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On the Analytic Continuation of Appell's Hypergeometric Function F_2 to Some Symmetric Domains in the Space $\mathbb{C}^2$

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Abstract: The paper considers the problem of representation and extension of Appell's hypergeometric functions by a special family of functions—branched continued fractions. Here, we establish new symmetric domains of the analytical continuation of Appell's hypergeometric function F_2 with real and complex parameters, using their branched continued fraction expansions whose elements are polynomials in the space $\mathbb{C}^2$. To do this, we used a technique that extends the domain of convergence of the branched continued fraction, which is already known for a small domain, to a larger domain, as well as the PC method to prove that it is also the domain of analytical continuation. A few examples are provided at the end to illustrate this.

Keywords: Appell's hypergeometric function; branched continued fraction; analytic continuation; convergence; approximation by rational functions

MSC: 33C65; 32A17; 32D99; 40A99; 41A20

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1. Introduction

This paper considers the Appell's hypergeometric function F_2 defined as (see, [1,2])

$$F_2(\alpha, \beta, \beta'; \gamma, \gamma'; \mathbf{z}) = \sum_{p,q=0}^{+\infty} \frac{(\alpha)_{p+q}(\beta)_p(\beta')_q}{(\gamma)_p(\gamma')_q} \frac{z_1^p}{p!} \frac{z_2^q}{q!}, \quad |z_1| + |z_2| < 1,$$

where $\alpha, \beta, \beta', \gamma, \gamma' \in \mathbb{C}$, $\gamma, \gamma' \notin \{0, -1, -2, \ldots\}$, $(\cdot)_k$ is the Pochhammer symbol, $\mathbf{z} = (z_1, z_2) \in \mathbb{C}^2$.

Appell's hypergeometric function F_2 surprisingly appears in various applications, in particular, in materials science for the compute of the canonical partition function of the model of heteropolymer in the form of a freely jointed chain [3], in probability theory and statistics for the study of the compound gamma bivariate distribution [4], in the theory of the quantum Hall effect for the explicit evaluation of the matrix elements of the Coulomb interaction of two-body [5], in the quantum field theory for the evaluation of Feynman integrals [6] and a two-loop diagram of the propagator-type (the so-called propagator seagull) [7], in the spectral theory of atom, molecule and plasma for the compute of multipole matrix elements [8].

Many works are devoted to the study of the Appell's hypergeometric function F_2 itself, in particular, to the establishment of recurrence relations [9,10], reduction and transformation formulas [11], to the construction of analytic continuations [12–14], integral representations [15,16] and asymptotic expansions [16–20]. We also note the work [21], which presents the *Mathematica* package `AppellF2.wl`, dedicated to the evaluation of the Appell's hypergeometric function F_2.

This paper discusses the representation and analytical extension of the Appell's hypergeometric function F_2 due a special family of functions—branched continued fractions.

Domains of analytical continuation will be symmetric domains of convergence of co-called confluent branched continued fractions.

Research in the direction mentioned above was started in [22], where a formal branched continued fraction expansion was constructed for the following ratio

$$\frac{F_2(\alpha,\beta,\beta';\gamma,\gamma';\mathbf{z})}{F_2(\alpha+1,\beta,\beta';\gamma+1,\gamma';\mathbf{z})}.$$

In [23], the following formal branched continued fraction expansion

$$\frac{F_2(\alpha,\beta,\beta';\beta,\gamma';\mathbf{z})}{F_2(\alpha+1,\beta,\beta';\beta,\gamma'+1;\mathbf{z})} = 1 - z_1 - \cfrac{h_1 z_2}{1 - \cfrac{h_2 z_2}{1 - z_1 - \cfrac{h_3 z_2}{1 - \cfrac{h_4 z_2}{1 - z_1 - \cfrac{h_5 z_2}{1 - \ddots}}}}}, \qquad (1)$$

where

$$h_{2k-1} = \frac{(\beta'+k-1)(\gamma'-\alpha+k-1)}{(\gamma'+2k-2)(\gamma'+2k-1)} \quad \text{and} \quad h_{2k} = \frac{(\alpha+k)(\gamma'-\beta'+k)}{(\gamma'+2k-1)(\gamma'+2k)}, \quad k \geq 1, \qquad (2)$$

was considered, and it was shown that

$$\Psi_{d,h} = \left\{\mathbf{z} \in \mathbb{C}^2 : \ z_1 \notin [1-d, +\infty), \ z_2 \notin \left[\frac{d}{4h}, +\infty\right)\right\}, \quad h > 0, \ 0 < d < 1,$$

is the domain of the analytical continuation of the function on the left side of (1) provided that $0 < h_k \leq h$ for all $k \geq 1$.

The following theorem holds:

Theorem 1. *Suppose that α, β', and γ' are complex constants such that*

$$|h_k| - \mathrm{Re}(h_k) \leq pq, \quad k \geq 1, \qquad (3)$$

where h_k, $k \geq 1$, are given in (2) herewith $\gamma' \notin \{0, -1, -2, \ldots\}$,

$$p > 0 \quad \text{and} \quad 0 < q < 1. \qquad (4)$$

Then, the following statements hold:

(A) The branched continued fraction

$$1 - z_1 + \cfrac{h_1 z_2}{1 + \cfrac{h_2 z_2}{1 - z_1 + \cfrac{h_3 z_2}{1 + \cfrac{h_4 z_2}{1 - z_1 + \cfrac{h_5 z_2}{1 + \ddots}}}}} \qquad (5)$$

converges uniformly on every compact subset of the domain

$$\Omega_{p,q}^{d,h} = \Omega_{p,q} \bigcup \Omega^{d,h}, \qquad (6)$$

where

$$\Omega_{p,q} = \left\{ \mathbf{z} \in \mathbb{C}^2 : \ \mathrm{Re}(z_1 e^{-(i/2)\arg(z_2)}) < (1-q)\cos\left(\frac{\arg(z_2)}{2}\right), \ |z_2| < \frac{1+\cos(\arg(z_2))}{4p} \right\} \tag{7}$$

and

$$\Omega^{d,h} = \left\{ \mathbf{z} \in \mathbb{C}^2 : \ |z_1| < \frac{1-d}{2}, \ |z_2| < \frac{d(1-d)}{2h} \right\}, \tag{8}$$

where

$$h = \sup_{k\in\mathbb{N}} |h_k| \quad \text{and} \quad 0 < d < 1, \tag{9}$$

to the function $f(\mathbf{z})$ holomorphic in the domain $\Omega_{p,q}^{d,h}$.

(B) The function $f(\mathbf{z})$ is an analytic continuation of the function

$$\frac{F_2(\alpha, \beta, \beta'; \beta, \gamma'; z_1, -z_2)}{F_2(\alpha+1, \beta, \beta'; \beta, \gamma'+1; z_1, -z_2)} \tag{10}$$

in the domain (6).

This paper is organized as follows. In Section 2, we give the necessary definitions and statements and prove Theorem 2, the result of which is a certain contribution to the theory of branched continued fractions. In the next section, we prove Theorem 1 and several important consequences from it. In Section 4, we give some examples. Finally, we collect our conclusions in Section 5.

2. Definitions and Auxiliary Results

Let us recall the necessary concepts of convergence in the theory of branched continued fractions (see, [24,25]).

