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On (p,q)-Analogs of the α-th Fractional Fourier Transform and Some (p,q)-Generalized Spaces

Shrideh Al-Omari [1,2,*] and Wael Salameh [3]

[1] Department of Mathematics, Faculty of Science, Al-Balqa Applied University, Salt 11134, Jordan
[2] Jadara University Research Center, Jadara University, Irbid 21110, Jordan
[3] Faculty of Information Technology, Abu Dhabi University, Abu Dhabi 59911, United Arab Emirates; wael.salameh@adu.ac.ae
* Correspondence: shridehalomari@bau.edu.jo

Abstract: In this article, the (p,q)-analogs of the α-th fractional Fourier transform are provided, along with a discussion of their characteristics in specific classes of (p,q)-generalized functions. Two spaces of infinitely (p,q)-differentiable functions are defined by introducing two (p,q)-differential symmetric operators. The (p,q)-analogs of the α-th fractional Fourier transform are demonstrated to be continuous and linear between the spaces under discussion. Next, theorems pertaining to specific convolutions are established. This leads to the establishment of multiple symmetric identities, which in turn requires the construction of (p,q)-generalized spaces known as (p,q)-Boehmians. Finally, in addition to deriving the inversion formulas, the generalized (p,q)- analogs of the α-th fractional Fourier transform are introduced, and their general properties are discussed.

Keywords: (p,q)-differentiable; α-th fractional Fourier transform; (p,q)-derivative operator; (p,q)-Boehmian; (p,q)-generalized functions; (p,q)-analogs

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1. Introduction

The core of q-calculus theory is the idea of deriving q-derivatives and q-integrals [1]. The q-calculus theory solves a wide range of symmetric problems, including sets of non-differentiable functions, integral transforms, Bessel functions, hypergeometric functions, beta functions, gamma functions, and many more (see, for more details, [2–6] and the references cited therein). It is a fundamental idea in many fields of physical science, such as mathematics, physics, high-energy nuclear physics, cosmic strings, and conformal quantum mechanics. It also addresses topics in number theory, combinatorics, quantum theory, physics, theory of relativity, orthogonal polynomials, and basic hypergeometric functions (see, e.g., [7,8]; see also [9,10]).

The q-calculus is the simplified form of the (p,q)-calculus when $p = 1$. Sadjang [11–13] carried out further research on the fundamental theorem of (p,q)-calculus as well as the (p,q)-integration, (p,q)-derivative, and (p,q)-Taylor formulae. Many researchers and developers of the (p,q)-Mathieu-type series, (p,q)-Hermite–Hadamard inequalities and (p,q)-Beta functions have produced more detailed work in [14–16]. Several scholars have also conducted more research on (p,q)-integral transformations. The characteristics of the (p,q)-analogs of the Laplace transform and their applications in the resolution of specific (p,q)-difference equations were investigated by Sadjang [12]. Later on, Jirakulchaiwong et al. [17] studied the (p,q)-analogs of Laplace-type integral transforms and gave characteristics that led to further applications.

The continuous linear forms known as generalized functions (distributions) are defined over sets of indefinitely smooth functions and have been widely used in applied physics and engineering problems [18]. Distributions are useful for characterizing physical phenomena as point charges and for smoothing out discontinuous functions. The recent generalized functions space, often called the space of Boehmians, has an algebraic structure analogous to the field of quotients [19]. When applied to function spaces, different spaces of Boehmians are produced from the structure, and multiplications are interpreted as convolutions [19–25]. Delta sequences with decreasing support in the origin are required while constructing Boehmian spaces. The uniqueness theorems, which are regarded as an uncertainty principle for Boehmian dynamics [26,27], were in reality the result of this idea. However, Boehmians allow different interpretations of such extended operators to form isomorphisms among the different Boehmian spaces because their definition is based on abstract algebraic notions [28].

In Section 2, we give a brief introduction to the (p,q)-theory of Boehmians and the (p,q)-calculus theory in this article. We prove a theory regarding (p,q)-convolutions and extract certain properties of the (p,q)-analogs of the α-th fractional Fourier transforms in Section 3. In Sections 4 and 5, we discuss two spaces of (p,q)-Boehmians. Section 6 examines a number of aspects of the generalized fractional integral operator, as well as its generalized inversion.

2. (p,q)-Calculus and (p,q)-Generalized Functions

The common ideas and symbols found in the (p,q)-calculus are summarized hereafter [11–13,15,17]. We consider q to be a fixed real number and $0 < q < p \leq 1$. The (p,q)-derivative is defined as the (p,q)-analog of the ψ derivative [29]

$$(D_{p,q}\psi)(\zeta) := \begin{cases} \dfrac{\psi(p\zeta) - \psi(q\zeta)}{p\zeta - q\zeta}, \text{ if } \zeta \neq 0, \\ \psi^{'}(0) \quad , \text{ if } \zeta = 0 \end{cases}. \tag{1}$$

If ψ is differentiable, then $\lim_{p,q\to 1} D_{p,q}\psi(\zeta) = \psi^{'}(\zeta)$. $[\zeta]_{p,q}$ and $\left([\zeta]_{p,q}\right)!$ must represent the (p,q)-numbers and (p,q)-factorials introduced by [30]

$$[\zeta]_{p,q} = \frac{p^{\zeta} - q^{\zeta}}{p - q} \text{ and } \left([\zeta]_{p,q}\right)! = \prod_{i=1}^{\zeta}[i]_{p,q}, [0]_{p,q} = 1, \tag{2}$$

respectively.

The product and division of two continuous functions, ψ_1 and ψ_2, meet the following respective (p,q)-analogs when taken as a (p,q)-derivative [29]

$$D_{p,q}(\psi_1\psi_2)(\zeta) = \psi_1(p\zeta)D_{p,q}\psi_2(\zeta) + \psi_2(q\zeta)D_{p,q}\psi_1(\zeta) \tag{3}$$

and

$$D_{p,q}\left(\frac{\psi_1}{\psi_2}\right)(\zeta) = \frac{\psi_2(p\zeta)D_{p,q}\psi_1(\zeta) - \psi_1(p\zeta)D_{p,q}\psi_2(\zeta)}{\psi_2(p\zeta)\psi_2(q\zeta)}. \tag{4}$$

Alternatively, they could be described as

$$D_{p,q}(\psi_1\psi_2)(\zeta) = \psi_1(q\zeta)D_{p,q}\psi_1(\zeta) + \psi_2(p\zeta)D_{p,q}\psi_1(\zeta) \tag{5}$$

and

$$D_{p,q}\left(\frac{\psi_1}{\psi_2}\right)(\zeta) = \frac{\psi_2(q\zeta)D_{p,q}\psi_1(\zeta) - \psi_1(q\zeta)D_{p,q}\psi_2(\zeta)}{\psi_2(p\zeta)\psi_2(q\zeta)}. \tag{6}$$

The (p,q)-integrals of a function ψ are defined by [29]

$$\int_0^x \psi(\zeta)d_{p,q}\zeta = (p-q)\zeta\sum_0^{\infty}\frac{q^i}{p^{i+1}}\psi\left(\zeta\frac{q^i}{p^{i+1}}\right), \left|\frac{p}{q}\right| > 1, \tag{7}$$

$$\int_0^{\infty} \psi(\zeta)d_{p,q}\zeta = (p-q)\sum_{-\infty}^{\infty}\frac{q^i}{p^{i+1}}\psi\left(\frac{q^i}{p^{i+1}}\right), \left|\frac{p}{q}\right| > 1, \tag{8}$$

when the sums are finite for real number ζ. The (p,q)-integration by parts is defined for functions ψ_1 and ψ_2 by [11]

$$\int_0^\infty \psi_1(\zeta)D_{p,q}\psi_2(\zeta)d_{p,q}\zeta = \psi_1(\zeta)\psi_2(\zeta)|_0^\infty - \int_0^\infty \psi_2(q\zeta)D_{p,q}\psi_1(\zeta)d_{p,q}\zeta. \quad (9)$$

