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On a Class of Generalized Multivariate Hermite–Humbert Polynomials via Generalized Fibonacci Polynomials

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Abstract: This paper offers a thorough examination of a unified class of Humbert's polynomials in two variables, extending beyond well-known polynomial families such as Gegenbauer, Humbert, Legendre, Chebyshev, Pincherle, Horadam, Kinnsy, Horadam–Pethe, Djordjević, Gould, Milovanović, Djordjević, Pathan, and Khan polynomials. This study's motivation stems from exploring polynomials that lack traditional nomenclature. This work presents various expansions for Humbert–Hermite polynomials, including those involving Hermite–Gegenbauer (or ultraspherical) polynomials and Hermite–Chebyshev polynomials. The proofs enhanced our understanding of the properties and interrelationships within this extended class of polynomials, offering valuable insights into their mathematical structure. This research consolidates existing knowledge while expanding the scope of Humbert's polynomials, laying the groundwork for further investigation and applications in diverse mathematical fields.

Keywords: Hermite polynomials; generalized humbert polynomials; generalized (p,q)-Fibonacci polynomials; generalized (p,q)-Lucas polynomials

MSC: 0A115; 05A19; 11B39; 33C45; 33C55; 33C99

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1. Introduction and Preliminaries

Special polynomials represent a distinctive class in mathematics, characterized by their inherent symmetries and unique properties that play pivotal roles in diverse mathematical areas. Their symmetrical structures often facilitate solving differential equations and analyzing orthogonal functions, making them tools for various fields. The symmetry inherent in these polynomials contributes to their wide-ranging applications, extending their utility to disciplines such as physics, engineering, and computer science, where they provide elegant solutions to complex problems.

Among the prominent examples of special polynomials are several well-known families. Legendre polynomials often appear in solving problems related to potential fields and wave equations, making them essential in electrostatics and quantum mechanics. Chebyshev polynomials are central to approximation theory and numerical analysis, particularly in minimizing errors in polynomial approximations. Hermite polynomials are crucial in quantum mechanics, especially in the study of the harmonic oscillator, where they describe the wave functions of quantum states. Humbert polynomials play an important role in combinatorial mathematics and the analysis of divided differences, contributing to the study of sequences and series. Appell polynomials are significant in the theory of special functions and modular forms, with applications extending to complex analysis and number theory. Touchard polynomials are instrumental in probability theory and combinatorics, particularly in the enumeration of permutations and partitions. These polynomials provide solutions to complex mathematical problems and are powerful tools for modeling and analysis across various scientific and engineering fields, where their inherent symmetry properties play a crucial role in simplifying computations and revealing underlying structures.

In 1965, Gould [1,2] made a significant contribution to the field of special functions by introducing a new class of polynomials, now referred to as Humbert polynomials, which he denoted as $\mathcal{P}_{\omega}(\theta, \nu_1, \nu_2, p, c)$. His work thoroughly analyzed these polynomials, delving into their explicit mathematical expressions, recurrence relations, and higher-order derivatives. Additionally, Gould explored their operational expansions and the inverse relations associated with them, offering a comprehensive framework for understanding these polynomials' behavior under various transformations and operations. Due to their ability to encompass several well-established families of polynomials through specific parameters, Humbert polynomials are particularly intriguing. For instance, when particular parameter values are selected, these polynomials can be reduced to Chebyshev, Gegenbauer, Kinney, Legendre, Liouville, or Pincherle polynomials. This unifying characteristic highlights the versatility of Humbert polynomials, making them a powerful tool in both theoretical research and practical applications.

The adaptability of the Humbert–Gould polynomials $\mathcal{P}_{\omega}(\theta, \nu_1, \nu_2, p, c)$ has attracted considerable attention from mathematicians, leading to a wealth of research aimed at expanding and generalizing their properties. Over the years, numerous scholars have contributed to this growing body of knowledge, bringing new insights and extensions to the original work. For example, Agarwal and Parihar [3] have provided valuable generalizations that broaden the scope of these polynomials in different mathematical contexts. Dilcher and Djordjević and their collaborators [4–8] have made significant strides in exploring the deeper properties of these polynomials, investigating their relationships with other special functions, and expanding their operational frameworks [9–14].

Researchers like He and Shiue [15,16] have furthered the understanding of Humbert–Gould polynomials by examining their applications in combinatorial mathematics and other areas. Horadam and their collaborators Mohan and Pethe have extensively studied these polynomials, focusing on their combinatorial aspects and connections to other mathematical structures. Khan and Pathan [17] have also played a key role in advancing the operational theory related to these polynomials, providing new methods for their manipulation and application [18–21].

Other notable contributions come from Milovanović and Djordjević [22], who have explored the algebraic properties of Humbert–Gould polynomials, and Dave [23], Liu [24], and Ma [25], who has worked on extending their applicability to various fields. Ramírez [26,27], Nalli and Haukkanen [28] Sinha [29], Shreshta [30], Wang, and the collaborative work of Wang and Wang [31,32] have all added to the rich tapestry of research surrounding these polynomials, each offering unique perspectives and insights that have furthered the mathematical community's understanding of these versatile functions.

Therefore, Gould's introduction of Humbert polynomials has opened up a vast area of mathematical inquiry, with numerous researchers building upon their foundational work. The ongoing exploration and generalization of these polynomials have deepened our understanding of their inherent properties and expanded their application across various domains, making them an essential subject of study in modern mathematics (see [1,2,12,13,15,24]).

The Humbert–Gould polynomials [1] are introduced via the generating function given by:

$$(c - m\nu_1\xi + \nu_2\xi^{\theta})^{p} = \sum_{\omega=0}^{\infty} \mathcal{P}_{\omega}(\theta, \nu_1, \nu_2, p, c)\xi^{\omega}, \tag{1}$$

where θ is a positive integer, and the remaining parameters are generally free of restrictions. The explicit form of the polynomial $\mathcal{P}_\omega(\theta, \nu_1, \nu_2, p, c)$ is expressed as follows (see [1]):

$$\mathcal{P}_\omega(\theta, \nu_1, \nu_2, p, c) = \sum_{\phi=0}^{\left[\frac{\omega}{\theta}\right]} \binom{p}{\phi} \binom{p-\phi}{\omega-\theta\phi} c^{p-\omega+(\theta-1)\phi} \nu_2^{\phi} (-\theta\nu_1)^{\omega-\theta\phi}, \tag{2}$$

where $[.]$ is the greatest integer.

In recent work, Wang and Wang [32] extended the concept by introducing generalized forms of the (p,q)-Fibonacci and (p,q)-Lucas polynomials. These are defined as follows.

Let $\theta \geq 2$ be a fixed positive integer, and consider two polynomials $p(\nu_1)$ and $q(\nu_1)$ with real coefficients. The generalized (p,q)-Fibonacci polynomials, denoted by $u_{\omega,\theta}(\nu_1)$ and (p,q)-Lucas polynomials $v_{\omega,\theta}(\nu_1)$, are defined through the generating functions (refer to [32]):

$$u_{\omega,\theta}(\nu_1) = p(\nu_1)u_{\omega-1,\theta}(\nu_1) + q(\nu_1)u_{\omega-\theta,\theta}(\nu_1), \ \ \omega \geq \theta, \tag{3}$$

with the initial conditions $u_{0,\theta}(\nu_1) = 0$, $u_{1,\theta}(\nu_1) = 1$, $u_{2,\theta}(\nu_1) = p(\nu_1) \cdots, u_{\theta-1,\theta}(\nu_1) = p^{\theta-2}(\nu_1)$. And

$$v_{\omega,\theta}(\nu_1) = p(\nu_1)v_{\omega-1,\theta}(\nu_1) + q(\nu_1)v_{\omega-\theta,\theta}(\nu_1), \ \ \omega \geq \theta \tag{4}$$

with the initial conditions $v_{0,\theta} = 2$, $v_{1,\theta} = p(\nu_1)$, $v_{2,\theta}(\nu_1) = p^2(\nu_1) \cdots, v_{\theta-1,\theta}(\nu_1) = p^{\theta-1}(\nu_1)$.

The generating functions for the sequences $(u_{\omega,\theta}(\nu_1))$ and $(v_{\omega,\theta}(\nu_1))$ are given by the following expressions [32]:

$$U_\theta(\nu_1, \xi) = \sum_{\omega=0}^{\infty} u_{\omega,\theta}(\nu_1)\xi^\omega = \frac{\xi}{1 - p(\nu_1)\xi - q(\nu_1)\xi^\theta}, \tag{5}$$

and

$$V_\theta(\nu_1, \xi) = \sum_{\omega=0}^{\infty} v_{\omega,\theta}(\nu_1)\xi^\omega = \frac{2 - p(\nu_1)\xi}{1 - p(\nu_1)\xi - q(\nu_1)\xi^\theta}. \tag{6}$$

These definitions imply the following relationship:

$$u_{\omega,\theta}(\nu_1) = u_{\omega+1,\theta}(\nu_1) + q(\nu_1)u_{\omega-\theta+1,\theta}(\nu_1), \quad \omega \geq \theta - 1. \tag{7}$$

It is important to note that the sequences of polynomials $(u_{\omega,\theta}(\nu_1))$ and $(v_{\omega,\theta}(\nu_1))$ satisfy the same recurrence relation of order θ, yet they differ in their initial conditions. These sequences are sometimes referred to as the generalized Lucas u-polynomial and the generalized v-polynomial sequences, respectively.

