symmetry

Article

The Application of Generalized Viscosity Implicit Midpoint Rule for Nonexpansive Mappings

Huancheng Zhang

Qinggong College, North China University of Science and Technology, Tangshan 063000, China; zhanghuancheng521@163.com; Tel.: +86-189-3157-1573

Abstract: This paper proposes new iterative algorithms by using the generalized viscosity implicit midpoint rule in Banach space, which is also a symmetric space. Then, this paper obtains strong convergence conclusions. Moreover, the results generalize the related conclusions of some researchers. Finally, this paper provides some examples to verify these conclusions. These conclusions further extend and enrich the relevant theory of symmetric space.

Keywords: nonexpansive mapping; generalized viscosity implicit midpoint rule; accretive operator; iterative algorithm

check for updates

Citation: Zhang, H. The Application of Generalized Viscosity Implicit Midpoint Rule for Nonexpansive Mappings. *Symmetry* **2024**, *16*, 1528. https://doi.org/10.3390/sym16111528

Academic Editors: Calogero Vetro and Sergei Odintsov

Received: 12 October 2024
Revised: 30 October 2024
Accepted: 11 November 2024
Published: 14 November 2024

1. Introduction

Definition 1 ([1]). *Let E be real Banach space and E^* be the dual space. $J : E \to 2^{E^*}$ is called the normalized duality mapping and defined by*

$$J(x) = \left\{ f \in E^* :< x, f >= \|x\|^2 = \|f\|^2 \right\},\ x \in E.$$

Definition 2 ([2]). *Let C be the nonempty set of E and for any $x, y \in C$.*

(1) *If $\|Sx - Sy\| \leq \|x - y\|$, then $S : C \to C$ is called nonexpansive mapping. Let $F(S)$ denote the fixed point set of S.*
(2) *If $\|fx - fy\| \leq k\|x - y\|$, $k \in [0, 1)$, then $f : C \to C$ is called contractive mapping.*

Definition 3 ([3]). *Let C be the nonempty set of E and for any $x, y \in C$.*

(1) *If there exists $j(x - y) \in J(x - y)$ such that $\langle Ax - Ay, j(x - y)\rangle \geq 0$, then $A : C \to E$ is called accretive operator.*
(2) *For any $r > 0$, if $R(I + rA) = E$, then A is called m-accretive operator.*
(3) *For any $r > 0$, if $J_r = (I + rA)^{-1}$, then $J_r : R(I + rA) \to D(A)$ is called the resolvent of m-accretive operator A.*

It is well known that J_r is nonexpansive mapping, and the fixed point set of J_r is the zero set of accretive operator A. Then the fixed point theory of nonexpansive mapping was used to solve the zero point problem of the accretive operator; see [1–6] and the references therein. It is well known that the implicit midpoint rule is a useful method for solving ordinary differential equations. Meanwhile, the viscosity iterative algorithm is very useful for finding solutions for variational inequality problems and the common fixed point of nonlinear operators; see [7–14] and the references therein.

Chang et al. [1] introduced the viscosity iterative algorithm for nonexpansive mapping and accretive operators, in 2009, as shown below.

$$\begin{cases} y_n = \beta_n x_n + (1 - \beta_n) S J_r x_n, \\ x_{n+1} = \alpha_n f x_n + (1 - \alpha_n) y_n. \end{cases}$$

Jung [15] proposed the following algorithm, in 2016:

$$x_{n+1} = J_{r_n}(\alpha_n f(x_n) + (1-\alpha_n)Sx_n), x_{n+1} = J_{r_n}(\alpha_n f(x_n) + (1-\alpha_n)Sx_n + e_n).$$

He proved that $\{x_n\}\{x_n\}$ converged strongly to $p \in F(S) \cap N(A)$. The results generalized related conclusions.

E is a real reflexive Banach space with a uniformly Gâteaux differentiable norm and C is a nonempty closed convex subset of E. Li [16] proposed a new iterative algorithm in 2017 and obtained strong convergence results:

$$\begin{cases} x_0 = x \in C, \\ y_n = \beta_n S J_{r_n}(e_n + x_n) + (1-\beta_n)x_n, \\ x_{n+1} = \alpha_n T(x_n) + (1-\alpha_n)y_n. \end{cases}$$

In the Hilbert space, Xu et al. [17] proposed the viscosity implicit midpoint rule:

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T\left(\frac{x_n + x_{n+1}}{2}\right),\ n \geq 0.$$

Under certain conditions of $\{\alpha_n\}$, they found that $\{x_n\}$ converged strongly to $q \in F(T)$, and q was the solution of variational inequality $\langle (I-f)q, x-q\rangle \geq 0$.

Luo et al. [18] extended the conclusions of Xu [17] from the Hilbert space to a uniformly smooth Banach space, in 2017:

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T\left(\frac{x_n + x_{n+1}}{2}\right),\ n \geq 0.$$

Under certain conditions of $\{\alpha_n\}$, they found that $\{x_n\}$ converged strongly to $p \in F(T)$, and p was the solution of variational inequality $\langle (I-f)p, x-p\rangle \geq 0$.

In the Hilbert space, Ke et al. [19] introduced the generalized viscosity implicit rule for nonexpansive mapping:

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T(s_n x_n + (1-s_n)x_{n+1}),\ n \geq 0.$$

Under some conditions of $\{\alpha_n\}$ and $\{s_n\}$, they found that $\{x_n\}$ converged strongly to $p \in F(T)$, and p was the solution of variational inequality $\langle (I-f)p, x-p\rangle \geq 0$.

In 2018, Zhang et al. [20] proposed two iterative algorithms by using the viscosity implicit midpoint rule in Banach space:

$$\begin{cases} y_n = \beta_n\left(\frac{x_n+x_{n+1}}{2}\right) + (1-\beta_n)J_{r_n}\left(\frac{x_n+x_{n+1}}{2}\right), \\ x_{n+1} = \alpha_n f x_n + (1-\alpha_n)Sy_n. \end{cases}$$

$$\begin{cases} y_n = \beta_n\left(\frac{x_n+x_{n+1}}{2}\right) + (1-\beta_n)J_{r_n}\left(\frac{x_n+x_{n+1}}{2} + e_n\right), \\ x_{n+1} = \alpha_n f x_n + (1-\alpha_n)Sy_n. \end{cases}$$

Under some conditions of $\{\alpha_n\}$, $\{\beta_n\}$ and $\{r_n\}$, they found that $\{x_n\}$ converged strongly to $q \in F(S) \cap N(A)$, and q was the solution of variational inequality $\left\langle (I-f)q, J_\phi(q-p)\right\rangle \leq 0$.

In Banach space, Zhang et al. [21] proposed an iterative algorithm by using the generalized viscosity implicit midpoint rule, in 2019:

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T(s_n x_n + (1-s_n)x_{n+1}),$$
$$x_{n+1} = \alpha_n x_n + \beta_n f(x_n) + \gamma_n T(s_n x_n + (1-s_n)x_{n+1}) + e_n.$$

Under some conditions of $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{s_n\}$, they found that $\{x_n\}$ converged strongly to $q \in F(T)$, and q was the solution of variational inequality $\langle (I-f)q, j(x-q)\rangle \leq 0$.

On the basis of the above research, this paper proposes new iterative algorithms by using the generalized viscosity implicit midpoint rule in Banach space to obtain a strong convergence conclusion. The results extend the previous results. In the end, this paper provides some examples to verify these conclusions.

2. Preliminaries

Definition 4 ([20]). *E is called uniformly convex, if there exists $\delta_\varepsilon > 0$ such that $\frac{\|x+y\|}{2} < 1 - \delta_\varepsilon$, where $\|x\| = \|y\| = 1$, $\|x - y\| \geq \varepsilon$, $\forall \varepsilon \in [0,2]$. $g : [0, +\infty) \to [0, +\infty)$ is a strictly increasing convex and continuous function with $g(0) = 0$. If g satisfies*

$$\|\lambda x + (1-\lambda)y\|^2 \leq \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)g(\|x-y\|), \tag{1}$$

then the Banach space is uniformly convex.

Definition 5 ([22]). *C is a nonempty set. If the distance function d satisfies $d(p,q) = d(q,p)$, $\forall (p,q) \in C$, then d is symmetric. C endowed with metric d forms a symmetric space.*

It is well known that the Banach space has symmetry.

Definition 6 ([23]). *For any $x, y \in U$ and $U = \{x \in E : \|x\| = 1\}$, if $\lim\limits_{t\to 0} \frac{\|x+ty\|-\|x\|}{t}$ exists, then E has a Gâteaux differentiable norm. For any $y \in U$, E has a uniformly Gâteaux differentiable norm, if $\lim\limits_{t\to 0} \frac{\|x+ty\|-\|x\|}{t}$ is attained uniformly for $x \in U$.*

As we all know, if E has a uniformly Gâteaux differentiable norm, so J is single valued and norm-to-weak* uniformly continuous on any bounded subset of E; see [23].

Definition 7 ([20]). *For any bounded closed convex subset D of C, where C is a closed convex subset of E, and D has at least two points and $diam(D)$ denotes the diameter of D. If there exists no diametral point, $x \in D$ such that $diam(D) > sup\{\|x - y\| | y \in D\}$, so C has normal structure.*

In order to prove the conclusions of this paper, we require the following lemmas.