We set $i(0) = 0$, $\mathfrak{I}_0 = \{0\}$, and

$$\mathfrak{I}_k = \{i(k) : \ i(k) = (i_1, i_2, \ldots, i_k), \ 1 \le i_r \le 2, \ 1 \le r \le k\}, \quad k \ge 1.$$

By Bodnar ([25], p. 15) for each $r \ge 1$ the symbol $\mathbf{u}^{(r)}$ denotes a vector in $\mathbb{C}^{2^r}$ with components $u_{j(r)}$, $j(r) \in \mathfrak{I}_r$; for each $r \ge 1$, $k \ge 1$ and for each multiindex $i(k) \in \mathfrak{I}_k$ the symbol $\mathbf{u}_{i(k)}^{(r)}$ is a vector in $\mathbb{C}^{2^r}$ with components $u_{i(k),j(r)}$, $i(k) \in \mathfrak{I}_k$, $1 \le j_s \le 2$, $1 \le s \le r$, $j_0 = i_k$, with the following order of components:

(i) $u_{n(r)} \prec u_{m(r)}$ $(u_{i(k),n(r)} \prec u_{i(k),m(r)})$, if $n(r) \prec m(r)$;

(ii) $n(r) \prec m(r)$, if $n_1 < m_1$ or there exists index s, $1 \le s < r$, such that $n_p = m_p$, $1 \le p \le s$, and $n_{s+1} < m_{s+1}$.

Let the ordered pair of sequences

$$\langle \{a_{i(k)}\}_{i(k)\in\mathfrak{I}_k,\, k\in\mathbb{N}}, \{b_{i(k)}\}_{i(k)\in\mathfrak{I}_k,\, k\in\mathbb{N}_0} \rangle,$$

of complex numbers such that:

(*) $a_{i(k)} \ne 0$ for all $i(k) \in \mathfrak{I}_k$, $k \ge 1$;

(**) if for $k \ge 1$ there exists a multiindex $i(k) \in \mathfrak{I}_k$ such that $b_{i(k)} = 0$, then $b_{i(k-1),j} \ne 0$ for $1 \le j \le 2$ and $j \ne i_k$,

gives rise to sequence $\{s_{i(k)}(\mathbf{w}_{i(k)}^{(1)})\}_{i(k)\in\mathfrak{I}_k,\, k\in\mathbb{N}_0}$ herewith $\mathbf{w}_0^{(1)} = \mathbf{w}^{(1)}$ and $\{S_k(\mathbf{w}^{(k+1)})\}_{k\in\mathbb{N}_0}$ of two-dimensional linear fractional transformations

$$s_0(\mathbf{w}^{(1)}) = b_0 + w_1 + w_2, \quad v_{i(k)} = s_{i(k)}(\mathbf{w}_{i(k)}^{(1)}) = \frac{a_{i(k)}}{b_{i(k)} + w_{i(k),1} + w_{i(k),2}}, \quad i(k) \in \mathcal{I}_k, \ k \ge 1,$$

and

$$S_0(\mathbf{w}^{(1)}) = s_0(\mathbf{w}^{(1)}), \quad S_k(\mathbf{w}^{(k+1)}) = S_{k-1}(\mathbf{v}^{(k)}), \quad k \geq 1,$$

and to a sequence $\{f_k\}_{k\in\mathbb{N}_0}$, given by

$$f_k = S_k(\mathbf{0}^{(k+1)}) \in \widehat{\mathbb{C}}, \quad k \geq 0,$$

where $\mathbf{0}^{(k+1)} = (0,0,\ldots,0)$ is a vector in $\mathbb{C}^{2^{k+1}}$.

Definition 1. *The ordered pair*

$$\langle\langle\{a_{i(k)}\}_{i(k)\in\mathfrak{I}_k,k\in\mathbb{N}}, \{b_{i(k)}\}_{i(k)\in\mathfrak{I}_k,k\in\mathbb{N}_0}\rangle, \{f_k\}_{k\in\mathbb{N}_0}\rangle$$

is the branched continued fraction denoted by symbols

$$b_0 + \sum_{i_1=1}^{2} \frac{a_{i(1)}}{b_{i(1)} + \sum_{i_2=1}^{2} \frac{a_{i(2)}}{b_{i(2)} + \ddots + \sum_{i_k=1}^{2} \frac{a_{i(k)}}{b_{i(k)} + \ddots}}}.$$

The numbers b_0, $a_{i(k)}$, and $b_{i(k)}$ are called elements of the branched continued fraction. The value

$$f_k = b_0 + \sum_{i_1=1}^{2} \frac{a_{i(1)}}{b_{i(1)} + \sum_{i_2=1}^{2} \frac{a_{i(2)}}{b_{i(2)} + \ddots + \sum_{i_k=1}^{2} \frac{a_{i(k)}}{b_{i(k)}}}},$$

is called the kth approximant of the branched continued fraction.

Note that a new and more general so-called sets approach to the concept of a branched continued fraction was proposed by Antonova in [26].

Next, considering the branched continued fractions, we admit confluent case where there are no constraints (*). Without reducing the generality, we will give the following definitions with approximant sequences $\{f_k\}_{k\in\mathbb{N}}$.

Definition 2. *A branched continued fraction*

$$\langle\langle\{a_{i(k)}\}_{i(k)\in\mathfrak{I}_k,k\in\mathbb{N}}, \{b_{i(k)}\}_{i(k)\in\mathfrak{I}_k,k\in\mathbb{N}_0}\rangle, \{f_k\}_{k\in\mathbb{N}}\rangle \tag{11}$$

converges if, at most, a finite number of its approximants don't make sense and if the limit of its sequence of approximants

$$\lim_{n\to\infty} f_n$$

exists and is finite.

Definition 3. *A branched continued fraction (11) converges absolutely if its sequence of approximants such that*

$$\sum_{n=1}^{\infty} |f_{n+1} - f_n| < +\infty.$$

Definition 4. *A branched continued fraction*

$$\langle\langle\{\widehat{a}_{i(k)}\}_{i(k)\in\mathfrak{I}_k,k\in\mathbb{N}}, \{\widehat{b}_{i(k)}\}_{i(k)\in\mathfrak{I}_k,k\in\mathbb{N}_0}\rangle, \{\widehat{f}_k\}_{k\in\mathbb{N}}\rangle$$

is a majorant of a branched continued fraction (11) if there exist a natural number n_0 and a positive constant M such that for $n \geq n_0$ and $k \geq 1$ the following relation holds

$$|f_{n+k} - f_n| \leq M|\widehat{f}_{n+k} - \widehat{f}_n|.$$

Again, without reducing the generality, let us put $b_0 = 0$.

Definition 5. *A convergence set Ω is a set $\Omega \neq \emptyset$ and $\Omega \subseteq \mathbb{C} \times \mathbb{C}$ such that: if $\langle a_{i(k)}, b_{i(k)} \rangle \in \Omega$ for all $i(k) \in \mathcal{I}_k$, $k \geq 1$, then a branched continued fraction*

$$\langle\langle\{a_{i(k)}\}_{i(k)\in\mathfrak{I}_k,k\in\mathbb{N}}, \{b_{i(k)}\}_{i(k)\in\mathfrak{I}_k,k\in\mathbb{N}}\rangle, \{f_k\}_{k\in\mathbb{N}}\rangle \tag{12}$$

converges.

Definition 6. *A uniform convergence set Ω is a convergence set to which there corresponds a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ of positive numbers depending only on Ω and converging to 0 such that*

$$|f_{n+k} - f_n| \leq \varepsilon_n, \quad n \geq 1,\ k \geq 1,$$

for every branched continued fraction (12) with all $\langle a_{i(k)}, b_{i(k)} \rangle \in \Omega$.

