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Modified Double Inertial Extragradient-like Approaches for Convex Bilevel Optimization Problems with VIP and CFPP Constraints

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Abstract: Convex bilevel optimization problems (CBOPs) exhibit a vital impact on the decision-making process under the hierarchical setting when image restoration plays a key role in signal processing and computer vision. In this paper, a modified double inertial extragradient-like approach with a line search procedure is introduced to tackle the CBOP with constraints of the CFPP and VIP, where the CFPP and VIP represent a common fixed point problem and a variational inequality problem, respectively. The strong convergence analysis of the proposed algorithm is discussed under certain mild assumptions, where it constitutes both sections that possess a mutual symmetry structure to a certain extent. As an application, our proposed algorithm is exploited for treating the image restoration problem, i.e., the LASSO problem with the constraints of fractional programming and fixed-point problems. The illustrative instance highlights the specific advantages and potential infect of the our proposed algorithm over the existing algorithms in the literature, particularly in the domain of image restoration.

Keywords: modified inertial subgradient extragradient method; variational inequality problem; pseudomonotone mapping; nonexpansive mapping; fixed point

MSC: 65Y05; 65K15; 68W10; 47H05; 47H10

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1. Introduction

Suppose $\emptyset \neq C \subset \mathcal{H}$ where C possesses both convexity and closedness and the real Hilbert space $\mathcal{H}$ has the inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. Let P_C be the nearest-point projection from $\mathcal{H}$ onto C. For a mapping $S : C \to C$, we use $\mathrm{Fix}(S)$ and $\mathbf{R}$ to indicate the fixed-point set of S and the real-number set, respectively. For an operator $A : \mathcal{H} \to \mathcal{H}$, we recall the classical variational inequality problem (VIP), i.e., the objective is to find $u^\dagger \in C$ such that $\langle Au^\dagger, v - u^\dagger \rangle \geq 0 \ \ \forall v \in C$, where $\mathrm{VI}(C, A)$ stands for the solution set of the VIP.

As far as we know, in 1976, the Korpelevich extragradient rule put forth in [1], is one of the most effective tools for tackling the VIP, i.e., for arbitrarily initial $u_0 \in C$, $\{u_n\}$ is the sequence fabricated by

$$\begin{cases} h_n = P_C(u_n - \ell A u_n), \\ u_{n+1} = P_C(u_n - \ell A h_n) \quad \forall n \geq 0, \end{cases} \tag{1}$$

with constant $\ell \in (0, \frac{1}{L})$. The research outcomes on the VIP are abundant and the Korpelevich extragradient rule has captured broad attention paid by numerous scholars. Moreover, they ameliorated this rule in different forms; refer to [2–18] and references therein, to name but a few.

In 2018, Thong and Hieu [18] first put forward the inertial subgradient extragradient approach, i.e., for any starting $u_1, u_0 \in \mathcal{H}$, $\{u_n\}$ is the sequence fabricated by

$$\begin{cases} q_n = u_n + \alpha_n(u_n - u_{n-1}), \\ h_n = P_C(q_n - \ell A q_n), \\ C_n = \{v \in \mathcal{H} : \langle q_n - \ell A q_n - h_n, v - h_n\rangle \le 0\}, \\ u_{n+1} = P_{C_n}(q_n - \ell A h_n) \quad \forall n \ge 1, \end{cases} \tag{2}$$

with constant $\ell \in (0, \frac{1}{L})$. Under mild restrictions, it was shown that $\{u_n\}$ is weakly convergent to a solution of the VIP. In 2020, the inertial subgradient extragradient-type approach was proposed in [14] for tackling the pseudomonotone VIP with Lipschitzian self-mapping on $\mathcal{H}$ and the common fixed-point problem (CFPP) of finite nonexpansive self-mappings $\{T_i\}_{i=1}^N$ on $\mathcal{H}$. Assume $\Omega = \bigcap_{i=1}^N \mathrm{Fix}(T_i) \cap \mathrm{VI}(C, A) \neq \emptyset$. Let $f : \mathcal{H} \to \mathcal{H}$ be of δ-contractivity with $0 \le \delta < 1$, and $F : \mathcal{H} \to \mathcal{H}$ be of both η-strong monotonicity and κ-Lipschitz continuity s.t. $\delta < \tau := 1 - \sqrt{1 - \rho(2\eta - \rho\kappa^2)}$ for $\rho \in (0, \frac{2\eta}{\kappa^2})$. Presume that $\{\ell_n\}, \{\gamma_n\}, \{\beta_n\}$ are the sequences in $(0, \infty)$ s.t. $\sum_{n=1}^\infty \beta_n = \infty$, $\beta_n \to 0$, $\ell_n = o(\beta_n)$, $0 < \liminf_{n\to\infty} \gamma_n \le \limsup_{n\to\infty} \gamma_n < 1$ and $\beta_n + \gamma_n < 1 \ \forall n$. Besides, we define $T_n := T_{n \bmod N} \ \forall n$, where $n \bmod N$ takes values in $\{1, 2, \ldots, N\}$.

Algorithm 1 (see [14], Algorithm 3.1). **Initialization**: Let $\lambda_1 > 0$, $\epsilon > 0$, $0 < \mu < 1$ and $u_1, u_0 \in \mathcal{H}$ be arbitrarily selected. For given u_n and u_{n-1}, select $\epsilon_n \in [0, \bar{\epsilon}_n]$, with

$$\bar{\epsilon}_n = \begin{cases} \min\{\epsilon, \frac{\ell_n}{\|u_n - u_{n-1}\|}\} & \text{if } u_n \neq u_{n-1}, \\ \epsilon & \text{otherwise.} \end{cases}$$

Iterations: Reckon u_{n+1} below:

Step 1. Reckon $q_n = T_n u_n + \epsilon_n(T_n u_n - T_n u_{n-1})$ and $y_n = P_C(q_n - \lambda_n A q_n)$.

Step 2. Put (half-space) $C_n := \{v \in \mathcal{H} : \langle q_n - \lambda_n A q_n - y_n, v - y_n\rangle \le 0\}$, and reckon $v_n = P_{C_n}(q_n - \lambda_n A y_n)$.

Step 3. Reckon $u_{n+1} = \beta_n f(u_n) + \gamma_n u_n + ((1 - \gamma_n)I - \beta_n \rho F)v_n$, and update

$$\lambda_{n+1} := \begin{cases} \min\{\lambda_n, \mu \frac{\|q_n - y_n\|^2 + \|v_n - y_n\|^2}{2\langle A q_n - A y_n, v_n - y_n\rangle}\} & \text{if } \langle A q_n - A y_n, v_n - y_n\rangle > 0, \\ \lambda_n & \text{otherwise.} \end{cases}$$

Set $n := n + 1$ and go to Step 1.

Under suitable conditions, it was proved that $\{u_n\}$ is strongly convergent to a point in Ω. On the other hand, we recall the bilevel optimization problem (BOP) (see [12]), i.e., the objective is to seek the minima below

$$\min_{x \in S_*} \omega(x), \tag{3}$$

in which $\omega : \mathcal{H} \to \mathbf{R}$ denotes a differentiable and strongly convex function and S_* stands for the nonempty solution set of the inner-level optimization problem below

$$\min_{x \in \mathcal{H}} \{f(x) + g(x)\}, \tag{4}$$

in which $f : \mathcal{H} \to \mathbf{R}$ is a differentiable and convex function, ∇f is L-Lipschitz continuous, and $g : \mathcal{H} \to \mathbf{R} \cup \{+\infty\}$ is a proper, convex and lower semi-continuous (l.s.c.) function. To the most of our knowledge, convex BOPs (CBOPs) display a crucial impact on the decision-making process under the hierarchical setting, while image restoration plays a critical role in signal processing and computer vision.

As well known from (3), $x^* \in \Lambda$ if and only if $x^* \in \mathrm{VI}(S_*, \nabla\omega)$, i.e., $x^* \in S_*$ solves the VIP: $\langle \omega(x^*), x - x^*\rangle \ge 0 \ \forall x \in S_*$, with Λ being the solution set of (3).

If there is the existence of a minimizer x^* of $f + g$, then the forward-backward operator $FB_\alpha := \mathrm{prox}_{\alpha g}(I - \alpha\nabla f)$ has a fixed point x^*, where ∇f denotes the gradient of f, $\alpha > 0$

denotes the stepsize and $\text{prox}_{\alpha g}$ denotes the proximity operator of g. That is, $x^* = FB_\alpha(x^*)$. If ∇f is ζ-inverse-strongly monotone with $\zeta > 0$ and $\alpha \in (0, 2\zeta]$, then the operator FB_α is nonexpansive. In 2016, in order to mitigate the constraints imposed by the Lipschitz condition on the gradient of f, Cruz and Nghia [19] introduced the linesearch procedure. They replaced the requirement of Lipschitz continuity for ∇f with more lenient hypotheses, as outlined below:

Hypothesis 1. *$f, g : \mathcal{H} \to \bar{\mathbf{R}} = (-\infty, \infty]$ are two proper convex l.s.c. functionals s.t.* $\text{dom}g \subset \text{dom}f$*;*

Hypothesis 2. *f is of differentiability on some open set covering* $\text{dom}g$*, the gradient of f possesses the uniform continuity on each bounded subset of* $\text{dom}f$*, and there holds the relation for ∇f to map each bounded set in* $\text{dom}g$ *to a bounded set in $\mathcal{H}$.*

In particular, Wattanataweekul et al. [12] designed the double inertial forward-backward viscosity algorithm with Linesearch C below, to solve the convex minimization problem (CMP) for the sum of both convex functions.

Linesearch C. Fix $x \in \text{dom}g,\ \theta \in (0,1),\ \delta > 0$ and $\sigma > 0$.
Input Let $\alpha = \sigma$.
When $\frac{\alpha}{2}\{\|\nabla f(FB_\alpha^2(x)) - \nabla f(FB_\alpha(x))\| + \|\nabla f(FB_\alpha(x)) - \nabla f(x)\|\}$
$> \delta(\|FB_\alpha^2(x) - FB_\alpha(x)\| + \|FB_\alpha(x) - x\|)$, **conduct**
$\alpha = \theta\alpha$.
End
Output α.

Assume that f and g satisfy the Hypotheses (H1)–(H2), and $\text{dom}f = \text{dom}g = \mathcal{H}$. Their double inertial forward-backward viscosity algorithm with Linesearch C is specified below.

Algorithm 2 (see [12], Algorithm 5). **Initialization**: Let $\{\mu_n\}, \{\rho_n\}, \{\gamma_n\}, \{\tau_n\} \subset \mathbf{R}^+$ be bounded sequences. Choose $x_1, x_0 \in \mathcal{H},\ \theta \in (0,1),\ \delta \in (0, \frac{1}{8})$ and $\sigma > 0$. Given a κ-contractive self-mapping F on $\mathcal{H}$ with $0 \le \kappa < 1$.

Iterations: For any n, reckon x_{n+1} below.