Lemma 1 ([24]). *Assume that for any $\lambda, \mu > 0$ and $x \in E$,*

$$J_\lambda x = J_\mu\Big(\frac{\mu}{\lambda}x + \Big(1 - \frac{\mu}{\lambda}\Big)J_\lambda x\Big).$$

Lemma 2 ([25]). *Let $\{a_n\}, \{b_n\}, \{c_n\}$ be three non-negative real sequences and satisfy*

$$a_{n+1} \leq (1-t_n)a_n + b_n + c_n, \forall n \geq 0,$$

where $\{t_n\} \subset (0,1)$. If $\sum\limits_{n=0}^{\infty} t_n = \infty$, $b_n = o(t_n)$ and $\sum\limits_{n=1}^{\infty} c_n < \infty$, then $\lim\limits_{n\to\infty} a_n = 0$.

Lemma 3 ([4,26]). *Assume that E is the real reflexive Banach space which has a uniformly Gâteaux differentiable norm, C is the nonempty closed convex subset of E with normal structure, $T : C \to C$ is the fixed contraction with $\tau \in (0,1)$ and $S : C \to C$ is the nonexpansive mapping which has a fixed point. For $t \in (0,1)$, $\{x_{S,T,t}\}$ is defined by*

$$x_{S,T,t} = tTx_t + (1-t)Sx_{S,T,t}.$$

So $\{x_t\}$ strongly converges to $x' \in F(S)$, which is the only solution of the variational inequality

$$\langle Tx' - x', j(x'-q)\rangle \geq 0, \ \forall q \in F(S).$$

Lemma 4 ([2]). *Let E be the Banach space and for $j(x+y) \in J(x+y)$, there exists*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle,\ \forall x, y \in E.$$

3. Results

Theorem 1. *Assume that E is a reflexive and uniformly convex Banach space with a uniformly Gâteaux differentiable norm, C is a nonempty closed convex subset of E with normal structure. $f: C \to C$ is contractive mapping with $k \in [0,1)$, A is the m-accretive operator in E and $S: C \to C$ is the nonexpansive mapping with $F(S) \cap N(A) \neq \varnothing$. For any $x_0 \in C$ and $n \geq 0$, $\{x_n\}$ is generated by*

$$\begin{cases} y_n = \beta_n x_n + (1-\beta_n) J_{r_n} x_{n+1}, \\ x_{n+1} = \alpha_n f x_n + (1-\alpha_n) S y_n, \end{cases} \tag{2}$$

where $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$ and $\{r_n\} \subset (0,1)$ satisfy the following conditions:

(i) $\sum_{n=1}^{\infty} |\beta_n - \beta_{n-1}| < \infty$;

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} a_n = 0$, $|\alpha_n - \alpha_{n-1}| = o(\alpha_n)$;

(iii) $\lim_{n\to\infty} r_n = r$, $\sum_{n=1}^{\infty} |r_n - r_{n-1}| < \infty$.

Then $\{x_n\}$ and $\{y_n\}$ strongly converge to $p \in F(S) \cap N(A)$ which is the only one solution of variational inequality $\big\langle (I-f)p, J_\phi(p-q) \big\rangle \leq 0$, $\forall q \in F(S) \cap N(A)$.

Proof. The proof process is divided into eleven steps.

Step 1: Show the boundedness of $\{x_n\}$ and $\{y_n\}$.

Taking $q \in F(S) \cap N(A)$, then we obtain

$$\begin{aligned} \|y_n - q\| &\leq \beta_n \|x_n - q\| + (1-\beta_n)\|J_{r_n} x_{n+1} - q\| \\ &\leq \beta_n \|x_n - q\| + (1-\beta_n)\|x_{n+1} - q\|, \end{aligned}$$

and then we obtain

$$\begin{aligned} \|x_{n+1} - q\| &\leq \alpha_n \|f x_n - q\| + (1-\alpha_n)\|S y_n - q\| \\ &\leq k\alpha_n \|x_n - q\| + \alpha_n \|fq - q\| + (1-\alpha_n)\|y_n - q\| \\ &\leq k\alpha_n \|x_n - q\| + \alpha_n \|fq - q\| + (1-\alpha_n)\beta_n \|x_n - q\| \\ &\quad + (1-\alpha_n)(1-\beta_n)\|x_{n+1} - q\|. \end{aligned}$$

It follows that

$$\begin{aligned} \|x_{n+1} - q\| &\leq \tfrac{k\alpha_n + (1-\alpha_n)\beta_n}{\alpha_n + \beta_n - \alpha_n\beta_n} \|x_n - q\| + \tfrac{\alpha_n}{\alpha_n + \beta_n - \alpha_n\beta_n} \|fq - q\| \\ &= \Big[1 - \tfrac{\alpha_n(1-k)}{\alpha_n + \beta_n - \alpha_n\beta_n}\Big] \|x_n - q\| + \tfrac{\alpha_n(1-k)}{\alpha_n + \beta_n - \alpha_n\beta_n} \tfrac{\|fq - q\|}{1-k} \\ &\leq \max\Big\{ \|x_0 - q\|, \tfrac{\|fq - q\|}{1-k} \Big\}. \end{aligned}$$

Then $\{x_n\}$ is bounded. So $\{y_n\}$, $\{f x_n\}$, $\{S x_n\}$, $\{J_{r_n} x_n\}$, $\{f y_n\}$, $\{J_{r_n} y_n\}$ and $\{S y_n\}$ are also bounded.

Step 2: Show that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.

From (2), we obtain

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|(1-\alpha_n)S y_n + \alpha_n f x_n - (1-\alpha_{n-1}) S y_{n-1} - \alpha_{n-1} f x_{n-1}\| \\ &\leq \alpha_n \|f x_n - f x_{n-1}\| + (1-\alpha_n)\|S y_n - S y_{n-1}\| + |\alpha_n - \alpha_{n-1}| \cdot \|f x_{n-1} - S y_{n-1}\| \\ &\leq k\alpha_n \|x_n - x_{n-1}\| + (1-\alpha_n)\|y_n - y_{n-1}\| + |\alpha_n - \alpha_{n-1}| \cdot \|f x_{n-1} - S y_{n-1}\|. \end{aligned} \tag{3}$$

From (2), we obtain

$$
\begin{aligned}
\|y_n - y_{n-1}\| &= \left\|(1-\beta_n)J_{r_n}x_{n+1} + \beta_n x_n - (1-\beta_{n-1})J_{r_{n-1}}x_n - \beta_{n-1}x_{n-1}\right. \\
&\le \beta_n\|x_n - x_{n-1}\| + (1-\beta_n)\left\|J_{r_n}x_{n+1} - J_{r_{n-1}}x_n\right\| + |\beta_n - \beta_{n-1}| \cdot \left\|x_{n-1} - J_{r_{n-1}}x_n\right\|.
\end{aligned} \tag{4}
$$

From Lemma 1, we obtain

$$
\begin{aligned}
\left\|J_{r_n}x_{n+1} - J_{r_{n-1}}x_n\right\| &= \left\|J_{r_{n-1}}\left(\tfrac{r_{n-1}}{r_n}x_{n+1} + \left(1 - \tfrac{r_{n-1}}{r_n}\right)J_{r_n}x_{n+1}\right) - J_{r_{n-1}}x_n\right\| \\
&\le \left\|\tfrac{r_{n-1}}{r_n}x_{n+1} + \left(1 - \tfrac{r_{n-1}}{r_n}\right)J_{r_n}x_{n+1} - x_n\right\| \\
&= \left\|\tfrac{r_{n-1}}{r_n}(x_{n+1} - x_n) + \left(1 - \tfrac{r_{n-1}}{r_n}\right)(J_{r_n}x_{n+1} - x_n)\right\| \\
&\le \tfrac{r_{n-1}}{r_n}\|x_{n+1} - x_n\| + \left|1 - \tfrac{r_{n-1}}{r_n}\right| \cdot \|x_{n+1} - x_n\| + \left|1 - \tfrac{r_{n-1}}{r_n}\right| \cdot \|J_{r_n}x_{n+1} - x_{n+1}\| \\
&\le \|x_{n+1} - x_n\| + \left|1 - \tfrac{r_{n-1}}{r_n}\right| \cdot \|J_{r_n}x_{n+1} - x_{n+1}\|.
\end{aligned} \tag{5}
$$

Taking (4) and (5) into (3), we have

$$
\begin{aligned}
\|x_{n+1} - x_n\| \le{}& k\alpha_n\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| \cdot \|fx_{n-1} - Sy_{n-1}\| + (1-\alpha_n)\beta_n\|x_n - x_{n-1}\| \\
&+ (1-\alpha_n)(1-\beta_n)\|x_{n+1} - x_n\| + (1-\alpha_n)|\beta_n - \beta_{n-1}| \cdot \left\|x_{n-1} - J_{r_{n-1}}x_n\right\| \\
&+ (1-\alpha_n)(1-\beta_n)\left|1 - \tfrac{r_{n-1}}{r_n}\right| \cdot \|x_{n+1} - J_{r_n}x_{n+1}\|.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\|x_{n+1} - x_n\| &\le \tfrac{k\alpha_n + (1-\alpha_n)\beta_n}{\alpha_n + \beta_n - \alpha_n\beta_n}\|x_n - x_{n-1}\| + \tfrac{|\alpha_n - \alpha_{n-1}|}{\alpha_n + \beta_n - \alpha_n\beta_n}M_1 \\
&\quad + \tfrac{(1-\alpha_n)|\beta_n - \beta_{n-1}| + (1-\alpha_n-\beta_n+\alpha_n\beta_n)\left|1 - \frac{r_{n-1}}{r_n}\right|}{\alpha_n + \beta_n - \alpha_n\beta_n}M_2 \\
&= \left[1 - \tfrac{\alpha_n(1-k)}{\alpha_n + \beta_n - \alpha_n\beta}\right]\|x_n - x_{n-1}\| + \tfrac{|\alpha_n - \alpha_{n-1}|}{\alpha_n + \beta_n - \alpha_n\beta}M_1 \\
&\quad + \tfrac{(1-\alpha_n)|\beta_n - \beta_{n-1}| + (1-\alpha_n-\beta_n+\alpha_n\beta_n)\left|1 - \frac{r_{n-1}}{r_n}\right|}{\alpha_n + \beta_n - \alpha_n\beta_n}M_2,
\end{aligned}
$$

where $M_1 = \max\{\|fx_{n-1} - Sy_{n-1}\|\}$ and $M_2 = \max\left\{\left\|x_{n-1} - J_{r_{n-1}}x_n\right\|, \|x_{n+1} - J_{r_n}x_{n+1}\|\right\}$.