The sequences $(u_{\omega,\theta}(\nu_1))$ and $(v_{\omega,\theta}(\nu_1))$ include several well-known polynomial sequences as particular cases. For example, when $\theta = 2$, these sequences simplify to the classical (p,q)-Fibonacci polynomials $u_\omega(\nu_1)$ and (p,q)-Lucas polynomials $v_\omega(\nu_1)$ (as defined in (see [19,21,26,27,31]. Further simplifications yield familiar polynomials associated with names such as Fibonacci, Lucas, Pell, Pell–Lucas, Jacobsthal, and Jacobsthal–Lucas, among others.

Recently, Wang and Wang [32] introduced the generalized the Humbert polynomials $u^{(r)}_{\omega+1,\theta}(\nu_1)$ as the convolutions of Fibonacci polynomials.

Definition 1. *For each complex number r, the generalized convolved (p,q)-Fibonacci polynomials, also known as the generalized Humbert polynomials $u^{(r)}_{\omega+1,\theta}(\nu_1)$, are defined by the generating function:*

$$(1 - p(\nu_1)\xi - q(\nu_1)\xi^\theta)^{-r} = \sum_{\omega=0}^{\infty} u^{(r)}_{\omega+1,\theta}(\nu_1)\frac{\xi^\omega}{\omega!}, \tag{8}$$

where θ is a positive integer.

This relationship underscores the connection between these generalized polynomial sequences and various well-known sequences in the mathematical literature.

The introduction of special functions with numerous indices and variables represents a noteworthy progression in the field of generalized special functions. These functions hold considerable importance, finding recognition in both practical applications and pure mathematical contexts. The demand for polynomials with multiple indices and variables arises from the necessity to address challenges across various mathematical disciplines, from the study of partial differential equations to abstract group theory. Recently, the polynomials represented by $\mathcal{F}_{\omega}^{[\theta]}(\nu_1, \nu_2, \cdots, \nu_\theta)$, known as multivariable Hermite Polynomials (MHP), were introduced in (see [33,34]) and are given by generating relation:

$$\exp\Big(\nu_1\xi + \nu_2\xi^2 + \cdots + \nu_\theta\xi^\theta\Big) = \sum_{\omega=0}^{\infty} \mathcal{F}_{\omega}^{[\theta]}(\nu_1, \nu_2, \cdots, \nu_\theta)\frac{\xi^\omega}{\omega!}, \tag{9}$$

with the operational rule:

$$\exp\left(\nu_2\frac{\partial^2}{\partial\nu_1{}^2} + \nu_3\,\frac{\partial^3}{\partial\nu_1{}^3} + \cdots + \nu_m\frac{\partial^\theta}{\partial\nu_1{}^\theta}\right)\nu_1^\omega = \mathcal{F}_{\omega}^{[\theta]}(\nu_1, \nu_2, \cdots, \nu_\theta), \tag{10}$$

and series representation:

$$\mathcal{F}_{\omega}^{[\theta]}(\nu_1, \nu_2, \cdots \nu_\theta) = \omega! \sum_{r=0}^{[\omega/\theta]} \frac{\nu_\theta^r\, \mathcal{F}_{\omega-\theta r}^{[\theta]}(\nu_1, \nu_2, \cdots, \nu_{\theta-1})}{r!\,(\omega-\theta r)!}. \tag{11}$$

This paper is structured as follows. Section 2 delves into the introduction of the generalized multivariate Humbert–Hermite polynomials, denoted as ${}_{\mathcal{F}}\mathcal{G}_{\omega+1,\theta}^{(r)}(\nu_1, \nu_2, \cdots, \nu_\theta)$. These polynomials are constructed by leveraging the framework of generalized (p,q)-Fibonacci polynomials. We provide a comprehensive exploration of their mathematical properties, including but not limited to their generating relations, explicit forms, and notable identities. This section also examines the underlying algebraic structure and potential applications of these polynomials in various mathematical contexts.

Section 3 shifts focus to the analysis and derivation of various expansions related to Hermite polynomials. Specifically, we obtain expansions for the Hermite–Chebyshev and Hermite–Gegenbauer polynomials. This section systematically explores the connections between these classical orthogonal polynomials, providing detailed proofs and discussing the implications of these expansions in broader mathematical and applied fields. Through these expansions, we highlight how these polynomials can be expressed in terms of other well-known polynomial families, thereby enriching the theory and application of Hermite-related polynomials. The paper concludes with some remarks.

2. Generalized Multivariate Hermite–Humbert Polynomials

In this section, we present the introduction of generalized multivariate Hermite–Humbert polynomials, denoted as ${}_{\mathcal{F}}\mathcal{G}_{\omega+1,\theta}^{(r)}(\nu_1, \nu_2, \cdots, \nu_\theta)$. These polynomials are developed within the framework of generalized (p,q)-Fibonacci polynomials, which serve as the foundational building blocks. By extending the concept of the classic Hermite–Humbert polynomials into a multivariate setting, we aim to explore the deeper structural relationships and functional properties that emerge in this generalized form. The introduction of these polynomials is not merely a theoretical exercise; it represents a significant expansion in the field of polynomial theory, with potential applications in areas such as combinatorics, number theory, and the study of special functions. This work seeks to provide a robust mathematical foundation for these polynomials, facilitating further exploration and application in both pure and applied mathematics.

We begin by formally defining the generalized multivariate Hermite–Humbert polynomials ${}_{\mathcal{F}}\mathcal{G}^{(r)}_{\omega+1,\theta}(\nu_1,\nu_2,\cdots,\nu_\theta)$, establishing the groundwork for a detailed examination of their properties. These properties include generating functions, recurrence relations, and identities that reveal the intricate connections between the multivariate Hermite–Humbert polynomials and the underlying (p,q)-Fibonacci polynomials. By deriving and analyzing these properties, we aim to uncover the mathematical significance of these polynomials, demonstrating how they generalize classical results and contribute to the broader understanding of polynomial sequences. This section serves as a critical foundation for the subsequent analysis, providing the essential tools and definitions required for the in-depth exploration of these polynomials and their applications in later sections.

Definition 2. *For each complex number r, the generalized convolved (p,q)-Fibonacci polynomials, also known as the generalized multivariate Hermite–Humbert polynomials ${}_{\mathcal{F}}\mathcal{G}^{(r)}_{\omega+1,\theta}(\nu_1,\nu_2,\cdots,\nu_\theta)$, are defined by the generating function:*

$$(1-p(\nu_1)\xi-q(\nu_1)\xi^\theta)^{-r}e^{\nu_2\xi+\nu_3\xi^2+\cdots+\nu_\theta\xi^\theta}=\sum_{\omega=0}^{\infty}{}_{\mathcal{F}}\mathcal{G}^{(r)}_{\omega+1,\theta}(\nu_1,\nu_2,\cdots,\nu_\theta)\frac{\xi^\omega}{\omega!}, \tag{12}$$

where θ is a positive integer, $r>0$, and the other parameters $\nu_1,\nu_2,\ldots,\nu_\theta$ are generally unrestricted.

Reduction to Known Results:

By setting $\nu_2=\cdots=\nu_\theta=0$ in Equation (12), we obtain a known result by Wang and Wang [32]:

Furthermore, by taking $\theta=2$, Equation (12) reduces to another known result by Pathan and Khan [20].

Special Cases and Connections:

When $r=1$, the polynomials ${}_{\mathcal{F}}\mathcal{G}^{(1)}_{\omega+1,\theta}(\nu_1,\nu_2,\cdots,\nu_\theta)$ become the generalized multivariate (p,q)-Fibonacci polynomials ${}_{\mathcal{F}}\nu^{(1)}_{\omega,\theta}(\nu_1,\nu_2,\cdots,\nu_\theta)$, that is:

$${}_{\mathcal{F}}\mathcal{G}^{(1)}_{\omega+1,\theta}(\nu_1,\nu_2,\cdots,\nu_\theta)={}_{\mathcal{F}}u^{(1)}_{\omega,\theta}(\nu_1,\nu_2,\cdots,\nu_\theta),\quad \text{for } \omega\geq 1. \tag{13}$$

By adjusting the generating function in Equation (12) with the substitutions $p(\nu_1)\to\frac{\theta\nu_1}{c}$, $q(\nu_1)\to-\frac{w\nu_2}{c}$, and $r\to-r$, where $c\neq 0$, we obtain:

$$(1-\frac{\theta\nu_1}{c}\xi+\frac{w\nu_2}{c}\xi^\theta)^{-r}e^{\nu_2\xi+\nu_3\xi^2+\cdots+\nu_\theta\xi^\theta}=c^{-r}\sum_{\omega=0}^{\infty}{}_{\mathcal{F}}\mathcal{P}_\omega(\theta,\nu_1,\nu_2,\cdots,\nu_\theta,r,c)\frac{\xi^\omega}{\omega!}. \tag{14}$$

where ${}_{\mathcal{F}}\mathcal{P}_\omega(\theta,\nu_1,\nu_2,\cdots,\nu_\theta,w,r,c)$ denotes the multivariate Humbert–Hermite–Gould polynomials. These polynomials can be further specialized into other polynomial families, such as the multivariate Hermite–Gegenbauer polynomials, Pincherle polynomials, and others, by appropriately choosing the parameters $\theta,\nu_1,\nu_2,\cdots,\nu_\theta,r$, and c.