The two types of (p,q)-exponential functions are defined by [12]

$$E_{p,q}(\zeta) = \sum_{j=0}^{\infty} \frac{q^{\frac{j(j-1)}{2}}\zeta^j}{\left([j]_{p,q}\right)!} \ (\zeta \in \mathbb{C}), \quad (10)$$

and

$$e_{p,q}(\zeta) = \sum_{j=0}^{\infty} \frac{p^{\frac{j(j-1)}{2}}\zeta^j}{\left([j]_{p,q}\right)!} \ (|\zeta| < 1). \quad (11)$$

In Equations (11) and (12), the q-exponential functions E_p and e_p, respectively, are obtained by substituting $p = 1$. Additionally, Ref. [11] provides (p,q)-derivatives of the (p,q)-analogs of the exponential function as

$$D_{p,q}e_{p,q}(k\zeta) = ke_{p,q}(kp\zeta) \text{ and } D_{p,q}E_{p,q}(k\zeta) = kE_{p,q}(kq\zeta), \ k \in \mathbb{R}. \quad (12)$$

Consequently,

$$D_{p,q}e_{p,q}(\zeta) = e_{p,q}(p\zeta) \text{ and } D_{p,q}E_{p,q}(\zeta) = E_{p,q}(q\zeta). \quad (13)$$

Further, from [12] (13), we recall that

$$D^n_{p,q}e_{p,q}(k\zeta) = k^n p^{\binom{n}{2}} e_{p,q}(kp^n\zeta) \text{ and } D^n_{p,q}E_{p,q}(k\zeta) = k^n q^{\binom{n}{2}} E_{p,q}(kq^n\zeta), n \in \mathbb{N}, k \in \mathbb{R}.$$

The (p,q)-gamma function of the first and second kinds are, respectively, defined by [11]

$$\Gamma_{p,q}(i) = p^{\frac{i(i-1)}{2}} \int_0^\infty \zeta^{i-1}E_{p,q}(-q\zeta)d_{p,q}\zeta. \quad (14)$$

Boehmians, driven by regular operators and introduced by Boehme [31], are among the newest generalizations of generalized functions. Numerous articles exist that link the expansion of Boehmians into several classes of tempered Boehmians, ultra Boehmians, integral transformations and other applications. Assume that Y is a subspace of a linear space X. Then, for any pair of elements $\psi \in (X, \circledast)$ and $\omega_1 \in (Y, \circledast)$, there are allocated the products $\circledast$ and $\circledast$ such that:

(*i*) For $\omega_1, \omega_2 \in Y$, we have $\omega_1 \circledast \omega_2 \in Y, \omega_1 \circledast \omega_2 = \omega_2 \circledast \omega_1$.
(*ii*) For $\psi \in X, \omega_1, \omega_2 \in Y$, we have $(\psi * \circledast \omega_1)\circledast\omega_2 = \psi \circledast (\omega_1 \circledast \omega_2)$.
(*iii*) $\psi_1, \psi_2 \in X, \omega_1 \in Y, r \in \mathbb{R} \Rightarrow$

$$(\psi_1 + \psi_2)\circledast\omega_1 = \psi_1\circledast\omega_1 + \psi_2\circledast\omega_1, r(\psi_1\circledast\omega_1) = (r\psi_1)\circledast\omega_1. \quad (15)$$

Let Δ represent a family of sequences that are part of Y. After that, Δ is regarded as a family of delta sequences if it satisfies both Δ_1 and Δ_2,

$$P_1 : \text{For } \psi_1, \psi_2 \in X, (x_n) \in \Delta \text{ and } \psi_1\circledast x_n = \psi_2\circledast x_n, \text{ we have } \psi_1 = \psi_2, \forall n \in \mathbb{N}.$$

$$P_2 : (y_n), (x_n) \in \Delta \Rightarrow (y_n \circledast x_n) \in \Delta.$$

If $S = \{((\psi_n), (x_n)), (\psi_n) \in X, (x_n) \in \Delta, \forall n \in \mathbb{N}\}$, then $((\psi_n), (x_n))$ is a pair of quotients of sequences in S iff

$$\psi_n \circledast x_m = \psi_m \circledast x_n, \quad (16)$$

for all natural numbers n and m. The pairs $((\psi_n),(y_n))$ and $((\kappa_n),(x_n))$ are equivalent pairs of quotients according to the notation $\sim$ iff

$$\psi_n \check{*} x_m = \kappa_m \check{*} y_n, \tag{17}$$

for all natural numbers n and m. In this regard, $\sim$ forms an equivalent relation on the set S, and therefore, $\frac{\psi_n}{y_n}$ constitutes an equivalence class named a Boehmian that we denote as B.

3. The α-th (p,q)-Fractional Fourier Transform and Its Convolution

A generalization of the classical Fourier integral operator into the fractional domains is the fractional Fourier integral operator [32]. Although it has been defined in several ways in the literature, the notion of rotations over an angle $\pi/2$ in the classical Fourier integral operator has been enlarged to give the most logical explanation of the fractional Fourier integral operator [33,34]. A rotation over an angle α is correlated with the fractional Fourier integral operator, whereas the typical Fourier integral operator corresponds with a rotation on the time–frequency plane and q-difference equations [23,35,36].

Let S be the Schwartz space of rapidly decreasing functions on $\mathbb{R}$, and $V(\mathbb{R})$ denotes the space [37]

$$V(\mathbb{R}) = \left\{ v \in S : v^{(k)}(0) = 0, k = 0,1,2,\ldots \right\}.$$

Then, the Lizorkin space $\Theta(\mathbb{R})$ is defined as

$$\Theta(\mathbb{R}) = \{\psi \in S : F\psi \in V(\mathbb{R})\}$$

where $F(\psi)$ is the Fourier transform of ψ. If $\psi \in S(\mathbb{R})$ and $\omega > 0$, then we have the following definition.

Definition 1 ([33]). *The α-th Fourier transform for a function ψ is defined for $0 < \alpha \leq 1$ by*

$$F_\alpha(\psi)(w) = \int_{-\infty}^{\infty} \psi(\zeta) e^{i\zeta w^{\frac{1}{\alpha}}} d\zeta. \tag{18}$$

The inverse transform of the α-th fractional Fourier transform F_α is given by [33]

$$\psi(\zeta) = \frac{1}{\sqrt{2\pi i}} \int_{-\infty}^{\infty} F_\alpha(\psi)(w) w^{\frac{1-\alpha}{\alpha}} e^{-i\zeta w^{\frac{1}{\alpha}}} dw.$$

Following [37], we introduce the following definition.

Definition 2. *An infinitely (p,q)-differentiable complex-valued function ψ over $\mathbb{R}$ is in $S^{v,r}_{p,q}$ if and only if*

$$\gamma_{r,p,q}(\psi) = \sup_{\zeta\in\mathbb{R}} \left| \zeta^r D^v_{p,q} \psi(\zeta) \right| < \infty, \tag{19}$$

for every choice of constants r and v.