Taking $t_n = \frac{\alpha_n(1-k)}{\alpha_n + \beta_n - \alpha_n\beta_n}$, then $t_n > \alpha_n(1-k)$. From $\sum\limits_{n=0}^{\infty}\alpha_n = \infty$, so $\sum\limits_{n=0}^{\infty}t_n = \infty$.

Taking $b_n = \frac{|\alpha_n - \alpha_{n-1}|}{\alpha_n + \beta_n - \alpha_n\beta_n}M_1$, then $\frac{b_n}{t_n} = \frac{|\alpha_n - \alpha_{n-1}|M_1}{\alpha_n(1-k)}$. From $|\alpha_n - \alpha_{n-1}| = o(\alpha_n)$, so $b_n = o(t_n)$.

Taking $c_n = \frac{(1-\alpha_n)|\beta_n - \beta_{n-1}| + (1-\alpha_n-\beta_n+\alpha_n\beta_n)\left|1 - \frac{r_{n-1}}{r_n}\right|}{\alpha_n + \beta_n - \alpha_n\beta_n}M_2$. From $\lim\limits_{n\to\infty} r_n = r$, so $c_n < N \cdot M_2\left(\frac{|r_n - r_{n-1}|}{r-\varepsilon} + |\beta_n - \beta_{n-1}|\right)(\forall\varepsilon > 0)$, where $N = \max\left\{\frac{1-\alpha_n}{\alpha_n + \beta_n - \alpha_n\beta_n}, \frac{1-\alpha_n-\beta_n+\alpha_n\beta_n}{\alpha_n + \beta_n - \alpha_n\beta_n}\right\}$. From $\sum\limits_{n=1}^{\infty}|\beta_n - \beta_{n-1}| < \infty$ and $\sum\limits_{n=1}^{\infty}|r_n - r_{n-1}| < \infty$, so $\sum\limits_{n=1}^{\infty}c_n < \infty$.

From Lemma 2, we have $\lim\limits_{n\to\infty}\|x_{n+1} - x_n\| = 0$.

Step 3: Show that $\lim\limits_{n\to\infty}\|x_n - J_{r_n}x_n\| = 0$.

From (1) and $\|\cdot\|^2$ is a convex function, then we find

$$
\begin{aligned}
\|x_{n+1} - q\|^2 &\le \alpha_n\|fx_n - q\|^2 + (1-\alpha_n)\|Sy_n - q\|^2 \\
&\le \alpha_n\|fx_n - q\|^2 + (1-\alpha_n)\|y_n - q\|^2 \\
&\le \alpha_n\|fx_n - q\|^2 + (1-\alpha_n)\beta_n\|x_n - q\|^2 + (1-\alpha_n)(1-\beta_n)\|J_{r_n}x_{n+1} - q\|^2 \\
&\quad - (1-\alpha_n)\beta_n(1-\beta_n)g(\|x_n - J_{r_n}x_{n+1}\|) \\
&\le \alpha_n\|fx_n - q\|^2 + (1-\alpha_n)\beta_n\|x_n - q\|^2 + (1-\alpha_n)(1-\beta_n)\|x_{n+1} - q\|^2 \\
&\quad - (1-\alpha_n)\beta_n(1-\beta_n)g(\|x_n - J_{r_n}x_{n+1}\|).
\end{aligned}
$$

It follows that

$$\begin{aligned}\|x_{n+1}-q\|^2 &\leq \tfrac{\beta_n-\alpha_n\beta_n}{\alpha_n+\beta_n-\alpha_n\beta_n}\|x_n-q\|^2+\tfrac{\alpha_n}{\alpha_n+\beta_n-\alpha_n\beta_n}\|fx_n-q\|^2\\ &\quad-\tfrac{\beta_n(1-\alpha_n)(1-\beta_n)}{\alpha_n+\beta_n-\alpha_n\beta_n}g(\|x_n-J_{r_n}x_{n+1}\|)\\ &=\Big(1-\tfrac{\alpha_n}{\alpha_n+\beta_n-\alpha_n\beta_n}\Big)\|x_n-q\|^2+\tfrac{\alpha_n}{\alpha_n+\beta_n-\alpha_n\beta_n}\|fx_n-q\|^2\\ &\quad-\tfrac{\beta_n(1-\alpha_n)(1-\beta_n)}{\alpha_n+\beta_n-\alpha_n\beta_n}g(\|x_n-J_{r_n}x_{n+1}\|)\\ &\leq \|x_n-q\|^2+\tfrac{\alpha_n}{\alpha_n+\beta_n-\alpha_n\beta_n}\|fx_n-q\|^2\\ &\quad-\tfrac{\beta_n(1-\alpha_n)(1-\beta_n)}{\alpha_n+\beta_n-\alpha_n\beta_n}g(\|x_n-J_{r_n}x_{n+1}\|).\end{aligned}$$

Then we have

$$\begin{aligned}&\tfrac{\beta_n(1-\alpha_n)(1-\beta_n)}{\alpha_n+\beta_n-\alpha_n\beta_n}g(\|x_n-J_{r_n}x_{n+1}\|)-\tfrac{\alpha_n}{\alpha_n+\beta_n-\alpha_n\beta_n}\|fx_n-q\|^2\\ &\leq \|x_n-q\|^2-\|x_{n+1}-q\|^2.\end{aligned}$$

If $\frac{\beta_n(1-\alpha_n)(1-\beta_n)}{\alpha_n+\beta_n-\alpha_n\beta_n}g(\|x_n-J_{r_n}x_{n+1}\|)\leq\frac{\alpha_n}{\alpha_n+\beta_n-\alpha_n\beta_n}\|fx_n-q\|^2$, so from $\lim_{n\to\infty}\alpha_n=0$ and the boundedness of $\{fx_n\}$, we find $\lim_{n\to\infty}g(\|x_n-J_{r_n}x_{n+1}\|)=0$.

If $\frac{\beta_n(1-\alpha_n)(1-\beta_n)}{\alpha_n+\beta_n-\alpha_n\beta_n}g(\|x_n-J_{r_n}x_{n+1}\|)>\frac{\alpha_n}{\alpha_n+\beta_n-\alpha_n\beta_n}\|fx_n-q\|^2$, so

$$\begin{aligned}&\sum_{n=0}^{H}\Big[\tfrac{\beta_n(1-\alpha_n)(1-\beta_n)}{\alpha_n+\beta_n-\alpha_n\beta_n}g(\|x_n-J_{r_n}x_{n+1}\|)-\tfrac{\alpha_n}{\alpha_n+\beta_n-\alpha_n\beta_n}\|fx_n-q\|^2\Big]\\ &\leq \|x_0-q\|^2-\|x_{H+1}-q\|^2\leq\|x_0-q\|^2.\end{aligned}$$

Then

$$\sum_{n=0}^{\infty}\bigg[\frac{\beta_n(1-\alpha_n)(1-\beta_n)}{\alpha_n+\beta_n-\alpha_n\beta_n}g(\|x_n-J_{r_n}x_{n+1}\|)-\frac{\alpha_n}{\alpha_n+\beta_n-\alpha_n\beta_n}\|fx_n-q\|^2\bigg]<\infty.$$

So we have

$$\lim_{n\to\infty}\bigg[\frac{\beta_n(1-\alpha_n)(1-\beta_n)}{\alpha_n+\beta_n-\alpha_n\beta_n}g(\|x_n-J_{r_n}x_{n+1}\|)-\frac{\alpha_n}{\alpha_n+\beta_n-\alpha_n\beta_n}\|fx_n-q\|^2\bigg]=0,$$

and then $\lim_{n\to\infty}g(\|x_n-J_{r_n}x_{n+1}\|)=0$.

From the property of g, so we find $\lim_{n\to\infty}\|x_n-J_{r_n}x_{n+1}\|=0$.

We also have

$$\begin{aligned}\|x_n-J_{r_n}x_n\| &\leq \|x_n-J_{r_n}x_{n+1}\|+\|J_{r_n}x_{n+1}-J_{r_n}x_n\|\\ &\leq \|x_n-J_{r_n}x_{n+1}\|+\|x_{n+1}-x_n\|.\end{aligned}$$

From step 2, we have $\lim_{n\to\infty}\|x_n-J_{r_n}x_n\|=0$.

Step 4: Show that $\lim_{n\to\infty}\|y_n-Sy_n\|=0$.

From (2), we find

$$\begin{aligned}\|y_n-Sy_n\| &\leq \beta_n\|x_n-Sy_n\|+(1-\beta_n)\|J_{r_n}x_{n+1}-Sy_n\|\\ &\leq \|x_n-Sy_n\|+(1-\beta_n)\|J_{r_n}x_{n+1}-x_n\|\\ &\leq \|x_{n+1}-Sy_n\|+\|x_n-x_{n+1}\|+(1-\beta_n)\|J_{r_n}x_{n+1}-x_n\|\\ &=\alpha_n\|fx_n-Sy_n\|+\|x_n-x_{n+1}\|+(1-\beta_n)\|J_{r_n}x_{n+1}-x_n\|.\end{aligned}$$

From steps 2 and step 3, the boundedness of $\{Sy_n\}$ and $\{fx_n\}$, and $\lim_{n\to\infty}\alpha_n=0$, we have $\lim_{n\to\infty}\|y_n-Sy_n\|=0$.