Polynomial Sequences and Representations:

By choosing particular values for θ, $p(\nu_1)$, and $q(\nu_1)$ in Equation (12), we can generate different polynomial sequences, as illustrated in Table 1. Furthermore, using the definitions of $\mathcal{F}^{[\theta]}_\omega(\nu_2,\nu_3,\cdots,\nu_\theta)$ and $u^{(r)}_{\omega+1,\theta}(\nu_1)$, we derive the following representation:

$${}_{\mathcal{F}}\mathcal{G}^{(r)}_{\omega+1,\theta}(\nu_1,\nu_2,\cdots,\nu_\theta)=\omega!\sum_{\phi=0}^{\omega}\frac{\mathcal{F}^{[\theta]}_\phi(\nu_2,\nu_3,\cdots,\nu_\theta)u^{(r)}_{\omega-\phi+1,\theta}(\nu_1)}{\phi!}. \tag{15}$$

Some significant special cases of this representation are detailed below. By substituting $q(\nu_1)\to-q(\nu_1)$ into Equation (15), we derive additional relationships and transformations of these polynomials, which are as follows:

$$ {}_{\mathcal{F}}\mathcal{G}^{(r)}_{\omega+1,\theta}(\nu_1,\nu_2,\nu_3,\cdots,\nu_\theta) = {}_{\mathcal{F}}C^{(r,\theta)}_{\omega}(\nu_1,\nu_2,\nu_3,\cdots,\nu_\theta) = \omega! \sum_{\phi=0}^{\omega} \frac{\mathcal{F}^{[\theta]}_{\phi}(\nu_2,\nu_3,\cdots,\nu_\theta) C^{(r,\theta)}_{\omega-\phi}(\nu_1)}{\phi!}. $$

where ${}_{\mathcal{F}}C^{(r,\theta)}_{\omega}(\nu_1,\nu_2,\nu_3,\cdots,\nu_\theta)$ are called the multivariate Hermite–Gegenbaurer polynomials:

$$ {}_{\mathcal{F}}C^{(1,\theta)}_{\omega}(\nu_1,\nu_2,\nu_3,\cdots,\nu_\theta) = {}_{\mathcal{F}}\nu^{\theta}_{\omega}(\nu_1,\nu_2,\nu_3,\cdots,\nu_\theta) = \omega! \sum_{\phi=0}^{\omega} \frac{\mathcal{F}^{[\theta]}_{\phi}(\nu_2,\nu_3,\cdots,\nu_\theta) u^{\theta}_{\omega-\phi}(\nu_1)}{\phi!}, $$

where ${}_{\mathcal{F}}u^{\theta}_{\omega}(\nu_1,\nu_2,\nu_3,\cdots,\nu_\theta)$ are called the multivariate Hermite–Chebyshev polynomials.

$$ {}_{\mathcal{F}}C^{1/2,\theta}_{\omega}(\nu_1,\nu_2,\nu_3,\cdots,\nu_\theta) = {}_{\mathcal{F}}\mathcal{P}^{\theta}_{\omega}(\nu_1,\nu_2,\nu_3,\cdots,\nu_\theta) = \omega! \sum_{\phi=0}^{\omega} \frac{\omega! \mathcal{F}^{[\theta]}_{\phi}(\nu_2,\nu_3,\cdots,\nu_\theta) \mathcal{P}^{\theta}_{\omega-\phi}(\nu_1)}{\phi!}, $$

where ${}_{\mathcal{F}}\mathcal{P}^{[\theta]}_{\omega}(\nu_1,\nu_2,\nu_3,\cdots,\nu_\theta)$ denotes the multivariate Hermite–Legendre polynomials.

As a special case, if we set $\nu_2 = \nu_3 = \cdots = 2\nu_1$, $\nu_m = -1$, and $q(\nu_1) \to -q(\nu_1)$ in Equation (12), the generalized multivariate Humbert–Hermite polynomial ${}_{\mathcal{F}}\mathcal{G}^{(r)}_{\omega+1,\theta}$ $(\nu_1,\nu_2,\nu_3,\cdots,\nu_\theta)$ simplifies to the Humbert–Hermite polynomial ${}_{\mathcal{F}}\mathcal{G}^{(r,)}_{\omega,\theta}(\nu_1)$ in one variable. Consequently, Equation (12) yields the following generating function:

$$ [1-p(\nu_1)\xi + q(\nu_1)\xi^{\theta}]^{-r} e^{2\nu_1\xi-\xi^{\theta}} = \sum_{\omega=0}^{\infty} {}_{\mathcal{F}}\mathcal{G}^{(r)}_{\omega,\theta}(\nu_1)\xi^{\omega}. \tag{16} $$

Furthermore, the Hermite–Gegenbauer (or ultraspherical) polynomials ${}_{\mathcal{F}}C^{(r)}_{\omega,2}(\nu_1)$ (which are denoted as ${}_{\mathcal{F}}C^{r}_{\omega}(\nu_1)$) in one variable, for nonnegative integer r, are given by:

$$ e^{2\nu_1\xi-\xi^2}(1-p(\nu_1)\xi + q(\nu_1)\xi^2)^{-r} = \sum_{\omega=0}^{\infty} {}_{\mathcal{F}}C^{r}_{\omega}(\nu_1)\frac{\xi^{\omega}}{\omega!}. \tag{17} $$

Letting $r = 1/2$ and $r = 1$, respectively, in (17) gives:

$$ e^{2\nu_1\xi-\xi^2}(1-p(\nu_1)\xi + q(\nu_1)\xi^2)^{-1/2} = \sum_{\omega=0}^{\infty} {}_{\mathcal{F}}\mathcal{P}_{\omega}(\nu_1)\frac{\xi^{\omega}}{\omega!}, \tag{18} $$

where ${}_{\mathcal{F}}\mathcal{P}_{\omega}(\nu_1)$ are Hermite–Legendre polynomials, and

$$ e^{2\nu_1\xi-\xi^2}(1-p(\nu_1)\xi + q(\nu_1)\xi^2)^{-1} = \sum_{\omega=0}^{\infty} {}_{\mathcal{F}}u_{\omega}(\nu_1)\frac{\xi^{\omega}}{\omega!}, \tag{19} $$

where ${}_{\mathcal{F}}u_{\omega}(\nu_1)$ represents the Hermite–Chebyshev polynomials.

Next, we derive the explicit expressions for the generalized multivariate Hermite–Humbert polynomials. We start with the following theorem.

Theorem 1. *Let $\omega \geq 0$ and $\phi \in \mathbb{N}$. Then:*

$$
{}_{\mathcal{F}}\mathcal{G}^{(r)}_{\omega+1,\theta}(\nu_1,\nu_2,\cdots,\nu_\theta) = \omega! \sum_{\delta=0}^{\omega} \sum_{\psi=0}^{[\frac{\omega-\delta}{\theta}]} \binom{-r}{N} \binom{N}{\psi} (-p)^{\omega-\delta-\theta\psi}(-q)^{\psi}
\times \mathcal{F}^{[\theta]}_{\delta}(\nu_2,\cdots,\nu_\theta)\frac{1}{\delta!}, \tag{20}
$$

where $N = \omega - \delta - (\theta-1)\psi$.

Proof. From (12), we have:

$$
\sum_{\omega=0}^{\infty} {}_{\mathcal{F}}\mathcal{G}^{(r)}_{\omega+1,\theta}(\nu_1,\nu_2,\cdots,\nu_\theta)\frac{\xi^\omega}{\omega!} = \sum_{\phi=0}^{\infty} \binom{-r}{\phi} (-p\xi - q\xi^\theta)^\phi \sum_{\delta=0}^{\infty} \mathcal{F}^{[\theta]}_{\delta}(\nu_2,\cdots,\nu_\theta)\frac{\xi^\delta}{\delta!}
$$

$$
= \sum_{\omega=0}^{\infty} \sum_{\psi=0}^{[\frac{\omega}{\theta}]} \binom{-r}{\omega-(\theta-1)\phi} \binom{\omega-(\theta-1)\psi}{\psi} (-p)^{\omega-\theta\psi}(-q)^{\psi}\xi^\omega
\times \sum_{\delta=0}^{\infty} \mathcal{F}^{[\theta]}_{\delta}(\nu_2,\cdots,\nu_\theta)\frac{\xi^\delta}{\delta!}
$$

$$
\sum_{\omega=0}^{\infty} {}_{\mathcal{F}}\mathcal{G}^{(r)}_{\omega+1,\theta}(\nu_1,\nu_2,\cdots,\nu_\theta)\frac{\xi^\omega}{\omega!} = \sum_{\omega=0}^{\infty} \sum_{\delta=0}^{\omega} \sum_{\psi=0}^{[\frac{\omega-\delta}{\theta}]} \binom{-r}{\omega-\delta-(\theta-1)\psi} \binom{\omega-\delta-(\theta-1)\psi}{\psi} (-p)^{\omega-\delta-\theta\psi}(-q)^{\psi}
\times \mathcal{F}^{[\theta]}_{\delta}(\nu_2,\cdots,\nu_\theta)\frac{\xi^\delta}{\delta!}.
$$

Hence, we complete the proof of the theorem. □

Remark 1. *On setting $\nu_2 = \cdots = \nu_\theta = 0$ in (20) we get the known result of Wang and Wang [32].*

Remark 2. *Adjusting $\theta = 2$, $\nu_\theta = -1$ and replacing ν_2 by $2\nu_2$ in (20), we get:*

$$
{}_{\mathcal{F}}\mathcal{G}^{(r)}_{\omega+1,2}(\nu_1,2\nu_2,-1) = \omega! \sum_{\delta=0}^{\omega} \sum_{\phi=0}^{[\frac{\omega-\delta}{2}]} \binom{-r}{\omega-\delta-\phi} \binom{\omega-\delta-\phi}{\phi} (-p)^{\omega-\delta-2\phi}(-q)^{\phi} \times \frac{\mathcal{F}_\delta(\nu_2)}{\delta!}. \tag{21}
$$

Theorem 2. *Let $\omega \geq 0$ and $\phi \in \mathbb{N}$. Then:*

$$
{}_{\mathcal{F}}\mathcal{G}^{(r)}_{\omega+1,\theta}(\nu_1,\nu_2,\cdots,\nu_\theta) = \sum_{\delta=0}^{\omega} \frac{\omega!\mathcal{F}^{[\theta]}_{\delta}(\nu_2,\cdots,\nu_\theta)}{\delta!} \sum_{i=0}^{[\frac{\omega-\delta}{2}]}
\times \sum_{0\leq j\leq i} \binom{r+i-1}{i} \binom{i}{j} \binom{M}{N} \left(\frac{p(\nu_1)}{2}\right)^{\omega-\delta-\theta\phi} q^\phi(\nu_1), \tag{22}
$$

where $M = \omega + 2r - \delta - (\theta-2)\phi - 1$, $N = \omega - \delta - 2i - (\theta-2)\phi$.

