}{12}\sum_{i=0}^{\infty} \frac{1}{4^{i+1}}\psi(2^i x) && (\text{for } \ell = 0),\\
\|f(x) - F(x)\| &\le \frac{1}{6}\sum_{i=0}^{\infty} 2^i \rho\left(\frac{1}{2^{i+1}}x\right)\\
&\quad + \frac{1}{6}\sum_{i=0}^{\infty} \frac{1}{8^{i+1}}\rho(2^i x) + \frac{1}{12}\sum_{i=0}^{\infty} \frac{1}{4^{i+1}}\psi(2^i x) && (\text{for } \ell = 1),\\
\|f(x) - F(x)\| &\le \frac{1}{6}\sum_{i=0}^{\infty} 2^i \rho\left(\frac{1}{2^{i+1}}x\right) + \frac{1}{12}\sum_{i=0}^{\infty} 4^i \psi\left(\frac{1}{2^{i+1}}x\right)\\
&\quad + \frac{1}{6}\sum_{i=0}^{\infty} \frac{1}{8^{i+1}}\rho(2^i x) + \frac{1}{12}\sum_{i=0}^{\infty} \frac{1}{16^{i+1}}\psi(2^i x) && (\text{for } \ell = 2),\\
\|f(x) - F(x)\| &\le \frac{1}{6}\sum_{i=1}^{\infty} 8^i \rho\left(\frac{1}{2^{i+1}}x\right) + \frac{1}{12}\sum_{i=0}^{\infty} 4^i \psi\left(\frac{1}{2^{i+1}}x\right)\\
&\quad + \frac{1}{12}\sum_{i=0}^{\infty} \frac{1}{16^{i+1}}\psi(2^i x) && (\text{for } \ell = 3),\\
\|f(x) - F(x)\| &\le \frac{1}{6}\sum_{i=1}^{\infty} 8^i \rho\left(\frac{1}{2^{i+1}}x\right) + \frac{1}{12}\sum_{i=1}^{\infty} 16^i \psi\left(\frac{1}{2^{i+1}}x\right) && (\text{for } \ell = 4)
\end{aligned}$$

for all $x \in V$*, where*

$$
\begin{aligned}
\rho(x) &= 5\varphi_e(0,x) + 4\varphi_e(x,x), \\
\psi(x) &= 5\varphi_e(0,x) + 4\varphi_e(x,x) + 46\varphi_e(0,0) && (\textit{for } m = 1), \\
\rho(x) &= \varphi_e(2x,x) + 4\varphi_e(x,x), \\
\psi(x) &= \varphi_e(2x,x) + 4\varphi_e(x,x) + \frac{5}{2}\varphi_e(0,0) && (\textit{for } m = 2), \\
\rho(x) &= \frac{1}{a^4 - a^2}\big(2\varphi_e((1-a)x,x) + 2\varphi_e((1+a)x,x) \\
&\qquad + \varphi_e((1+2a)x,x) + \varphi_e((1-2a)x,x) \\
&\qquad + \varphi_e(x,3x) + 2a^2\varphi_e(2x,x) \\
&\qquad + (4a^2-3)\varphi_e(x,x) + a^2\varphi_e(2x,2x) \\
&\qquad + 2a^2\varphi_e(x,2x)\big), \\
\psi(x) &= \frac{6}{a^4 - a^2}\big(2\varphi_e(ax,x) + 2a^2\varphi_e(x,x) \\
&\qquad + 2(a^2-1)\varphi_e(0,x) + \varphi_e(0,2x) + 12\varphi_e(0,0)\big) && (\textit{for } m = 3), \\
\rho(x) &= \varphi_e(-x,x) + 5\varphi_e(-2x,x), \\
\psi(x) &= \varphi_e(-x,x) + 5\varphi_e(-2x,x) && (\textit{for } m = 4)
\end{aligned}
$$

for all $x \in V$*.*

Using Example 2, we have Hyers–Ulam–Rassias stability of the functional equations $D_m f = 0$.

Example 3. *Let* $\theta \geq 0$ *and let* p *be a positive real number with* $p \notin \{1,2,3,4\}$*. Let* $f : X \to Y$ *be a mapping satisfying* $f(0) = 0$ *and*

$$\|D_m f(x,y)\| \leq \theta\big(\|x\|^p + \|y\|^p\big)$$

for all x, y*. Then, there exists a unique mapping* $F : X \to Y$ *such that* $D_m F(x,y) = 0$ *for all* x, y *and*