The dense subspace of $S^{v,r}_{p,q}$ denoted by $D^v_{r,p,q}(\mathbb{R})$ consists of those (p,q)-differentiable functions of compact supports over $\mathbb{R}$ such that

$$\sup_{\zeta\in\mathbb{R}} \left| D^v_{p,q} \psi(\zeta) \right| < \infty. \tag{20}$$

Definition 3. *The (p,q)-analog of the α-th Fourier transform of a function ψ of the first type is defined for $0 < \alpha \leq 1$ by*

$$F^{\alpha,p}_q(w) = \int_{-\infty}^{\infty} \psi(\zeta) E_{p,q}\left(i\zeta w^{\frac{1}{\alpha}}\right) d_{p,q}\zeta, \tag{21}$$

whereas the α-th (p,q)*-analog of the Fourier transform of a function ψ of the second type is defined for* $0 < \alpha \leq 1$ *by*

$$\breve{F}_q^{\alpha,p}(\psi)(w) = \int_{-\infty}^{\infty} \psi(\zeta) e_{p,q}\left(i\zeta w^{\frac{1}{\alpha}}\right) d_{p,q}\zeta. \quad (22)$$

Theorem 1. *Let* ψ, ψ_1 *and* ψ_2 *be functions of certain exponential growth conditions. Then, the following statements hold true.*

(i) *(Linearity) For real numbers* α_1, α_1 *we have*

$$\begin{aligned} F_q^{\alpha,p}(\alpha_1\psi_1 + \alpha_2\psi_2)(w) &= \alpha_1 F_q^{\alpha,p}(\psi_1)(w) + \alpha_2 F_q^{\alpha,p}(\psi_2)(w). \\ \breve{F}_q^{\alpha,p}(\alpha_1\psi_1 + \alpha_2\psi_2)(w) &= \alpha_1 \breve{F}_q^{\alpha,p}(\psi_1)(w) + \alpha_2 \breve{F}_q^{\alpha,p}(\psi_2)(w). \end{aligned}$$

(ii) *(Scaling) For a real number β, we have*

$$F_q^{\alpha,p}(\psi(\beta\zeta))(w) = \frac{1}{\beta} F_q^{\alpha,p}(\psi(\zeta))\left(\frac{w}{\beta^\alpha}\right). \quad \breve{F}_q^{\alpha,p}(\psi(\beta\zeta))(w) = \frac{1}{\beta} \breve{F}_q^{\alpha,p}(\psi(\zeta))\left(\frac{w}{\beta^\alpha}\right).$$

Proof. The proof of (i) follows from the definition of the (p,q)-integral. To prove (ii), let $z = \beta\zeta \Rightarrow d_{p,q}\zeta = \frac{1}{\beta} d_{p,q}z$. Then, inserting the given substitution under the integral sign yields

$$\begin{aligned} F_q^{\alpha,p}(\psi(\beta))(w) &= \int_0^{\infty} \psi(\beta\zeta) E_{p,q}\left(i\zeta w^{\frac{1}{\alpha}}\right) d_{p,q}\zeta \\ &= \int_0^{\infty} \psi(z) E_{p,q}\left(i\frac{z}{\beta} w^{\frac{1}{\alpha}}\right) \frac{d_{p,q}z}{\beta} \\ &= \frac{1}{\beta} \int_0^{\infty} \psi(z) E_{p,q}\left(iz\left(\frac{w}{\beta^\alpha}\right)^{\frac{1}{\alpha}}\right) d_{p,q}z \\ &= \frac{1}{\beta} F_q^{\alpha,p}(\psi(\zeta))\left(\frac{w}{\beta^\alpha}\right). \end{aligned}$$

□

The method used for the proof of the first part is also applicable to the second part. This concludes the proof of the theorem.

Theorem 2. *Let* $0 < \alpha \leq 1$*. Then, the* (p,q)*-analog of the α-th fractional Fourier transform of a function ψ of the first type, assuming the following properties:*

(i) $F_q^{\alpha,p}(\psi(\zeta - x))(w) = E_{p,q}\left(ixw^{\frac{1}{\alpha}}\right) F_q^{\alpha,p}(\psi)(w).$

(ii) $F_q^{\alpha,p}(D_{p,q}\psi(\zeta))(w) = -iw^{\frac{1}{\alpha}} q^{-1} F_q^{\alpha,p}(\psi)(q^{-\alpha}w).$

(iii) $F_q^{\alpha,p}\left(D_{p,q}^n\psi(\zeta)\right)(w) = -iq^{-1}w^{\frac{1}{\alpha}} F_q^{\alpha,p}\left(D_{p,q}^{n-1}f\right)(w).$

(iv) $F_q^{\alpha,p}\left(D_{p,q}^n\psi(\zeta)\right)(w) = \left(-iq^{-1}w^{\frac{1}{\alpha}}\right)^n F_q^{\alpha,p}(\psi)(w).$

(v) $D_{p,q}\left(F_q^{\alpha,p}(\psi(\zeta))\right)(w) = i\frac{1}{\alpha} F_q^{\alpha,p}(\zeta\psi)(q^\alpha w).$

Proof. (i) From Definition 3, we have

$$F_q^{\alpha,p}(\psi)(w) = \int_{-\infty}^{\infty} \psi(\zeta - x) E_{p,q}\left(i\zeta w^{\frac{1}{\alpha}}\right) d_{p,q}\zeta. \quad (23)$$

Using the change of variable $v = \zeta - x$ gives $d_{p,q}v = d_{p,q}\zeta$. Hence, we derive

$$\begin{aligned} F_q^{\alpha,p}(\psi)(w) &= \int_{-\infty}^{\infty} \psi(v) E_{p,q}\left(i(v+x)w^{\frac{1}{\alpha}}\right) d_{p,q}v \\ &= \int_{-\infty}^{\infty} \psi(v) E_{p,q}\left(i(v+x)w^{\frac{1}{\alpha}}\right) d_{p,q}v \\ &= E_{p,q}\left(ixw^{\frac{1}{\alpha}}\right) \int_{-\infty}^{\infty} \psi(v) E_{p,q}\left(ivw^{\frac{1}{\alpha}}\right) d_{p,q}v \\ &= E_{p,q}\left(ixw^{\frac{1}{\alpha}}\right) F_q^{\alpha,p}(\psi)(w). \end{aligned}$$

(ii) With the aid of Equations (1) and (21), we obtain

$$F_q^{\alpha,p}\left(D_{p,q}\psi(\zeta)\right)(w) = \int_{-\infty}^{\infty} D_{p,q}\psi(\zeta) E_{p,q}\left(i\zeta w^{\frac{1}{\alpha}}\right) d_{p,q}\zeta. \tag{24}$$

Hence, the (p,q)-integration by parts (3) and the fact that $\psi \in S^{v}_{r.p,q}(\mathbb{R})$, which gives $\psi(\zeta) E_{p,q}\left(i\zeta p w^{\frac{1}{\alpha}}\right)\Big|_{-\infty}^{\infty} = 0$, yield

$$\begin{aligned} F_q^{\alpha,p}\left(D_{p,q}\psi(\zeta)\right)(w) &= \psi(\zeta) E_{p,q}\left(i\zeta p w^{\frac{1}{\alpha}}\right)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \psi(q\zeta) D_{p,q} E_{p,q}\left(i\zeta w^{\frac{1}{\alpha}}\right) d_{p,q}\zeta \\ &= -iw^{\frac{1}{\alpha}} \int_{-\infty}^{\infty} \psi(q\zeta) E_{p,q}\left(i\zeta q w^{\frac{1}{\alpha}}\right) d_{p,q}\zeta. \end{aligned}$$

Using Equation (21) and altering the variables so that $q\zeta = z$ reveal

$$\begin{aligned} F_q^{\alpha,p}\left(D_{p,q}\psi\right)(w) &= -iw^{\frac{1}{\alpha}} q^{-1} \int_{-\infty}^{\infty} \psi(t) E_{p,q}\left(i\zeta q q^{-1} w^{\frac{1}{\alpha}}\right) d_{p,q}\zeta \\ &= -iw^{\frac{1}{\alpha}} q^{-1} \int_{-\infty}^{\infty} \psi(\zeta) E_{p,q}\left(i\zeta w^{\frac{1}{\alpha}}\right) d_{p,q}\zeta \\ &= -iw^{\frac{1}{\alpha}} q^{-1} \int_{-\infty}^{\infty} \psi(\zeta) E_{p,q}\left(i\zeta w^{\frac{1}{\alpha}}\right) d_{p,q}\zeta \\ &= -iw^{\frac{1}{\alpha}} q^{-1} F_q^{\alpha,p}(\psi)(w). \end{aligned}$$