Reasoning similarly as in the proof of Theorem 2 [24], we will prove the following result:

Theorem 2. *Suppose that $m_{0,k}$, $k \geq 1$, are constants such that*

$$0 < m_{0,k} \leq 1, \quad k \geq 1. \tag{13}$$

Then, the following statements hold:
(A) The branched continued fraction

$$1 - z_{1,0} + \cfrac{m_{0,1}z_{0,1}}{1 + \cfrac{m_{0,2}(1-m_{0,1})z_{0,2}}{1 - (1-m_{0,2})z_{1,2} + \cfrac{m_{0,3}(1-m_{0,2})z_{0,3}}{1 + \cfrac{m_{0,4}(1-m_{0,3})z_{0,4}}{1 - (1-m_{0,4})z_{1,4} + \cfrac{m_{0,5}(1-m_{0,4})z_{0,5}}{1 + \ddots}}}}} \tag{14}$$

converges absolutely and uniformly for

$$|z_{1,2k}| \leq \frac{1}{2}, \quad |z_{0,2k+1}| \leq \frac{1}{2} \quad \text{and} \quad |z_{0,2k+2}| \leq 1, \quad k \geq 0; \tag{15}$$

(B) The values of the branched continued fraction (14) and of its approximants are in the closed disk

$$|w - 1| \leq 1. \tag{16}$$

Proof. Let us show that the majorant of branched continued fraction (14) is

$$\frac{1}{2}-\cfrac{\frac{m_{0,1}}{2}}{1-\cfrac{m_{0,2}(1-m_{0,1})}{1-\frac{(1-m_{0,2})}{2}-\cfrac{\frac{m_{0,3}(1-m_{0,2})}{2}}{1-\cfrac{m_{0,4}(1-m_{0,3})}{1-\frac{(1-m_{0,4})}{2}-\cfrac{\frac{m_{0,5}(1-m_{0,4})}{2}}{1-\ddots}}}}}. \tag{17}$$

We set

$$Q_n^{(n)}=\widehat{Q}_n^{(n)}=1,\quad n\geq 1, \tag{18}$$

and

$$\begin{aligned} Q_k^{(n)} &= 1-(1-m_{0,k})\delta(k)\\ &+\cfrac{m_{0,k+1}(1-m_{0,k})z_{0,k+1}}{1+\ddots-(1-m_{0,n-2})\delta(n-2)+\cfrac{m_{0,n-1}(1-m_{0,n-2})z_{0,n-1}}{1-(1-m_{0,n-1})\delta(n-1)+m_{0,n}(1-m_{0,n-1})z_{0,n}}},\\ \widehat{Q}_k^{(n)} &= 1-(1-m_{0,k})\widehat{\delta}(k)\\ &-\cfrac{m_{0,k+1}(1-m_{0,k})\frac{1+2\widehat{\delta}(k+1)}{2}}{1-\ddots-(1-m_{0,n-2})\widehat{\delta}(n-2)-\cfrac{m_{0,n-1}(1-m_{0,n-2})\frac{1+2\widehat{\delta}(n-1)}{2}}{1-(1-m_{0,n-1})\widehat{\delta}(n-1)-m_{0,n}(1-m_{0,n-1})\frac{1+2\widehat{\delta}(n)}{2}}}, \end{aligned}$$

where, $n\geq 2$, $1\leq k\leq n-1$, and, for $x\in\mathbb{N}$,

$$\delta(x)=\begin{cases} z_{1,x}, & \text{if } x \text{ is even},\\ 0, & \text{if } x \text{ is odd},\end{cases}\qquad \widehat{\delta}(x)=\begin{cases} 1/2, & \text{if } x \text{ is even},\\ 0, & \text{if } x \text{ is odd}.\end{cases} \tag{19}$$

Then

$$Q_k^{(n)}=1-(1-m_{0,k})\delta(k)+\frac{m_{0,k+1}(1-m_{0,k})z_{0,k+1}}{Q_{k+1}^{(n)}}, \tag{20}$$

$$\widehat{Q}_k^{(n)}=1-(1-m_{0,k})\widehat{\delta}(k)-\frac{m_{0,k+1}(1-m_{0,k})(1+2\widehat{\delta}(k+1))}{2\widehat{Q}_{k+1}^{(n)}}, \tag{21}$$

where $n\geq 2$, $1\leq k\leq n-1$.

Thus, for each $n\geq 1$ we write the nth approximants of (14) and (17) as

$$f_n=1-z_{1,0}+\frac{m_{0,1}z_{0,1}}{Q_1^{(n)}},\quad \widehat{f}_n=\frac{1}{2}-\frac{m_{0,1}}{2\widehat{Q}_1^{(n)}},$$

respectively.

Using relations (13), (15), and (18)–(21), for an arbitrary $n\geq 1$ by induction on k, $1\leq k\leq n$, we show that

$$|Q_k^{(n)}| \geq \widehat{Q}_k^{(n)} \geq m_{0,k}. \tag{22}$$

For $k = n$ inequalities (22) are obvious (see, (13) and (18)). By induction hypothesis that (22) hold for $k = r + 1$, where $r + 1 \leq n$, we prove (22) for $k = r$. Indeed, use of inequalities (15) and (19)–(21) lead to

$$\begin{aligned} |Q_r^{(n)}| &= \left| 1 - (1 - m_{0,r})\delta(r) + \frac{m_{0,r+1}(1 - m_{0,r})z_{0,r+1}}{Q_{r+1}^{(n)}} \right| \\ &\geq 1 - (1 - m_{0,r})|\delta(r)| - \frac{m_{0,r+1}(1 - m_{0,r})|z_{0,r+1}|}{|Q_{r+1}^{(n)}|} \\ &\geq 1 - (1 - m_{0,r})\widehat{\delta}(r) - \frac{m_{0,r+1}(1 - m_{0,r})(1 + 2\widehat{\delta}(r+1))}{2\widehat{Q}_{r+1}^{(n)}} \\ &= \widehat{Q}_r^{(n)}. \end{aligned}$$

It is easy to see that

$$\widehat{\delta}(r) + \frac{1 + 2\widehat{\delta}(r+1)}{2} = 1.$$

By virtue of estimates (22), $\widehat{Q}_{r+1}^{(n)} \neq 0$. Therefore, replacing $m_{0,r+1}$ by $\widehat{Q}_{r+1}^{(n)}$, inequalities (22) are obtained for $k = r$.

From (13) and (22) it follows that $Q_k^{(n)} \neq 0$ and $\widehat{Q}_k^{(n)} > 0$ for all $n \geq 1$ and $1 \leq k \leq n$. Applying the method suggested in ([25], p. 28) and the relations (18) and (20) we find the formula for the difference of two approximants of the branched continued fraction (14). For $n \geq 1$ and $k \geq 1$ on the first step we obtain

$$\begin{aligned} f_{n+k} - f_n &= 1 - z_{1,0} + \frac{m_{0,1}z_{0,1}}{Q_1^{(n+k)}} - \left(1 - z_{1,0} + \frac{m_{0,1}z_{0,1}}{Q_1^{(n)}}\right) \\ &= -\frac{m_{0,1}z_{0,1}}{Q_1^{(n+k)}Q_1^{(n)}}(Q_1^{(n+k)} - Q_1^{(n)}). \end{aligned}$$

Let r be arbitrary integer number, moreover $1 \leq r \leq n - 1$, $n \geq 2$. Then for $n \geq 2$ and $k \geq 1$ we have

$$\begin{aligned} Q_r^{(n+k)} - Q_r^{(n)} &= 1 - (1 - m_{0,r})\delta(r) + \frac{m_{0,r+1}(1 - m_{0,r})z_{0,r+1}}{Q_{r+1}^{(n+k)}} \\ &\quad - \left(1 - (1 - m_{0,r})\delta(r) + \frac{m_{0,r+1}(1 - m_{0,r})z_{0,r+1}}{Q_{r+1}^{(n)}}\right) \\ &= -\frac{m_{0,r+1}(1 - m_{0,r})z_{0,r+1}}{Q_{r+1}^{(n+k)}Q_{r+1}^{(n)}}(Q_{r+1}^{(n+k)} - Q_{r+1}^{(n)}). \end{aligned} \tag{23}$$

Applying recurrence relation (23) and taking into account that

$$Q_n^{(n+k)} - Q_n^{(n)} = -(1 - m_{0,n})\delta(n) + \frac{m_{0,n+1}(1 - m_{0,n})z_{0,n+1}}{Q_{n+1}^{(n+k)}},$$

after nth step we obtain

$$f_{n+k} - f_n = (-1)^{n+1}\left((1 - m_{0,n})\delta(n) - \frac{m_{0,n+1}(1 - m_{0,n})z_{0,n+1}}{Q_{n+1}^{(n+k)}}\right)\prod_{r=1}^{n}\frac{m_{0,r}(1 - m_{0,r-1})z_{0,r}}{Q_r^{(n+k)}Q_r^{(n)}},$$

where $m_{0,0} = 0$.