Step 1. Reckon $w_n = x_n - \theta_n(x_{n-1} - x_n)$ with $\theta_n = \begin{cases} \min\{\mu_n, \frac{\gamma_n \tau_n}{\|x_{n-1} - x_n\|}\} & \text{if } x_{n-1} \neq x_n, \\ \mu_n & \text{otherwise.} \end{cases}$

Step 2. Reckon $z_n = \text{prox}_{\alpha_n g}(I - \alpha_n \nabla f) w_n$ and $y_n = \text{prox}_{\alpha_n g}(I - \alpha_n \nabla f) z_n$, with α_n=Linesearch C $(w_n, \sigma, \theta, \delta)$.

Step 3. Reckon $u_n = y_n + \delta_n(y_n - x_{n-1})$ with $\delta_n = \begin{cases} \min\{\rho_n, \frac{\gamma_n \tau_n}{\|y_n - x_{n-1}\|}\} & \text{if } y_n \neq x_{n-1}, \\ \rho_n & \text{otherwise.} \end{cases}$

Step 4. Reckon $x_{n+1} = \gamma_n F(x_n) + (1 - \gamma_n) u_n$. Set $n := n + 1$ and go to Step 1.

The strong convergence result for Algorithm 2 was established in [12]. As a consequence, they obtained an algorithm for solving the CBOP (3)–(4). Inspired by the research works in [12,14], we devise a modified double inertial extragradient-like algorithm with Linesearch C for solving the CBOP with VIP and CFPP constraints. The strong convergence result for the proposed algorithm is proved under certain appropriate assumptions, where the proposed algorithm consists of both sections that possess a mutual symmetry structure to a certain extent. As an application, our proposed algorithm is invoked to deal with the image restoration problem, i.e., the LASSO problem with the constraints of fractional programming and fixed-point problems. The illustrative instance highlights the specific advantages and potential influence of our proposed algorithm over the existing algorithms in the literature, particularly in the domain of image restoration.

The structure of the article is sketched below: In Section 2, we release certain basic tools and terminologies for later usage. Section 3 discusses and analyzes the strong convergence of the proposed algorithm. Finally, in Section 4, our main result is invoked to deal with the image restoration problem, i.e., the LASSO problem with the constraints of fractional programming and fixed-point problems.

Our algorithm is more advantageous and more flexible than Algorithm 5 in [12] because it involves solving the VIP for the Lipschitzian pseudomonotone operator and the CFPP of finite nonexpansive operators. Our result improves and extends the corresponding results in [12,14,18].

Lastly, it is worth addressing that the existing method (see [12]) is most closely relevant to our suggested method, that is, the double inertial forward-backward viscosity algorithm with Linesearch C for tackling a CBOP (see [12]) is developed into the modified double inertial extragradient-like algorithm with Linesearch C for tackling a CBOP with CFPP and VIP constraints, where this VIP implicates a Lipschitzian pseudomonotone operator and this CFPP involves a finite family of nonexpansive mappings. It is noteworthy that the double inertial forward-backward viscosity algorithm with Linesearch C for settling the CBOP (see [12]) is invalid for tackling the CBOP with CFPP and VIP constraints due to the reasons below: (i) the first constraint imposed on the CBOP is the VIP for Lipschitzian pseudomonotone operator and (ii) the second constraint imposed on the CBOP is the CFPP of finite nonexpansive mappings. Therefore, there is no way for the double inertial forward-backward viscosity algorithm with Linesearch C to treat the CBOP with CFPP and VIP constraints. In this work, it is a natural motivation that the double inertial forward-backward viscosity algorithm with Linesearch C for tackling the CBOP is developed into the modified double inertial extragradient-like algorithm with Linesearch C for tackling the CBOP with CFPP and VIP constraints.

2. Preliminaries

Suppose $\emptyset \neq C \subset \mathcal{H}$ throughout, with C being of both convexity and closedness in $\mathcal{H}$. For a given $\{h_n\} \subset \mathcal{H}$, we use $h_n \to h$ (resp., $h_n \rightharpoonup h$) to denote the strong (resp., weak) convergence of $\{h_n\}$ to h. Let $T : C \to \mathcal{H}$ be a mapping. T is termed as being nonexpansive if $\|Tx - Ty\| \leq \|x - y\| \ \forall x, y \in C$. In addition, $T : C \to H$ is termed to be

(i) L-Lipschitzian or L-Lipschitz continuous iff $\exists L > 0$ s.t. $\|Th - Tu\| \leq L\|h - u\| \ \forall h, u \in C$;

(ii) monotone iff $\langle Th - Tu, h - u\rangle \geq 0 \ \forall h, u \in C$;

(iii) pseudomonotone iff $\langle Th, u - h\rangle \geq 0 \Rightarrow \langle Tu, u - h\rangle \geq 0 \ \forall h, u \in C$;

(iv) of $\breve{\alpha}$-strong monotonicity iff $\exists \breve{\alpha} > 0$ s.t. $\langle Th - Tv, h - v\rangle \geq \breve{\alpha}\|h - v\|^2 \ \forall h, v \in C$;

(v) of $\breve{\beta}$-inverse-strong monotonicity iff $\exists \breve{\beta} > 0$ s.t. $\langle Th - Tv, h - v\rangle \geq \breve{\beta}\|Th - Tv\|^2 \ \forall h, v \in C$;

(vi) of sequentially weak continuity if $\forall \{h_n\} \subset C$, there holds the relation: $h_n \rightharpoonup h \Rightarrow Th_n \rightharpoonup Th$.

One can clearly see that the monotonicity implies the pseudomonotonicity but the reverse implication is false. It is easily known that $\forall h \in \mathcal{H}$, $\exists | z \in C$ s.t. $\|h - z\| \leq \|h - x\| \ \ \forall x \in C$. We define $P_C h = z \ \forall h \in \mathcal{H}$. Then, P_C is known as the nearest-point projection from $\mathcal{H}$ onto C.

Lemma 1 ([20])**.** *For each $v, x \in \mathcal{H}$, $y \in C$, $s \in [0, 1]$, there are the relations below:*

(i) $\|P_C v - P_C x\|^2 \leq \langle P_C v - P_C x, v - x\rangle$;

(ii) $\langle v - P_C v, y - P_C v\rangle \leq 0$;

(iii) $\|v - P_C v\|^2 + \|y - P_C v\|^2 \leq \|v - y\|^2$;

(iv) $\|v - x\|^2 = \|v\|^2 - \|x\|^2 - 2\langle v - x, x\rangle$;

(v) $\|sv + (1 - s)x\|^2 = s\|v\|^2 + (1 - s)\|x\|^2 - s(1 - s)\|v - x\|^2$.

Lemma 2 ([7])**.** *For $0 < \beta \leq \alpha$ and $h \in \mathcal{H}$, there are the relations below:*

$$\frac{\|h - P_C(h - \alpha Ah)\|}{\alpha} \leq \frac{\|h - P_C(h - \beta Ah)\|}{\beta} \text{ and } \|h - P_C(h - \beta Ah)\| \leq \|h - P_C(h - \alpha Ah)\|.$$

Lemma 3 ([6])**.** *Let $A : C \to \mathcal{H}$ be pseudomonotone and continuous. Given an $h^\dagger \in C$. Then, $\langle Ah^\dagger, h - h^\dagger\rangle \geq 0 \ \forall h \in C \ \Leftrightarrow \ \langle Ah, h - h^\dagger\rangle \geq 0 \ \forall h \in C$.*

Lemma 4 ([21]). *Let $\{a_n\}$ be a sequence in $[0,\infty)$ s.t. $a_{n+1} \le (1-\mu_n)a_n + \mu_n b_n$ $\forall n \ge 1$, where $\{\mu_n\}$ and $\{b_n\}$ are two real sequences s.t. (i) $\{\mu_n\} \subset [0,1]$ and $\sum_{n=1}^{\infty} \mu_n = \infty$, and (ii) $\limsup_{n\to\infty} b_n \le 0$ or $\sum_{n=1}^{\infty} |\mu_n b_n| < \infty$. Then, $\lim_{n\to\infty} a_n = 0$.*

Lemma 5 ([20]). *Demiclosedness principle. Let $T : C \to C$ be a nonexpansive mapping with $\mathrm{Fix}(T) \ne \emptyset$. Then, $I - T$ is demiclosed at zero, that is, if $\{h_n\} \subset C$ s.t. $h_n \rightharpoonup h \in C$ and $(I-T)h_n \to 0$, then $(I-T)h = 0$, where I is the identity mapping of $\mathcal{H}$.*

On the other hand, the terminology of nearest-point projection is extended to the notion below.

Let $g : \mathcal{H} \to \bar{\mathbf{R}}$ be a proper convex l.s.c. function. According to [22,23], one knows that the mapping prox_g, which is termed as the proximity operator associated with g, is formulated below:

$$\mathrm{prox}_g(x) := \mathrm{argmin}_{y\in\mathcal{H}} g(y) + \frac{1}{2}\|x-y\|^2.$$

Meanwhile, it is also of formulation $\mathrm{prox}_g = (I + \partial g)^{-1}$, in which ∂g denotes the subdifferential of g, written as $\partial g(x) := \{u \in \mathcal{H} : \langle u, v - x\rangle \le g(v) - g(x) \ \forall v \in \mathcal{H}\} \ \forall x \in \mathcal{H}$.

We present some connections between the proximity and subdifferential operators. For $\alpha > 0$ and $u \in \mathcal{H}$, then $\mathrm{prox}_{\alpha g} = (I + \alpha\partial g)^{-1} : \mathcal{H} \to \mathrm{dom} g$, and $(u - \mathrm{prox}_{\alpha g}(u))/\alpha \in \partial g(\mathrm{prox}_{\alpha g}(u))$.

Lemma 6 ([24]). *Given a proper convex l.s.c. function $g : \mathcal{H} \to \bar{\mathbf{R}}$, and two sequences $\{h_n\}, \{u_n\} \subset \mathcal{H}$ are considered such that $u_n \in \partial g(h_n)$ $\forall n \ge 1$. If $h_n \rightharpoonup h$ and $u_n \to u$, then $u \in \partial g(h)$.*

Lemma 7 ([25]). *Presume that $\{\mathbf{\Phi}_n\}$ is a real sequence that does not decrease at infinity in the sense that, $\exists \{\mathbf{\Phi}_{n_k}\} \subset \{\mathbf{\Phi}_n\}$ s.t. $\mathbf{\Phi}_{n_k} < \mathbf{\Phi}_{n_k+1}$ $\forall k \ge 1$. If the sequence $\{\varphi(n)\}_{n\ge n_0}$ of integers is defined as $\varphi(n) = \max\{k \le n : \mathbf{\Phi}_k < \mathbf{\Phi}_{k+1}\}$, with integer $n_0 \ge 1$ fulfilling $\{k \le n_0 : \mathbf{\Phi}_k < \mathbf{\Phi}_{k+1}\} \ne \emptyset$, then the following holds:*

(i) $\varphi(n_0) \le \varphi(n_0+1) \le \cdots$ and $\varphi(n) \to \infty$;

(ii) $\mathbf{\Phi}_{\varphi(n)} \le \mathbf{\Phi}_{\varphi(n)+1}$ and $\mathbf{\Phi}_n \le \mathbf{\Phi}_{\varphi(n)+1}$ $\forall n \ge n_0$.