(iii) Utilizing the definition of $F_q^{\alpha,p}$ in conjunction with the (p,q)-integration by parts suggests that

$$\begin{aligned} F_q^{\alpha,p}\left(D_{p,q}^{n}\psi(\zeta)\right)(w) &= \int_{-\infty}^{\infty} D_{p,q}^{n}\psi(\zeta) E_{p,q}\left(i\zeta w^{\frac{1}{\alpha}}\right) d_{p,q}\zeta \\ &= D_{p,q}^{n-1}\psi(\zeta) E_{p,q}\left(i\zeta p w^{\frac{1}{\alpha}}\right)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} D_{p,q}^{n-1}\psi(q\zeta) D_{p,q} E_{p,q}\left(i\zeta w^{\frac{1}{\alpha}}\right) d_{p,q}\zeta \\ &= D_{p,q}^{n-1}\psi(\zeta) E_{p,q}\left(i\zeta p w^{\frac{1}{\alpha}}\right)\Big|_{-\infty}^{\infty} - iw^{\frac{1}{\alpha}} \int_{-\infty}^{\infty} D_{p,q}^{n-1}\psi(q\zeta) E_{p,q}\left(itw^{\frac{1}{\alpha}}\right) d_{p,q}\zeta \\ &= -iq^{-1} w^{\frac{1}{\alpha}} \int_{-\infty}^{\infty} D_{p,q}^{n-1}\psi(q\zeta) E_{p,q}\left(i\zeta w^{\frac{1}{\alpha}}\right) d_{p,q}\zeta \\ &= -iq^{-1} w^{\frac{1}{\alpha}} F_q^{\alpha,p}\left(D_{p,q}^{n-1}\psi(\zeta)\right)(w). \end{aligned}$$

(iv) This part follows by proceeding n-times using the (p,q)-integration by parts for Part (iii).
(v) Using the definitions of the integral $F_q^{\alpha,p}$ and the (p,q)-derivative, we derive

$$
\begin{aligned}
D_{p,q}\left(F_q^{\alpha,p}(\psi(\zeta))\right)(w) &= \int_{-\infty}^{\infty}\psi(\zeta)D_{p,q}^{w}E_{p,q}\left(i\zeta w^{\frac{1}{\alpha}}\right)d_{p,q}\zeta \\
&= i\frac{1}{\alpha}\int_{-\infty}^{\infty}\zeta\psi(\zeta)E_{p,q}\left(i\zeta q w^{\frac{1}{\alpha}}\right)d_{p,q}\zeta \\
&= i\frac{1}{\alpha}\int_{-\infty}^{\infty}\zeta\psi(\zeta)E_{p,q}\left(i\zeta (q^{\alpha} w)^{\frac{1}{\alpha}}\right)d_{p,q}\zeta \\
&= i\frac{1}{\alpha}F_q^{\alpha,p}(\zeta\psi)(q^{\alpha}w).
\end{aligned}
$$

The proof is ended. □

The theorem that follows has a proof that is quite similar to the previous theorem.

Theorem 3. *Let $0 < \alpha \leq 1$. Then, the (p,q)-analog of the α-th fractional Fourier transform of a function ψ of the second type assumes the following properties:*

(i) $\breve{F}_q^{\alpha,p}(\psi(\zeta - x))(w) = E_{p,q}\left(ixw^{\frac{1}{\alpha}}\right)\breve{F}_q^{\alpha,p}(\psi)(w)$.

(ii) $\breve{F}_q^{\alpha,p}(D_{p,q}\psi)(w) = -iw^{\frac{1}{\alpha}}q^{-1}\breve{F}_q^{\alpha,p}(\psi)(q^{-\alpha}w)$.

(iii) $\breve{F}_q^{\alpha,p}\left(D_{p,q}^{n}\psi\right)(w) = -iqw^{\frac{1}{\alpha}}\breve{F}_q^{\alpha,p}\left(D_{p,q}^{n-1}\psi\right)(w)$.

(iv) $\breve{F}_q^{\alpha,p}\left(D_{p,q}^{n}\psi\right)(w) = \left(-iq^{-1}w^{\frac{1}{\alpha}}\right)^{n}\breve{F}_q^{\alpha,p}\left(D_{p,q}^{n-1}\psi\right)(w)$.

(v) $D_{p,q}\left(\breve{F}_q^{\alpha,p}(\psi(\zeta))\right)(w) = i\frac{1}{\alpha}\breve{F}_q^{\alpha,p}(\zeta\psi(\zeta))(q^{\alpha}w)$.

Theorem 4. *(Convolution Theorems) Let ψ_1 and ψ_2 be functions belonging to $S_{p,q}^{v,r}$. Then, the convolution theorems for $F_q^{\alpha,p}$ and $\breve{F}_q^{\alpha,p}$ are given by*

$$(i)\ F_q^{\alpha,p}(\psi_1 \check{*} \psi_2)(w) = \left(F_q^{\alpha,p}\psi_1\right)(w)\left(F_q^{\alpha,p}\psi_2\right)(w),$$

$$(ii)\ \breve{F}_q^{\alpha,p}(\psi_1 \check{*} \psi_2)(w) = \left(\breve{F}_q^{\alpha,p}\psi_1\right)(w)\left(\breve{F}_q^{\alpha,p}\psi_2\right)(w),$$

where the convolution product $\psi_1 \check{*} \psi_2$ is given by

$$(\psi_1 \check{*} \psi_2)(\zeta) = \int_{-\infty}^{\infty}\psi_1(z)\psi_2(\zeta - z)d_{p,q}z,\ \zeta > 0. \tag{25}$$

Proof. To prove the first part, from definitions of the (p,q)-analog of the α-th fractional Fourier transform and the convolution product, we have

$$
\begin{aligned}
F_q^{\alpha,p}(\psi_1 \check{*} \psi_2)(w) &= \int_{-\infty}^{\infty}(\psi_1 \check{*} \psi_2)(\zeta)E_{p,q}\left(i\zeta w^{\frac{1}{\alpha}}\right)d_{p,q}\zeta \\
&= \int_{-\infty}^{\infty}E_{p,q}\left(i\zeta w^{\frac{1}{\alpha}}\right)\int_{-\infty}^{\infty}\psi_1(z)\psi_2(\zeta - z)d_q z d_{p,q}\zeta.
\end{aligned}
$$

Therefore, by allowing $\zeta - z = y$, we obtain $d_{p,q}\zeta = d_{p,q}y$. Hence, computations yield

$$
\begin{aligned}
F_q^{\alpha,p}(\psi_1 \check{*} \psi_2)(w) &= \int_{-\infty}^{\infty}E_{p,q}\left(i(z+y)w^{\frac{1}{\alpha}}\right)\int_{-\infty}^{\infty}\psi_1(z)\psi_2(y)d_q z d_{p,q}y \\
&= \int_{-\infty}^{\infty}E_{p,q}\left(i(z+y)w^{\frac{1}{\alpha}}\right)\int_{-\infty}^{\infty}\psi_1(z)\psi_2(y)d_q z d_{p,q}y \\
&= \int_{-\infty}^{\infty}g(y)E_{p,q}\left(iyw^{\frac{1}{\alpha}}\right)d_{p,q}y\left(\int_{-\infty}^{\infty}\psi_1(z)E_{p,q}\left(izw^{\frac{1}{\alpha}}\right)d_q z\right) \\
&= F_q^{\alpha,p}\psi_1(w)F_q^{\alpha,p}\psi_2(w).
\end{aligned}
$$