Step 5: Show that $\lim_{n\to\infty}\|x_n-y_n\|=0$.

From (2), we find

$$\begin{aligned}\|x_n - y_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - Sy_n\| + \|Sy_n - y_n\| \\ &= \|x_n - x_{n+1}\| + \alpha_n\|fx_n - Sy_n\| + \|Sy_n - y_n\|.\end{aligned}$$

From step 2 and step 4, $\lim_{n\to\infty} \alpha_n = 0$ and the boundedness of $\{Sy_n\}$ and $\{fx_n\}$, we have $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Step 6: Show that $\lim_{n\to\infty} \|y_n - J_{r_n}y_n\| = 0$.

Using the results of step 3 and 5, we obtain

$$\begin{aligned}\|y_n - J_{r_n}y_n\| &\le \|y_n - x_n\| + \|x_n - J_{r_n}x_n\| + \|J_{r_n}x_n - J_{r_n}y_n\| \\ &\le \|y_n - x_n\| + \|x_n - J_{r_n}x_n\| + \|x_n - y_n\|.\end{aligned}$$

So we have $\lim_{n\to\infty} \|y_n - J_{r_n}y_n\| = 0$.

Step 7: Show that $\lim_{n\to\infty} \|x_n - Sx_n\| = 0$.

Using the results of step 4 and 5, we obtain

$$\begin{aligned}\|x_n - Sx_n\| &\le \|x_n - y_n\| + \|y_n - Sy_n\| + \|Sy_n - Sx_n\| \\ &\le \|x_n - y_n\| + \|y_n - Sy_n\| + \|y_n - x_n\|.\end{aligned}$$

So we have $\lim_{n\to\infty} \|x_n - Sx_n\| = 0$.

Step 8: Show that $\lim_{n\to\infty} \|y_n - J_r y_n\| = 0$.

Using the results of step 6 and Lemma 1, we obtain

$$\begin{aligned}\|y_n - J_r y_n\| &\le \|y_n - J_{r_n}y_n\| + \|J_{r_n}y_n - J_r y_n\| \\ &= \|y_n - J_{r_n}y_n\| + \left\|J_r\left(\frac{r}{r_n}y_n + \left(1 - \frac{r}{r_n}\right)J_{r_n}y_n\right) - J_r y_n\right\| \\ &\le \|y_n - J_{r_n}y_n\| + \left|1 - \frac{r}{r_n}\right| \cdot \|J_{r_n}y_n - y_n\|.\end{aligned}$$

From $\lim_{n\to\infty} r_n = r$, we get $\lim_{n\to\infty} \|y_n - J_r y_n\| = 0$.

Step 9: Show that $\lim_{n\to\infty} \|x_n - J_r x_n\| = 0$.

Using the results of step 5 and step 8, we obtain

$$\begin{aligned}\|x_n - J_r x_n\| &\le \|x_n - y_n\| + \|y_n - J_r y_n\| + \|J_r y_n - J_r x_n\| \\ &\le 2\|x_n - y_n\| + \|y_n - J_r y_n\|.\end{aligned}$$

So we have $\lim_{n\to\infty} \|x_n - J_r x_n\| = 0$.

Step 10: Show that $\limsup_{n\to\infty} \langle (I - f)p, J(p - x_n)\rangle = 0$.

Let $\{x_t\}$ be defined by $x_t = tfx_t + (1-t)Sx_t$. From Lemma 3, we find that $\{x_t\}$ strongly converges to $p \in P_{F(S)\cap N(A)}fp$, and p is also the unique solution of the variational inequality $\langle (I - f)p, J(p - q)\rangle \le 0, \forall q \in F(S) \cap N(A)$.

We have

$$\begin{aligned}\|x_t - x_n\|^2 &= (1-t)\langle Sx_t - x_n, J(x_t - x_n)\rangle + t\langle fx_t - x_n, J(x_t - x_n)\rangle \\ &\le (1-t)\|Sx_t - x_t\| \cdot \|x_t - x_n\| + (1-t)\|x_t - x_n\|^2 \\ &+ t\langle fx_t - x_t, J(x_t - x_n)\rangle + t\|x_t - x_n\|^2 \\ &= (1-t)\|Sx_t - x_t\| \cdot \|x_t - x_n\| + \|x_t - x_n\|^2 \\ &+ t\langle fx_t - x_t, J(x_t - x_n)\rangle.\end{aligned}$$

It follows that $\langle x_t - fx_t, J(x_t - x_n)\rangle \leq \frac{1-t}{t}\|Sx_t - x_t\| \cdot \|x_t - x_n\|$. According to step 7, we find $\limsup\limits_{n\to\infty}\langle (I-f)p, J(p - x_n)\rangle = 0$.

Step 11: Show that $\lim\limits_{n\to\infty}\|x_n - p\| = 0$.

According to Lemma 4, we obtain

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\leq \Big(1-\alpha_n)^2\|Sy_n - p\|^2 + 2\alpha_n\langle fx_n - p, J(x_{n+1} - p)\rangle \\
&\leq \Big(1-\alpha_n)^2\|y_n - p\|^2 + 2k\alpha_n\|x_n - p\| \cdot \|x_{n+1} - p\| \\
&\quad +2\alpha_n\langle fp - p, J(x_{n+1} - p)\rangle \\
&\leq (1-\alpha_n)^2(\beta_n\|x_n - p\| + (1-\beta_n)\|x_{n+1} - p\|)^2 \\
&\quad +2k\alpha_n\|x_n - p\| \cdot \|x_{n+1} - p\| + 2\alpha_n\langle fp - p, J(x_{n+1} - p)\rangle \\
&= (1-\alpha_n)^2\beta_n^2\|x_n - p\|^2 + (1-\alpha_n)^2(1-\beta_n)^2\|x_{n+1} - p\|^2 \\
&\quad +2\big(\beta_n(1-\alpha_n)^2(1-\beta_n) + k\alpha_n\big)\|x_n - p\| \cdot \|x_{n+1} - p\| \\
&\quad +2\alpha_n\langle fp - p, J(x_{n+1} - p)\rangle \\
&\leq (1-\alpha_n)^2\beta_n^2\|x_n - p\|^2 + (1-\alpha_n)^2(1-\beta_n)^2\|x_{n+1} - p\|^2 \\
&\quad +\big(\beta_n(1-\alpha_n)^2(1-\beta_n) + k\alpha_n\big)\Big(\|x_n - p\|^2 + \|x_{n+1} - p\|^2\Big) \\
&\quad +2\alpha_n\langle fp - p, J(x_{n+1} - p)\rangle.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\leq \frac{\beta_n^2(1-\alpha_n)^2 + (1-\alpha_n)^2\beta_n(1-\beta_n) + k\alpha_n}{1-(1-\alpha_n)^2(1-\beta_n)^2 - \beta_n(1-\alpha_n)^2(1-\beta_n) - k\alpha_n}\|x_n - p\|^2 \\
&\quad + \frac{2\alpha_n}{1-(1-\alpha_n)^2(1-\beta_n)^2 - \beta_n(1-\alpha_n)^2(1-\beta_n) - k\alpha_n}\langle fp - p, J(x_{n+1} - p)\rangle \\
&= \left[1 - \frac{1-\big(1-\alpha_n)^2 - 2k\alpha_n}{1-(1-\alpha_n)^2(1-\beta_n) - k\alpha_n}\right]\|x_n - p\|^2 \\
&\quad + \frac{2\alpha_n}{1-(1-\alpha_n)^2(1-\beta_n) - k\alpha_n}\langle fp - p, J(x_{n+1} - p)\rangle.
\end{aligned}$$

Taking $t_n = \frac{1-\big(1-\alpha_n)^2 - 2k\alpha_n}{1-(1-\alpha_n)^2(1-\beta_n) - k\alpha_n}$. From $\lim\limits_{n\to\infty}\alpha_n = 0$, we have

$$t_n \geq 1 - \Big(1-\alpha_n)^2 - 2k\alpha_n = (2 - 2k - \alpha_n)\alpha_n \geq (2 - 2k - \varepsilon)\alpha_n \ (\forall\varepsilon > 0).$$

From $\sum\limits_{n=0}^{\infty}\alpha_n = \infty$, we obtain $\sum\limits_{n=0}^{\infty}t_n = \infty$.
Taking $b_n = \frac{2\alpha_n}{1-(1-\alpha_n)^2(1-\beta_n) - k\alpha_n}\langle fp - p, J(x_{n+1} - p)\rangle$, then we have

$$\frac{b_n}{t_n} = \frac{2\langle fp - p, J(x_{n+1} - p)\rangle}{2 - 2k - \alpha_n}.$$

From $\lim\limits_{n\to\infty}\alpha_n = 0$ and step 10, we obtain $b_n = o(t_n)$.
Let $c_n = 0$, so we have $\sum\limits_{n=0}^{\infty}c_n < \infty$.
According to Lemma 2, we have $\lim\limits_{n\to\infty}\|x_n - p\| = 0$.
We also have

$$\|y_n - p\| \leq \|y_n - x_n\| + \|x_n - p\|.$$

From step 5, we find $\lim\limits_{n\to\infty}\|y_n - p\| = 0$. This completes the proof. □

Theorem 2. *Assume that E is a reflexive and uniformly convex Banach space with uniformly Gâteaux differentiable norm, C is nonempty closed convex subset of E with normal structure.* $f : C \to C$ *is contractive mapping with* $k \in [0,1)$, *A is an m-accretive operator in E and*

$S: C \to C$ is nonexpansive mapping with $F(S) \cap N(A) \neq$. For any $x_0 \in C$ and $n \geq 0$, $\{x_n\}$ is generated by

$$\begin{cases} y_n = \beta_n x_n + (1-\beta_n) J_{r_n} x_{n+1} + e_n, \\ x_{n+1} = \alpha_n f x_n + (1-\alpha_n) S y_n, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}, \{r_n\} \subset (0,1)$ and $\{e_n\} \subset E$ satisfy some conditions:

(i) $\sum\limits_{n=1}^{\infty} |\beta_n - \beta_{n-1}| < \infty$;

(ii) $\sum\limits_{n=0}^{\infty} \alpha_n = \infty, \lim\limits_{n\to\infty} a_n = 0, |\alpha_n - \alpha_{n-1}| = o(\alpha_n)$;

(iii) $\lim\limits_{n\to\infty} r_n = r, \sum\limits_{n=1}^{\infty} |r_n - r_{n-1}| < \infty$;

(iv) $\|e_n\| = o(\alpha_n)$.