3. Convergence Analysis

In what follows, we introduce and analyze a modified double inertial extragradient-like approach with Linesearch C, to resolve the convex minimization problem (CMP) for the sum of both convex functions, with the VIP and CFPP constraints. The strong convergence outcome for the suggested approach is acquired. Whereby, we derive a new algorithm for tackling the CBOP with VIP and CFPP constraints. From now on, let $\mathrm{dom} f = \mathrm{dom} g = \mathcal{H}$, and suppose f and g fulfill the requirements (H1)-(H2). Moreover, assume always that the following holds:

A is L-Lipschitzian pseudomonotone self-mapping on $\mathcal{H}$ satisfying $\liminf_{n\to\infty} \|Ah_n\| \ge \|Ah\|$ $\forall \{h_n\} \subset C$ s.t. $h_n \rightharpoonup h$;

$\{T_i\}_{i=1}^N$ is a finite family of nonexpansive self-mappings on $\mathcal{H}$ s.t. $\Omega = \bigcap_{i=1}^N \mathrm{Fix}(T_i) \cap \mathrm{VI}(C,A) \cap S_* \ne \emptyset$.

In addition, let the sequence $\{T_n\}$ be defined as in Algorithm 1, i.e., $T_n := T_{n \bmod N}$ for each $n \ge 1$. Next, we first present a modified double inertial extragradient-like algorithm with Linesearch C as follows.

Algorithm 3. Initial Step: Let $\{\mu_n\}, \{\rho_n\}, \{\beta_n\}, \{\gamma_n\}, \{\tau_n\} \subset \mathbf{R}^+$ be bounded sequences. Choose $x_1, x_0 \in \mathcal{H}$, $\sigma, \lambda_1 > 0$, $0 < \delta < 1/8$ and $0\theta, \mu < 1$. Given a κ-contractive self-mapping F on $\mathcal{H}$ with $0 \le \kappa < 1$.

Iterative Steps: For any n, reckon x_{n+1} below.

Step 1. Reckon $w_n = x_n - \theta_n(x_{n-1} - x_n)$ with $\theta_n = \begin{cases} \min\{\mu_n, \frac{\gamma_n\tau_n}{\|x_{n-1}-x_n\|}\} & \text{if } x_{n-1} \neq x_n, \\ \mu_n & \text{otherwise.} \end{cases}$

Step 2. Reckon $z_n = \text{prox}_{\alpha_n g}(I - \alpha_n \nabla f)w_n$ and $y_n = \text{prox}_{\alpha_n g}(I - \alpha_n \nabla f)z_n$, with

$$\alpha_n = \text{Linesearch C } (w_n, \sigma, \theta, \delta).$$

Step 3. Reckon $q_n = T_n y_n + \delta_n(T_n y_n - T_n x_{n-1})$ with

$$\delta_n = \begin{cases} \min\{\rho_n, \frac{\gamma_n\tau_n}{\|y_n - x_{n-1}\|}\} & \text{if } y_n \neq x_{n-1}, \\ \rho_n & \text{otherwise.} \end{cases} \quad (5)$$

Step 4. Reckon $u_n = P_C(q_n - \lambda_n A q_n)$ and $v_n = P_{C_n}(q_n - \lambda_n A u_n)$, with

$$C_n := \{v \in \mathcal{H} : \langle q_n - \lambda_n A q_n - u_n, v - u_n \rangle \leq 0\}.$$

Step 5. Reckon $x_{n+1} = \gamma_n F(x_n) + \beta_n x_n + (1 - \beta_n - \gamma_n)v_n$, and update

$$\lambda_{n+1} := \begin{cases} \min\{\lambda_n, \mu \frac{\|q_n - u_n\|^2 + \|v_n - u_n\|^2}{2\langle A q_n - A u_n, v_n - u_n \rangle}\} & \text{if } \langle A q_n - A u_n, v_n - u_n \rangle > 0, \\ \lambda_n & \text{otherwise.} \end{cases} \quad (6)$$

Set $n := n + 1$ and go to Step 1.

Condition 3. *Presume that* $\{\tau_n\}, \{\gamma_n\}, \{\beta_n\}, \{\alpha_n\} \subset \mathbf{R}^+$ *are such that the following hold:*
(C1) $0 < a_1 \leq \alpha_n \leq a_2 < 1$;
(C2) $\gamma_n \in (0, 1)$, $\gamma_n \to 0$ *and* $\sum_{n=1}^{\infty} \gamma_n = \infty$;
(C3) $\beta_n + \gamma_n \leq 1$, $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$ *and* $\tau_n \to 0$.

Remark 1. *It is easy to see that, from the definitions of* θ_n, δ_n *we obtain that* $\lim_{n\to\infty} \frac{\theta_n}{\gamma_n} \times \|x_n - x_{n-1}\| = 0$ *and* $\lim_{n\to\infty} \frac{\delta_n}{\gamma_n} \|y_n - x_{n-1}\| = 0$. *Indeed, we have* $\theta_n \|x_n - x_{n-1}\| \leq \gamma_n \tau_n$ *and* $\delta_n \|y_n - x_{n-1}\| \leq \gamma_n \tau_n \ \ \forall n \geq 1$, *which together with* $\lim_{n\to\infty} \tau_n = 0$ *imply that* $\frac{\theta_n}{\gamma_n} \|x_n - x_{n-1}\| \leq \tau_n \to 0$ *and* $\frac{\delta_n}{\gamma_n} \|y_n - x_{n-1}\| \leq \tau_n \to 0$ *as* $n \to \infty$.

In order to show the strong convergence of Algorithm 3, we need several lemmas below. The first lemma can be found in [12], Lemma 3.1

Lemma 8. *Let* $\{x_n\}$ *be the sequence generated by Algorithm 3 and* $p \in \mathcal{H}$. *Then,*

$$\begin{aligned} \|w_n - p\|^2 - \|y_n - p\|^2 \ &\geq 2\alpha_n[(f+g)(y_n) + (f+g)(z_n) - 2(f+g)(p)] \\ &+ (1 - 8\delta)(\|w_n - z_n\|^2 + \|z_n - y_n\|^2) \quad \forall n \geq 1. \end{aligned}$$

Lemma 9. *Suppose* $\{\lambda_n\}$ *is fabricated in (6). Then, the following hold: (i)* $\{\lambda_n\}$ *is nonincreasing and (ii)* $\min\{\lambda_1, \frac{\mu}{L}\} =: \lambda \leq \lambda_n \ \forall n$.

Proof. By (6) we first obtain $\lambda_{n+1} \leq \lambda_n \ \forall n$. Also, it is evident that

$$\left.\begin{array}{c} \frac{1}{2}(\|q_n - u_n\|^2 + \|v_n - u_n\|^2) \geq \|q_n - u_n\|\|v_n - u_n\| \\ \langle A q_n - A u_n, v_n - u_n \rangle \leq L\|q_n - u_n\|\|v_n - u_n\| \end{array}\right\} \Rightarrow \lambda_{n+1} \geq \min\{\lambda_n, \frac{\mu}{L}\}.$$

□

Remark 2. *In case* $q_n = u_n$ *or* $A u_n = 0$, *one has* $u_n \in \text{VI}(C, A)$. *In fact, by Lemmas 2 and 9, when* $q_n = u_n$ *or* $A u_n = 0$, *we obtain*

$$0 = \|u_n - P_C(q_n - \lambda_n A q_n)\| \geq \|u_n - P_C(u_n - \lambda A u_n)\|.vspace6pt$$

We are now ready to show several lemmas, which are vital to discuss the strong convergence of our algorithm.

Lemma 10. *For the sequences* $\{q_n\}, \{u_n\}, \{v_n\}$ *fabricated in Algorithm 3, one has*

$$\|v_n - q\|^2 \le \|q_n - q\|^2 - (1 - \mu\frac{\lambda_n}{\lambda_{n+1}})\|q_n - u_n\|^2 - (1 - \mu\frac{\lambda_n}{\lambda_{n+1}})\|v_n - u_n\|^2 \ \ \forall q \in \Omega. \quad (7)$$

Proof. We first assert that

$$2\langle Aq_n - Au_n, v_n - u_n\rangle \le \frac{\mu}{\lambda_{n+1}}\|q_n - u_n\|^2 + \frac{\mu}{\lambda_{n+1}}\|v_n - u_n\|^2 \ \ \forall n \ge 1. \quad (8)$$

Indeed, if $\langle Aq_n - Au_n, v_n - u_n\rangle \le 0$, (8) is valid. On the contrary, by (6) one has (8). Also, let $q \in \Omega \subset C \subset C_n$. It can be readily seen that

$$\begin{aligned}\|v_n - q\|^2 &= \|P_{C_n}(q_n - \lambda_n Au_n) - P_{C_n}q\|^2 \le \langle v_n - q, q_n - \lambda_n Au_n - q\rangle \\ &= \tfrac{1}{2}\|v_n - q\|^2 + \tfrac{1}{2}\|q_n - q\|^2 - \tfrac{1}{2}\|v_n - q_n\|^2 - \langle v_n - q, \lambda_n Au_n\rangle.\end{aligned}$$

This means that

$$\|v_n - q\|^2 \le \|q_n - q\|^2 - \|v_n - q_n\|^2 - 2\langle v_n - q, \lambda_n Au_n\rangle. \quad (9)$$

According to $q \in \mathrm{VI}(C, A)$, one obtains $\langle Aq, y - q\rangle \ge 0 \ \forall y \in C$. Because A is of pseudomonotonicity on C, one has $\langle Ay, y - q\rangle \ge 0 \ \forall q \in C$. Setting $y := u_n \in C$ one obtains $\langle Au_n, q - u_n\rangle \le 0$. As a result,

$$\langle Au_n, q - v_n\rangle = \langle Au_n, q - u_n\rangle + \langle Au_n, u_n - v_n\rangle \le \langle Au_n, u_n - v_n\rangle. \quad (10)$$

Combining (9) and (10), one obtains

$$\|v_n - q\|^2 \le \|q_n - q\|^2 - \|v_n - u_n\|^2 - \|u_n - q_n\|^2 + 2\langle q_n - \lambda_n Au_n - u_n, v_n - u_n\rangle. \quad (11)$$

Since $u_n = P_C(q_n - \lambda_n Aq_n)$ and $v_n \in C_n$, we have

$$\begin{aligned}2\langle q_n - \lambda_n Au_n - u_n, v_n - u_n\rangle &= 2\langle q_n - \lambda_n Aq_n - u_n, v_n - u_n\rangle + 2\lambda_n\langle Aq_n - Au_n, v_n - u_n\rangle \\ &\le 2\lambda_n\langle Aq_n - Au_n, v_n - u_n\rangle,\end{aligned}$$

which together with (8), implies that

$$2\langle q_n - \lambda_n Au_n - u_n, v_n - u_n\rangle \le \mu\frac{\lambda_n}{\lambda_{n+1}}\|q_n - u_n\|^2 + \mu\frac{\lambda_n}{\lambda_{n+1}}\|v_n - u_n\|^2. \quad (12)$$