Similar evidence supports the second part. This ends the proof. □

Theorem 5. *(Convolution Theorems) Let ψ_1 and ψ_2 be functions belonging to $S^{v}_{r.p,q}(\mathbb{R})$. Then, the convolution theorems for $F^{\alpha,p}_q$ and $\breve{F}^{\alpha,p}_q$ are given by*

$$(i)F^{\alpha,p}_q\left(\psi_1 \times^{\alpha}_q \psi_2\right)(w) = \left(F^{\alpha,p}_q\psi_1\right)(w)\left(F^{\alpha,p}_q\psi_2\right)(w),$$
$$(ii)\breve{F}^{\alpha,p}_q\left(\psi_1 \times^{\alpha}_q \psi_2\right)(w) = \left(\breve{F}^{\alpha,p}_q\psi_1\right)(w)\left(\breve{F}^{\alpha,p}_q\psi_2\right)(w),$$

where

$$\left(\psi_1 \times^{\alpha}_q \psi_2\right)(\zeta) = \int_0^{\infty} \psi_1(z)\psi_2(z-\zeta)d_{p,q}z, \;\; \zeta \geq 0. \tag{26}$$

Proof. This theorem's proof is comparable to that of the preceding theorem. However, we have

$$\begin{aligned} F^{\alpha,p}_q\left(\psi_1 \times^{\alpha}_q \psi_2\right)(w) &= \int_{-\infty}^{\infty}\left(\psi_1 \times^{\alpha}_q \psi_2\right)(\zeta)E_{p,q}\left(i\zeta w^{\frac{1}{\alpha}}\right)d_{p,q}\zeta \\ &= \int_{-\infty}^{\infty}\int_0^{\infty}\psi_1(z)\psi_2(z-\zeta)d_qzE_{p,q}\left(i\zeta w^{\frac{1}{\alpha}}\right)d_{p,q}\zeta \\ &= \int_{-\infty}^{\infty}\int_0^{\infty}\psi_1(z)\psi_2(z-\zeta)d_qzE_{p,q}\left(i\zeta w^{\frac{1}{\alpha}}\right)d_{p,q}\zeta \end{aligned}$$

Altering the variable as $u = z - \zeta$ gives

$$\begin{aligned} F^{\alpha,p}_q\left(\psi_1 \times^{\alpha}_q \psi_2\right)(w) &= \int_{-\infty}^{\infty}\int_0^{\infty}\psi_1(z)\psi_2(u)d_qzE_{p,q}\left(i(z-u)w^{\frac{1}{\alpha}}\right)d_{p,q}u \\ &= \int_{-\infty}^{\infty}\psi_2(u)E_{p,q}\left(-iuw^{\frac{1}{\alpha}}\right)d_{p,q}u\int_0^{\infty}\psi_1(z)E_{p,q}\left(izw^{\frac{1}{\alpha}}\right)d_qz \\ &= F^{\alpha,p}_q(\psi_2)(-w)\int_0^{\infty}\psi_1(z)E_{p,q}\left(izw^{\frac{1}{\alpha}}\right)d_qz. \end{aligned}$$

This ends the proof. □

Definition 4. *Let u be a locally integrable function on (a,∞). Then, the* (p,q)*-Riemann–Liouville integral of order α, $0 < \alpha \leq 1$, of the function u is given by*

$$_aI^{\alpha}_x u(x) = \frac{1}{\Gamma_{p,q}(\alpha)}\int_a^{\infty}(x-\zeta)^{\alpha-1}_{p,q}u(\zeta)d_{p,q}\zeta, \tag{27}$$

and for a locally integrable function u on $(-\infty,b)$, we have

$$_aI^{\alpha}_x u(x) = \frac{1}{\Gamma_{p,q}(\alpha)}\int_{-\infty}^{b}(x-\zeta)^{\alpha-1}_{p,q}u(\zeta)d_{p,q}\zeta. \tag{28}$$

Inserting $a = -\infty$ in (27) *and $b = \infty$ in* (28)*, we obtain the* (p,q)*-analogs for the Weyl fractional integrals of order α. However, we insert a proof for the simple result.*

Theorem 6. *Let u be a function belonging to $S^{v,r}_{p,q}$, the Lizorkin space. For $0 < \alpha \leq 1$, $0 < \beta \leq 1, \zeta > 0$ and $w \neq 0$, we have*

$$(i)\;F^{\alpha,p}_q\left({}_aI^{\beta}_{\infty}u\right)(w) = F^{\alpha,p}_q\left(\frac{\zeta^{\beta-1}}{\Gamma_{p,q}(\beta)}\right)(w)\left(F^{\alpha,p}_q u\right)(w).$$

$$(ii)\;\breve{F}^{\alpha,p}_q\left({}_aI^{\beta}_{\infty}u\right)(w) = \breve{F}^{\alpha,p}_q\left(\frac{\zeta^{\beta-1}}{\Gamma_{p,q}(\beta)}\right)(w)\left(\breve{F}^{\alpha,p}_q u\right)(w).$$

Proof. To prove the first part, we, by the definition of the (p,q)-Riemann–Liouville integral of order β, $0 < \beta \leq 1, \zeta > 0$, have

$$F_q^{\alpha,p}\left({}_aI_\infty^\beta u\right)(w) = F_q^{\alpha,p}\left(\frac{\zeta^{\beta-1}}{\Gamma_{p,q}(\beta)} \check{*} u(\zeta)\right)(w).$$

Applying Theorem 4 gives

$$F_q^{\alpha,p}\left({}_aI_\infty^\beta u\right)(w) = F_q^{\alpha,p}\left(\frac{\zeta^{\beta-1}}{\Gamma_{p,q}(\beta)}\right)(w)\left(F_q^{\alpha,p}u\right)(w).$$

The proof of the second part is omitted as it is similar. This ends the proof of our result. □

4. The (*p*,*q*)-Space $\beta_{p,q}^v\left(S_{p,q}^{v,r}, D_{p,q}^{v,r}, \check{*}, \Lambda_p^{\alpha,q}\right)$

In the present section, we aim to establish the (*p*,*q*)-analog of a recent space of generalized function (namely, the (p,q)-Boehmian space) with the sets $S_{p,q}^{v,r}, D_{p,q}^{v,r}, \check{*}, \Lambda_p^{\alpha,q}$. Therefore, we introduce a class of (p,q)-delta sequences as follows:

Definition 5. *Let $\Delta_p^{\alpha,q}$ denote the set of subsequences $(x_n)_1^\infty$ of the (p,q)-space $D_{p,q}^{v,r}$ such that the following hold:*

$$\int_{-\infty}^{\infty} x_n(\zeta) d_{p,q}\zeta = 1, (\forall n \in \mathbb{N}), \tag{29}$$

$$\|x_n\|_{p,q} = \int_{-\infty}^{\infty} |x_n(\zeta)| d_{p,q}\zeta < M, (M \in \mathbb{R}, n \in \mathbb{N}), \tag{30}$$

$$supp(x_n(\zeta)) \to 0 \text{ as } n \to \infty, A > 0. \tag{31}$$

We prove the subsequent theorem.

Theorem 7. *The class $\left(\Lambda_p^{\alpha,q}, \check{*}\right)$ forms a class of* (p,q)*-delta identities.*

Proof. Here, we show that $(x_n \check{*} y_n) \in \Lambda_p^{\alpha,q}$ for all $(x_n), (y_n) \in \Lambda_p^{\alpha,q}$. As the proofs of Equations (30) and (31) are straightforward, it suffices to show that Equation (29) holds. By applying the (*p*,*q*)-convolution theorem for $w = 0$, we have

$$F_q^{\alpha,p}(x_n \check{*} y_n)(0) = \left(F_q^{\alpha,p}x_n\right)(0)\left(F_q^{\alpha,p}y_n\right)(0). \tag{32}$$

Therefore, by applying Equation (21) for both sides of Equation (32), we derive

$$\int_{-\infty}^{\infty} (x_n \check{*} y_n)(\zeta) d_{p,q}\zeta = \left(\int_{-\infty}^{\infty} x_n(\zeta) d_{p,q}\zeta\right)\left(\int_{-\infty}^{\infty} y_n(\zeta) d_{p,q}\zeta\right).$$