Then $\{x_n\}$ and $\{y_n\}$ strongly converge to $p \in F(S) \cap N(A)$ which is the only solution of variational inequality $\langle (I-f)p, J_\phi(p-q) \rangle \leq 0, \forall q \in F(S) \cap N(A)$.

Proof. Let

$$\begin{cases} w_n = \beta_n z_n + (1-\beta_n) J_{r_n} z_{n+1}, \\ z_{n+1} = \alpha_n f z_n + (1-\alpha_n) S w_n. \end{cases}$$

Then we have

$$\begin{aligned} \|x_{n+1} - z_{n+1}\| &\leq (1-\alpha_n)\|y_n - w_n\| + k\alpha_n \|x_n - z_n\| \\ &\leq (1-\alpha_n)(\beta_n \|x_n - z_n\| + (1-\beta_n)\|x_{n+1} - z_{n+1}\| + \|e_n\|) + k\alpha_n \|x_n - z_n\| \\ &= (k\alpha_n + (1-\alpha_n)\beta_n)\|x_n - z_n\| + (1-\alpha_n)(1-\beta_n)\|x_{n+1} - z_{n+1}\| + (1-\alpha_n)\|e_n\|. \end{aligned}$$

It follows that

$$\begin{aligned} \|x_{n+1} - z_{n+1}\| &\leq \tfrac{k\alpha_n + (1-\alpha_n)\beta_n}{\alpha_n + \beta_n - \alpha_n\beta_n} \|x_n - z_n\| + \tfrac{1-\alpha_n}{\alpha_n + \beta_n - \alpha_n\beta_n} \|e_n\| \\ &= \left[1 - \tfrac{\alpha_n(1-k)}{\alpha_n + \beta_n - \alpha_n\beta_n}\right] \|x_n - z_n\| + \tfrac{1-\alpha_n}{\alpha_n + \beta_n - \alpha_n\beta_n} \|e_n\|. \end{aligned}$$

Taking $t_n = \frac{\alpha_n(1-k)}{\alpha_n + \beta_n - \alpha_n\beta_n}$, then $t_n \geq \alpha_n(1-k)$. From $\sum\limits_{n=0}^{\infty} \alpha_n = \infty$, we have $\sum\limits_{n=0}^{\infty} t_n = \infty$.

Taking $b_n = \frac{1-\alpha_n}{\alpha_n + \beta_n - \alpha_n\beta_n} \|e_n\|$, then $\frac{b_n}{t_n} = \frac{(1-\alpha_n)\|e_n\|}{\alpha_n(1-k)}$. From $\|e_n\| = o(\alpha_n)$, we have $b_n = o(t_n)$.

Let $c_n = 0$, so we have $\sum\limits_{n=0}^{\infty} c_n < \infty$.

According to Lemma 2, we have $\lim\limits_{n\to\infty} \|x_n - z_n\| = 0$. According to Theorem 1, we find that $\{w_n\}$ and $\{z_n\}$ strongly converge to $p \in F(S) \cap N(A)$ which is the only solution of variational inequality $\langle (I-f)p, J_\phi(p-q) \rangle \leq 0, \forall q \in F(S) \cap N(A)$. Then $\{y_n\}$ and $\{x_n\}$ also strongly converge to $p \in F(S) \cap N(A)$. This completes the proof. □

Theorem 3. *Assume that E is a reflexive and uniformly convex Banach space with a uniformly Gâteaux differentiable norm, C is a nonempty closed convex subset of E with normal structure. $f: C \to C$ is contractive mapping with $k \in [0,1)$, A is an m-accretive operator in E and $S: C \to C$ is nonexpansive mapping with $F(S) \cap N(A) \neq$. For any $x_0 \in C$ and $n \geq 0$, $\{x_n\}$ is generated by*

$$\begin{cases} y_n = \beta_n x_n + (1-\beta_n) J_{r_n}(x_{n+1} + e_n), \\ x_{n+1} = \alpha_n f x_n + (1-\alpha_n) S y_n, \end{cases} \quad (6)$$

where $\{\alpha_n\}, \{\beta_n\}, \{r_n\} \subset (0,1)$ and $\{e_n\} \subset E$ satisfy some conditions:

(i) $\sum\limits_{n=1}^{\infty} |\beta_n - \beta_{n-1}| < \infty$;

(ii) $\sum\limits_{n=0}^{\infty} \alpha_n = \infty,\ \lim\limits_{n\to\infty} a_n = 0,\ |\alpha_n - \alpha_{n-1}| = o(\alpha_n)$;

(iii) $\lim_{n\to\infty} r_n = r$, $\sum_{n=1}^{\infty} |r_n - r_{n-1}| < \infty$;

(iv) $\sum_{n=1}^{\infty} \|e_n\| < \infty$.

Then $\{x_n\}$ *and* $\{y_n\}$ *strongly converge to* $p \in F(S) \cap N(A)$ *which is the only solution of variational inequality* $\langle (I-f)p, J_\phi(p-q) \rangle \le 0, \forall q \in F(S) \cap N(A)$.

Proof. The proof process is divided into eleven steps.

Step 1: Show the boundedness of $\{x_n\}$ and $\{y_n\}$.

Taking $q \in F(S) \cap N(A)$, then we obtain

$$\begin{aligned}\|y_n - q\| &\le (1-\beta_n)\|J_{r_n}(x_{n+1}+e_n) - q\| + \beta_n\|x_n - q\| \\ &\le (1-\beta_n)\|x_{n+1}+e_n - q\| + \beta_n\|x_n - q\| \\ &\le (1-\beta_n)\|x_{n+1} - q\| + (1-\beta_n)\|e_n\| + \beta_n\|x_n - q\|.\end{aligned}$$

and then we obtain

$$\begin{aligned}\|x_{n+1} - q\| &\le \alpha_n\|fx_n - q\| + (1-\alpha_n)\|Sy_n - q\| \\ &\le k\alpha_n\|x_n - q\| + \alpha_n\|fq - q\| + (1-\alpha_n)\|y_n - q\| \\ &\le k\alpha_n\|x_n - q\| + \alpha_n\|fq - q\| + (1-\alpha_n)\beta_n\|x_n - q\| \\ &+(1-\alpha_n)(1-\beta_n)\|x_{n+1} - q\| + (1-\alpha_n)(1-\beta_n)\|e_n\|.\end{aligned}$$

It follows that

$$\begin{aligned}\|x_{n+1} - q\| &\le \tfrac{(1-\alpha_n)\beta_n + k\alpha_n}{\alpha_n+\beta_n-\alpha_n\beta_n}\|x_n - q\| + \tfrac{2\alpha_n}{\alpha_n+\beta_n-\alpha_n\beta_n}\|fq - q\| \\ &+ \tfrac{(1-\alpha_n)(1-\beta_n)}{\alpha_n+\beta_n-\alpha_n\beta_n}\|e_n\| \\ &\le \left[1 - \tfrac{\alpha_n(1-k)}{\alpha_n+\beta_n-\alpha_n\beta_n}\right]\|x_n - q\| + \tfrac{\alpha_n(1-k)}{\alpha_n+\beta_n-\alpha_n\beta_n}\tfrac{\|fq-q\|}{1-k} \\ &+N\cdot\|e_n\| \\ &\le \max\left\{\|x_0 - q\|, \tfrac{\|fq-q\|}{1-k} + N\cdot\|e_n\|\right\}.\end{aligned}$$

Then $\{x_n\}$ is bounded. So $\{y_n\}$, $\{fx_n\}$, $\{Sx_n\}$, $\{J_{r_n}x_n\}$, $\{fy_n\}$, $\{J_{r_n}y_n\}$ and $\{Sy_n\}$ are also bounded.

Step 2: Show that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$.