Table 1. Special cases of the generalized multivariate Hermite–Humbert polynomials.

	$p(\nu_1)$	$q(\nu_1)$	m	r	Generating Function	$\sum_{n=0}^{\infty} {}_{\mathcal{F}}\mathcal{G}^{(r)}_{\omega+1,\theta}(\nu_1,\nu_2,\cdots,\nu_\theta)\frac{\xi^\omega}{\omega!}$	Polynomials
(1)	1	1	2	r	$(1-\xi-\xi^2)^{-r}e^{\nu_2\xi+\nu_3\xi^2+\cdots+\nu_\theta\xi^\theta}$	${}_{\mathcal{F}}F^{(r)}_{\omega+1}(\nu_2,\nu_3,\cdots,\nu_\theta\xi^\theta)$	multivariate Hermite–Fibonacci–Hoggatt
(2)	k	1	2	r	$(1-k\xi-\xi^2)^{-r}e^{\nu_2\xi+\nu_3\xi^2+\cdots+\nu_\theta\xi^\theta}$	${}_{\mathcal{F}}F^{(r)}_{\phi,\omega+1}(\nu_2,\cdots,\nu_\theta)$	multivariate Hermite–Fibonacci–Ramèrez
(3)	1	ν	2	r	$(1-\xi-\nu\xi^2)^{-r}e^{\nu_2\xi+\nu_3\xi^2+\cdots+\nu_\theta\xi^\theta}$	${}_{\mathcal{F}}D^{(r)}_{\omega+1}(\nu,\nu_2,\cdots,\nu_\theta)$	multivariate Hermite–Dilcher
(4)	2ν	1	2	r	$(1-2\nu\xi-\xi^2)^{-r}e^{\nu_2\xi+\nu_3\xi^2+\cdots+\nu_\theta\xi^\theta}$	${}_{\mathcal{F}}\mathbb{P}^{(r)}_{\omega+1}(\nu,\nu_2,\cdots,\nu_\theta)$	multivariate Hermite–Pell–Horadam–Mohan
(5)	2ν	-1	2	$\frac{1}{2}$	$(1-2\nu\xi-\xi^2)^{-\frac{1}{2}}e^{\nu_2\xi+\nu_3\xi^2+\cdots+\nu_\theta\xi^\theta}$	${}_{\mathcal{F}}\mathcal{P}_{\omega}(\nu,\nu_2,\cdots,\nu_\theta)$	multivariate Hermite–Legendre
(6)	2ν	-1	2	r	$(1-2\nu\xi-\xi^2)^{-r}e^{\nu_2\xi+\nu_3\xi^2+\cdots+\nu_\theta\xi^\theta}$	${}_{\mathcal{F}}C^{(r)}_{\omega}(\nu,\nu_2,\cdots,\nu_\theta)$	multivariate Hermite–Gegenbauer
(7)	2ν	1 -2ν	2	r	$(1-2\nu\xi+(2\nu-1)\xi^2)^{-r}e^{\nu_2\xi+\nu_3\xi^2+\cdots+\nu_\theta\xi^\theta}$	${}_{\mathcal{F}}S^{(r)}_{\omega}(\nu,\nu_2,\cdots,\nu_\theta)$	multivariate Hermite–Sinha
(8)	2ν	p	2	r	$(1-2\nu\xi-p\xi^2)^{-r}e^{\nu_2\xi+\nu_3\xi^2+\cdots+\nu_\theta\xi^\theta}$	${}_{\mathcal{F}}F^{(r)}_{\omega}(\nu,\nu_2,\cdots,\nu_\theta)$	multivariate Hermite–Fibonacci–Liu
(9)	$h(\nu)$	1	2	r	$(1-h(\nu)\xi-\xi^2)^{-r}e^{\nu_2\xi+\nu_3\xi^2+\cdots+\nu_\theta\xi^\theta}$	${}_{\mathcal{F}}F^{(r)}_{h,\omega+1}(\nu,\nu_2,\cdots,\nu_\theta)$	multivariate Hermite–Fibonacci–Ramèrez
(10)	$p(\nu_1)$	$q(\nu_1)$	2	r	$(1-p(\nu_1)\xi-q(\nu_1)\xi^2)^{-r}e^{\nu_2\xi+\nu_3\xi^2+\cdots+\nu_\theta\xi^\theta}$	${}_{\mathcal{F}}u^{(r)}_{\omega+1}(\nu_1,\nu_2,\cdots,\nu_\theta)$	multivariate Hermite–Fibonacci–Wang
(11)	$\frac{2q}{p^2}$	$\frac{1}{p^2}$	2	$\frac{1}{2}$	$p(p^2-2q\xi-\xi^2)^{-\frac{1}{2}}e^{\nu_2\xi+\nu_3\xi^2+\cdots+\nu_\theta\xi^\theta}$	${}_{\mathcal{F}}f^{(r)}_{\omega+1,p,q}(\nu_2,\cdots,\nu_\theta)$	Hermite–Lioville
(12)	2ν	-1	3	r	$(1-2\nu\xi+\xi^3)^{-r}e^{\nu_2\xi+\nu_3\xi^2+\cdots+\nu_\theta\xi^\theta}$	${}_{\mathcal{F}}\mathcal{P}^{(r)}_{\omega+1}(\nu,\nu_2,\cdots,\nu_\theta)$	multivariate Hermite–Horadam–Pethe
(13)	3ν	-1	3	$\frac{1}{2}$	$(1-3\nu\xi+\xi^3)^{-\frac{1}{2}}e^{\nu_2\xi+\nu_3\xi^2+\cdots+\nu_\theta\xi^\theta}$	${}_{\mathcal{F}}\mathbb{P}_{\omega}(\nu,\nu_2,\cdots,\nu_\theta)$	multivariate Hermite–Pincherle
(14)	3ν	-1	3	r	$(1-3\nu\xi+\xi^3)^{-r}e^{\nu_2\xi+\nu_3\xi^2+\cdots+\nu_\theta\xi^\theta}$	${}_{\mathcal{F}}\mathbb{P}^{(r)}_{\omega}(\nu,\nu_2,\cdots,\nu_\theta)$	multivariate Hermite–Pincherle–Humbert
(15)	1	2ν	θ	r	$(1-\xi-2\nu\xi^\theta)^{-r}e^{\nu_2\xi+\nu_3\xi^2+\cdots+\nu_\theta\xi^\theta}$	${}_{\mathcal{F}}J^{(r)}_{\omega+1,\theta}(\nu,\nu_2,\cdots,\nu_\theta)$	multivariate Gould–Hopper–Jacobsthal–Djordjević
(16)	ν	-1	θ	r	$(1-\nu\xi+\xi^\theta)^{-r}e^{\nu_2\xi+\nu_3\xi^2+\cdots+\nu_\theta\xi^\theta}$	${}_{\mathcal{F}}V^{(r)}_{\omega+1,\theta}(\nu,\nu_2,\cdots,\nu_\theta)$	multivariate Hermite–Chebyshev–Djordjević
(17)	ν	-2	θ	r	$(1-\nu\xi+2\xi^\theta)^{-r}e^{\nu_2\xi+\nu_3\xi^2+\cdots+\nu_\theta\xi^\theta}$	${}_{\mathcal{F}}a^{(r)}_{\omega+1,\theta}(\nu,\nu_2,\cdots,\nu_\theta)$	multivariate Hermite–Fermat–Djordjević
(18)	2ν	-1	θ	r	$(1-2\nu\xi+\xi^\theta)^{-r}e^{\nu_2\xi+\nu_3\xi^2+\cdots+\nu_\theta\xi^\theta}$	${}_{\mathcal{F}}\mathcal{P}^{(r)}_{\omega,\theta}(\nu,\nu_2,\cdots,\nu_\theta)$	multivariate Hermite–Milovanović–Djordjević
(19)	$\theta\nu$	-1	θ	$\frac{1}{\theta}$	$(1-\theta\nu\xi+\xi^\theta)^{-\frac{1}{\theta}}e^{\nu_2\xi+\nu_3\xi^2+\cdots+\nu_\theta\xi^\theta}$	${}_{\mathcal{F}}\mathcal{P}^{(r)}_{\omega,\theta}(\nu,\nu_2,\cdots,\nu_\theta)$	multivariate Hermite–Kinney
(20)	$\theta\nu$	-1	θ	r	$(1-\theta\nu\xi+\xi^\theta)^{-r}e^{\nu_2\xi+\nu_3\xi^2+\cdots+\nu_\theta\xi^\theta}$	${}_{\mathcal{F}}\prod^{(r)}_{\omega,\theta}(\nu,\nu_2,\cdots,\nu_\theta)$	multivariate Hermite–Humbert
(21)	$\frac{\theta\nu}{c}$	$-\frac{\nu}{c}$	θ	-r	$c^{-r}(c-\theta\nu\xi+\nu\xi^\theta)^{-r}e^{\nu_2\xi+\nu_3\xi^2+\cdots+\nu_\theta\xi^\theta}$	$c^{-r}{}_{\mathcal{F}}\mathcal{P}_{\omega}(\theta,\nu,\nu_2,\cdots,\nu_\theta,r,c)$	multivariate Hermite–Humbert–Gould
(22)	$2+\nu$	-1	θ	r	$(1-(2+\nu)\xi+\xi^\theta)^{-r}e^{\nu_2\xi+\nu_3\xi^2+\cdots+\nu_\theta\xi^\theta}$	${}_{\mathcal{F}}B^{(r)}_{\omega+1,\theta}(\nu,\nu_2,\cdots,\nu_\theta)$	multivariate Hermite–Morgan–Voyce–Djordjević