Using (19) and (22), we get the following

$$
\begin{aligned}
|f_{n+k} - f_n| &\le \prod_{r=1}^{n} \frac{m_{0,r}(1 - m_{0,r-1})|z_{0,r}|}{|Q_r^{(n+k)}||Q_r^{(n)}|} \\
&\quad \times \left((1 - m_{0,n})|\delta(n)| + \frac{m_{0,n+1}(1 - m_{0,n})|z_{0,n+1}|}{|Q_{n+1}^{(n+k)}|} \right) \\
&\le \frac{1}{2^n} \prod_{r=1}^{n} \frac{m_{0,r}(1 - m_{0,r-1})(1 + 2\widehat{\delta}_{0,r})}{\widehat{Q}_r^{(n+k)}\widehat{Q}_r^{(n)}} \\
&\quad \times \left((1 - m_{0,n})\widehat{\delta}(n) + \frac{m_{0,n+1}(1 - m_{0,n})(1 + 2\widehat{\delta}_{0,n+1})}{2\widehat{Q}_{n+1}^{(n+k)}} \right) \\
&= -(\widehat{f}_{n+k} - \widehat{f}_n),
\end{aligned}
$$

where $n \ge 1$, $k \ge 1$, and $m_{0,0} = 0$. Thus,

$$
|f_{n+k} - f_n| \le \widehat{f}_n - \widehat{f}_{n+k}, \quad n \ge 1,\ k \ge 1. \tag{24}
$$

It follows that the sequence $\{\widehat{f}_n\}_{n\in\mathbb{N}}$ is monotonically decreasing and due to inequalities (22) is bounded from below. Indeed, for $n \ge 1$ we have

$$
\begin{aligned}
\widehat{f}_n &= \frac{1}{2} - \frac{m_{0,1}}{2\widehat{Q}_1^{(n)}} \\
&\ge 0.
\end{aligned}
$$

Therefore, there exists a limit

$$
\widehat{f} = \lim_{n\to\infty} \widehat{f}_n.
$$

Now, using the relation (24), we obtain for $k \ge 1$

$$
\begin{aligned}
\sum_{n=1}^{k} |f_{n+1} - f_n| &\le -\sum_{n=1}^{k} (\widehat{f}_{n+1} - \widehat{f}_n) \\
&= \frac{1 - m_{0,1}}{2} - \widehat{f}_{k+1}.
\end{aligned}
$$

If $k \to \infty$ it follows that the branched continued fraction (14) converges absolutely and uniformly for $z_{1,2k}$ and $z_{0,k}$, $k \ge 1$, which satisfies the inequalities (15). This proves (A).

Finally, by (22) for any $n \ge 1$ we obtain

$$
\begin{aligned}
|f_n - 1| &\le |z_{1,0}| + \frac{m_{0,1}|z_{0,1}|}{|Q_1^{(n)}|} \\
&\le \frac{1}{2} + \frac{1}{2} \\
&= 1,
\end{aligned}
$$

which proves (B). □

Note that Theorem 2 is an analogue to Theorem 11.1 in [27]. Moreover, it can be proved in another way, using a generalization of the Sleshinsky-Pringsheim criterion ([28], Proposition 1).

We will also need the convergence continuation theorem, which follows from Theorem 2.17 [25] (see also ([27], Theorem 24.2)).

Theorem 3 (Convergence Continuation Theorem)**.** *Suppose that $\{f_n(\mathbf{z})\}_{n\in\mathbb{N}}$ is a sequence of functions holomorphic in the domain Ω, $\Omega \subset \mathbb{C}^2$, uniformly bounded on every compact subset of Ω. Suppose that this sequence converges at each point of the set Θ, $\Theta \subset \Omega$, which is the real neighborhood of the point $\mathbf{z}^0$ in Ω. Then the sequence $\{f_n(\mathbf{z})\}_{n\in\mathbb{N}}$ converges uniformly on every compact subset of the domain Ω to a function holomorphic in this domain.*

Next, we recall the necessary concepts of the PC method in the theory of branched continued fractions, which will be used to establish the analytical continuation of the function (see, [29,30]).

Let

$$L(\mathbf{z}) = \sum_{p,q=0}^{+\infty} a_{p,q} z_1^p z_2^q,$$

where $a_{p,q} \in \mathbb{C}$, $p \geq 0$, $q \geq 0$, $\mathbf{z} \in \mathbb{C}^2$, be a formal double power series at $\mathbf{z} = \mathbf{0}$. Let $f(\mathbf{z})$ be function holomorphic in a neighbourhood of the origin $\mathbf{z} = \mathbf{0}$. Let the mapping $\Lambda : f(\mathbf{z}) \to \Lambda(f)$ associate with $f(\mathbf{z})$ its Taylor expansion in a neighbourhood of the origin.

Definition 7. *A sequence $\{f_n(\mathbf{z})\}_{n\in\mathbb{N}}$ of functions holomorphic at the origin corresponds to a formal double power series $L(\mathbf{z})$ at $\mathbf{z} = \mathbf{0}$ if*

$$\lim_{n\to+\infty} \lambda(L - \Lambda(f_n)) = +\infty,$$

where λ is the function defined as follows: $\lambda : \mathbb{L} \to \mathbb{Z}_{\geq 0} \cup \{+\infty\}$; if $L(\mathbf{z}) \equiv 0$ then $\lambda(L) = +\infty$; if $L(\mathbf{z}) \not\equiv 0$ then $\lambda(L) = n$, where n is the smallest degree of homogeneous terms for which $a_{p,q} \neq 0$, that is $n = p + q$.

If $\{f_n(\mathbf{z})\}_{n\in\mathbb{N}}$ corresponds at $\mathbf{z} = \mathbf{0}$ to a formal double power series $L(\mathbf{z})$, then the order of correspondence of $f_n(\mathbf{z})$ is defined to be

$$\nu_n = \lambda(L - \Lambda(f_n)).$$

By the definition of λ, the series $L(\mathbf{z})$ and $\Lambda(f_n)$ agree for all homogeneous terms up to and including degree $(\nu_n - 1)$.

Definition 8. *A branched continued fraction whose elements are polynomials in the space $\mathbb{C}^2$ corresponds to a formal double power series $L(\mathbf{z})$ at $\mathbf{z} = \mathbf{0}$ if its sequence of approximants corresponds to $L(\mathbf{z})$.*

Finally, we present the well-known Weierstrass' theorem ([31], p. 23) and the principle of analytic continuation ([32], p. 39).

Theorem 4 (Weierstrass' Theorem)**.** *Suppose that $\{f_n(\mathbf{z})\}_{n\in\mathbb{N}}$ is a sequence of holomorphic functions in a domain Ω, $\Omega \subset \mathbb{C}^2$, converges to a function $f(\mathbf{z})$ uniformly on each compact subset in the domain Ω. Then $f(\mathbf{z})$ is a holomorphic in Ω, and for any $p \geq 0$, $q \geq 0$,*

$$\frac{\partial^{p+q} f_n(\mathbf{z})}{\partial z_1^p \partial z_2^q} \to \frac{\partial^{p+q} f(\mathbf{z})}{\partial z_1^p \partial z_2^q} \quad as \quad n \to +\infty$$

on each compact subset in the domain Ω.