Therefore, substituting (12) for (11), we infer that inequality (7) holds. □

Lemma 11. *Suppose* $\{x_n\}$ *is fabricated in Algorithm 3. Assume* $\lim_{n\to\infty} \tau_n = 0$. *Then* $\{x_n\}$ *is bounded.*

Proof. Let $q \in \Omega$. Using Lemma 8, we obtain

$$\begin{aligned}\|w_n - q\|^2 - \|y_n - q\|^2 &\ge 2\alpha_n[(f+g)(y_n) + (f+g)(z_n) - 2(f+g)(q)] \\ &\quad + (1 - 8\delta)(\|w_n - z_n\|^2 + \|z_n - y_n\|^2) \\ &\ge (1 - 8\delta)(\|w_n - z_n\|^2 + \|z_n - y_n\|^2) \ge 0.\end{aligned} \quad (13)$$

Thanks to $w_n = x_n - \theta_n(x_{n-1} - x_n)$, we deduce that

$$\|y_n - q\| \le \|w_n - q\| \le \|x_n - q\| + \theta_n\|x_{n-1} - x_n\|. \quad (14)$$

This along with the definition of q_n, leads to

$$\begin{aligned} \|q_n - q\| &\leq \|T_n y_n - q\| + \delta_n \|T_n y_n - T_n x_{n-1}\| \\ &\leq \|y_n - q\| + \delta_n \|x_{n-1} - y_n\| \\ &\leq \|x_n - q\| + \theta_n \|x_{n-1} - x_n\| + \delta_n \|x_{n-1} - y_n\|. \end{aligned} \tag{15}$$

On the other hand, using (7) we obtain

$$\|v_n - q\|^2 \leq \|q_n - q\|^2 - (1 - \mu \frac{\lambda_n}{\lambda_{n+1}}) \|q_n - u_n\|^2 - (1 - \mu \frac{\lambda_n}{\lambda_{n+1}}) \|v_n - u_n\|^2.$$

Because $1 - \mu \frac{\lambda_n}{\lambda_{n+1}} \to 1 - \mu > 0$ as $n \to \infty$, we might assume $1 - \mu \frac{\lambda_n}{\lambda_{n+1}} > 0 \ \forall n$. Therefore,

$$\|v_n - q\| \leq \|q_n - q\| \quad \forall n. \tag{16}$$

According to Remark 1, one has that $\frac{\theta_n}{\gamma_n} \|x_{n-1} - x_n\| \to 0$ and $\frac{\delta_n}{\gamma_n} \|x_{n-1} - y_n\| \to 0$ as $n \to \infty$. As a result, $\exists M_1 > 0$ s.t.

$$\frac{\theta_n}{\gamma_n} \|x_{n-1} - x_n\| + \frac{\delta_n}{\gamma_n} \|x_{n-1} - y_n\| \leq M_1 \quad \forall n \geq 1. \tag{17}$$

Combining (15)–(17), we obtain

$$\begin{aligned} &\|v_n - q\| \leq \|q_n - q\| \\ &\leq \|x_n - q\| + \theta_n \|x_n - x_{n-1}\| + \delta_n \|y_n - x_{n-1}\| \\ &= \|x_n - q\| + \gamma_n [\tfrac{\theta_n}{\gamma_n} \|x_n - x_{n-1}\| + \tfrac{\delta_n}{\gamma_n} \|y_n - x_{n-1}\|] \\ &\leq \|x_n - q\| + \gamma_n M_1 \quad \forall n \geq 1. \end{aligned} \tag{18}$$

Using the definition of x_{n+1} and (18), we have

$$\begin{aligned} &\|x_{n+1} - q\| \\ &\leq \gamma_n \|F(x_n) - q\| + \beta_n \|x_n - q\| + (1 - \beta_n - \gamma_n) \|v_n - q\| \\ &\leq \gamma_n \|F(x_n) - F(q)\| + \gamma_n \|F(q) - q\| + \beta_n \|x_n - q\| + (1 - \beta_n - \gamma_n) \|v_n - q\| \\ &\leq \gamma_n \kappa \|x_n - q\| + \gamma_n \|F(q) - q\| + \beta_n \|x_n - q\| + (1 - \beta_n - \gamma_n) \|v_n - q\| \\ &\leq \gamma_n \kappa \|x_n - q\| + \gamma_n \|F(q) - q\| + \beta_n \|x_n - q\| + (1 - \beta_n - \gamma_n) [\|x_n - q\| + \gamma_n M_1] \\ &\leq [1 - \gamma_n (1 - \kappa)] \|x_n - q\| + \gamma_n [M_1 + \|F(q) - q\|] \\ &= [1 - \gamma_n (1 - \kappa)] \|x_n - q\| + \gamma_n (1 - \kappa) \cdot \tfrac{M_1 + \|F(q) - q\|}{1 - \kappa} \\ &\leq \max\{\|x_n - q\|, \tfrac{M_1 + \|F(q) - q\|}{1 - \kappa}\}. \end{aligned}$$

By induction, we obtain $\|x_n - q\| \leq \max\{\|x_1 - q\|, \frac{M_1 + \|F(q) - q\|}{1 - \kappa}\} \ \forall n$. As a result, $\{x_n\}$ is of boundedness. Consequently, $\{F(x_n)\}, \{y_n\}, \{z_n\}, \{q_n\}, \{u_n\}, \{v_n\}$ and $\{w_n\}$ all are of boundedness. □

Lemma 12. *Let $\{q_n\}, \{y_n\}, \{x_n\}, \{u_n\}$ and $\{w_n\}$ be fabricated in Algorithm 3. Suppose $x_{n+1} - x_n \to 0$, $q_n - y_n \to 0$, $q_n - u_n \to 0$, $y_n - w_n \to 0$, and $\exists \{q_{n_k}\} \subset \{q_n\}$ s.t. $q_{n_k} \rightharpoonup z \in \mathcal{H}$. Then, z lies in Ω provided Condition 3 holds.*

Proof. From Algorithm 3, we obtain $q_n - y_n = T_n y_n - y_n + \delta_n (T_n y_n - T_n x_{n-1}) \ \forall n \geq 1$, and hence

$$\begin{aligned} \|T_n y_n - y_n\| &= \|q_n - y_n - \delta_n (T_n y_n - T_n x_{n-1})\| \\ &\leq \|q_n - y_n\| + \delta_n \|T_n y_n - T_n x_{n-1}\| \\ &\leq \|q_n - y_n\| + \gamma_n \cdot \tfrac{\delta_n}{\gamma_n} \|y_n - x_{n-1}\|. \end{aligned}$$

Using Remark 1 and the assumption $q_n - y_n \to 0$, we have

$$\lim_{n \to \infty} \|y_n - T_n y_n\| = 0. \tag{19}$$

Also, from $u_n = P_C(q_n - \lambda_n A q_n)$, we have $\langle q_n - \lambda_n A q_n - u_n, v - u_n \rangle \le 0 \ \forall v \in C$, and hence

$$\frac{1}{\lambda_n}\langle q_n - u_n, v - u_n \rangle + \langle Aq_n, u_n - q_n \rangle \le \langle Aq_n, v - q_n \rangle \quad \forall v \in C. \tag{20}$$

Thanks to Lipschitz's condition on A, $\{Aq_{n_k}\}$ is of boundedness. Noticing $\lambda_n \ge \min\{\lambda_1, \frac{\mu}{L}\}$, one deduces from (20) that $\liminf_{k\to\infty}\langle Aq_{n_k}, v - q_{n_k}\rangle \ge 0$. Meanwhile, it is clear that $\langle Au_n, v - u_n \rangle = \langle Au_n - Aq_n, v - q_n \rangle + \langle Aq_n, v - q_n \rangle + \langle Au_n, q_n - u_n \rangle$. Since $q_n - u_n \to 0$, one obtains $Aq_n - Au_n \to 0$. This along with (20) arrives at $\liminf_{k\to\infty}\langle Au_{n_k}, v - u_{n_k}\rangle \ge 0$.

Let us assert $\|y_n - T_l y_n\| \to 0 \ \forall l \in \{1, \dots, N\}$. In fact, since $\|x_n - w_n\| = \theta_n \|x_{n-1} - x_n\| = \gamma_n \cdot \frac{\theta_n}{\gamma_n}\|x_{n-1} - x_n\| \to 0$, we deduce from $y_n - w_n \to 0$ that

$$\|x_n - y_n\| \le \|x_n - w_n\| + \|w_n - y_n\| \to 0 \quad (n \to \infty),$$

and hence

$$\|y_n - y_{n+1}\| \le \|y_n - x_n\| + \|x_n - x_{n+1}\| + \|x_{n+1} - y_{n+1}\| \to 0 \quad n \to \infty.$$

It is clear that for $m = 1, \dots, N$,

$$\begin{aligned}\|y_n - T_{n+m}y_n\| &\le \|y_n - y_{n+m}\| + \|y_{n+m} - T_{n+m}y_{n+m}\| + \|T_{n+m}y_{n+m} - T_{n+m}y_n\| \\ &\le 2\|y_n - y_{n+m}\| + \|y_{n+m} - T_{n+m}y_{n+m}\|.\end{aligned}$$

So, using (19) and $y_n - y_{n+1} \to 0$ one obtains $\lim_{n\to\infty}\|y_n - T_{n+m}y_n\| = 0$ for $m = 1, \dots, N$. This immediately implies that

$$\lim_{n\to\infty}\|y_n - T_l y_n\| = 0 \quad \text{for } l = 1, \dots, N. \tag{21}$$

Next, we select $\{\breve{\epsilon}_k\} \subset (0,1)$ s.t. $\breve{\epsilon}_k \downarrow 0$. For each k, one denotes by $m_k(\ge 1)$ the smallest number satisfying

$$\langle Au_{n_j}, v - u_{n_j}\rangle + \breve{\epsilon}_k \ge 0 \quad \forall j \ge m_k. \tag{22}$$