Proof. Let $p(\nu_1) = p$ and $q(\nu_1) = q$ in Equation (12) and we have:

$$\sum_{\omega=0}^{\infty} {}_{\mathcal{F}}\mathcal{G}_{\omega+1,\theta}^{(r)}(\nu_1, \nu_2, \cdots, \nu_\theta)\frac{\xi^\omega}{\omega!} = (1 - p\xi - q\xi^\theta)^{-r} e^{\nu_2\xi, \cdots, \nu_\theta\xi^\theta}.$$

Now:

$$\begin{aligned}
(1 - p\xi - q\xi^\theta)^{-r} &= \left(1 - p\xi + \left(\frac{p\xi}{2}\right)^2 - \left(\frac{p\xi}{2}\right)^2 - q\xi^\theta\right)^{-r} \\
&= \left(1 - \frac{p\xi}{2}\right)^{-2r} \left(1 - \frac{(\frac{p\xi}{2})^2 + q\xi^\theta}{(1 - \frac{p\xi}{2})^2}\right)^{-r} \\
&= \sum_{i=0}^{\infty} \binom{-r}{i} \left\{\left(\frac{p\xi}{2}\right)^2 + q\xi^\theta\right\}^i \left(1 - \frac{p\xi}{2}\right)^{-2r-2i} \\
= \sum_{i=0}^{\infty} \binom{-r}{i} (-1)^i \left(\frac{p\xi}{2}\right)^{2i} &\sum_{j=0}^{i} \binom{i}{j} \left\{\frac{q\xi^\theta}{(\frac{p\xi}{2})^2}\right\} \sum_{\phi=0}^{\infty} \binom{-2r-2i}{\phi} (-1)^\phi \left(\frac{p\xi}{2}\right)^\phi \\
= \sum_{i=0}^{\infty} \sum_{j=0}^{i} \sum_{\phi=0}^{\infty} &\binom{r+i-1}{i} \binom{i}{j} \binom{2r+2i+\phi-1}{\phi} \left(\frac{p}{2}\right)^{2i-2j+\phi} q^j \xi^{2i-2j+\phi+\theta j}.
\end{aligned} \tag{23}$$

From (23), we have:

$$\begin{aligned}
&\sum_{\omega=0}^{\infty} {}_{\mathcal{F}}\mathcal{G}_{\omega+1,\theta}^{(r)}(\nu_1, \nu_2, \cdots, \nu_\theta)\frac{\xi^\omega}{\omega!} \\
&= \sum_{\omega=0}^{\infty} \sum_{i=0}^{[\frac{\omega}{2}]} \sum_{0 \le j \le i} \binom{r+i-1}{i} \binom{i}{j} \binom{\omega+2r-(\theta-2)j-1}{\omega-2i-(\theta-2)j} \left(\frac{p(\nu_1)}{2}\right)^{\omega-\theta} q^j(\nu_1)\xi^\omega \\
&\quad \times \sum_{\delta=0}^{\infty} \mathcal{F}_\delta^{[\theta]}(\nu_2, \cdots, \nu_\theta)\frac{\xi^\delta}{\delta!} \\
&= \sum_{\omega=0}^{\infty} \sum_{\delta=0}^{\omega} \frac{\mathcal{F}_\delta^{[\theta]}(\nu_2, \cdots, \nu_\theta)}{\delta!} \sum_{i=0}^{[\frac{\omega-\delta}{2}]} \sum_{0 \le j \le i} \binom{r+i-1}{i} \binom{i}{j} \binom{M}{N} \left(\frac{p(\nu_1)}{2}\right)^{\omega-\delta-\theta j} q^j(\nu_1)\xi^\omega.
\end{aligned}$$

By matching the coefficients of ξ^ω on both sides, we arrive at the desired result, Equation (22). □

Remark 3. *If we set $\nu_2 = \cdots = \nu_\theta = 0$ in Equation (22), we get to know the result of Wang and Wang [32].*

3. On Expansions of Multivariate Hermite–Chebyshev and Multivariate Hermite–Gegenbaurer Polynomials

In this section, we focus on proving several important theorems related to the expansions of multivariate Hermite–Gegenbauer and multivariate Hermite–Chebyshev polynomials in three variables. These expansions are crucial for understanding the deeper connections between various families of orthogonal polynomials and their applications in mathematical analysis and theoretical physics.

We begin our exploration by examining Equations (12) and (14), along with a specific case of (12) where $r = 1$ and $q(\nu_1) \to -q(\nu_1)$. The following equation:

$$(1 - p(\nu_1)\xi + q(\nu_1)\xi^\theta)^{-1} e^{\nu_2\xi + \nu_3\xi^2 + \cdots + \nu_\theta\xi^\theta} = \sum_{\omega=0}^{\infty} {}_{\mathcal{F}}u_{\omega,\theta}(\nu_1, \nu_2, \cdots, \nu_\theta)\frac{\xi^\omega}{\omega!}, \tag{24}$$

is not just a mathematical expression, but a powerful tool that will be used to derive a series of subsidiary results in the following theorem. These results are significant because they offer insights into the structure and relationships of multivariate polynomials, which have broad implications in areas such as combinatorics, approximation theory, and mathematical physics.

Understanding these expansions allows us to bridge the gap between different polynomial families, providing a unified framework that can be applied to solve complex problems in various domains. The theorems we prove here are not only of theoretical interest but also pave the way for practical applications, making them a valuable addition to the existing body of knowledge in the field.

Theorem 3. *Let $\omega \geq 0$ and $\phi \in \mathbb{N}$. Then:*

$$\sum_{\delta=0}^{\omega} \frac{\mathcal{F}_{\delta}^{[\theta]}(\phi\nu_2, \cdots, \phi\nu_\theta) u_{\omega+1-\delta,\theta}^{(r\phi)}(\nu_1)}{\delta!}$$

$$= \sum_{\omega_1+\omega_2+\cdots+\omega_\phi=\omega} \frac{{}_{\mathcal{F}}\mathcal{G}_{\omega_1+1,\theta}^{(r)}(\nu_1, \nu_2, \cdots, \nu_\theta)\, {}_{\mathcal{F}}\mathcal{G}_{\omega_2+1,\theta}^{(r)}(\nu_1, \nu_2, \cdots, \nu_\theta) \cdots {}_{\mathcal{F}}\mathcal{G}_{\omega_\phi+1,\theta}^{(r)}(\nu_1, \nu_2, \cdots, \nu_\theta)}{(\omega_1+1)!(\omega_2+1)!\cdots(\omega_\phi+1)!}. \quad (25)$$

Proof. Rewrite the (12) as:

$$\left[(1 - p(\nu_1)\xi - q(\nu_1)\xi^\theta)^{-r} e^{\nu_2\xi + \nu_3\xi^2 + \cdots + \nu_\theta\xi^\theta}\right]^\phi$$

$$= (1 - p(\nu_1)\xi - q(\nu_1)\xi^\theta)^{-r\phi} e^{\phi\nu_2\xi + \phi\nu_3\xi^2 + \cdots + \phi\nu_\theta\xi^\theta} = \left[\sum_{\omega=0}^{\infty} {}_{\mathcal{F}}\mathcal{G}_{\omega+1,\theta}^{(r)}(\nu_1, \nu_2, \cdots, \nu_\theta) \frac{\xi^\omega}{\omega!}\right]^\phi.$$

Using (9), we can write:

$$e^{\phi\nu_2\xi + \phi\nu_3\xi^2 + \cdots + \phi\nu_\theta\xi^\theta} = \sum_{\delta=0}^{\infty} \mathcal{F}_{\delta}^{[\theta]}(\phi\nu_2, \cdots, \phi\nu_\theta) \frac{\xi^\delta}{\delta!}.$$