Theorem 5 (Principle of Analytic Continuation)**.** *Suppose that $f_1(\mathbf{z})$ and $f_2(\mathbf{z})$ are functions holomorphic in the domains Ω_1, $\Omega_1 \subset \mathbb{C}^2$, and Ω_2, $\Omega_2 \subset \mathbb{C}^2$, respectively, and suppose that $\Omega_1 \cap \Omega_2$ is a domain. Next, suppose that in the real neighborhood of the point $\mathbf{z}^0$, $\mathbf{z}^0 \in \Omega_1 \cap \Omega_2$, the functions $f_1(\mathbf{z})$ and $f_2(\mathbf{z})$ coincide. Then these functions are an analytic continuation of one another, i.e., there is a unique function $f(\mathbf{z})$ that is holomorphic in $\Omega_1 \cup \Omega_2$ and coincides with $f_1(\mathbf{z})$ in Ω_1 and with $f_2(\mathbf{z})$ in Ω_2.*

3. Convergence and Analytical Continuation

Proof of Theorem 1. We prove (A). Let

$$Q_n^{(n)}(\mathbf{z}) = 1, \quad n \geq 1, \tag{25}$$

and

$$Q_{2k-1}^{(2n)}(\mathbf{z}) = 1 + \cfrac{h_{2k}z_2}{1 - z_1 + \cfrac{h_{2k+1}z_2}{1 + \ddots - z_1 + \cfrac{h_{2n-1}z_2}{1 + h_{2n}z_2}}},$$

$$Q_{2k-2}^{(2n)}(\mathbf{z}) = 1 - z_1 + \cfrac{h_{2k-1}z_2}{1 + \cfrac{h_{2k}z_2}{1 + \ddots - z_1 + \cfrac{h_{2n-1}z_2}{1 + h_{2n}z_2}}}$$

$$Q_{2k-1}^{(2n+1)}(\mathbf{z}) = 1 + \cfrac{h_{2k}z_2}{1 - z_1 + \cfrac{h_{2k+1}z_2}{1 + \ddots + \cfrac{h_{2n}z_2}{1 - z_1 + h_{2n+1}z_2}}},$$

$$Q_{2k}^{(2n+1)}(\mathbf{z}) = 1 - z_1 + \cfrac{h_{2k+1}z_2}{1 + \cfrac{h_{2k+2}z_2}{1 + \ddots + \cfrac{h_{2n}z_2}{1 - z_1 + h_{2n+1}z_2}}},$$

where $n \geq 1$, $1 \leq k \leq n$. Then the following relations hold

$$Q_{2k-1}^{(2n)}(\mathbf{z}) = 1 + \frac{h_{2k}z_2}{Q_{2k}^{(2n)}(\mathbf{z})}, \quad Q_{2k-2}^{(2n)}(\mathbf{z}) = 1 - z_1 + \frac{h_{2k-1}z_2}{Q_{2k-1}^{(2n)}(\mathbf{z})}, \tag{26}$$

and

$$Q_{2k-1}^{(2n+1)}(\mathbf{z}) = 1 + \frac{h_{2k}z_2}{Q_{2k}^{(2n+1)}(\mathbf{z})}, \quad Q_{2k}^{(2n+1)}(\mathbf{z}) = 1 - z_1 + \frac{h_{2k+1}z_2}{Q_{2k+1}^{(2n+1)}(\mathbf{z})}, \tag{27}$$

where $n \geq 1$, $1 \leq k \leq n$, and, thus, for each $n \geq 1$ we write the nth approximants of (5) as

$$f_n(\mathbf{z}) = 1 - z_1 + \frac{h_1 z_2}{Q_1^{(n)}(\mathbf{z})}. \tag{28}$$

Let n be an arbitrary natural number, $\arg(z_2) = \alpha$, and $\mathbf{z}$ be an arbitrary fixed point from (7). By induction on k, $1 \leq k \leq n$, we show the following

$$\mathrm{Re}(Q_{2k-1}^{(2n)}(\mathbf{z})e^{-i\alpha/2}) > \frac{\cos(\alpha/2)}{2} \geq c > 0 \tag{29}$$

and

$$\mathrm{Re}(Q_{2k-1}^{(2n+1)}(\mathbf{z})e^{-i\alpha/2}) > \frac{\cos(\alpha/2)}{2} \geq c > 0. \tag{30}$$

By virtue of an arbitrary fixed point $\mathbf{z}$, $\mathbf{z} \in \Omega_{p,q}$ it follows that for its arbitrary neighborhood, there exists δ, $0 < \delta \leq \pi/2$, such that $|\alpha/2| \leq \pi/2 - \delta$ and, thus,

$$\cos(\alpha/2) \geq \cos(\pi/2 - \delta) = \sin(\delta) = 2c > 0.$$

Let us prove the inequalities (29). From (26) we have

$$Q_{2k-1}^{(2n)}(\mathbf{z})e^{-i\alpha/2} = e^{-i\alpha/2} + \frac{h_{2k}z_2e^{-i\alpha}}{Q_{2k}^{(2n)}(\mathbf{z})e^{-i\alpha/2}}, \tag{31}$$

where $1 \le k \le n$, and

$$Q_{2k}^{(2n)}(\mathbf{z})e^{-i\alpha/2} = e^{-i\alpha/2} - \frac{z_1e^{-2i\alpha/2}}{e^{-i\alpha/2}} + \frac{h_{2k+1}z_2e^{-i\alpha}}{Q_{2k+1}^{(2n)}(\mathbf{z})e^{-i\alpha/2}}, \tag{32}$$

where $1 \le k \le n-1$. Using (3), (7), (25), (31), and Corollary 2 in [29], for $k = n$ we obtain

$$\begin{aligned}
\mathrm{Re}(Q_{2n-1}^{(2n)}(\mathbf{z})e^{-i\alpha/2}) &\ge \cos(\alpha/2) - \frac{(|h_{2n}| - \mathrm{Re}(h_{2n}))|z_2|}{2\,\mathrm{Re}(Q_{2n}^{(2n)}(\mathbf{z})e^{-i\alpha/2})} \\
&> \cos(\alpha/2) - \frac{2pq}{2\cos(\alpha/2)}\frac{1+\cos(\alpha)}{4p} \\
&> \cos(\alpha/2) - \frac{\cos(\alpha/2)}{2} \\
&= \frac{\cos(\alpha/2)}{2}.
\end{aligned}$$

Let the inequalities (29) hold for $k = r+1$ such that $r+1 \le n$. Then, by (3), (7), Corollary 2 in [29], and the induction hypothesis, from (31) and (32) for $k = r$ we have

$$\begin{aligned}
\mathrm{Re}(Q_{2r}^{(2n)}(\mathbf{z})e^{-i\alpha/2}) &\ge \cos(\alpha/2) - \mathrm{Re}(z_1e^{-i\alpha/2}) - \frac{(|h_{2r+1}| - \mathrm{Re}(h_{2r+1}))|z_2|}{2\,\mathrm{Re}(Q_{2r+1}^{(2n)}(\mathbf{z})e^{-i\alpha/2})} \\
&> \cos(\alpha/2) - (1-q)\cos(\alpha/2) - \frac{2pq}{2\cos(\alpha/2)}\frac{1+\cos(\alpha)}{4p} \\
&= \cos(\alpha/2) - (1-q)\cos(\alpha/2) - \frac{q\cos(\alpha/2)}{2} \\
&= \frac{q\cos(\alpha/2)}{2}
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Re}(Q_{2r-1}^{(2n)}(\mathbf{z})e^{-i\alpha/2}) &\ge \cos(\alpha/2) - \frac{(|h_{2r}| - \mathrm{Re}(h_{2r}))|z_2|}{2\,\mathrm{Re}(Q_{2r}^{(2n)}(\mathbf{z})e^{-i\alpha/2})} \\
&> \cos(\alpha/2) - \frac{2pq}{2q\cos(\alpha/2)}\frac{1+\cos(\alpha)}{4p} \\
&= \cos(\alpha/2) - \frac{\cos(\alpha/2)}{2} \\
&= \frac{\cos(\alpha/2)}{2},
\end{aligned}$$

respectively.