According to (6), we find

$$\begin{aligned}\|x_{n+1} - x_n\| &= \|\alpha_n fx_n + (1-\alpha_n)Sy_n - \alpha_{n-1}fx_{n-1} - (1-\alpha_{n-1})Sy_{n-1}\| \\ &\le \alpha_n\|fx_n - fx_{n-1}\| + (1-\alpha_n)\|Sy_n - Sy_{n-1}\| + |\alpha_n - \alpha_{n-1}|\cdot\|fx_{n-1} - Sy_{n-1}\| \\ &\le k\alpha_n\|x_n - x_{n-1}\| + (1-\alpha_n)\|y_n - y_{n-1}\| + |\alpha_n - \alpha_{n-1}|\cdot\|fx_{n-1} - Sy_{n-1}\|.\end{aligned} \tag{7}$$

From (6), we obtain

$$\begin{aligned}\|y_n - y_{n-1}\| \le \beta_n\|x_n - x_{n-1}\| &+ (1-\beta_n)\left\|J_{r_n}(x_{n+1}+e_n) - J_{r_{n-1}}(x_n + e_{n-1})\right\| \\ &+|\beta_n - \beta_{n-1}|\cdot\left\|x_{n-1} - J_{r_{n-1}}(x_n + e_{n-1})\right\|.\end{aligned} \tag{8}$$

From Lemma 1, we have

$$\begin{aligned}&\left\|J_{r_n}(x_{n+1}+e_n) - J_{r_{n-1}}(x_n + e_{n-1})\right\| \\ &= \left\|J_{r_{n-1}}\left(\tfrac{r_{n-1}}{r_n}(x_{n+1}+e_n) + \left(1 - \tfrac{r_{n-1}}{r_n}\right)J_{r_n}(x_{n+1}+e_n)\right) - J_{r_{n-1}}(x_n + e_{n-1})\right\| \\ &\le \left\|\left(1 - \tfrac{r_{n-1}}{r_n}\right)J_{r_n}(x_{n+1}+e_n) - (x_n + e_{n-1}) + \tfrac{r_{n-1}}{r_n}(x_{n+1}+e_n)\right\| \\ &\le \left|1 - \tfrac{r_{n-1}}{r_n}\right|\cdot\|J_{r_n}(x_{n+1}+e_n) - (x_n + e_{n-1})\| + \tfrac{r_{n-1}}{r_n}\|x_{n+1} - x_n + e_n - e_{n-1}\| \\ &\le \left|1 - \tfrac{r_{n-1}}{r_n}\right|\cdot\|J_{r_n}(x_{n+1}+e_n) - (x_{n+1}+e_n)\| + \|x_{n+1} - x_n + e_n - e_{n-1}\|.\end{aligned} \tag{9}$$

Taking (8) and (9) into (7), we have

$$\begin{aligned}\|x_{n+1}-x_n\| \le{}& k\alpha_n\|x_n-x_{n-1}\| + (1-\alpha_n)\beta_n\|x_n-x_{n-1}\| + (1-\alpha_n)|\beta_n-\beta_{n-1}|M_3\\ &+(1-\alpha_n)(1-\beta_n)\left|1-\frac{r_{n-1}}{r_n}\right|M_4 + (1-\alpha_n)(1-\beta_n)\|x_{n+1}-x_n\|\\ &+(1-\alpha_n)(1-\beta_n)\|e_n-e_{n-1}\| + |\alpha_n-\alpha_{n-1}|M_1.\end{aligned}$$

It follows that

$$\begin{aligned}\|x_{n+1}-x_n\| \le{}& \left[1-\frac{\alpha_n(1-k)}{\alpha_n+\beta_n-\alpha_n\beta_n}\right]\|x_n-x_{n-1}\| + \frac{|\alpha_n-\alpha_{n-1}|}{\alpha_n+\beta_n-\alpha_n\beta_n}M_1\\ &+|\beta_n-\beta_{n-1}|N\cdot M_3 + \frac{|r_n-r_{n-1}|}{r_n}N\cdot M_4 + (\|e_n\|+\|e_{n-1}\|)N.\end{aligned}$$

where $M_3=\max\{\|x_{n-1}-J_{r_{n-1}}(x_n+e_{n-1})\|\}$, $M_4=\max\{\|J_{r_n}(x_{n+1}+e_n)-(x_{n+1}+e_n)\|\}$.

Taking $t_n=\frac{\alpha_n(1-k)}{\alpha_n+\beta_n-\alpha_n\beta_n}$, then $t_n>\alpha_n(1-k)$. From $\sum\limits_{n=0}^{\infty}\alpha_n=\infty$, so $\sum\limits_{n=0}^{\infty}t_n=\infty$.

Taking $b_n=\frac{|\alpha_n-\alpha_{n-1}|}{\alpha_n+\beta_n-\alpha_n\beta_n}M_1$, then $\frac{b_n}{t_n}=\frac{|\alpha_n-\alpha_{n-1}|M_1}{\alpha_n(1-k)}$. From $|\alpha_n-\alpha_{n-1}|=o(\alpha_n)$, so $b_n=o(t_n)$.

Let $c_n=|\beta_n-\beta_{n-1}|N\cdot M_3+\frac{|r_n-r_{n-1}|}{r_n}N\cdot M_4+(\|e_n\|+\|e_{n-1}\|)N$, then
$c_n<|\beta_n-\beta_{n-1}|N\cdot M_3+\frac{|r_n-r_{n-1}|}{r+\varepsilon}N\cdot M_4+(\|e_n\|+\|e_{n-1}\|)N(\forall\varepsilon>0)$.

From $\sum\limits_{n=1}^{\infty}|\beta_n-\beta_{n-1}|<\infty$, $\sum\limits_{n=1}^{\infty}\|e_n\|<\infty$, $\lim\limits_{n\to\infty}r_n=r$ and $\sum\limits_{n=1}^{\infty}|r_n-r_{n-1}|<\infty$, so $\sum\limits_{n=1}^{\infty}c_n<\infty$.

According to Lemma 2, so we have $\lim\limits_{n\to\infty}\|x_{n+1}-x_n\|=0$.

Step 3: Show that $\lim\limits_{n\to\infty}\|x_n-J_{r_n}x_n\|=0$.

From (1) and $\|\cdot\|^2$ is a convex function and, then we obtain

$$\begin{aligned}\|x_{n+1}-q\|^2 \le{}& (1-\alpha_n)\|Sy_n-q\|^2+\alpha_n\|fx_n-q\|^2\\ \le{}& (1-\alpha_n)\|y_n-q\|^2+\alpha_n\|fx_n-q\|^2\\ \le{}& \beta_n(1-\alpha_n)\|x_n-q\|^2+(1-\alpha_n)(1-\beta_n)\|J_{r_n}(x_{n+1}+e_n)-q\|^2\\ &+\alpha_n\|fx_n-q\|^2\\ ={}& \alpha_n\|fx_n-q\|^2+\beta_n(1-\alpha_n)\|x_n-q\|^2\\ &+(1-\alpha_n)(1-\beta_n)\left\|J_{\frac{r_n}{2}}\left(\frac{1}{2}(x_{n+1}+e_n)+\frac{1}{2}J_{r_n}(x_{n+1}+e_n)\right)-q\right\|^2\\ \le{}& \alpha_n\|fx_n-q\|^2+\beta_n(1-\alpha_n)\|x_n-q\|^2+\frac{(1-\alpha_n)(1-\beta_n)}{2}\|x_{n+1}+e_n-q\|^2\\ &+\frac{(1-\alpha_n)(1-\beta_n)}{2}\|J_{r_n}(x_{n+1}+e_n)-q\|^2\\ &-\frac{(1-\alpha_n)(1-\beta_n)}{4}g(\|x_{n+1}+e_n-J_{r_n}(x_{n+1}+e_n)\|)\\ \le{}& \alpha_n\|fx_n-q\|^2+\beta_n(1-\alpha_n)\|x_n-q\|^2+(1-\alpha_n)(1-\beta_n)\|x_{n+1}+e_n-q\|^2\\ &-\frac{(1-\alpha_n)(1-\beta_n)}{4}g(\|x_{n+1}+e_n-J_{r_n}(x_{n+1}+e_n)\|)\\ \le{}& \alpha_n\|fx_n-q\|^2+\beta_n(1-\alpha_n)\|x_n-q\|^2\\ &+(1-\alpha_n)(1-\beta_n)\left(\|x_{n+1}-q\|^2+2\langle e_n,J(x_{n+1}+e_n-q)\rangle\right)\\ &-\frac{(1-\alpha_n)(1-\beta_n)}{4}g(\|x_{n+1}+e_n-J_{r_n}(x_{n+1}+e_n)\|).\end{aligned}$$

It follows that

$$\begin{aligned}\|x_{n+1}-q\|^2 \le{}& \left(1-\frac{\alpha_n}{\alpha_n+\beta_n-\alpha_n\beta_n}\right)\|x_n-q\|^2+\frac{\alpha_n}{\alpha_n+\beta_n-\alpha_n\beta_n}\|fx_n-q\|^2\\ &+2N\|e_n\|\cdot\|x_{n+1}+e_n-q\|-\frac{(1-\alpha_n)(1-\beta_n)}{4}g(\|x_{n+1}+e_n-J_{r_n}(x_{n+1}+e_n)\|)\\ \le{}& \|x_n-q\|^2+\frac{\alpha_n}{\alpha_n+\beta_n-\alpha_n\beta_n}\|fx_n-q\|^2+2N\|e_n\|\cdot\|x_{n+1}+e_n-q\|\\ &-\frac{(1-\alpha_n)(1-\beta_n)}{4}g(\|x_{n+1}+e_n-J_{r_n}(x_{n+1}+e_n)\|).\end{aligned}$$

Then we have

$$\frac{(1-\alpha_n)(1-\beta_n)}{4} g(\|x_{n+1}+e_n-J_{r_n}(x_{n+1}+e_n)\|) - \frac{\alpha_n}{\alpha_n+\beta_n-\alpha_n\beta_n}\|fx_n-q\|^2$$
$$-2N\|e_n\|\cdot\|x_{n+1}+e_n-q\| \le \|x_n-q\|^2-\|x_{n+1}-q\|^2.$$

If

$$\frac{(1-\alpha_n)(1-\beta_n)}{4} g(\|x_{n+1}+e_n-J_{r_n}(x_{n+1}+e_n)\|) \le \frac{\alpha_n}{\alpha_n+\beta_n-\alpha_n\beta_n}\|fx_n-q\|^2$$
$$+2N\|e_n\|\cdot\|x_{n+1}+e_n-q\|.$$

so from $\lim_{n\to\infty}\alpha_n=0$, step 1 and $\sum_{n=0}^{\infty}\|e_n\|<\infty$, we have $\lim_{n\to\infty} g(\|x_{n+1}+e_n-J_{r_n}(x_{n+1}+e_n)\|)=0$.