Now:

$$\sum_{\omega=0}^{\infty} u_{\omega+1,\theta}^{(r\phi)}(\nu_1)\xi^\omega \sum_{\delta=0}^{\infty} \mathcal{F}_{\delta}^{[\theta]}(\phi\nu_2, \cdots, \phi\nu_\theta) \frac{\xi^\delta}{\delta!}$$

$$= \sum_{\omega=0}^{\infty} \sum_{\omega_1+\omega_2+\cdots+\omega_\phi=\omega} \frac{{}_{\mathcal{F}}\mathcal{G}_{\omega_1+1,\theta}^{(r)}(\nu_1, \nu_2, \cdots, \nu_\theta)\, {}_{\mathcal{F}}\mathcal{G}_{\omega_2+1,\theta}^{(r)}(\nu_1, \nu_2, \cdots, \nu_\theta) \cdots {}_{\mathcal{F}}\mathcal{G}_{\omega_\phi+1,\theta}^{(r)}(\nu_1, \nu_2, \cdots, \nu_\theta)}{(\omega_1+1)!(\omega_2+1)!\cdots(\omega_\phi+1)!} \xi^\omega.$$

$$\sum_{\omega=0}^{\infty} \sum_{\delta=0}^{\omega} \frac{\mathcal{F}_{\delta}^{[\theta]}(\phi\nu_2, \cdots, \phi\nu_\theta) \nu_{\omega+1-\delta}^{(r\phi,\theta)}(\nu_1)}{\delta!} \xi^\omega$$

$$= \sum_{\omega=0}^{\infty} \sum_{\omega_1+\omega_2+\cdots+\omega_\phi=\omega} \frac{{}_{\mathcal{F}}\mathcal{G}_{\omega_1+1,\theta}^{(r)}(\nu_1, \nu_2, \cdots, \nu_\theta)\, {}_{\mathcal{F}}\mathcal{G}_{\omega_2+1,\theta}^{(r)}(\nu_1, \nu_2, \cdots, \nu_\theta) \cdots {}_{\mathcal{F}}\mathcal{G}_{\omega_\phi+1,\theta}^{(r)}(\nu_1, \nu_2, \cdots, \nu_\theta)}{(\omega_1+1)!(\omega_2+1)!\cdots(\omega_\phi+1)!} \xi^\omega.$$

Which completes the proof of the result. □

Remark 4. *Letting $r = 1$, $q(\nu_1) \to -q(\nu_1)$ in (25), we have the following.*

Corollary 1. *For $\phi \in \mathbb{N}$ and $\nu_1, \nu_2, \cdots, \nu_\theta \in \mathbb{C}$, then:*

$$\sum_{\delta=0}^{\omega} \frac{\mathcal{F}_{\delta}^{\theta}(\phi\nu_2, \cdots, \phi\nu_\theta) C_{\omega-\delta,\theta}^{\phi}(\nu_1)}{\delta!}$$

$$= \sum_{\omega_1+\omega_2+\cdots+\omega_\phi=\omega} \frac{{}_{\mathcal{F}}v^{[\theta]}_{\omega_1}(\nu_1,\nu_2,\cdots,\nu_\theta)\,{}_{\mathcal{F}}u^{\theta}_{\omega_2}(\nu_1,\nu_2,\cdots,\nu_\theta)\cdots{}_{\mathcal{F}}u^{\theta}_{n_\phi}(\nu_1,\nu_2,\cdots,\nu_\theta)}{\omega_1!\omega_2!\cdots\omega_\phi!}. \tag{26}$$

Corollary 2. *For $r=0$ in (25) with $\phi\in\mathbb{N}$ and $\nu_1,\cdots,\nu_\theta\in\mathbb{C}$, then:*

$$\frac{\mathcal{F}^{[\theta]}_{\omega}(\phi\nu_2,\cdots,\phi\nu_\theta)}{\omega!} = \sum_{\omega_1+\omega_2+\cdots+\omega_\phi=\omega} \frac{\mathcal{F}^{[\theta]}_{\omega_1}(\nu_1,\nu_2,\cdots,\nu_\theta)\mathcal{F}^{[\theta]}_{\omega_2}(\nu_1,\nu_2,\cdots,\nu_\theta)\cdots\mathcal{F}^{[\theta]}_{\omega_\phi}(\nu_1,\nu_2,\cdots,\nu_\theta)}{\omega_1!\omega_2!\cdots\omega_\phi!}. \tag{27}$$

Theorem 4. *Let $\omega\geq 0$. Then:*

$$\mathcal{F}^{[\theta]}_{\omega}(\phi\nu_1,\phi\nu_2,\cdots,\phi\nu_\theta)$$
$$= \omega! \sum_{\omega_1+\omega_2+\cdots+\omega_\phi=\omega} \frac{\mathcal{F}^{[\theta]}_{\omega_1}(\nu_1,\nu_2,\cdots,\nu_\theta)\mathcal{F}^{[\theta]}_{\omega_2}(\nu_1,\nu_2,\cdots,\nu_\theta)\ldots\mathcal{F}^{[\theta]}_{\omega_\phi}(\nu_1,\nu_2,\cdots,\nu_\theta)}{(\omega_1+1)!(\omega_2+1)!\cdots(\omega_\phi+1)!}. \tag{28}$$

Proof. The definition of $\mathcal{F}^{[\theta]}_{\omega}(\nu_1,\nu_2,\cdots,\nu_\theta)$ can be written as:

$$\left[e^{\nu_1\xi+\nu_2\xi^2+\cdots+\nu_\theta\xi^\theta}\right]^{\phi} = e^{\phi\nu_1\xi+\phi\nu_2\xi^2+\cdots+\phi\nu_\theta\xi^\theta}$$
$$= \left[\sum_{\omega=0}^{\infty}\mathcal{F}^{[\theta]}_{\omega}(\nu_1,\nu_2,\cdots,\nu_\phi)\frac{\xi^\omega}{\omega!}\right]^{\phi}.$$

Using [35], we can write:

$$e^{\phi Y\xi+\phi Z\xi^\theta} = \sum_{\delta=0}^{\infty}\mathcal{F}^{[\theta]}_{\delta}(\phi Y,\phi Z)\frac{\xi^\delta}{\delta!}$$

$$\sum_{n=0}^{\infty}\mathcal{F}^{[\theta]}_{\omega}(\phi\nu_1,\phi\nu_2,\cdots,\phi\nu_\theta)\frac{\xi^\omega}{\omega!}$$
$$= \sum_{\omega=0}^{\infty}\sum_{\omega_1+\omega_2+\cdots+\omega_\phi=\omega} \frac{\mathcal{F}^{[\theta]}_{\omega_1}(\nu_1,\nu_2,\cdots,\nu_\theta)\mathcal{F}^{[\theta]}_{\omega_2}(\nu_1,\nu_2,\cdots,\nu_\theta)\ldots\mathcal{F}^{[\theta]}_{\omega_\phi}(\nu_1,\nu_2,\cdots,\nu_\theta)}{(\omega_1+1)!(\omega_2+1)!\cdots(\omega_\phi+1)!}\xi^\omega.$$

Hence, we complete the proof of the result. □

Remark 5. *Adjusting $\theta=2$, $r=0$, $\nu_\theta=-1$, $\nu_2\to 2\nu$ in (28), it reduces to the known result of Batahan and Shehata [35].*

Corollary 3. *For $\phi\in\mathbb{N}$ and $\nu\in\mathbb{C}$, then:*

$$\sum_{p=0}^{[\frac{\omega}{2}]}\frac{(-\phi)^p(2\phi\nu)^{\omega-2p}}{(\omega-2p)p!} = \sum_{\omega_1+\omega_2+\cdots+\omega_\phi=\omega}\frac{\mathcal{F}_{\omega_1}(\nu)\mathcal{F}_{\omega_2}(\nu)\cdots\mathcal{F}_{\omega_\phi}(\nu)}{\omega_1!\omega_2!\cdots\omega_\phi!}. \tag{29}$$

Theorem 5. *Let $\omega\geq 0$. Then:*

$$\sum_{\delta=0}^{[\frac{\omega}{\theta}]}\frac{(-1)^\delta(r\phi)_{\omega-(\theta-1)\delta}(p(X))^{\omega-\theta\delta}(q(X))^\delta}{\delta!\,(\omega-\theta\delta)!}$$
$$= \sum_{\omega_1+\omega_2+\cdots+\omega_\phi=\omega} u^{(r)}_{\omega_1+1,\theta}(X)u^{(r)}_{\omega_2+1,\theta}(X)\cdots u^{(r)}_{\omega_\phi+1,\theta}(X), \tag{30}$$

where $X=\sum_{i=0}^{\phi}\nu_i$.

Proof. By applying the power series expansion of $[1 - p(X)\xi - q(X)\xi^{\theta}]^{-r}$ and arranging the series appropriately, we obtain:

$$[1 - p(X)\xi - q(X)\xi^{\theta}]^{-r\phi} = \sum_{\omega=0}^{\infty} \sum_{\delta=0}^{[\frac{\omega}{\theta}]} \frac{(-1)^{\delta}(r\phi)_{\omega-(\theta-1)\delta}(p(X))^{\omega-\theta\delta}(q(X))^{\delta}}{\delta!\ (\omega-\theta\delta)!}\xi^{\omega}.$$

Additionally, we can express this as follows:

$$[1 - p(X)\xi - q(X)\xi^{\theta}]^{-r\phi} = \Big[[1 - p(X)\xi - q(X)\xi^{\theta}]^{-r}\Big]^{\phi} = \left[\sum_{\omega=0}^{\infty} u^{(r)}_{\omega+1,\theta}(X)\xi^{\omega}\right]^{\phi}$$

$$= \sum_{\omega=0}^{\infty} \sum_{\omega_1+\omega_2+\cdots+\omega_\phi=\omega} u^{(r)}_{\omega_1+1,\theta}(X)u^{(r)}_{\omega_2+1,\theta}(X)\cdots u^{(r)}_{n_\phi+1,\theta}(X)\xi^{\omega}. \tag{31}$$

Now, by equating the coefficients of ξ on both sides of the resulting equation, we obtain the desired result. □