Since (x_n) and (y_n) are (p,q)-delta sequences in $\Lambda_p^{\alpha,q}$ it follows that

$$\int_{-\infty}^{\infty} (x_n \check{*} y_n)(\zeta) d_{p,q}\zeta = 1.$$

The proof is ended. □

Theorem 8. *The product $\check{*}$ is commutative in $S_{p,q}^{v,r}$, i.e., $\kappa \check{*} \psi = \psi \check{*} \kappa$.*

Proof. We by the (*p*,*q*)-convolution theorem, (Theorem 4(ii)) have that

$$F_q^{\alpha,p}(\kappa \check{*} \psi)(w) = \left(F_q^{\alpha,p}\kappa\right)(w)\left(F_q^{\alpha,p}\psi\right)(w). \tag{33}$$

The right-hand side of Equation (33) can be interchanged to yield

$$F_q^{\alpha,p}(\kappa \circledast \psi)(w) = \left(F_q^{\alpha,p}\psi\right)(w)\left(F_q^{\alpha,p}\kappa\right)(w)$$

Hence, once again by the (p,q)-convolution theorem (Theorem 4(ii)), we obtain

$$F_q^{\alpha,p}(\kappa \circledast \psi)(w) = F_q^{\alpha,p}(\psi \circledast \kappa)(w).$$

Thus, applying the inverse $F_q^{\alpha,p}$ transform to both sides ends the proof of our result. □

Theorem 9. *Let $\kappa, \psi, \varphi \in S_{p,q}^{v,r}$; then, the associative law holds: $\kappa \circledast (\psi \circledast \varphi) = (\kappa \circledast \psi) \circledast \varphi$.*

Proof. By applying the $F_q^{\alpha,p}$ transform to $\kappa \circledast (\psi \circledast \varphi)$ and using (Theorem 4(ii)) four times, we obtain

$$\begin{aligned}
F_q^{\alpha,p}(\kappa \circledast (\psi \circledast \varphi))(w) &= \left(F_q^{\alpha,p}\kappa\right)(w)\left(F_q^{\alpha,p}(\psi \circledast \varphi)\right)(w) \\
&= \left(F_q^{\alpha,p}\kappa\right)(w)\left(F_q^{\alpha,p}\psi\right)(w)\left(F_q^{\alpha,p}\varphi\right)(w) \\
&= \left(F_q^{\alpha,p}\kappa \circledast \psi\right)(w)\left(F_q^{\alpha,p}\varphi\right)(w) \\
&= F_q^{\alpha,p}((\kappa \circledast \psi) \circledast \varphi)(w).
\end{aligned}$$

Hence, by applying the inverse $F_q^{\alpha,p}$ transform to both sides, we reach the given result. □

Theorem 10. *If $\psi \in D_{p,q}^{v,r}$ and $\kappa, \kappa_n, \varphi \in S_{p,q}^{v,r}, \kappa_n \to \kappa$, as $n \to \infty$, then we have*

(i) $(\kappa + \varphi) \circledast \psi = \kappa \circledast \psi + \varphi \circledast \psi$.
(ii) $\kappa_n \circledast \psi \to \kappa \circledast \psi$ *as* $n \to \infty$.
(iii) $\lambda(\kappa \circledast \psi) = (\lambda \kappa \circledast \psi)$, *for some* $\lambda \in \mathbb{C}$.

Simple computations provide proof for this theorem. We removed the information as a result.

Theorem 11. *If $\kappa \in S_{p,q}^{v,r}$ and $\psi \in D_{p,q}^{v,r}$, then $\kappa \circledast \psi \in S_{p,q}^{v,r}$.*

The definitions of $S_{p,q}^{v,r}$ and $D_{p,q}^{v,r}$ provide the proof for this theorem. We therefore removed the details.

Theorem 12. *Let $(x_n) \in \Lambda_p^{\alpha,q}$ and $\kappa \in S_{p,q}^{v,r}$; then $\kappa \circledast x_n \to \kappa$ as $n \to \infty$.*

Proof. With reference to Equation (19), we arrive at

$$\begin{aligned}
\left|\zeta^r D_{p,q}^v(\kappa \circledast x_n - \kappa)(\zeta)\right| &= \left|\zeta^r D_{p,q}^v\left(\int_{-\infty}^{\infty} \kappa_z(\zeta) - \kappa(\zeta)\right) x_n(z) d_{p,q} z\right| \\
&\leq M \int_K \left|\zeta^r D_{p,q}^v(\Phi_z(\zeta)\kappa_z(\zeta) - \kappa(\zeta))\right| d_{p,q} z \to 0
\end{aligned}$$

as $n \to \infty$ where $\Phi_z(\zeta) = \kappa_z(\zeta) - \kappa(\zeta)$, $\kappa_z(t) = \kappa(t - z)$, M is positive constant such that $|x_n| \leq M$ and K is bounded subset in $\mathbb{R}$ such that $supp(x_n) \subseteq K$ for all $n \in \mathbb{N}$.

The proof is finished. □

The space $\beta_{p,q}^v \equiv \beta_{p,q}^v\left(S_{p,q}^{v,r}, D_{p,q}^{v,r}, \circledast, \Lambda_p^{\alpha,q}\right)$ of (p,q)-Boehmians is thereby defined. The sequences (κ_n, x_n) and (θ_n, y_n) are equivalent, $(\kappa_n, x_n) \sim (\theta_m, y_m)$, in $\beta_{p,q}^v$ if

$$\kappa_n \circledast y_m = \theta_m \circledast x_n (\forall m, n \in \mathbb{N}). \tag{34}$$

Indeed, $\sim$ defines an equivalence relation on $\beta^{v}_{p,q}$ and the class containing (κ_n, x_n) is an equivalence in $\beta^{v}_{p,q}$ denoted as

$$\frac{\kappa_n}{x_n} \tag{35}$$

which we call (*p*,*q*)-Boehmian. Some (*p*,*q*)-embedding between $S^{v,r}_{p,q}$ and $\beta^{v}_{p,q}$ is expressed as

$$\kappa \to \frac{\kappa \check{*} x_n}{x_n},$$

for all $m, n \in \mathbb{N}$. If $\frac{\kappa_n}{x_n} \in \beta^{v}_{r,p,q}$ and $\varepsilon \in S^{v,r}_{p,q}$, then

$$\left(\frac{\kappa_n}{x_n}\right) \check{*} \varepsilon = \frac{\kappa_n \check{*} \varepsilon}{x_n}.$$

In the following section, we aim to construct a space of ranges for the α-th fractional (*p*,*q*)-Fourier transforms.

5. The (*p*,*q*)-Space $\beta^{\alpha}_{f}\left(S^{\alpha,q}_{p,f}, D^{\alpha,q}_{p,f}, \odot, \Lambda^{\alpha,q}_{p,f}\right)$ of Ultra-Boehmians

To define the class of (*p*,*q*)-ultra-Boehmians, we let $S^{\alpha,q}_{p,f}$ and $D^{\alpha,q}_{p,f}$ be the fraction spaces of $F^{\alpha,p}_{q}$ of all members of $S^{v,r}_{p,q}$ and $D^{v,r}_{p,q}$, over $\mathbb{R}$, respectively. In that manner, we let $\Lambda^{\alpha,q}_{p,f}$ be the fractional set of all $F^{\alpha,p}_{q}$ transforms of all sequences in $\Lambda^{\alpha}_{p,q}$. Then, we present a product on $S^{\alpha,q}_{p,f}$ as follows:

$$\left(U_f \odot V_f\right)(w) = U_f(w) V_f(w), \tag{36}$$

for $U_f \in S^{\alpha,q}_{p,f}$ and $V_f \in D^{\alpha,q}_{p,f}$. Then, we are in a position to establish the following theorem.