Because of the decreasing property of $\{\breve{\epsilon}_k\}$, we obtain the increasing property of $\{m_k\}$. For simplicity, $\{u_{n_{m_k}}\}$ is still written as $\{u_{m_k}\}$. From $q_n - u_n \to 0$ and $q_{n_k} \rightharpoonup z$ it is easy to see that $\|Az\| \le \liminf_{k\to\infty}\|Au_{n_k}\|$. In case $Az = 0$, one has that z lies in $\mathrm{VI}(C, A)$. In case $Az \ne 0$, from $\{u_{m_k}\} \subset \{u_{n_k}\}$ we might assume $Au_{m_k} \ne 0 \ \forall k$. Hence, one sets $h_{m_k} = \frac{Au_{m_k}}{\|Au_{m_k}\|^2}$. As a result, $\langle Au_{m_k}, h_{m_k}\rangle = 1$. Thus, by (22) one obtains $\langle Au_{m_k}, x + \varepsilon_k h_{m_k} - u_{m_k}\rangle \ge 0$. Because A is of pseudomonotonicity, one has $\langle A(v + \breve{\epsilon}_k h_{m_k}), v + \breve{\epsilon}_k h_{m_k} - u_{m_k}\rangle \ge 0$, hence arriving at

$$\langle Av, v - u_{m_k}\rangle \ge \langle Av - A(v + \breve{\epsilon}_k h_{m_k}), v + \breve{\epsilon}_k h_{m_k} - u_{m_k}\rangle - \breve{\epsilon}_k\langle Av, h_{m_k}\rangle \quad \forall k. \tag{23}$$

Let us show $\lim_{k\to\infty}\breve{\epsilon}_k h_{m_k} = 0$. In fact, using $q_{n_k} \rightharpoonup z$ and $q_n - u_n \to 0$, one obtains $u_{n_k} \rightharpoonup z$. Thus, $\{u_n\} \subset C$ implies that z lies in C. Note, that $\{u_{m_k}\} \subset \{u_{n_k}\}$ and $\varepsilon_k \downarrow 0$. Therefore,

$$0 \le \limsup_{k\to\infty}\|\breve{\epsilon}_k h_{m_k}\| = \limsup_{k\to\infty}\frac{\breve{\epsilon}_k}{\|Au_{m_k}\|} \le \frac{\limsup_{k\to\infty}\breve{\epsilon}_k}{\liminf_{k\to\infty}\|Au_{n_k}\|} = 0.$$

As a result, $\breve{\epsilon}_k u_{m_k} \to 0$.

In what follows, one claims that z lies in Ω. In fact, using $q_n - y_n \to 0$ and $q_{n_k} \rightharpoonup z$, one obtains $y_{n_k} \rightharpoonup z$. By (21) one has $y_{n_k} - T_l y_{n_k} \to 0 \ \forall l \in \{1, \dots, N\}$. Because Lemma 5 implies that $I - T_l$ is demiclosed at zero, one obtains $z \in \mathrm{Fix}(T_l)$. Therefore, one obtains $z \in \bigcap_{m=1}^N \mathrm{Fix}(T_m)$. Additionally, as $k \to \infty$, one obtains that the right-hand side of (23) converges to 0 due to the fact that A is uniformly continuous, the sequences $\{q_{m_k}\}, \{h_{m_k}\}$ are bounded and $\lim_{k\to\infty} \breve{\epsilon}_k h_{m_k} = 0$. As a result, $\langle Av, v - z\rangle = \liminf_{k\to\infty}\langle Av, v - u_{m_k}\rangle \geq 0\ \forall v \in C$. Using Lemma 3 one has that z lies in $\mathrm{VI}(C, A)$. As a result, $z \in \bigcap_{m=1}^{N} \mathrm{Fix}(T_m) \cap \mathrm{VI}(C, A)$.

Finally, let us show $z \in S_*$. Indeed, from (13) and $w_n - y_n \to 0$ it follows that

$$\begin{aligned}&(1-8\delta)(\|w_n - z_n\|^2 + \|z_n - y_n\|^2)\\ &\leq \|w_n - p\|^2 - \|y_n - p\|^2\\ &\leq \|w_n - y_n\|(\|w_n - p\| + \|y_n - p\|) \to 0 \quad (n\to\infty),\end{aligned}$$

which hence arrives at

$$\lim_{n\to\infty}\|w_n - z_n\| = \lim_{n\to\infty}\|z_n - y_n\| = 0, \tag{24}$$

and

$$\|z_n - q_n\| \leq \|z_n - y_n\| + \|y_n - q_n\| \to 0 \quad (n\to\infty).$$

Based on the hypothesis (H2), we obtain

$$\|\nabla f(w_{n_k}) - \nabla f(z_{n_k})\| \to 0 \quad (n\to\infty).$$

Using the condition (C1) and (24), we obtain

$$\lim_{k\to\infty}\left\|\frac{w_{n_k} - z_{n_k}}{\alpha_{n_k}} + \nabla f(z_{n_k}) - \nabla f(w_{n_k})\right\| = 0. \tag{25}$$

Thanks to $z_{n_k} = \mathrm{prox}_{\alpha_{n_k} g}(I - \alpha_{n_k}\nabla f)w_{n_k}$, we have

$$w_{n_k} - \alpha_{n_k}\nabla f(w_{n_k}) \in z_{n_k} + \alpha_{n_k}\partial g(z_{n_k}),$$

which hence yields

$$\frac{w_{n_k} - z_{n_k}}{\alpha_{n_k}} + \nabla f(z_{n_k}) - \nabla f(w_{n_k}) \in \partial g(z_{n_k}) + \nabla f(z_{n_k}) = \partial(f+g)(z_{n_k}). \tag{26}$$

Using Lemma 6 we conclude from (25), (26) and $z_{n_k} \rightharpoonup z$ that $0 \in \partial(f+g)(z)$. As a result, $z \in S_*$. Consequently, $z \in \bigcap_{i=1}^{N}\mathrm{Fix}(T_i) \cap \mathrm{VI}(C, A) \cap S^* = \Omega$. □

Theorem 1. *Let $\{x_n\}$ be fabricated in Algorithm 3. If Condition 3 holds, then $\{x_n\}$ strongly converges to an element $x^* \in \Omega$, where $x^* = P_\Omega F(x^*)$.*

Proof. First of all, by the Banach Contraction Principle, one knows that there is only a fixed point x^* of $P_\Omega \circ F$ in $\mathcal{H}$. So, there is only a solution $x^* \in \Omega$ of the VIP

$$\langle (I - F)x^*, v - x^*\rangle \geq 0 \quad \forall v \in \Omega. \tag{27}$$

In what follows, one divides the remainder of the proofs into a few claims.

Claim 1. We show that

$$\begin{aligned}&\{(1-8\delta)(\|z_n - w_n\|^2 + \|y_n - z_n\|^2) + (1 - \mu\tfrac{\lambda_n}{\lambda_{n+1}})(\|q_n - u_n\|^2\\ &\quad + \|v_n - u_n\|^2)\}(1-\beta_n) + (1-\beta_n)\beta_n\|v_n - x_n\|^2\\ &\leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \theta_n\|x_{n-1} - x_n\|M_2 + \delta_n\|x_{n-1} - y_n\|M_3 + \gamma_n M_4\end{aligned} \tag{28}$$

for some $M_i > 0, i = 2, 3, 4$.

Indeed, using Lemma 11, $\{x_n\}$ is bounded, and hence $\{F(x_n)\}$ and $\{q_n\}$ are of boundedness. From (13) and the definition of w_n, one has

$$\begin{aligned} \|y_n - x^*\|^2 &\le \|w_n - x^*\|^2 - (1-8\delta)(\|w_n - z_n\|^2 + \|z_n - y_n\|^2) \\ &= \|x_n - x^*\|^2 + \theta_n^2\|x_n - x_{n-1}\|^2 + 2\theta_n\langle x_n - x^*, x_n - x_{n-1}\rangle \\ &\quad - (1-8\delta)(\|z_n - w_n\|^2 + \|y_n - z_n\|^2) \\ &\le \|x_n - x^*\|^2 + \theta_n\|x_{n-1} - x_n\|[2\|x_n - x^*\| + \theta_n\|x_{n-1} - x_n\|] \\ &\quad - (1-8\delta)(\|z_n - w_n\|^2 + \|y_n - z_n\|^2). \end{aligned} \tag{29}$$

Since $q_n = T_n y_n + \delta_n(T_n y_n - T_n x_{n-1})$, by (29) one obtains

$$\begin{aligned} \|q_n - x^*\|^2 &\le [\|T_n y_n - x^*\| + \delta_n\|T_n y_n - T_n x_{n-1}\|]^2 \\ &\le [\|y_n - x^*\| + \delta_n\|y_n - x_{n-1}\|]^2 \\ &= \|y_n - x^*\|^2 + \delta_n\|x_{n-1} - y_n\|[2\|y_n - x^*\| + \delta_n\|x_{n-1} - y_n\|] \\ &\le \|x_n - x^*\|^2 + \theta_n\|x_{n-1} - x_n\|[2\|x_n - x^*\| + \theta_n\|x_{n-1} - x_n\|] \\ &\quad - (1-8\delta)(\|z_n - w_n\|^2 + \|y_n - z_n\|^2) + \delta_n\|x_{n-1} - y_n\| \\ &\quad \times [2\|y_n - x^*\| + \delta_n\|x_{n-1} - y_n\|], \end{aligned} \tag{30}$$

which together with (7), arrives at

$$\begin{aligned} &\|x_{n+1} - x^*\|^2 = \|\gamma_n(F(x_n) - v_n) + \beta_n(x_n - x^*) + (1-\beta_n)(v_n - x^*)\|^2 \\ &\le \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\|v_n - x^*\|^2 - (1-\beta_n)\beta_n\|v_n - x_n\|^2 \\ &\quad + 2\gamma_n\langle F(x_n) - v_n, x_{n+1} - x^*\rangle \\ &\le \beta_n\|x_n - x^*\|^2 + (1-\beta_n)[\|q_n - x^*\|^2 - (1 - \mu\tfrac{\lambda_n}{\lambda_{n+1}})(\|q_n - u_n\|^2 + \|v_n - u_n\|^2)] \\ &\quad - (1-\beta_n)\beta_n\|v_n - x_n\|^2 + 2\gamma_n\langle F(x_n) - v_n, x_{n+1} - x^*\rangle \\ &\le \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\{\|x_n - x^*\|^2 + \|x_{n-1} - x_n\|\theta_n[2\|x_n - x^*\| + \|x_{n-1} - x_n\|\theta_n] \\ &\quad - (1-8\delta)(\|z_n - w_n\|^2 + \|y_n - z_n\|^2) + \delta_n\|x_{n-1} - y_n\|[2\|y_n - x^*\| + \|x_{n-1} - y_n\|\delta_n] \\ &\quad - (1 - \mu\tfrac{\lambda_n}{\lambda_{n+1}})(\|q_n - u_n\|^2 + \|v_n - u_n\|^2)\} - (1-\beta_n)\beta_n\|v_n - x_n\|^2 \\ &\quad + 2\gamma_n\langle F(x_n) - v_n, x_{n+1} - x^*\rangle \\ &\le \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\{\|x_n - x^*\|^2 + \|x_{n-1} - x_n\|\theta_n M_2 \\ &\quad - (1-8\delta)(\|w_n - z_n\|^2 + \|z_n - y_n\|^2) + \|x_{n-1} - y_n\|\delta_n M_3 \\ &\quad - (1 - \mu\tfrac{\lambda_n}{\lambda_{n+1}})(\|q_n - u_n\|^2 + \|v_n - u_n\|^2)\} - \beta_n(1-\beta_n)\|x_n - v_n\|^2 + \gamma_n M_4 \\ &\le \|x_n - x^*\|^2 - (1-\beta_n)\{(1-8\delta)(\|z_n - w_n\|^2 + \|y_n - z_n\|^2) \\ &\quad + (1 - \mu\tfrac{\lambda_n}{\lambda_{n+1}})(\|q_n - u_n\|^2 + \|v_n - u_n\|^2)\} - (1-\beta_n)\beta_n \\ &\quad \times \|v_n - x_n\|^2 + \theta_n\|x_{n-1} - x_n\|M_2 + \delta_n\|x_{n-1} - y_n\|M_3 + \gamma_n M_4 \end{aligned}$$

where $\sup_{n\ge1}[2\|x_n - x^*\| + \|x_{n-1} - x_n\|\theta_n] \le M_2$, $\sup_{n\ge1}[2\|y_n - x^*\| + \|x_{n-1} - y_n\|\delta_n] \le M_3$ and $\sup_{n\ge1} 2\|F(x_n) - v_n\|\|x_{n+1} - x^*\| \le M_4$ for some $M_i > 0, i = 2,3,4$.