$$
\begin{aligned}
\|f(x) - F(x)\| &\leq \left(\frac{\alpha_m}{6(2-2^p)} + \frac{\beta_m}{12(4-2^p)}\right)\theta\|x\|^p && (\textit{for } 0 < p < 1), \\
\|f(x) - F(x)\| &\leq \left(\frac{\alpha_m}{6(2^p-2)} + \frac{\beta_m}{12(4-2^p)} + \frac{\alpha_m}{6(8-2^p)}\right)\theta\|x\|^p && (\textit{for } 1 < p < 2), \\
\|f(x) - F(x)\| &\leq \left(\frac{\alpha_m}{6(2^p-2)} + \frac{\beta_m}{12(2^p-4)} + \frac{\alpha_m}{6(8-2^p)} + \frac{\beta_m}{12(16-2^p)}\right)\theta\|x\|^p && (\textit{for } 2 < p < 3), \\
\|f(x) - F(x)\| &\leq \left(\frac{\beta_m}{12(2^p-4)} + \frac{4\alpha_m}{3\cdot 2^p(2^p-8)} + \frac{\beta_m}{12(16-2^p)}\right)\theta\|x\|^p && (\textit{for } 3 < p < 4), \\
\|f(x) - F(x)\| &\leq \left(\frac{4\alpha_m}{3\cdot 2^p(2^p-8)} + \frac{4\beta_m}{3\cdot 2^p(2^p-16)}\right)\theta\|x\|^p && (\textit{for } p > 4)
\end{aligned}
$$

for all $x \in X$, where

$$\begin{aligned} \alpha_1 &= 13, & \beta_1 &= 13, \\ \alpha_2 &= 9 + 2^p, & \beta_2 &= 9 + 2^p, \\ \alpha_3 &= \frac{1}{a^4 - a^2}\left(2|a-1|^p + 2|a+1|^p + |2a-1|^p + |2a+1|^p + 3^p + 6a^2 2^p + 12a^2 + 1\right), \\ \beta_3 &= \frac{6}{a^4 - a^2}\left(2a^p + 6a^2 + 2^p\right), \\ \alpha_4 &= 7 + 5 \cdot 2^p, & \beta_4 &= 7 + 5 \cdot 2^p. \end{aligned}$$

Example 4. *Let $f : V \to Y$ be an arbitrary mapping, r a fixed positive rational number with $r \neq 1$, and $F : V \to Y$ a generalized polynomial mapping of degree n with $f(0) = F(0)$. If there exists a function $\phi : V \setminus \{0\} \to [0, \infty)$ that satisfies*

$$\|f(x) - F(x)\| \le \sum_{i=0}^{\infty} \frac{1}{r^i}\phi(r^i x) < \infty \ \text{ or} \tag{25}$$

$$\|f(x) - F(x)\| \le \sum_{i=0}^{\infty} \frac{1}{r^{(\ell+1)i}}\phi(r^i x) + \sum_{i=0}^{\infty} r^{\ell i}\phi\left(\frac{1}{r^i}x\right) < \infty \ \text{ or} \tag{26}$$

$$\|f(x) - F(x)\| \le \sum_{i=0}^{\infty} r^{ni}\phi\left(\frac{1}{r^i}x\right) < \infty \tag{27}$$

for all $x \in V \setminus \{0\}$ and for some $\ell \in \{1, 2, \dots, n-1\}$, then F is a uniquely determined generalized polynomial mapping of degree n.

Proof. Let $\tilde{f}, F' : V \to Y$ be the mappings defined by $\tilde{f}(x) = f(x) - f(0)$ and $F'(x) = F(x) - F(0)$. Then, $f(x) - F(x) = \tilde{f}(x) - F'(x)$ and we can apply Corollary 6 to $\tilde{f}$ and F'. So, we can obtain unique F' that satisfies either (17), (18), or (19). It means that we can have unique F that satisfies either (25), (26), or (27). □

5. Conclusions

Considering Hyers–Ulam stability of functional equations, it is generally difficult to prove the uniqueness of the stability function with conditions similar to Găvruţa condition. The uniqueness theorems of this paper obtained through direct calculation can be applied to various functional equations. As an application of Theorem 5 and Corollary 6, we considered Examples 2 and 3 to obtain generalized stability of the functional equation $D_m f = 0$, and here, we have the uniqueness of the stability mapping F.

For future research, we can apply Theorem 5 and Corollary 6 to the functional equations in [13–17,19–23] obtained Hyers–Ulam–Rassias stability and we can obtain the uniqueness of the stability mapping F with conditions similar to Găvruţa condition.

Author Contributions: Conceptualization, Y.-H.L.; methodology, S.-M.J., Y.-H.L. and J.R.; validation, S.-M.J., Y.-H.L. and J.R.; formal analysis, S.-M.J., Y.-H.L. and J.R.; writing—original draft preparation, S.-M.J., Y.-H.L. and J.R.; writing—review and editing, S.-M.J., Y.-H.L. and J.R.; supervision, J.R.; funding acquisition, J.R. All authors have read and agreed to the published version of the manuscript.

Funding: This work was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIT) (No. 2021R1A2C109489611).

Data Availability Statement: Data is contained within the article.

Conflicts of Interest: The authors declare no conflict of interest.

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