Remark 6. *When setting $r = 1$ and $q(X) \to -q(X)$ in Theorem 5, the result simplifies to:*

$$\sum_{\delta=0}^{[\frac{\omega}{\theta}]} \frac{(-1)^{\delta}(\phi)_{\omega-(\theta-1)\delta}(p(x))^{\omega-\theta\delta}(q(X))^{\delta}}{\delta!\ (\omega-\theta\delta)!}$$

$$= \sum_{\omega_1+\omega_2+\cdots+\omega_\phi=\omega} \nu^{\theta}_{\omega_1}(X)\nu^{\theta}_{\omega_2}(X)\cdots\nu^{\theta}_{\omega_\phi}(X). \tag{32}$$

In a similar manner, we can define the generalized (p, q)-Lucas polynomials as follows:

Definition 3. *For any complex number r, the generalized convolved (p, q)-Lucas polynomials, also referred to as the generalized multivariate Hermite–Humbert polynomials ${}_{\mathcal{F}}v^{(r)}_{\omega,\theta}(\nu_1, \nu_2, \cdots, \nu_\theta)$, are specified by:*

$$\left(\frac{2 - p(\nu_1)\xi}{1 - p(\nu_1)\xi - q(\nu_1)\xi^{\theta}}\right)^{r} e^{\nu_2\xi+\nu_3\xi^2+\cdots+\nu_\theta\xi^\theta} = \sum_{\omega=0}^{\infty} {}_{\mathcal{F}}v^{(r)}_{\omega,\theta}(\nu_1, \nu_2, \cdots, \nu_\theta)\frac{\xi^{\omega}}{\omega!}, \tag{33}$$

where $\theta \in \mathbb{N}$, $r > 0$, and the remaining parameters are generally unrestricted.

By setting $\nu_2 = \cdots = \nu_\theta = 0$ in Equation (33), it simplifies to the known result by Wang and Wang [32] as follows:

$$\left(\frac{2 - p(\nu_1)\xi}{1 - p(\nu_1)\xi - q(\nu_1)\xi^{\theta}}\right)^{r} = \sum_{\omega=0}^{\infty} v^{(r)}_{\omega,\theta}(\nu_1)\xi^{\omega}. \tag{34}$$

Theorem 6. *Let $\omega \geq 0$. Then:*

$$\sum_{\delta=0}^{\omega} v^{r\phi}_{\omega-\delta,\theta}(\nu_1)\mathcal{F}^{[\theta]}_{\delta}(\phi\nu_2, \phi\nu_3, \cdots, \phi\nu_\theta)\frac{1}{\delta!}$$

$$= \sum_{\omega_1+\omega_2+\cdots+\omega_\phi=\omega} \frac{{}_{\mathcal{F}}v^{(r)}_{\omega_1+1,\theta}(\nu_1, \nu_2, \cdots, \nu_\theta)\,{}_{\mathcal{F}}v^{(r)}_{\omega_2+1,\theta}(\nu_1, \nu_2, \cdots, \nu_\theta)\cdots{}_{\mathcal{F}}v^{(r)}_{\omega_\phi+1,\theta}(\nu_1, \nu_2, \cdots, \nu_\theta)}{(\omega_1+1)!(\omega_2+1)!\cdots(\omega_\phi+1)!}. \tag{35}$$

Proof. Utilizing expressions (9) and (33), it follows that:

$$\left[\left(\frac{2 - p(\nu_1)\xi}{1 - p(\nu_1)\xi - q(\nu_1)\xi^{\theta}}\right)^{r} e^{\nu_2\xi+\nu_3\xi^2+\cdots+\nu_\theta\xi^\theta}\right]^{\phi} = \left(\sum_{\omega=0}^{\infty} {}_{\mathcal{F}}v^{(r)}_{\omega,\theta}(\nu_1, \nu_2, \cdots, \nu_\theta)\frac{\xi^{\omega}}{\omega!}\right)^{\phi}$$

$$\left(\frac{2-p(\nu_1)\xi}{1-p(\nu_1)\xi-q(\nu_1)\xi^\theta}\right)^{\phi r} e^{\phi\nu_2\xi+\phi\nu_3\xi^2+\cdots+k\nu_\theta\xi^\theta} = \left(\sum_{\omega=0}^{\infty} {}_{\mathcal{F}}v_{\omega,\theta}^{(r)}(\nu_1,\nu_2,\cdots,\nu_\theta)\frac{\xi^\omega}{\omega!}\right)^\phi$$

$$\sum_{\omega=0}^{\infty} v_{\omega,\theta}^{r\phi}(\nu_1)\xi^\omega \sum_{\delta=0}^{\infty} \mathcal{F}_\delta^{[\theta]}(\phi\nu_2,\phi\nu_3,\cdots,\nu_\theta)\frac{\xi^\delta}{\delta!}$$

$$= \sum_{\omega=0}^{\infty} \sum_{\omega_1+\omega_2+\cdots+\omega_\phi=\omega} \frac{{}_{\mathcal{F}}v_{\omega_1+1,\theta}^{(r)}((\nu_1,\nu_2,\cdots,\nu_\theta))\,{}_{\mathcal{F}}v_{\omega_2+1,\theta}^{(r)}((\nu_1,\nu_2,\cdots,\nu_\theta))\cdots {}_{\mathcal{F}}v_{\omega_\phi+1,\theta}^{(r)}((\nu_1,\nu_2,\cdots,\nu_\theta))}{(\omega_1+1)!(\omega_2+1)!\cdots(\omega_\phi+1)!}\xi^\omega$$

$$\sum_{\omega=0}^{\infty}\sum_{\delta=0}^{\omega} v_{\omega-\delta,\theta}^{r\phi}(\nu_1)\mathcal{F}_\delta^{[\theta]}((\phi\nu_2,k\nu_3,\cdots,k\nu_\theta))\frac{\xi^\omega}{\delta!}$$

$$= \sum_{\omega=0}^{\infty} \sum_{\omega_1+\omega_2+\cdots+\omega_\phi=\omega} \frac{{}_{\mathcal{F}}v_{\omega_1+1,\theta}^{(r)}((\nu_1,\nu_2,\cdots,\nu_\theta))\,{}_{\mathcal{F}}v_{\omega_2+1,\theta}^{(r)}((\nu_1,\nu_2,\cdots,\nu_\theta))\cdots {}_{\mathcal{F}}v_{\omega_k+1,\theta}^{(r)}((\nu_1,\nu_2,\cdots,\nu_\theta))}{(\omega_1+1)!(\omega_2+1)!\cdots(\omega_\phi+1)!}\xi^\omega.$$

By matching the coefficients of ξ^n on both sides, we obtain Equation (35). □

Remark 7. *When $\nu_2=\cdots=\nu_\theta=0$ is set in Theorem 6, the result simplifies to:*

$$v_{\omega,\theta}^{r\phi}(\nu_1) = \sum_{\omega_1+\omega_2+\cdots+\omega_\phi=\omega} \frac{v_{\omega_1+1,\theta}^{(r)}(\nu_1)v_{\omega_2+1,\theta}^{(r)}(\nu_1)\cdots v_{\omega_\phi+1,\theta}^{(r)}(\nu_1)}{(\omega_1+1)!(\omega_2+1)!\cdots(\omega_\phi+1)!}. \tag{36}$$

Theorem 7. *Let $\omega \geq 0$. Then:*

$$\sum_{\omega_1+\omega_2+\cdots+\omega_\phi=\omega} \frac{{}_{\mathcal{F}}v_{\omega_1,\theta}(\nu_1,\nu_2,\cdots,\nu_\theta)\,{}_{\mathcal{F}}v_{\omega_2,\theta}(\nu_1,\nu_2,\cdots,\nu_\theta)\cdots {}_{\mathcal{F}}v_{\omega_\phi,\theta}(\nu_1,\nu_2,\cdots,\nu_\theta)}{\omega_1!\omega_2!\cdots\omega_\phi!}$$

$$= \sum_{i=0}^{\omega}\binom{r}{\omega-i}2^{r-\omega+i}(-p(\nu_1))^{\omega-i}\,{}_{\mathcal{F}}u_{i+1,\theta}^{(r)}(\nu_1,r\nu_2,r\nu_3,\cdots,r\nu_\theta)\frac{1}{i!}. \tag{37}$$

Proof. The expression (33) can be rewritten as:

$$\left[\left(\frac{2-p(\nu_1)\xi}{1-p(\nu_1)\xi-q(\nu_1)\xi^\theta}\right)e^{\nu_2\xi+\nu_3\xi^2+\cdots+\nu_\theta\xi^\theta}\right]^r = \left(\sum_{\omega=0}^{\infty} {}_{\mathcal{F}}v_{\omega,\theta}(\nu_1,\nu_2,\cdots,\nu_\theta)\frac{\xi^\omega}{\omega!}\right)^r$$

$$(2-p(\nu_1)\xi)^r(1-p(\nu_1)\xi-q(\nu_1)\xi^\theta)^{-r}e^{r\nu_2\xi+r\nu_3\xi^2+\cdots+r\nu_\theta\xi^\theta} = \left(\sum_{\omega=0}^{\infty} {}_{\mathcal{F}}v_{\omega,\theta}(\nu_1,\nu_2,\cdots,\nu_\theta)\frac{\xi^\omega}{\omega!}\right)^r$$

$$\sum_{\omega=0}^{r}\binom{r}{\omega}2^{r-\omega}(-p(\nu_1))^\omega\xi^\omega\sum_{i=0}^{\infty} {}_{\mathcal{F}}u_{i+1,\theta}^{(r)}(\nu_1,r\nu_2,r\nu_3,\cdots,r\nu_\theta)\frac{\xi^i}{i!}$$

$$= \sum_{\omega=0}^{\infty}\sum_{\omega_1+\omega_2+\cdots+\omega_\phi=\omega} \frac{{}_{\mathcal{F}}v_{\omega_1,\theta}(\nu_1,\nu_2,\cdots,\nu_\theta)\,{}_{\mathcal{F}}v_{\omega_2,\theta}(\nu_1,\nu_2,\cdots,\nu_\theta)\cdots {}_{\mathcal{F}}v_{\omega_\phi,\theta}(\nu_1,\nu_2,\cdots,\nu_\theta)}{\omega_1!\omega_2!\cdots\omega_\phi!}\xi^\omega.$$