Similarly, we obtain the inequalities (30).

Thus, $Q_1^{(n)}(\mathbf{z}) \neq 0$ for $n \ge 1$ and for all $\mathbf{z} \in \Omega_{p,q}$, i.e., that each approximant (28) is a holomorphic function in the domain (7).

Let Γ be an arbitrary compact subset of (7). Then there exists an open bi-disk

$$\Theta_r = \{\mathbf{z} \in \mathbb{C}^2 : \ |z_1| < r, \ |z_2| < r\}$$

of radius r, $r > 0$, such that $\Gamma \subset \Theta_r$. Moreover, for any $n \geq 1$ and $\mathbf{z} \in \Omega_{p,q} \bigcap \Theta_r$ from (25), (29), and (30) we have

$$\begin{aligned} |f_n(\mathbf{z})| &\leq 1 + r + \frac{|h_1|r}{\mathrm{Re}(Q_1^{(n)}(\mathbf{z})e^{-i\alpha/2})} \\ &< 1 + r + \frac{2hr}{\cos(\alpha/2)} \\ &= C(\Omega_{p,q} \bigcap \Theta_r), \end{aligned}$$

where h is defined in (9), i.e., the sequence $\{f_n(\mathbf{z})\}_{n\in\mathbb{N}}$ is uniformly bounded on every compact subset of $\Omega_{p,q}$.

It is obvious that for every l such that

$$0 < l < \min\left\{\frac{1-d}{2}, \frac{d(1-d)}{2h}, 1-q, \frac{1}{4p}\right\}, \tag{33}$$

where p and q are defined in (4), h and d are defined in (9), the domain

$$\Upsilon_l = \{\mathbf{z} \in \mathbb{R}^2 :\ 0 < z_1 < l,\ 0 < z_2 < l\}$$

contained in (7), in particular $\Upsilon_{l/2} \subset \Omega_{p,q}$. Using (9) and (33), for any $\mathbf{z} \in \Upsilon_l$, $\Upsilon_l \subset \Omega_{p,q}$, we obtain that

$$0 < z_1 < \frac{1-d}{2}$$

and for any $k \geq 1$

$$\begin{aligned} |h_k z_2| &< \frac{hd(1-d)}{2h} \\ &= \frac{d(1-d)}{2}, \end{aligned}$$

i.e., the elements of (5) satisfy the Theorem 1, with $m_{0,k} = d$, $k \geq 1$. It means that branched continued fraction (5) converges for all $\mathbf{z} \in \Upsilon_l$, $\Upsilon_l \subset \Omega_{p,q}$. Therefore, according to the Theorem 3, the convergence of (5) is uniform on compact subsets of the domain (7).

From (A) of Theorem 1, with $m_{0,k} = d$, $k \geq 1$, $0 < d < 1$, it follows that the branched continued fraction (5) converges for all $\mathbf{z} \in \Omega^{d,h}$, where $\Omega^{d,h}$ is defined by (8), and from (B) of the same theorem it follows that all approximants of (5) lie in the closed disk (16) if $\mathbf{z} \in \Omega^{d,h}$. Hence, by Theorem 3, the branched continued fraction (5) converges uniformly on compact subsets of the domain (8). This is a complete proof of (A).

Now, we prove (B). Let

$$\begin{aligned} G_{2n}^{(2n)}(\mathbf{z}) &= \frac{F_2(\alpha+n,\beta,\beta'+n;\beta,\gamma'+2n;z_1,-z_2)}{F_2(\alpha+n+1,\beta,\beta'+n;\beta,\gamma'+2n+1;z_1,-z_2)}, \\ G_{2n+1}^{(2n+1)}(\mathbf{z}) &= \frac{F_2(\alpha+n+1,\beta,\beta'+n;\beta,\gamma'+2n+1;z_1,-z_2)}{F_2(\alpha+n+1,\beta,\beta'+n+1;\beta,\gamma'+2n+2;z_1,-z_2)}, \end{aligned}$$

and

$$G_{2k-1}^{(2n)}(\mathbf{z}) = 1 + \cfrac{h_{2k}z_2}{1 - z_1 + \cfrac{h_{2k+1}z_2}{1 + \ddots - z_1 + \cfrac{h_{2n-1}z_2}{G_{2n}^{(2n)}(\mathbf{z})}}},$$

$$G_{2k-2}^{(2n)}(\mathbf{z}) = 1 - z_1 + \cfrac{h_{2k-1}z_2}{1 + \cfrac{h_{2k}z_2}{1 + \ddots - z_1 + \cfrac{h_{2n-1}z_2}{G_{2n}^{(2n)}(\mathbf{z})}}},$$

$$G_{2k-1}^{(2n+1)}(\mathbf{z}) = 1 + \cfrac{h_{2k}z_2}{1 - z_1 + \cfrac{h_{2k+1}z_2}{1 + \ddots + \cfrac{h_{2n}z_2}{G_{2n+1}^{(2n+1)}(\mathbf{z})}}},$$

$$G_{2k}^{(2n+1)}(\mathbf{z}) = 1 - z_1 + \cfrac{h_{2k+1}z_2}{1 + \cfrac{h_{2k+2}z_2}{1 + \ddots + \cfrac{h_{2n}z_2}{G_{2n+1}^{(2n+1)}(\mathbf{z})}}},$$

where $n \geq 1$ and $1 \leq k \leq n$. Then

$$G_{2k-1}^{(2n)}(\mathbf{z}) = 1 + \frac{h_{2k}z_2}{G_{2k}^{(2n)}(\mathbf{z})}, \quad G_{2k-2}^{(2n)}(\mathbf{z}) = 1 - z_1 + \frac{h_{2k-1}z_2}{G_{2k-1}^{(2n)}(\mathbf{z})}, \tag{34}$$

and

$$G_{2k-1}^{(2n+1)}(\mathbf{z}) = 1 + \frac{h_{2k}z_2}{G_{2k}^{(2n+1)}(\mathbf{z})}, \quad G_{2k}^{(2n+1)}(\mathbf{z}) = 1 - z_1 + \frac{h_{2k+1}z_2}{G_{2k+1}^{(2n+1)}(\mathbf{z})}, \tag{35}$$

where $n \geq 1$, $1 \leq k \leq n$.

It follows (see also [22]) that for each $n \geq 1$

$$\begin{aligned}\frac{F_2(\alpha, \beta, \beta'; \beta, \gamma'; z_1, -z_2)}{F_2(\alpha + 1, \beta, \beta'; \beta, \gamma' + 1; z_1, -z_2)} &= 1 - z_1 + \cfrac{h_1 z_2}{1 + \cfrac{h_2 z_2}{1 + \ddots - z_1 + \cfrac{h_{2n-1}z_2}{1 + \cfrac{h_{2n}z_2}{G_{2n}^{(2n)}(\mathbf{z})}}}} \\ &= 1 - z_1 + \frac{h_1 z_2}{G_1^{(2n)}(\mathbf{z})}\end{aligned}$$

and

$$\begin{aligned}\frac{F_2(\alpha, \beta, \beta'; \beta, \gamma'; z_1, -z_2)}{F_2(\alpha + 1, \beta, \beta'; \beta, \gamma' + 1; z_1, -z_2)} &= 1 - z_1 + \cfrac{h_1 z_2}{1 + \cfrac{h_2 z_2}{1 + \ddots + \cfrac{h_{2n}z_2}{1 - z_1 + \cfrac{h_{2n+1}z_2}{G_{2n+1}^{(2n+1)}(\mathbf{z})}}}} \\ &= 1 - z_1 + \frac{h_1 z_2}{G_1^{(2n+1)}(\mathbf{z})}.\end{aligned}$$