If

$$\frac{(1-\alpha_n)(1-\beta_n)}{4} g(\|x_{n+1}+e_n-J_{r_n}(x_{n+1}+e_n)\|) \ge \frac{\alpha_n}{\alpha_n+\beta_n-\alpha_n\beta_n}\|fx_n-q\|^2$$
$$+2N\|e_n\|\cdot\|x_{n+1}+e_n-q\|.$$

so

$$\sum_{n=0}^{H}\Big[\frac{(1-\alpha_n)(1-\beta_n)}{4} g(\|x_{n+1}+e_n-J_{r_n}(x_{n+1}+e_n)\|) - \frac{\alpha_n}{\alpha_n+\beta_n-\alpha_n\beta_n}\|fx_n-q\|^2$$
$$-2N\|e_n\|\cdot\|x_{n+1}+e_n-q\|\Big] \le \|x_0-q\|^2-\|x_{H+1}-q\|^2 \le \|x_0-q\|^2.$$

Then

$$\sum_{n=0}^{\infty}\Big[\frac{(1-\alpha_n)(1-\beta_n)}{4} g(\|x_{n+1}+e_n-J_{r_n}(x_{n+1}+e_n)\|) - \frac{\alpha_n}{\alpha_n+\beta_n-\alpha_n\beta_n}\|fx_n-q\|^2$$
$$-2N\|e_n\|\cdot\|x_{n+1}+e_n-q\|\Big] < \infty.$$

So we have

$$\lim_{n\to\infty}\Big[\frac{(1-\alpha_n)(1-\beta_n)}{4} g(\|x_{n+1}+e_n-J_{r_n}(x_{n+1}+e_n)\|) - \frac{\alpha_n}{\alpha_n+\beta_n-\alpha_n\beta_n}\|fx_n-q\|^2$$
$$-2N\|e_n\|\cdot\|x_{n+1}+e_n-q\|\Big] = 0.$$

and then $\lim_{n\to\infty} g(\|x_{n+1}+e_n-J_{r_n}(x_{n+1}+e_n)\|)=0$.

According to the property of g, we obtain $\lim_{n\to\infty}\|x_{n+1}+e_n-J_{r_n}(x_{n+1}+e_n)\|=0$.

We also have

$$\begin{aligned}\|x_n-J_{r_n}x_n\| &\le \|x_n-x_{n+1}\|+\|x_{n+1}+e_n-J_{r_n}(x_{n+1}+e_n)\|+\|e_n\| \\ &\quad +\|J_{r_n}(x_{n+1}+e_n)-J_{r_n}x_n\| \\ &\le \|x_n-x_{n+1}\|+\|x_{n+1}+e_n-J_{r_n}(x_{n+1}+e_n)\|+\|e_n\|+\|x_{n+1}+e_n-x_n\| \\ &\le 2\|x_n-x_{n+1}\|+\|x_{n+1}+e_n-J_{r_n}(x_{n+1}+e_n)\|+2\|e_n\|.\end{aligned}$$

According to $\sum_{n=0}^{\infty}\|e_n\|<\infty$ and step 2, we have $\lim_{n\to\infty}\|x_n-J_{r_n}x_n\|=0$.

Step 4: Show that $\lim_{n\to\infty}\|y_n-Sy_n\|=0$.

From (6), we obtain

$$\begin{aligned}\|y_n-Sy_n\| &\le \beta_n\|x_n-Sy_n\|+(1-\beta_n)\|J_{r_n}(x_{n+1}+e_n)-Sy_n\| \\ &\le \beta_n\|x_n-x_{n+1}\|+\beta_n\|x_{n+1}-Sy_n\|+(1-\beta_n)\|J_{r_n}(x_{n+1}+e_n)-(x_{n+1}+e_n)\| \\ &\quad +(1-\beta_n)\|x_{n+1}+e_n-Sy_n\| \\ &\le \beta_n\|x_n-x_{n+1}\|+\|x_{n+1}-Sy_n\|+(1-\beta_n)\|J_{r_n}(x_{n+1}+e_n)-(x_{n+1}+e_n)\| \\ &\quad +(1-\beta_n)\|e_n\| \\ &= \beta_n\|x_n-x_{n+1}\|+\alpha_n\|fx_n-Sy_n\|+(1-\beta_n)\|J_{r_n}(x_{n+1}+e_n)-(x_{n+1}+e_n)\| \\ &\quad +(1-\beta_n)\|e_n\|.\end{aligned}$$

From step 1, step 2 and step 3, $\lim_{n\to\infty}\alpha_n=0$ and $\sum_{n=0}^{\infty}\|e_n\|<\infty$, we have $\lim_{n\to\infty}\|y_n-Sy_n\|=0$.

Step 5: Show that $\lim_{n\to\infty}\|x_n - y_n\| = 0$.

Using the results of step 2 and step 4, we obtain

$$\begin{aligned}\|x_n - y_n\| &= \|x_n - x_{n+1} + x_{n+1} - Sy_n + Sy_n - y_n\| \\ &\leq \|x_n - x_{n+1}\| + \alpha_n\|fx_n - Sy_n\| + \|Sy_n - y_n\|.\end{aligned}$$

According to $\lim_{n\to\infty}\alpha_n = 0$ and the boundedness of $\{fx_n\}$ and $\{Sy_n\}$, we have $\lim_{n\to\infty}\|x_n - y_n\| = 0$.

Step 6: Show that $\lim_{n\to\infty}\|y_n - J_{r_n}y_n\| = 0$.

Using the results of step 3 and step 5, we obtain

$$\begin{aligned}\|y_n - J_{r_n}y_n\| &\leq \|y_n - x_n\| + \|x_n - J_{r_n}x_n\| + \|J_{r_n}x_n - J_{r_n}y_n\| \\ &\leq \|y_n - x_n\| + \|x_n - J_{r_n}x_n\| + \|x_n - y_n\|.\end{aligned}$$

So we have $\lim_{n\to\infty}\|y_n - J_{r_n}y_n\| = 0$.

Step 7: Show that $\lim_{n\to\infty}\|x_n - Sx_n\| = 0$.

Using the results of step 4 and step 5, we obtain

$$\begin{aligned}\|x_n - Sx_n\| &\leq \|x_n - y_n\| + \|y_n - Sy_n\| + \|Sy_n - Sx_n\| \\ &\leq \|x_n - y_n\| + \|y_n - Sy_n\| + \|y_n - x_n\|.\end{aligned}$$

So we have $\lim_{n\to\infty}\|x_n - Sx_n\| = 0$.

Step 8: Show that $\lim_{n\to\infty}\|y_n - J_r y_n\| = 0$.

Using the results of step 6 and Lemma 1, we obtain

$$\begin{aligned}\|y_n - J_r y_n\| &\leq \|y_n - J_{r_n}y_n\| + \|J_{r_n}y_n - J_r y_n\| \\ &= \|y_n - J_{r_n}y_n\| + \left\|J_r\left(\tfrac{r}{r_n}y_n + \left(1 - \tfrac{r}{r_n}\right)J_{r_n}y_n\right) - J_r y_n\right\| \\ &\leq \|y_n - J_{r_n}y_n\| + \left|1 - \tfrac{r}{r_n}\right| \cdot \|J_{r_n}y_n - y_n\|.\end{aligned}$$

From $\lim_{n\to\infty} r_n = r$, we have $\lim_{n\to\infty}\|y_n - J_r y_n\| = 0$.

Step 9: Show that $\lim_{n\to\infty}\|x_n - J_r x_n\| = 0$.

Using the results of step 5 and step 8, we obtain

$$\begin{aligned}\|x_n - J_r x_n\| &\leq \|x_n - y_n\| + \|y_n - J_r y_n\| + \|J_r y_n - J_r x_n\| \\ &\leq \|x_n - y_n\| + \|y_n - J_r y_n\| + \|y_n - x_n\|.\end{aligned}$$

So we have $\lim_{n\to\infty}\|x_n - J_r x_n\| = 0$.

Step 10: Show that $\limsup_{n\to\infty}\langle (I-f)p, J(p - x_n)\rangle = 0$.

According to Theorem 1, we find that $\{x_t\}$ strongly converges to $p \in P_{F(S)\cap N(A)}fp$ which is the only solution of variational inequality $\langle (I-f)p, J(p-q)\rangle \leq 0, \forall q \in F(S) \cap N(A)$.

We obtain

$$\begin{aligned}\|x_t - x_n\|^2 &= (1-t)\langle Sx_t - x_n, J(x_t - x_n)\rangle + t\langle fx_t - x_n, J(x_t - x_n)\rangle \\ &\leq (1-t)\|Sx_t - x_t\| \cdot \|x_t - x_n\| + (1-t)\|x_t - x_n\|^2 \\ &\quad + t\langle fx_t - x_t, J(x_t - x_n)\rangle + t\|x_t - x_n\|^2 \\ &= (1-t)\|Sx_t - x_t\| \cdot \|x_t - x_n\| + \|x_t - x_n\|^2 + t\langle fx_t - x_t, J(x_t - x_n)\rangle.\end{aligned}$$

It follows that $\langle x_t - f x_t, J(x_t - x_n)\rangle \le \frac{1-t}{t}\|Sx_t - x_t\| \cdot \|x_t - x_n\|$. According to step 1 and step 7, we have $\limsup_{n\to\infty}\langle (I-f)p, J(p-x_n)\rangle = 0$.