By equating the coefficients of ξ^ω on both sides, we obtain Equation (37). □

Remark 8. *By setting $\nu_2=\cdots=\nu_\theta=0$ in Theorem 7, the result simplifies to the following:*

$$\sum_{\omega_1+\omega_2+\cdots+\omega_\phi=\omega} v_{\omega_1,\theta}(\nu_1)v_{\omega_2,\theta}(\nu_1)\cdots v_{\omega_\phi,\theta}(\nu_1) = \sum_{i=0}^{\omega}\binom{r}{\omega-i}2^{r-\omega+i}(-p(\nu_1))^{\omega-i}u_{i+1,\theta}^{(r)}(\nu_1). \tag{38}$$

4. Conclusions

This paper has thoroughly investigated the generalized multivariable Humbert–Hermite polynomials, represented by the notation ${}_{\mathcal{F}}\mathcal{G}^{(r)}_{\omega+1,\theta}(\nu_1, \nu_2, \cdots, \nu_\theta)$. Building on the generalized (p, q)-Fibonacci polynomials, Section 2 has revealed key mathematical properties of these polynomials, including their generating relations, explicit forms, and significant identities. Additionally, the section has explored these polynomials' algebraic structure and potential mathematical applications. Section 3 expanded the discussion by focusing on the derivation and analysis of expansions related to Hermite polynomials, particularly Hermite–Chebyshev and Hermite–Gegenbauer polynomials. This section has systematically examined the connections between these classical orthogonal polynomials, providing detailed proof and discussing the broader implications of these expansions. The ability to express these polynomials in terms of other well-known families enhances the theory and application of Hermite-related polynomials.

Future observations could focus on extending the derived expansions of Hermite-related polynomials, particularly Hermite–Chebyshev and Hermite–Gegenbauer polynomials, to more generalized families or multivariable settings. This could deepen the understanding of their orthogonal properties and recurrence relations in broader contexts. Additionally, exploring numerical methods for efficiently computing these expansions could be valuable, particularly in applications such as quantum mechanics or signal processing. Further research might uncover new combinatorial interpretations or connections between these polynomials and algebraic identities, enhancing their use in approximation theory, especially for solving differential equations and optimization problems.

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References

1. Gould, H.W. Inverse series relation and other expansions involving Humbert polynomials. *Duke Math. J.* **1965**, *32*, 697–711. [CrossRef]
2. Gould, H.W.; Hooper, A.T. Operational formulas connected with two generalizations of Hermite polynomials. *Duke Math. J.* **1962**, *29*, 51–63. [CrossRef]
3. Agarwal, R.; Parihar, H.S. On certain generalized polynomial system associated with Humbert polynomials. *Sci. Ser. A Math. Sci. (N. S.)* **2012**, *23*, 31–44.
4. Dilcher, K. A generalization of Fibonacci polynomials and a representation of Gagenbauer polynomials of integer order. *Fibonacci Quart.* **1987**, *25*, 300–303. [CrossRef]
5. Djorjević, G.B. A generalization of Gegenbauer polynomial with two variables. *Indian J. Pure Appl. Math.* **2024**, *in press*.
6. Djorjević, G.B. Polynomials related to generalized Chebyshev polynomials. *Filomat* **2009**, *23*, 279–290. [CrossRef]
7. Djorjević, G.B.; Djordjevic, S.S. Convolutions of the generalized Morgan-Voyce polynomials. *Appl. Math. Comput.* **2015**, *259*, 106–115.
8. Djorjević, G.B.; Srivastava, H.M. Incomplete generalized Jacobsthal and Jacobsthal-Lucas numbers. *Math. Comput. Model.* **2005**, *42*, 1049–1056. [CrossRef]
9. Guan, H.; Khan, W.A.; Kizilates, C. On generalized bivariate (p,q)-Bernoulli-Fibonacci and generalized bivariate (p,q)-Bernoulli-Lucas polynomials. *Symmetry* **2023**, *15*, 943. [CrossRef]
10. Andrews, L.C. *Special Functions for Engineers and Mathematicians*; Macmillan Co.: New York, NY, USA, 1985.

11. Abramowitz, M.; Stegun, I.A. *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*; Reprint of the 1972 Edition; Dover Publications, Inc.: New York, NY, USA, 1992.
12. Bell, E.T. Exponential polynomials. *Ann. Math.* **1934**, *35*, 258–277. [CrossRef]
13. Comtet, L. *Advanced Combinatorics*; D. Reidel Publishing Co.: Dordrecht, The Netherlands, 1974.
14. Zhang, C.; Khan, W.A.; Kizilates, C. On (p,q)-Fibonacci and (p,q)-Lucas polynomials associated with Changhee numbers with their applications. *Symmetry* **2023**, *15*, 851. [CrossRef]
15. He, T.-X.; Shiue, P.J.-S. On sequences of numbers and polynomials defined by linear recurrence relations of order 2. *Int. J. Math. Math. Sci.* **2009**, *2009*, 709386. [CrossRef]
16. He, T.-X.; Shiue, P.J.-S. Sequences of non-Gegenbauer-Humbert polynomials meet the generalized Gegenebauer-Humbert polynomials. *Int. Sch. Res. Netw. ISRN Algebra* **2011**, *2011*, 268096. [CrossRef]
17. Khan, W.A.; Pathan, M.A. On a class of Humbert-Hermite polynomials. *Novi Sad J. Math.* **2021**, *51*, 1–11.
18. Cheon, G.-S.; Kim, H.; Shapiro, L.W. A generalization of Lucas polynomials sequence. *Discret. Appl. Math.* **2009**, *157*, 920–927. [CrossRef]
19. Lee, G.; Asci, M. Some properties of the (p,q)-Fibonacci and (p,q)-Lucas polynomials. *J. Appl. Math.* **2012**, *2012*, 264842. [CrossRef]
20. Pathan, M.A.; Khan, W.A. On a class of generalized Humberts-Hermite polynomials via generalized Fibonacci polynomials. *Turk. J. Math.* **2022**, *46*, 929–945. [CrossRef]
21. Wang, J. Some new results for the (p,q)-Fibonacci and Lucas polynomials. *Adv. Differ. Equ.* **2014**, *2014*, 64. [CrossRef]
22. Milovanović, G.V.; Djordjevixcx, G.B. On some properties of Humbert's polynomials. *Fibonacci Quart.* **1987**, *25*, 356–360. [CrossRef]
23. Dave, C.K. Another generalization of Gegenbauer polynomials. *J. Indian Acad. Math.* **1978**, *2*, 42–45.
24. Liu, G. Formulas for convolution Fibonacci numbers and polynomials. *Fibonacci Quart.* **2002**, *40*, 352–357. [CrossRef]
25. Ma, S.-M. Identities involving generalized Fibonacci-type polynomials. *Appl. Math. Comput.* **2011**, *217*, 9297–9301. [CrossRef]
26. Ramírez, J.L. On convolved generalized Fibonacci and Lucas polynomials. *Appl. Math. Comput.* **2014**, *229*, 208–213. [CrossRef]
27. Ramírez, J.L. Some properties of convolved k-Fibonacci numbers. *ISRN Combin.* **2013**, *2013*, 759641. [CrossRef]
28. Nalli, A.; Haukkanen, P. On generalized Fibonacci and Lucas polynomials. *Chaos Solitons Fractals* **2009**, *42*, 3179–3186. [CrossRef]
29. Sinha, S.K. On a polynomial associated with Gegenbauer polynomials. *Proc. Nat. Acad. Sci. India* **1989**, *59*, 439–455.
30. Shrestha, N.B. Polynomial associated with Legendre polynomials. *Nepali Math. Sci. Rep. Triv. Univ.* **1977**, *2*, 1.
31. Wang, W.; Wang, H. Some results on convolved (p,q)-Fibonacci polynomials. *Integral Transform. Spec. Funct.* **2015**, *26*, 340–356. [CrossRef]
32. Wang, W.; Wang, H. Generalized Humbert polynomials via generalized Fibonacci polynomials. *Appl. Math. Comput.* **2017**, *307*, 204–216. [CrossRef]
33. Dattoli, G.; Torre, A. Exponential operators, quasi-monomials and generalized polynomials. *Radiat. Phys. Chem.* **2000**, *57*, 21–26. [CrossRef]
34. Dattoli, G.; Maimo, G.; Torre, A.; Cesarano, C. Generalized Hermite polynomials and super-Gaussian forms. *J. Math. Anal. Appl.* **1996**, *233*, 597–609. [CrossRef]
35. Batahan, R.S.; Shehata, A. Hermite-Chebyshev polynomials with their generalization form. *J. Math. Sci. Adv. Appl.* **2014**, *29*, 47–59.