Since $G_k^{(n)}(\mathbf{0}) = 1$ and $Q_k^{(n)}(\mathbf{0}) = 1$ for any $1 \le k \le n$, $n \ge 1$, then the $1/G_k^{(n)}(\mathbf{z})$ and $1/Q_k^{(n)}(\mathbf{z})$ have formal Taylor expansions in a neighborhood of the origin. It is clear that $G_k^{(n)}(\mathbf{z}) \not\equiv 0$ and $Q_k^{(n)}(\mathbf{z}) \not\equiv 0$ for all $1 \le k \le n$ and $n \ge 1$. Applying the method suggested in ([25], p. 28) and (25)–(27), (34), and (35), for each $n \ge 1$ we have

$$\frac{F_2(\alpha,\beta,\beta';\beta,\gamma';z_1,-z_2)}{F_2(\alpha+1,\beta,\beta';\beta,\gamma'+1;z_1,-z_2)} - f_{2n-1}(\mathbf{z}) = -\frac{h_{2n}z_2^{2n}}{G_{2n}^{(2n)}(\mathbf{z})}\prod_{r=1}^{2n-1}\frac{h_r}{G_r^{(2n)}(\mathbf{z})Q_r^{(2n-1)}(\mathbf{z})}$$

and

$$\frac{F_2(\alpha,\beta,\beta';\beta,\gamma';z_1,-z_2)}{F_2(\alpha+1,\beta,\beta';\beta,\gamma'+1;z_1,-z_2)} - f_{2n}(\mathbf{z}) = -z_2^{2n}\left(z_1 - \frac{h_{2n+1}z_2}{G_{2n+1}^{(2n+1)}(\mathbf{z})}\right)\prod_{r=1}^{2n}\frac{h_r}{G_r^{(2n+1)}(\mathbf{z})Q_r^{(2n)}(\mathbf{z})}.$$

Hence, in a neighborhood of origin for any $n \ge 1$, we have

$$\Lambda\left(\frac{F_2(\alpha,\beta,\beta';\beta,\gamma';z_1,-z_2)}{F_2(\alpha+1,\beta,\beta';\beta,\gamma'+1;z_1,-z_2)}\right) - \Lambda(f_n) = \sum_{\substack{p+q\ge n+1\\ p\ge 0,\ q\ge 0}} a_{p,q}^{(n)} z_1^p z_2^q,$$

where $a_{p,q}^{(n)}$, $p \ge 0$, $q \ge 0$, $p+q \ge n+1$, are some coefficients. It follows that

$$\lambda\left(\Lambda\left(\frac{F_2(\alpha,\beta,\beta';\beta,\gamma';z_1,-z_2)}{F_2(\alpha+1,\beta,\beta';\beta,\gamma'+1;z_1,-z_2)}\right) - \Lambda(f_n)\right) = n+1$$

tends monotonically to $+\infty$ as $n \to +\infty$.

Thus, the branched continued fraction (5) corresponds at $\mathbf{z} = \mathbf{0}$ to a formal double power series

$$\Lambda\left(\frac{F_2(\alpha,\beta,\beta';\beta,\gamma';z_1,-z_2)}{F_2(\alpha+1,\beta,\beta';\beta,\gamma'+1;z_1,-z_2)}\right).$$

Let Δ be a neighborhood of the origin contained (6), and in which

$$\Lambda\left(\frac{F_2(\alpha,\beta,\beta';\beta,\gamma';z_1,-z_2)}{F_2(\alpha+1,\beta,\beta';\beta,\gamma'+1;z_1,-z_2)}\right) = \sum_{p,q=0}^{+\infty} a_{p,q} z_1^p z_2^q. \tag{36}$$

From (A), it follows that the sequence of approximants of (5) converges uniformly on each compact subset of Δ to function $f(\mathbf{z})$ holomorphic in the domain Δ. Then, by Theorem 3, for arbitrary $p+q$, $p \ge 0$, $q \ge 0$, we have

$$\frac{\partial^{p+q} f_n(\mathbf{z})}{\partial z_1^p \partial z_2^q} \to \frac{\partial^{p+q} f(\mathbf{z})}{\partial z_1^p \partial z_2^q} \quad \text{as} \quad n \to +\infty$$

on each compact subset of Δ. And now, due to the above proven, the expansion of each approximant of (5) into formal double power series and series (36) agree for all homogeneous terms up to and including degree n. Then, for any $p+q$, $p \ge 0$, $q \ge 0$, we have

$$\lim_{n\to+\infty}\left(\frac{\partial^{p+q} f_n}{\partial z_1^p \partial z_2^q}(\mathbf{0})\right) = \frac{\partial^{p+q} f}{\partial z_1^p \partial z_2^q}(\mathbf{0}) = p!q!a_{p,q}.$$

Hence,

$$f(\mathbf{z}) = \sum_{p,q=0}^{+\infty}\frac{1}{p!q!}\left(\frac{\partial^{p+q} f}{\partial z_1^p \partial z_2^q}(\mathbf{0})\right)z_1^p z_2^q = \sum_{p,q=0}^{+\infty} a_{p,q} z_1^p z_2^q$$

for all $\mathbf{z} \in \Delta$.

Finally, (B) follows from Theorem 4. □

Note that in the same way the domains of the analytical continuation of the ratios of Horn's hypergeometric functions H_4, H_6, and H_7 were obtained in the works [33–35], respectively. Another approach using the PF Method (see, [29]) is applied in [36].

Corollary 1. *Suppose that β' and γ' are complex constants such that satisfy inequality (3), where*

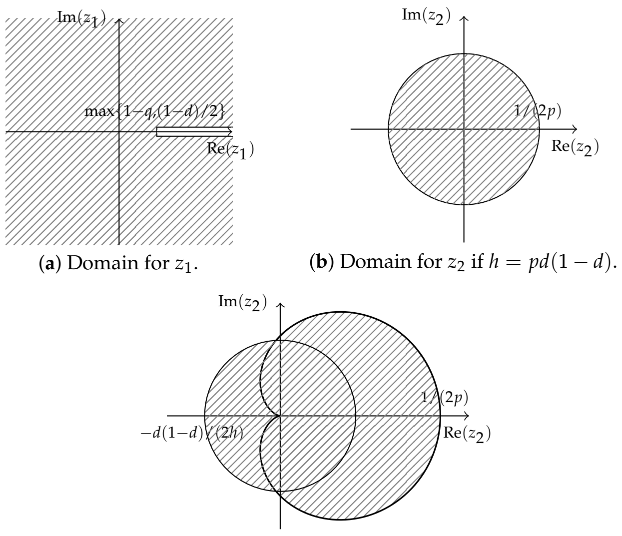
$$h_1 = \frac{\beta'}{\gamma'}, \quad h_{2k} = \frac{k(\gamma' - \beta' + k - 1)}{(\gamma' + 2k - 2)(\gamma' + 2k - 1)}, \quad \textit{and} \quad h_{2k+1} = \frac{(\beta' + k)(\gamma' + k - 1)}{(\gamma' + 2k - 1)(\gamma' + 2k)}, \quad k \geq 1, \tag{37}$$

herewith $\gamma' \notin \{0, -1, -2, \ldots\}$, and where $p > 0$ and $0 < q < 1$. Then the branched continued fraction

$$\cfrac{1}{1 - z_1 + \cfrac{\frac{\beta'}{\gamma'} z_2}{1 + \cfrac{\frac{(\gamma' - \beta')}{\gamma'(\gamma'+1)} z_2}{1 - z_1 + \cfrac{\frac{(\beta'+1)\gamma'}{(\gamma'+1)(\gamma'+2)} z_2}{1 + \cfrac{\frac{2(\gamma' - \beta' + 1)}{(\gamma'+2)(\gamma'+3)} z_2}{1 + \ddots}}}}} \tag{38}$$

converges uniformly on every compact subset of the domain (6) to the function $f(\mathbf{z})$ holomorphic in this domain, in addition, the function $f(\mathbf{z})$ is an analytic continuation of the function $F_2(1, \beta, \beta'; \beta, \gamma'; z_1, -z_2)$ in the domain $\Omega_{p,q}^{h,l}$.