Claim 2. We show that

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\le [1 - \gamma_n(1-\kappa)]\|x_n - x^*\|^2 + \gamma_n(1-\kappa)\{\tfrac{\theta_n\|x_n - x_{n-1}\|}{\gamma_n} \cdot \tfrac{M_2}{1-\kappa} \\ &\quad + \tfrac{\delta_n\|y_n - x_{n-1}\|}{\gamma_n} \cdot \tfrac{M_3}{1-\kappa} + \tfrac{2}{1-\kappa}\langle F(x^*) - x^*, x_{n+1} - x^*\rangle\}. \end{aligned} \tag{31}$$

Indeed, noticing $x_{n+1} = \gamma_n F(x_n) + \beta_n x_n + (1-\beta_n-\gamma_n)v_n$, we deduce from (16) and (30) that

$$\begin{aligned}
&\|x_{n+1}-x^*\|^2 = \|\gamma_n(F(x_n)-x^*)+\beta_n(x_n-x^*)+(1-\beta_n-\gamma_n)(v_n-x^*)\|^2 \\
&= \|\gamma_n(F(x_n)-F(x^*))+\beta_n(x_n-x^*)+(1-\beta_n-\gamma_n)(v_n-x^*)+\gamma_n(F(x^*)-x^*)\|^2 \\
&\le \|\gamma_n(F(x_n)-F(x^*))+\beta_n(x_n-x^*)+(1-\beta_n-\gamma_n)(v_n-x^*)\|^2 \\
&\quad + 2\gamma_n\langle F(x^*)-x^*, x_{n+1}-x^*\rangle \\
&\le \gamma_n\|F(x_n)-F(x^*)\|^2+\beta_n\|x_n-x^*\|^2+(1-\beta_n-\gamma_n)\|v_n-x^*\|^2 \\
&\quad + 2\gamma_n\langle F(x^*)-x^*, x_{n+1}-x^*\rangle \\
&\le \gamma_n\kappa\|x_n-x^*\|^2+\beta_n\|x_n-x^*\|^2+(1-\beta_n-\gamma_n)\|q_n-x^*\|^2 \\
&\quad + 2\gamma_n\langle F(x^*)-x^*, x_{n+1}-x^*\rangle \\
&\le \gamma_n\kappa\|x_n-x^*\|^2+\beta_n\|x_n-x^*\|^2+(1-\beta_n-\gamma_n)[\|x_n-x^*\|^2+\theta_n\|x_n-x_{n-1}\|M_2 \\
&\quad + \delta_n\|y_n-x_{n-1}\|M_3]+2\gamma_n\langle F(x^*)-x^*, x_{n+1}-x^*\rangle \\
&\le [1-\gamma_n(1-\kappa)]\|x_n-x^*\|^2+\theta_n\|x_n-x_{n-1}\|M_2+\delta_n\|y_n-x_{n-1}\|M_3 \\
&\quad + 2\gamma_n\langle F(x^*)-x^*, x_{n+1}-x^*\rangle \\
&= [1-\gamma_n(1-\kappa)]\|x_n-x^*\|^2+\gamma_n(1-\kappa)\{\tfrac{\theta_n\|x_n-x_{n-1}\|}{\gamma_n}\cdot\tfrac{M_2}{1-\kappa}+\tfrac{\delta_n\|y_n-x_{n-1}\|}{\gamma_n}\cdot\tfrac{M_3}{1-\kappa} \\
&\quad + \tfrac{2}{1-\kappa}\langle F(x^*)-x^*, x_{n+1}-x^*\rangle\}.
\end{aligned}$$

Claim 3. We show that $x_n \to x^* \in \Omega$, which is only a solution of VIP (27). In fact, setting $\mathbf{\Phi}_n = \|x_n - x^*\|^2$, one can derive $\mathbf{\Phi}_n \to 0$ in both aspects below.

Aspect 1. Presume that $\exists$ (integer) $n_0 \ge 1$ s.t. there holds the nonincreasing property of $\{\mathbf{\Phi}_n\}$. One then has that $\mathbf{\Phi}_n \to d < +\infty$ and $\mathbf{\Phi}_n - \mathbf{\Phi}_{n+1} \to 0$ as $n \to \infty$. From (28) and (17) one obtains

$$\begin{aligned}
&(1-\beta_n)\{(1-8\delta)(\|w_n-z_n\|^2+\|z_n-y_n\|^2)+(1-\mu\tfrac{\lambda_n}{\lambda_{n+1}})(\|q_n-u_n\|^2 \\
&\quad +\|v_n-u_n\|^2)\}+\beta_n(1-\beta_n)\|x_n-v_n\|^2 \\
&\le \|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2+\theta_n\|x_n-x_{n-1}\|M_2+\delta_n\|y_n-x_{n-1}\|M_3+\gamma_nM_4 \\
&= \mathbf{\Phi}_n-\mathbf{\Phi}_{n+1}+\gamma_n[\tfrac{\theta_n}{\gamma_n}\|x_n-x_{n-1}\|M_2+\tfrac{\delta_n}{\gamma_n}\|y_n-x_{n-1}\|M_3]+\gamma_nM_4 \\
&\le \mathbf{\Phi}_n-\mathbf{\Phi}_{n+1}+\gamma_nM_1(M_2+M_3)+\gamma_nM_4.
\end{aligned} \tag{32}$$

Since $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$, $\gamma_n \to 0$, $\mathbf{\Phi}_n - \mathbf{\Phi}_{n+1} \to 0$ and $\lim_{n\to\infty}(1-\mu\frac{\lambda_n}{\lambda_{n+1}}) = 1-\mu > 0$, we infer from (32) that

$$\lim_{n\to\infty}\|w_n-z_n\| = \lim_{n\to\infty}\|z_n-y_n\| = 0$$

and

$$\lim_{n\to\infty}\|q_n-u_n\| = \lim_{n\to\infty}\|v_n-u_n\| = \lim_{n\to\infty}\|x_n-v_n\| = 0. \tag{33}$$

Thus, we conclude that

$$\begin{aligned}
\|v_n-q_n\| &\le \|v_n-u_n\|+\|u_n-q_n\| \to 0 \ \ (n\to\infty), \\
\|x_n-q_n\| &\le \|x_n-v_n\|+\|v_n-q_n\| \to 0 \ \ (n\to\infty), \\
\|y_n-w_n\| &\le \|y_n-z_n\|\|z_n-w_n\| \to 0 \ \ (n\to\infty), \\
\|v_n-y_n\| &\le \|v_n-x_n\|+\|x_n-w_n\|+\|w_n-y_n\| \\
&\le \|v_n-x_n\|+\theta_n\|x_n-x_{n-1}\|+\|w_n-y_n\| \to 0 \quad (n\to\infty),
\end{aligned} \tag{34}$$

and hence

$$\|q_n-y_n\| \le \|q_n-v_n\|+\|v_n-y_n\| \to 0 \ \ (n\to\infty). \tag{35}$$

Thanks to $x_{n+1} = \gamma_n F(x_n) + \beta_n x_n + (1-\beta_n-\gamma_n)v_n$, we have

$$\begin{aligned}
\|x_{n+1}-x_n\| &\le \gamma_n\|F(x_n)-x_n\|+(1-\beta_n-\gamma_n)\|v_n-x_n\| \\
&\le \gamma_n\|F(x_n)-x_n\|+\|v_n-x_n\| \to 0 \quad (n\to\infty).
\end{aligned} \tag{36}$$

Thanks to the boundedness of $\{x_n\}$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle F(x^*) - x^*, x_n - x^*\rangle = \lim_{k\to\infty}\langle F(x^*) - x^*, x_{n_k} - x^*\rangle. \quad (37)$$

Since $\mathcal{H}$ is reflexive and $\{x_n\}$ is bounded, we might assume that $x_{n_k} \rightharpoonup \tilde{x}$. Hence, from (37) we obtain

$$\begin{aligned}\limsup_{n\to\infty}\langle F(x^*) - x^*, x_n - x^*\rangle &= \lim_{k\to\infty}\langle F(x^*) - x^*, x_{n_k} - x^*\rangle \\ &= \langle F(x^*) - x^*, \tilde{x} - x^*\rangle.\end{aligned} \quad (38)$$

Because $x_{n+1} - x_n \to 0,\ q_n - y_n \to 0,\ q_n - u_n \to 0,\ y_n - w_n \to 0$ and $q_{n_k} \rightharpoonup \tilde{x}$ (due to $q_n - x_n \to 0$), by Lemma 12 we infer that $\tilde{x} \in \Omega$. Hence from (27) and (38) we obtain

$$\limsup_{n\to\infty}\langle F(x^*) - x^*, x_n - x^*\rangle = \langle F(x^*) - x^*, \tilde{x} - x^*\rangle \le 0, \quad (39)$$

which immediately leads to

$$\begin{aligned}&\limsup_{n\to\infty}\langle F(x^*) - x^*, x_{n+1} - x^*\rangle \\ &= \limsup_{n\to\infty}[\langle F(x^*) - x^*, x_{n+1} - x_n\rangle + \langle F(x^*) - x^*, x_n - x^*\rangle] \\ &\le \limsup_{n\to\infty}[\|F(x^*) - x^*\|\|x_{n+1} - x_n\| + \langle F(x^*) - x^*, x_n - x^*\rangle] \le 0.\end{aligned} \quad (40)$$

Note, that $\{\gamma_n(1-\kappa)\} \subset [0,1],\ \sum_{n=1}^{\infty}\gamma_n(1-\kappa) = \infty$, and

$$\begin{aligned}\limsup_{n\to\infty}\{&\tfrac{\theta_n\|x_n - x_{n-1}\|}{\gamma_n}\cdot\tfrac{M_2}{1-\kappa} + \tfrac{\delta_n\|y_n - x_{n-1}\|}{\gamma_n}\cdot\tfrac{M_3}{1-\kappa} \\ &+ \tfrac{2}{1-\kappa}\langle F(x^*) - x^*, x_{n+1} - x^*\rangle\} \le 0.\end{aligned}$$

Therefore, using Lemma 4 we deduce from (31) that $x_n \to x^*$ as $n \to \infty$.