Theorem 13. *Let* $U_f, \left(U_{f,n}\right)_{n=1}^{\infty}, H_f, V_f \in S^{\alpha,q}_{p,f}, U_{f,n} \to U_f$ *as* $n \to \infty$ *and* $Y_f \in D^{\alpha,q}_{p,f}$. *Then, the following identities hold.*

(i) $\left(U_f + V_f\right) \odot^{q}_{p} Y_f = U_f \odot Y_f + V_f \odot Y_f,$
(ii) $U_{f,n} \odot Y_f \to U_f \odot Y_f$ *as* $U_{f,n} \to U_f$ *as* $n \to \infty,$
(iii) $U_f \odot V_f = V_f \odot U_f,$
(iv) $U_f \odot \left(V_f \odot H_f\right) = \left(U_f \odot V_f\right) \odot H_f,$
(v) $\eta\left(U_f \odot V_f\right) = \left(\eta U_f \odot V_f\right), \eta \in \mathbb{C}.$

Proof. The proofs for (i) and (ii) are simple since they resemble the proofs provided to the space $\beta^{v}_{p,q}\left(S^{v,r}_{p,q}, D^{v,r}_{p,q}, \check{*}, \Lambda^{\alpha}_{p,q}\right)$. □

Proof. (iii) Let $\kappa, \psi \in S^{v,r}_{p,q}$ be such that $U_f = F^{\alpha,p}_{q}\kappa$ and $V_f = F^{\alpha,p}_{q}\psi$; then, by Equation (36), we have

$$\left(U_f \odot V_f\right)(w) = U_f(w) V_f(w) = \left(F^{\alpha,p}_{q}\kappa\right)(w)\left(F^{\alpha,p}_{q}\psi\right)(w).$$

Hence, using Theorem 4(*ii*) gives

$$\left(U_f \odot V_f\right)(w) = F^{\alpha,p}_{q}(\kappa \check{*} \psi)(w) \in S^{\alpha,q}_{p,f}.$$

Since $\kappa \check{*} \psi = \psi \check{*} \kappa$ in $\beta^{v}_{p,q}$, it follows again from Equation (36) that

$$\left(U_f \odot V_f\right)(w) = F^{\alpha,p}_{q}(\psi \check{*} \kappa)(w) = \left(V_f \odot U_f\right)(w) \in S^{\alpha,q}_{p,f}. \tag{37}$$

□

Proof. The proof of (iv) is similar to that of (iii), whereas the proof of (v) is straightforward. The proof is therefore ended. □

Theorem 14. *Let* $(\tilde{\varphi}_n), (\varphi_n) \in \Lambda^{\alpha,q}_{p,f}$ *and* $U_f \in S^{\alpha,q}_{p,f}$, *then* $(\tilde{\varphi}_n \odot \varphi_n) \in \Lambda^{\alpha,q}_{p,f}$ *and* $\lim_{n\to\infty} U_f \odot \tilde{\varphi}_n = U_f$.

Proof. Let $(x_n), (y_n) \in \Lambda^{\alpha}_{p,q}$ be such that $F^{\alpha,p}_q x_n = \tilde{\varphi}_n$ and $F^{\alpha,p}_q y_n = \varphi_n, \forall n \in \mathbb{N}$. Then, by Equation (36), we have

$$(\tilde{\varphi}_n \odot \varphi_n)(w) = \tilde{\varphi}_n(w)\varphi_n(w) = F^{\alpha,p}_q (x_n \check{*} y_n)(w) \,.$$

Hence, $(\tilde{\varphi}_n \check{*} \varphi_n)$ belongs to $\Lambda^{\alpha,q}_{p,f}$ since $(x_n \check{*} y_n)$ belongs to $\Lambda^{\alpha}_{p,q}$. It is also possible to develop a comparable proof for the second part of the theorem. The proof of $\lim_{n\to\infty} U_f \odot \tilde{\varphi}_n = U_f$ is straightforward.

This ends the proof. □

The space $\beta^{\alpha}_f \equiv \beta^{\alpha}_f \left(S^{\alpha,q}_{p,f}, D^{\alpha,q}_{p,f}, \odot, \Lambda^{\alpha,q}_{p,f} \right)$ of ultra-Boehmians is obtained.

It is clear from the context that the set of all (p,q)-delta sequences extends the set of all delta sequences given in [38] as p and q tend to 1. Moreover, as q tends to 1 the spaces $\beta^{\alpha}_f \equiv \beta^{\alpha}_f \left(S^{\alpha,q}_{p,f}, D^{\alpha,q}_{p,f}, \odot, \Lambda^{\alpha,q}_{p,f} \right)$ and $\beta^{v}_{p,q} \equiv \beta^{v}_{p,q} \left(S^{v,r}_{p,q}, D^{v,r}_{p,q}, \check{*}, \Lambda^{\alpha,q}_{p} \right)$ of (p,q) give new spaces of Boehmians.

Two sequences $\left(U_{f,n}, X_{f,n} \right)$ and $\left(\Phi_{f,n}, Y_{f,n} \right)$ from β^{α}_f are equivalent, $\left(U_{f,n}, X_{f,n} \right) \sim_{p,q} \left(\Phi_{f,n}, Y_{f,n} \right)$, iff

$$U_{f,n} \odot Y_{f,m} = \Phi_{f,m} \odot X_{f,n} (\forall m,n \in \mathbb{N}).$$

Indeed, $\sim_{p,q}$ defines an equivalence relation on β^{α}_f. An ultra-Boehmian in β^{α}_f is written as

$$\frac{U_{f,n}}{X_{f,n}} \tag{38}$$

where $U_{f,n} = F^{\alpha,p}_q \kappa_n \in S^{\alpha,q}_{p,f}$ and $X_{f,n} = F^{\alpha,p}_q x_n \in \Lambda^{\alpha,q}_{p,f}$. An embedding between $S^{\alpha,q}_{p,f}$ and β^{α}_f is expressed as

$$y \to \frac{y \odot X_{f,n}}{X_{f,n}}, (\forall m,n \in \mathbb{N}).$$

If $\dfrac{U_{f,n}}{X_{f,n}} \in \beta^{\alpha}_f$ and X $\in S^{\alpha,q}_{p,f}$, then it follows that

$$\left(\frac{U_{f,n}}{X_{f,n}} \right) \odot X = \frac{U_{f,n} \odot X}{X_{f,n}}. \tag{39}$$

The notions of addition, convergence, and scalar multiplication in β^{α}_f are comparable to those of the first space.

Definition 6. *Let* $(x_n) \in \Lambda^{\alpha}_{p,q}$ *and* $(\kappa_n) \in S^{v,r}_{p,q}$; *then, the generalized* α*-th* (p,q)*-fractional Fourier operator* $X^{\alpha,p}_q$ *of a* (p,q)*-Boehmian* $\dfrac{\kappa_n}{x_n}$ *in* $\beta^{v}_{p,q}$ *can be drawn as*

$$X^{\alpha,p}_q \left(\frac{\kappa_n}{x_n} \right) = \frac{F^{\alpha,p}_q \kappa_n}{F^{\alpha,p}_q x_n}, \tag{40}$$

where $F^{\alpha,p}_q \kappa_n \in S^{\alpha,q}_{p,f}$ *and* $F^{\alpha,p}_q x_n \in \Lambda^{\alpha,q}_{p,f}$, *which is indeed a member of* β^{α}_f.

6. Inversion and Characteristics

This section discusses some properties of the generalized α-th (p,q)-fractional Fourier operator $X_q^{\alpha,p}$. In order to demonstrate the well-definedness of $F_q^{\alpha,p}$, we have the following theorem.