Graphical illustrations of domains for variables z_1 and z_2 in (6) are shown in Figure 1a–c.



(**a**) Domain for z_1.

(**b**) Domain for z_2 if $h = pd(1-d)$.

(**c**) Domain for z_2 if $h > pd(1-d)$.

Figure 1. Domains for variables z_1 and z_2 in (6).

By using Theorem 1, we obtain the following result:

Theorem 6. *Suppose that* α, β', *and* γ' *are real constants such that*

$$0 < h_k \le h, \quad k \ge 1, \tag{39}$$

where h_k, $k \ge 1$, *are given in (2) herewith* $\gamma' \notin \{0, -1, -2, \ldots\}$, h *is a positive number. Then the branched continued fraction (5) converges uniformly on every compact subset of the domain*

$$\Phi_h = \left\{ \mathbf{z} \in \mathbb{C}^2 : \; z_1 \notin [1, +\infty), \; z_2 \notin \left(-\infty, -\frac{1}{8h} \right] \right\} \tag{40}$$

to the function $f(\mathbf{z})$ *holomorphic in this domain, in addition, the function* $f(\mathbf{z})$ *is an analytic continuation of the function (10) in the domain* Φ_h.

Proof. If $h_k > 0$ for all $k \ge 1$, then it is clear that inequality (3) is true for all $p > 0$. Let Γ be an arbitrary compact subset of the domain (40). Since $0 < q < 1$ and $0 < d < 1$, the inclusions $\Gamma \subseteq \Omega_{p,q}^{d,h} \subseteq \Phi_h$ hold for $d = 1/2$ and certain fairly small p and q for which the set $\Omega_{p,q}^{d,h}$ is the domain (6). Thus, this theorem is a direct corollary of Theorem 1. □

Corollary 2. *Suppose that* α, β', *and* γ' *are real constants such that satisfy inequality (39), where* h_k, $k \ge 1$, *are given in (37) herewith* $\gamma' \notin \{0, -1, -2, \ldots\}$, h *is a positive number. Then the branched continued fraction (38) converges uniformly on every compact subset of the domain (40) to the function* $f(\mathbf{z})$ *holomorphic in this domain, in addition, the function* $f(\mathbf{z})$ *is an analytic continuation of the function* $F_2(1, \beta, \beta'; \beta, \gamma'; \mathbf{z})$ *in the domain* Φ_h.

Note that (40) is the Cartesian product of the plane cut along the real axis from 1 to $+\infty$ and the plane cut along the real axis from $-1/(8h)$ to $-\infty$, where h is a positive number satisfying (39).

By using Theorem 5, we directly obtain the following result from Theorem 2.2 in [23] and Theorem 6:

Theorem 7. *Suppose that* α, β', *and* γ' *are real constants such that satisfy inequality (39), where* h_k, $k \ge 1$, *are given in (2) herewith* $\gamma' \notin \{0, -1, -2, \ldots\}$, h *is a positive number. Then the branched continued fraction (1) converges uniformly on every compact subset of the domain*

$$\Xi_h = \left\{ \mathbf{z} \in \mathbb{C}^2 : \; z_1 \notin [1, +\infty), \; z_2 \notin \left[\frac{1}{4h}, +\infty \right) \right\} \tag{41}$$

to the function $f(\mathbf{z})$ *holomorphic in the this domain, in addition, the function* $f(\mathbf{z})$ *is an analytic continuation of the function on the left side of (1) in the domain* Ξ_h.

Corollary 3. *Suppose that* α, β', *and* γ' *are real constants such that satisfy inequality (39), where* h_k, $k \ge 1$, *are given in (37) herewith* $\gamma' \notin \{0, -1, -2, \ldots\}$, h *is a positive number. Then the branched continued fraction*

$$\cfrac{1}{1 - z_1 - \cfrac{\frac{\beta'}{\gamma'} z_2}{1 - \cfrac{\frac{(\gamma' - \beta')}{\gamma'(\gamma'+1)} z_2}{1 - z_1 - \cfrac{\frac{(\beta'+1)\gamma'}{(\gamma'+1)(\gamma'+2)} z_2}{1 - \cfrac{\frac{2(\gamma' - \beta' + 1)}{(\gamma'+2)(\gamma'+3)} z_2}{1 - \ddots}}}}}$$

converges uniformly on every compact subset of the domain (41) to the function $f(\mathbf{z})$ holomorphic in the this domain, in addition, the function $f(\mathbf{z})$ is an analytic continuation of the function $F_2(1,\beta,\beta';\beta,\gamma';\mathbf{z})$ in the domain Ξ_h.

4. Examples

As an example, by Corollary 3 we get

$$\begin{aligned}\ln\left(1+\frac{z_2}{1+z_1}\right) &= z_2F_2(1,\beta,1;\beta,2;-z_1,-z_2)\\ &= \cfrac{z_2}{1+z_1+\cfrac{\frac{1}{2}z_2}{1+\cfrac{\frac{1}{6}z_2}{1+z_1+\cfrac{\frac{1}{3}z_2}{1+\cfrac{\frac{1}{5}z_2}{1+\ddots}}}}},\end{aligned} \tag{42}$$

where the branched continued fraction converges and represents a single-valued branch of the analytic function on the left side of (42) in the domain

$$\Xi_{1/2} = \left\{\mathbf{z}\in\mathbb{C}^2:\ z_1\notin(-\infty,-1],\ z_2\notin\left(-\infty,-\frac{1}{2}\right]\right\}.$$

One more example, by Corollary 3 we obtain

$$\begin{aligned}\arctan\sqrt{\frac{z_2}{1+z_1}} &= \sqrt{z_2(1+z_1)}F_2\left(1,\beta,\frac{1}{2};\beta,\frac{3}{2};-z_1,-z_2\right)\\ &= \cfrac{\sqrt{z_2(1+z_1)}}{1+z_1+\cfrac{\frac{1}{3}z_2}{1+\cfrac{\frac{4}{15}z_2}{1+z_1+\cfrac{\frac{9}{35}z_2}{1+\cfrac{\frac{16}{63}z_2}{1+\ddots}}}}},\end{aligned} \tag{43}$$

where the branched continued fraction converges and represents a single-valued branch of the analytic function on the left side of (43) in the domain

$$\Xi_{1/3} = \left\{\mathbf{z}\in\mathbb{C}^2:\ z_1\notin(-\infty,-1],\ z_2\notin\left(-\infty,-\frac{3}{4}\right]\right\}.$$

5. Conclusions

In this paper, we discussed the representation and extension of the analytic functions due branched continued fractions as a special family of functions. Our results are new symmetric domains of analytical extension of the Appell's hypergeometric function F_2 with certain conditions on its real and complex parameters. In particular, we obtained the domain of analytical continuation, which is the Cartesian product of the plane cut along the real axis from 1 to $+\infty$ and the plane cut along the real axis from $-1/(4h)$ to $-\infty$, where and h is a positive number satisfying (39). However, the problem of establishing the domains of the analytical extension of the Appell's hypergeometric functions F_2 with arbitrary parameters remains open.

The results of the study of branched continued fraction expansions of the Appel's hypergeometric functions F_1, F_3 and F_4 can be found in [37–40].

Further studies of branched continued fraction expansions consist in the use of new parabolic [41–43] and angular [44,45] domains of convergence of branched continued fractions. Other directions of research are truncation errors analysis [46–50] and computational stability [51–53]. Finally, taking into account the efficiency of approximation of functions by branched continued fractions [23,34,36] and the breadth of application of hypergeometric function, the applied direction of research is natural and intriguing.

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