Aspect 2. Suppose that $\exists\{\mathbf{\Phi}_{n_k}\} \subset \{\mathbf{\Phi}_n\}$ s.t. $\mathbf{\Phi}_{n_k} < \mathbf{\Phi}_{n_k+1}\ \forall k \in \mathcal{N}$, where $\mathcal{N}$ is the set of all positive integers. Define the mapping $\varphi : \mathcal{N} \to \mathcal{N}$ by

$$\varphi(n) := \max\{k \le n : \mathbf{\Phi}_k < \mathbf{\Phi}_{k+1}\}.$$

By Lemma 7, we obtain

$$\mathbf{\Phi}_{\varphi(n)} \le \mathbf{\Phi}_{\varphi(n)+1} \quad \text{and} \quad \mathbf{\Phi}_n \le \mathbf{\Phi}_{\varphi(n)+1}.$$

From (28) and (17) we obtain

$$\begin{aligned}&(1-\beta_{\varphi(n)})\{(1-8\delta)(\|w_{\varphi(n)} - z_{\varphi(n)}\|^2 + \|z_{\varphi(n)} - y_{\varphi(n)}\|^2) + (1-\mu\tfrac{\lambda_{\varphi(n)}}{\lambda_{\varphi(n)+1}})(\|q_{\varphi(n)} - u_{\varphi(n)}\|^2 \\ &\quad + \|v_{\varphi(n)} - u_{\varphi(n)}\|^2)\} + \beta_{\varphi(n)}(1-\beta_{\varphi(n)})\|x_{\varphi(n)} - v_{\varphi(n)}\|^2 \\ &\le \mathbf{\Phi}_{\varphi(n)} - \mathbf{\Phi}_{\varphi(n)+1} + \theta_{\varphi(n)}\|x_{\varphi(n)} - x_{\varphi(n)-1}\|M_2 + \delta_{\varphi(n)}\|y_{\varphi(n)} - x_{\varphi(n)-1}\|M_3 + \gamma_{\varphi(n)}M_4 \\ &\le \mathbf{\Phi}_{\varphi(n)} - \mathbf{\Phi}_{\varphi(n)+1} + \gamma_{\varphi(n)}M_1(M_2 + M_3) + \gamma_{\varphi(n)}M_4.\end{aligned} \quad (41)$$

This hence implies that

$$\lim_{n\to\infty}\|z_{\varphi(n)} - w_{\varphi(n)}\| = \lim_{n\to\infty}\|y_{\varphi(n)} - z_{\varphi(n)}\| = 0$$

and

$$\lim_{n\to\infty}\|q_{\varphi(n)} - u_{\varphi(n)}\| = \lim_{n\to\infty}\|v_{\varphi(n)} - u_{\varphi(n)}\| = \lim_{n\to\infty}\|x_{\varphi(n)} - v_{\varphi(n)}\| = 0.$$

So it follows that

$$\lim_{n\to\infty}\|q_{\varphi(n)} - v_{\varphi(n)}\| = \lim_{n\to\infty}\|q_{\varphi(n)} - x_{\varphi(n)}\| = \lim_{n\to\infty}\|y_{\varphi(n)} - v_{\varphi(n)}\| = 0.$$

Applying the analogous reasonings to those in the proofs of Aspect 1, one obtains

$$\lim_{n\to\infty}\|w_{\varphi(n)}-y_{\varphi(n)}\|=\lim_{n\to\infty}\|q_{\varphi(n)}-y_{\varphi(n)}\|=\lim_{n\to\infty}\|x_{\varphi(n)+1}-x_{\varphi(n)}\|=0,$$

and

$$\limsup_{n\to\infty}\langle F(x^*)-x^*,x_{\varphi(n)+1}-x^*\rangle\le 0.$$

In what follows, using (31) one obtains

$$\begin{aligned}\gamma_{\varphi(n)}(1-\kappa)\mathbf{\Phi}_{\varphi(n)} &\le \mathbf{\Phi}_{\varphi(n)}-\mathbf{\Phi}_{\varphi(n)+1}+\gamma_{\varphi(n)}(1-\kappa)\{\tfrac{\|x_{\varphi(n)-1}-x_{\varphi(n)}\|\theta_{\varphi(n)}}{\gamma_{\varphi(n)}}\cdot\tfrac{M_2}{1-\kappa}\\ &\quad+\tfrac{\|x_{\varphi(n)-1}-y_{\varphi(n)}\|\delta_{\varphi(n)}}{\gamma_{\varphi(n)}}\cdot\tfrac{M_3}{1-\kappa}+\tfrac{2}{1-\kappa}\langle F(x^*)-x^*,x_{\varphi(n)+1}-x^*\rangle\},\end{aligned}$$

which hence arrives at

$$\begin{aligned}\limsup_{n\to\infty}\mathbf{\Phi}_{\varphi(n)} &\le \limsup_{n\to\infty}\{\tfrac{\theta_{\varphi(n)}\|x_{\varphi(n)}-x_{\varphi(n)-1}\|}{\gamma_{\varphi(n)}}\cdot\tfrac{M_2}{1-\kappa}\\ &\quad+\tfrac{\delta_{\varphi(n)}\|y_{\varphi(n)}-x_{\varphi(n)-1}\|}{\gamma_{\varphi(n)}}\cdot\tfrac{M_3}{1-\kappa}+\tfrac{2}{1-\kappa}\langle F(x^*)-x^*,x_{\varphi(n)+1}-x^*\rangle\}\le 0.\end{aligned}$$

Thus, $\lim_{n\to\infty}\mathbf{\Phi}_{\varphi(n)}=0$. Also, it is easily known that

$$\begin{aligned}&\mathbf{\Phi}_{\varphi(n)+1}-\mathbf{\Phi}_{\varphi(n)}\\ &=-2\langle x_{\varphi(n)}-x_{\varphi(n)+1},x_{\varphi(n)}-x^*\rangle+\|x_{\varphi(n)}-x_{\varphi(n)+1}\|^2\\ &\le 2\|x_{\varphi(n)}-x_{\varphi(n)+1}\|\|x_{\varphi(n)}-x^*\|+\|x_{\varphi(n)}-x_{\varphi(n)+1}\|^2.\end{aligned} \tag{42}$$

Thanks to $\mathbf{\Phi}_n\le\mathbf{\Phi}_{\varphi(n)+1}$, we obtain

$$\begin{aligned}&\mathbf{\Phi}_n\le\mathbf{\Phi}_{\varphi(n)+1}\\ &\le\mathbf{\Phi}_{\varphi(n)}+2\|x_{\varphi(n)}-x_{\varphi(n)+1}\|\sqrt{\mathbf{\Phi}_{\varphi(n)}}+\|x_{\varphi(n)}-x_{\varphi(n)+1}\|^2\to 0\quad(n\to\infty).\end{aligned}$$

As a result, $\mathbf{\Phi}_n\to 0$ $(n\to\infty)$. □

In the forthcoming discussion, we let $\mathcal{H}=\mathbf{R}^n$ and introduce specific assumptions regarding the mappings f,g, and ω, that are pertinent to problem (3)–(4).

(B1) $f,g:\mathbf{R}^n\to(-\infty,\infty)$ are proper convex l.s.c. functions, with ∇f being uniformly continuous;

(B2) $\omega:\mathbf{R}^n\to(-\infty,\infty)$ is of strong convexity possessing parameter σ, where the gradient of ω is L_ω-Lipschitzian, and $s\in(0,\frac{2}{L_\omega+\sigma})$.

Based on the stated assumptions, we propose the modified double inertial extragradient-like algorithm with Linesearch C (Algorithm 4) to solve problem (4)–(4) with VIP and CFPP constraints.

Algorithm 4. Initial Step: Let $\{\mu_n\},\{\rho_n\},\{\beta_n\},\{\gamma_n\},\{\tau_n\}\subset\mathbf{R}^+$ be bounded sequences. Choose $x_1,x_0\in\mathbf{R}^n$, $\sigma,\lambda_1>0$, $0<\delta<\frac{1}{8}$ and $0<\theta,\mu<1$.

Iterative Steps: For any n, reckon x_{n+1} below.

Step 1. Reckon $w_n=x_n-\theta_n(x_{n-1}-x_n)$ with $\theta_n=\begin{cases}\min\{\mu_n,\frac{\gamma_n\tau_n}{\|x_{n-1}-x_n\|}\} & \text{if } x_{n-1}\ne x_n,\\ \mu_n & \text{otherwise.}\end{cases}$

Step 2. Reckon $z_n=\mathrm{prox}_{\alpha_n g}(I-\alpha_n\nabla f)w_n$ and $y_n=\mathrm{prox}_{\alpha_n g}(I-\alpha_n\nabla f)z_n$, with

$$\alpha_n\text{=Linesearch C }(w_n,\sigma,\theta,\delta).$$

Step 3. Reckon $q_n=T_ny_n+\delta_n(T_ny_n-T_nx_{n-1})$ with

$$\delta_n=\begin{cases}\min\{\rho_n,\frac{\gamma_n\tau_n}{\|y_n-x_{n-1}\|}\} & \text{if } y_n\ne x_{n-1},\\ \rho_n & \text{otherwise.}\end{cases}$$

Step 4. Reckon $u_n = P_C(q_n - \lambda_n A q_n)$ and $v_n = P_{C_n}(q_n - \lambda_n A u_n)$, with

$$C_n := \{v \in \mathcal{H} : \langle q_n - \lambda_n A q_n - u_n, v - u_n \rangle \le 0\}.$$

Step 5. Reckon $x_{n+1} = \gamma_n (I - s\nabla\omega)(x_n) + \beta_n x_n + (1 - \beta_n - \gamma_n) v_n$, and update

$$\lambda_{n+1} := \begin{cases} \min\{\lambda_n, \mu \frac{\|q_n - u_n\|^2 + \|v_n - u_n\|^2}{2\langle A q_n - A u_n, v_n - u_n \rangle}\} & \text{if } \langle A q_n - A u_n, v_n - u_n \rangle > 0, \\ \lambda_n & \text{otherwise.} \end{cases}$$

Set $n := n + 1$ and go to Step 1.

We provide a lemma, which is vital to our forthcoming results.