Theorem 15. *The generalized α-th (p,q)-fractional Fourier operator $X_q^{\alpha,p} : \beta^v_{p,q} \to \beta^\alpha_f$ is well-defined.*

Proof. Let us assume that $\frac{\kappa_n}{x_n} = \frac{\theta_n}{y_n} \in \beta^v_{r,p,q}$. Then, the notion of quotients of sequences in $\beta^v_{p,q}$ implies that $\kappa_n \circledast y_m = \theta_m \circledast x_n$, $(m,n \in \mathbb{N})$. Hence, applying $F_q^{\alpha,p}$ to both sides of the previous equation gives

$$F_q^{\alpha,p}(\kappa_n \circledast y_m) = F_q^{\alpha,p}(\theta_m \circledast x_n), (m,n \in \mathbb{N}). \tag{41}$$

Consequently, the (*p*,*q*)-convolution theorem (Theorem 4) says

$$\left(F_q^{\alpha,p}\kappa_n\right)\left(F_q^{\alpha,p}y_m\right) = \left(F_q^{\alpha,p}\theta_m\right)\left(F_q^{\alpha,p}x_n\right).$$

As an alternative, this might be written by using the operation $\odot$ as

$$\left(F_q^{\alpha,p}\kappa_n\right)\odot\left(F_q^{\alpha,p}y_m\right) = \left(F_q^{\alpha,p}\theta_m\right)\odot\left(F_q^{\alpha,p}x_n\right).$$

Also as a result of Equation (56) and the concept of quotients in β^α_f, we arrive at

$$\frac{F_q^{\alpha,p}\kappa_n}{F_q^{\alpha,p}x_n} = \frac{F_q^{\alpha,p}\theta_n}{F_q^{\alpha,p}y_n}, (m,n \in \mathbb{N}). \tag{42}$$

Thus, using Equation (40), we reached to the conclusion that

$$X_q^{\alpha,p}\left(\frac{\kappa_n}{x_n}\right) = X_q^{\alpha,p}\left(\frac{\theta_n}{y_n}\right), (m,n \in \mathbb{N}).$$

This ends the proof. □

Theorem 16. *The generalized α-th (p,q)-fractional Fourier operator $X_q^{\alpha,p} : \beta^v_{p,q} \to \beta^\alpha_f$ is linear. Proof of this theorem follows from the concept of addition of the (p,q)-Boehmians spaces. Hence, it has been deleted.*

Theorem 17. *Let $\frac{\kappa_n}{x_n} \in \beta^v_{p,q}$, $\frac{\kappa_n}{x_n} = 0$, then $X_q^{\alpha,p}\left(\frac{\kappa_n}{x_n}\right) = 0$.*

Proof of this theorem is straightforward. Details are, therefore, omitted.

Theorem 18. *Let $\frac{\kappa_n}{x_n}, \frac{\theta_n}{y_n} \in \beta^v_{p,q}$; then, we have*

$$X_q^{\alpha,p}\left(\frac{\kappa_n}{x_n}\circledast\frac{\theta_n}{y_n}\right)(w) = X_q^{\alpha,p}\left(\frac{\kappa_n}{x_n}\right)X_q^{\alpha,p}\left(\frac{\theta_n}{y_n}\right).$$

Proof. Let $\frac{\kappa_n}{x_n}, \frac{\theta_n}{y_n} \in \beta^v_{p,q}$ be given . Then, by employing $\circledast$, we obtain

$$X_q^{\alpha,p}\left(\frac{\kappa_n}{x_n}\circledast\frac{\theta_n}{y_n}\right) = X_q^{\alpha,p}\left(\frac{\kappa_n \circledast \theta_n}{x_n \circledast y_n}\right).$$

Hence, the (*p*,*q*)-convolution Theorem, Theorem 4, reveals

$$X_q^{\alpha,p}\left(\frac{\kappa_n}{x_n}\check{*}\frac{\theta_n}{y_n}\right)(w) = X_q^{\alpha,p}\left(\frac{\kappa_n}{x_n}\right)X_q^{\alpha,p}\left(\frac{\theta_n}{y_n}\right).$$

The proof is complete. □

Definition 7. *If* $\frac{U_{f,n}}{X_{f,n}} \in \beta_f^\alpha$, $\frac{U_{f,n}}{X_{f,n}} = \frac{F_q^{\alpha,p}\kappa_n}{F_q^{\alpha,p}x_n}$, *then we introduce the inverse operator of* $X_q^{\alpha,p}$, $\left(X_q^{\alpha,p}\right)^{-1} : \beta_f^\alpha \to \beta_{p,q}^v$, *as*

$$\left(X_q^{\alpha,p}\right)^{-1}\left(\frac{U_{f,n}}{X_{f,n}}\right) = \frac{\kappa_n}{x_n}, \tag{43}$$

for each $(x_n) \in \Lambda_{p,q}^\alpha$.

Theorem 19. *The inverse generalized α-th* (p,q)*-fractional Fourier operator* $\left(X_q^{\alpha,p}\right)^{-1} : \beta_f^\alpha \to \beta_{p,q}^v$ *is well-defined and linear.*

Proof. Let $\frac{U_{f,n}}{X_{f,n}} = \frac{V_{f,n}}{Y_{f,n}}$ in β_f^α, where $\frac{U_{f,n}}{X_{f,n}} = \frac{F_q^{\alpha,p}\kappa_n}{F_q^{\alpha,p}x_n}$ and $\frac{V_{f,n}}{Y_{f,n}} = \frac{F_q^{\alpha,p}\theta_n}{F_q^{\alpha,p}y_n}$. Then,

$$F_q^{\alpha,p}\kappa_n \odot F_q^{\alpha,p}y_m = F_q^{\alpha,p}\theta_m \odot F_q^{\alpha,p}x_n \tag{44}$$

for some $(\theta_n), (\kappa_n)$ in $S_{p,q}^{v,r}$. By using Theorem 4, we derive

$$X_q^{\alpha,p}(\kappa_n\check{*}y_m) = X_q^{\alpha,p}(\theta_m\check{*}x_n)(m,n \in \mathbb{N}).$$

Therefore, Applying the inversion formula in Equation (43), we get $\kappa_n\check{*}y_m = \theta_m\check{*}x_n(m,n \in \mathbb{N})$. Hence, we have

$$\frac{\kappa_n}{x_n} = \frac{\theta_n}{y_n}.$$

To show that $\left(X_q^{\alpha,p}\right)^{-1}$ is linear, let $\frac{U_{f,n}}{X_{f,n}} = \frac{F_q^{\alpha,p}\kappa_n}{F_q^{\alpha,p}x_n}, \frac{V_{f,n}}{Y_{f,n}} = \frac{F_q^{\alpha,p}\theta_n}{F_q^{\alpha,p}y_n}$ be members in β_f^α; then, by the addition of β_f^α and the (*p*,*q*)-convolution theorem, we write

$$\left(X_q^{\alpha,p}\right)^{-1}\left(\frac{U_{f,n}}{X_{f,n}} + \frac{V_{f,n}}{Y_{f,n}}\right) = \left(X_q^{\alpha,p}\right)^{-1}\left(\frac{F_q^{\alpha,p}(\kappa_n\check{*}y_n + \theta_n\check{*}x_n)}{F_q^{\alpha,p}(x_n\check{*}y_n)}\right).$$

Therefore, considering the inversion formula in Equation (43), we assert that

$$\left(X_q^{\alpha,p}\right)^{-1}\left(\frac{U_{f,n}}{X_{f,n}} + \frac{V_{f,n}}{Y_{f,n}}\right) = \frac{\kappa_n\check{*}y_n + \theta_n\check{*}x_n}{x_n\check{*}y_n}.$$

Hence, addition in $\beta_{p,q}^v$ finishes the proof of the theorem. □

7. Conclusions

Two (*p*,*q*)-analogs of the *α*-th fractional Fourier transform in the post-quantum calculus have been demonstrated and expanded into a class of (*p*,*q*)-generalized functions known as (*p*,*q*)-Boehmians. The generalized results are identified as a generalization of the traditional results of Romero et al. [33]. Additionally, the paper has looked into different sets of (*p*,*q*)-delta sequences, (*p*,*q*)-convolution products, and (*p*,*q*)-classes of Boehmians. As a result, the created sets of (p,q)-Boehmians were thoroughly examined using the generalized α-th (p,q)-Fourier transform and its inversion formula. Numerous findings about the generalized integral and its inverse formula were discovered.

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