Lemma 13 ([26])**.** *Suppose $\omega : \mathbf{R}^n \to (-\infty, \infty)$ is of strong convexity with $\sigma > 0$ and the gradient of ω is Lipschitzian with constant L_ω. Choose $s \in (0, \frac{2}{\sigma + L_\omega})$ arbitrarily. Then, the mapping $S_s = I - s\nabla\omega$ is a contractive map s.t.*

$$\|(I - s\nabla\omega)u - (I - s\nabla\omega)v\| \le \sqrt{1 - \frac{2s\sigma L_\omega}{\sigma + L_\omega}} \|u - v\| \quad \forall u, v \in \mathbf{R}^n.$$

Theorem 2. *Suppose $\{x_n\}$ is fabricated in Algorithm 4 and let $\mho$ be the solution set of problem (3)–(4) with VIP and CFPP constraints and $x^* = P_\Omega(I - s\nabla\omega)(x^*)$. Then $\{x_n\}$ converges strongly to $x^* \in \mho$ provided all conditions in Theorem 1 are fulfilled.*

Proof. Consider $F = I - s\nabla\omega$. According to Lemma 13, F acts as a contractive map. Using Theorem 1, we conclude that $x_n \to x^* \in \Omega$, where $x^* = P_\Omega F(x^*)$. Therefore, for each $v \in \Omega$, one has

$$0 \ge \langle F(x^*) - x^*, v - x^* \rangle = \langle x^* - s\nabla\omega(x^*) - x^*, v - x^* \rangle = \langle -s\nabla\omega(x^*), v - x^* \rangle.$$

This immediately yields

$$\langle \nabla\omega(x^*), v - x^* \rangle \ge 0 \quad \forall v \in \Omega.$$

As a result, $x^* \in \mho$. Therefore, we obtain $x_n \to x^* \in \mho$ by Theorem 1. □

4. An Application

In this section, our Algorithm 4 is applied to find a solution to the LASSO problem with constraints of fractional programming and fixed-point problems. Since the accurate solution of this problem is unknown, one employs $\|x_n - x_{n+1}\|$ to estimate the error of the n-th iterate, which shows the utility of verifying whether the suggested algorithm converges to the solution as well or not.

First, recall some preliminaries. We set a mapping

$$\Gamma(x) := Mx + q,$$

that is found in [27] and was discussed by numerous scholars for applicable examples (see, e.g., [28]), with

$$M = BB^T + D + G \quad \text{and} \quad B, D, G \in \mathbf{R}^{m \times m}$$

where D is skew-symmetric and G is diagonal matrix, for which diagonal entries are nonnegative (hence M is positive semidefinite), and q is an element in $\mathbf{R}^m$. The feasible $C \subset \mathbf{R}^m$ is of both closedness and convexity, and formulated below

$$C := \{x \in \mathbf{R}^m : Hx \le d\},$$

with $H \in \mathbf{R}^{l \times m}$ and the vector d being nonnegative. It is not hard to find that Γ is of both β-(strong) monotonicity and L-Lipschitz continuity with $\beta = \min\{\text{eig}(M)\}$ and $L = \max\{\text{eig}(M)\}$.

As far as we know, image reconstruction implicates invoking varied matters to meliorate the quality of images. This encompasses tasks, e.g., image deblurring or deconvolution, which aim to repair any blurriness emerging in an image and rehabilitate it to a clearer and more visually appealing status. The attempt to reconstruct the image comes back to the 1950s and has been exploited in different areas, e.g., consumer photography, image/video decoding, and scientific exploration [29,30]. From mathematical viewpoint, image reconstruction is usually formulated below:

$$v = \mathcal{A}x + \check{b}, \tag{43}$$

where $v \in \mathbf{R}^m$ represents the observed image, $\mathcal{A} \in \mathbf{R}^{m\times n}$ denotes the blurring matrix, $x \in \mathbf{R}^n$ denotes the original image, and $\check{b}$ is noise. To attain our aim of rehabilitating the optimal valid image $\bar{x} \in \mathbf{R}^n$ that meets (43), we are devoted to tackling the least squares problem (44) while minimizing the impact of $\check{b}$. Via doing so, we can make sure that our technique is efficient and superior for acquiring the desired outcomes. This technique seeks to minimize the squared discrepancy between v and $\mathcal{A}x$ with the goal of ameliorating the reconstruction procedure and strengthening the image quality

$$\min_x \|v - \mathcal{A}x\|_2^2, \tag{44}$$

where $\|\cdot\|_2$ is the spectral norm. Varied iterative processes can be utilized to evaluate the solution shown in (44). It is noteworthy that (44) causes a challenge because it lies in the category of ill-posed problems. In the case when the number of unknown variables goes over the number of observations, it commonly arrives at an unstable norm. This is a vital issue that can pose varied problems. This issue has been broadly explored and recorded in varied research, e.g., [31,32]. Regularization approaches have been suggested to resolve the challenge of improving the least squares problem. In particular, Tikhonov regularizing technique becomes a crucial method that ameliorates the accuracy and stableness of solutions. Via this approach, we can attain our goal of resolving problems in the most effective and superior way possible

$$\min_x \{\|v - \mathcal{A}x\|_2^2 + \zeta\|Lx\|_2\}, \tag{45}$$

where $\zeta > 0$ is a constant, which is termed the regularization parameter, and $\|\cdot\|_2$ is the spectral norm. Besides, the Tikhonov matrix is represented by $L \in \mathbf{R}^{m\times n}$ with a default configuration viewing L as the identity matrix. The least absolute shrinkage and selection operator (LASSO), invented in Tibshirani [33], is a prominent way to tackle (43). Denoting by S_* the solution set of the minimization problem below

$$\min_x \{\|v - \mathcal{A}x\|_2^2 + \zeta\|x\|_1\}, \tag{46}$$

we aim at seeking a point $x^* \in S_*$ s.t.

$$x^* = \arg\min_{x\in S_*} \frac{1}{2}\|x\|_2^2. \tag{47}$$

Next, in our illustrative instance, we explore and apply Algorithm 4 for tackling the CBOP with constraints of fractional programming and fixed point problems. We set $\omega(x) = \frac{1}{2}\|x\|_2^2$, $f(x) = \|v - \mathcal{A}x\|_2^2$ and $g(x) = \zeta\|x\|_1$ with $\zeta = 5 \times 10^{-5}$. In this case, the observed images under consideration are blurred ones.

For convenience, let $m = n = l = 4$. We give the operator A. Consider the following fractional programming problem:

$$\begin{array}{l} \min g(x) = \frac{x^T Q x + a^T x + a_0}{b^T x + b_0}, \\ \text{subject to } x \in X := \{x \in \mathbf{R}^4 : b^T x + b_0 > 0\}, \end{array}$$

where

$$Q = \begin{pmatrix} 5 & -1 & 2 & 0 \\ -1 & 5 & -1 & 3 \\ 2 & -1 & 3 & 0 \\ 0 & 3 & 0 & 5 \end{pmatrix}, \ a = \begin{pmatrix} 1 \\ -2 \\ -2 \\ 1 \end{pmatrix}, \ b = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \ a_0 = -2, \ b_0 = 4.$$

It is easy to verify that Q is symmetric and positive definite in $\mathbf{R}^4$ and consequently g is pseudoconvex on $X = \{x \in \mathbf{R}^4 : b^T x + b_0 > 0\}$. Then,

$$Ax := \nabla g(x) = \frac{(b^T x + b_0)(2Qx + a) - b(x^T Qx + a^T x + a_0)}{(b^T x + b_0)^2}.$$

It is easily known that A is pseudomonotone (see [34] for more details). Now, we give a nonexpansive mapping $T_1 : \mathbf{R}^4 \to C$ defined by $T_1 x = P_C x \ \ \forall x \in \mathbf{R}^4$.

Let starting points x_1, x_0 be randomly selected in $\mathbf{R}^4$. Take $F(x) = (I - 0.01\nabla\omega)x$, $\mu = 0.3$, $\gamma_n = \tau_n = \frac{1}{n+1}$, $\mu_n = \rho_n = 0.1$, $\beta_n = \frac{1}{3}$, $\sigma = 2$, $\theta = 0.9$, $\delta = 0.1$,

$$\theta_n = \begin{cases} \min\{0.1, \frac{1/(n+1)^2}{\|x_{n-1} - x_n\|}\} & \text{if } x_{n-1} \neq x_n, \\ 0.1 & \text{otherwise,} \end{cases}$$

$\alpha_n =$ Linesearch C $(w_n, 2, 0.9, 0.1)$, and

$$\delta_n = \begin{cases} \min\{0.1, \frac{1/(n+1)^2}{\|x_{n-1} - y_n\|}\} & \text{if } x_{n-1} \neq y_n, \\ 0.1 & \text{otherwise.} \end{cases}$$

As a result, Algorithm 4 is rephrased as follows:

$$\begin{cases} w_n = x_n - \theta_n(x_{n-1} - x_n), \\ z_n = \mathrm{prox}_{\alpha_n g}(I - \alpha_n \nabla f) w_n, \\ y_n = \mathrm{prox}_{\alpha_n g}(I - \alpha_n \nabla f) z_n, \\ q_n = T_1 y_n - \delta_n(T_1 x_{n-1} - T_1 y_n), \\ u_n = P_C(q_n - \lambda_n A q_n), \\ v_n = P_{C_n}(q_n - \lambda_n A u_n), \\ x_{n+1} = \frac{1}{n+1} \cdot \frac{99}{100} x_n + \frac{1}{3} x_n + (\frac{n}{n+1} I - \frac{1}{3} I) v_n \quad \forall n \geq 1, \end{cases}$$

in which λ_n and C_n are selected as in Algorithm 4 for every n. Therefore, Theorem 2 guarantees that $\{x_n\}$ is convergent to a solution of the LASSO problem with constraints of the fractional programming problem and the fixed-point problem of T_1.

In the end, it is worth mentioning that, there have been many works that deal with the problem of designing an algorithm to solve (46) see, e.g., [35,36] and the references wherein. Moreover, some of them are able to solve globally the problem using non-convexity assumptions.

5. Conclusions

This article is focused on designing and analyzing iterative algorithms to tackle convex bilevel optimization problem (CBOP) with CFPP and VIP constraints, with the CFPP and VIP representing a common fixed point problem and a variational inequality problem, respectively. Here, the CFPP implicates finite nonexpansive mappings and the VIP involves a Lipschitzian pseudomonotone mapping in a real Hilbert space.

To the best of our awareness, the CBOP reveals a prominent role in the decision-making process under the hierarchical setting, when image reconstruction exhibits a vital effect on signal processing and computer vision. With the help of the subgradient extragradient and forward-backward viscosity methods, we have designed a novel double inertial extragradient-like approach with Linesearch C for tackling the CBOP with the CFPP and

VIP constraints. Under certain appropriate conditions, we have proved that the sequence fabricated by the proposed algorithm is strongly convergent to a solution of the CBOP with CFPP and VIP constraints, where the proposed algorithm consists of both sections which possess a mutual symmetry structure to a certain extent. As an application, our proposed algorithm is exploited for treating the image restoration problem, i.e., the LASSO problem with the constraints of fractional programming and fixed-point problems. The illustrative instance highlights the specific advantages and potential influence of our proposed algorithm over the existing algorithms in the literature, particularly in the domain of image restoration. Finally, it is worth mentioning that a section of our subsequent investigation is concentrated on establishing the strong convergence outcome for the modification of our devised method with quasi-inertial Tseng's extragradient steps (see [13]) and adaptive step sizes.

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