Step 11: Show that $\lim_{n\to\infty}\|x_n - p\| = 0$.

According to Lemma 4, we obtain

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\le \Big(1-\alpha_n)^2\|Sy_n - p\|^2 + 2\alpha_n\langle f x_n - p, J(x_{n+1}-p)\rangle \\
&\le \Big(1-\alpha_n)^2\|y_n - p\|^2 + 2k\alpha_n\|x_n - p\|\cdot\|x_{n+1}-p\| + 2\alpha_n\langle fp - p, J(x_{n+1}-p)\rangle \\
&\le \Big(1-\alpha_n)^2((1-\beta_n)\|x_{n+1}-p\| + (1-\beta_n)\|e_n\| + \beta_n\|x_n - p\|)^2 \\
&\quad +2k\alpha_n\|x_n - p\|\cdot\|x_{n+1}-p\| + 2\alpha_n\langle fp - p, J(x_{n+1}-p)\rangle \\
&= \beta_n^2(1-\alpha_n)^2\|x_n - p\|^2 + (1-\alpha_n)^2\Big(1-\beta_n)^2\|x_{n+1}-p\|^2 \\
&\quad +\Big(1-\alpha_n)^2(1-\beta_n)\|e_n\|^2 + 2\beta_n(1-\alpha_n)^2(1-\beta_n)\|x_n - p\|\cdot\|x_{n+1}-p\| \\
&\quad +2\beta_n(1-\alpha_n)^2(1-\beta_n)\|x_n - p\|\cdot\|e_n\| + 2k\alpha_n\|x_n - p\|\cdot\|x_{n+1}-p\| \\
&\quad +2(1-\alpha_n)^2(1-\beta_n)^2\|x_{n+1}-p\|\cdot\|e_n\| + 2\alpha_n\langle fp - p, J(x_{n+1}-p)\rangle \\
&\le \beta_n^2(1-\alpha_n)^2\|x_n - p\|^2 + (1-\alpha_n)^2(1-\beta_n)^2\|x_{n+1}-p\|^2 \\
&\quad +\Big(\beta_n(1-\alpha_n)^2(1-\beta_n) + k\alpha_n\Big)\Big(\|x_n - p\|^2 + \|x_{n+1}-p\|^2\Big) \\
&\quad +2(1-\alpha_n)^2(1-\beta_n)\|e_n\|(\beta_n\|x_n - p\| + (1-\beta_n)\|x_{n+1}-p\|) \\
&\quad +2\alpha_n\langle fp - p, J(x_{n+1}-p)\rangle + \Big(1-\alpha_n)^2(1-\beta_n)\|e_n\|^2.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\le \frac{\big(1-\alpha_n)^2\beta_n^2 + \big(1-\alpha_n)^2\beta_n(1-\beta_n) + k\alpha_n}{1-(1-\alpha_n)^2(1-\beta_n)^2 - (1-\alpha_n)^2\beta_n(1-\beta_n) - k\alpha_n}\|x_n - p\|^2 \\
&\quad + \frac{2\alpha_n}{1-(1-\alpha_n)^2(1-\beta_n)^2 - (1-\alpha_n)^2\beta_n(1-\beta_n) - k\alpha_n}\langle fp - p, J(x_{n+1}-p)\rangle \\
&\quad + \frac{\big(1-\alpha_n)^2(1-\beta_n)\|e_n\|}{1-(1-\alpha_n)^2(1-\beta_n)^2 - (1-\alpha_n)^2\beta_n(1-\beta_n) - k\alpha_n}(2\beta_n\|x_n - p\| + 2(1-\beta_n)\|x_{n+1}-p\| + \|e_n\|) \\
&= \left[1 - \frac{1-\big(1-\alpha_n)^2 - 2k\alpha_n}{1-(1-\alpha_n)^2(1-\beta_n) - k\alpha_n}\right]\|x_n - p\|^2 \\
&\quad + \frac{2\alpha_n}{1-(1-\alpha_n)^2(1-\beta_n) - k\alpha_n}\langle fp - p, J(x_{n+1}-p)\rangle \\
&\quad + \frac{\big(1-\alpha_n)^2(1-\beta_n)\|e_n\|}{1-(1-\alpha_n)^2(1-\beta_n) - k\alpha_n}(2\beta_n\|x_n - p\| + 2(1-\beta_n)\|x_{n+1}-p\| + \|e_n\|).
\end{aligned}$$

Taking $t_n = \frac{1-\big(1-\alpha_n)^2 - 2k\alpha_n}{1-(1-\alpha_n)^2(1-\beta_n) - k\alpha_n}$. From $\lim_{n\to\infty}\alpha_n = 0$, we have

$$t_n \ge 1 - \Big(1-\alpha_n)^2 - 2k\alpha_n = (2-2k-\alpha_n)\alpha_n \ge (2-2k-\varepsilon)\alpha_n\ (\forall\varepsilon > 0).$$

From $\sum_{n=0}^{\infty}\alpha_n = \infty$, we obtain $\sum_{n=0}^{\infty}t_n = \infty$.

Taking $b_n = \frac{2\alpha_n}{1-(1-\alpha_n)^2(1-\beta_n) - k\alpha_n}\langle fp - p, J(x_{n+1}-p)\rangle$, then we have

$$\frac{b_n}{t_n} = \frac{2\langle fp - p, J(x_{n+1}-p)\rangle}{2-2k-\alpha_n}.$$

From $\lim_{n\to\infty}\alpha_n = 0$ and step 10, we obtain $b_n = o(t_n)$.

Let $c_n = \frac{\big(1-\alpha_n)^2(1-\beta_n)\|e_n\|}{1-(1-\alpha_n)^2(1-\beta_n) - k\alpha_n}(2\beta_n\|x_n - p\| + 2(1-\beta_n)\|x_{n+1}-p\| + \|e_n\|)$, then $c_n \le 2L\Big(\beta_n\|x_n - p\| + (1-\beta_n)\|x_{n+1}-p\| + \frac{\|e_n\|}{2}\Big)\|e_n\|$, where $L = \max\left\{\frac{\big(1-\alpha_n)^2(1-\beta_n)}{1-(1-\alpha_n)^2(1-\beta_n) - k\alpha_n}\right\}$. From the boundedness of $\{x_n\}$, $\{\|e_n\|\}$ and $\{\beta_n\}$ and $\sum_{n=0}^{\infty}\|e_n\| < \infty$, we obtain $\sum_{n=0}^{\infty}c_n < \infty$.

According to Lemma 2, we have $\lim_{n\to\infty}\|x_n - p\| = 0$.

We also have

$$\|y_n - p\| \leq \|y_n - x_n\| + \|x_n - p\|.$$

From step 5, we obtain $\lim\limits_{n\to\infty}\|y_n - p\| = 0$. This completes the proof. □

4. Numerical Examples

We provide some numerical examples to verify conclusions.

Example 1. *For any $x \in R$, assume $J_{r_n}x = \frac{r_n x}{3}$, $f(x) = \frac{x}{7}$ and $S(x) = \frac{x}{5}$, then $F(S) = \{0\}$. Assume $\alpha_n = \frac{1}{n}$, $\beta_n = \frac{n}{2n+2}$ and $r_n = 1 - \frac{1}{n}$. From Theorem 1, they satisfy these conditions. $\{x_n\}$ is generated by (2). So we find that $\{x_n\}$ strongly converges to 0.*

From (2), we have

$$x_{n+1} = \frac{-21n^3 + 51n^2 + 30n}{203n^3 + 210n^2 + 21n - 14}x_n. \tag{10}$$

Let $x_1 = 1$ in (10) and then we obtain the desired results; see Figure 1.

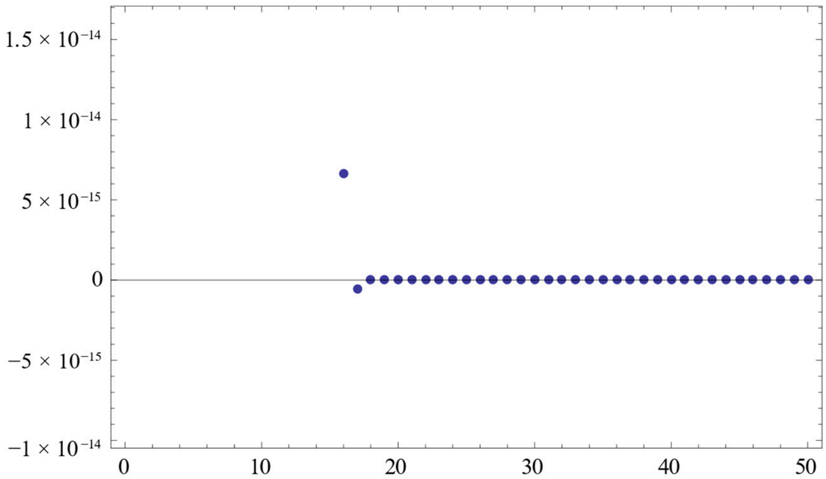

Figure 1. Numerical results.

Example 2. *For any $x \in R$, assume $J_{r_n}x = \frac{r_n x}{3}$, $f(x) = \frac{x}{7}$ and $S(x) = \frac{x}{5}$, then $F(S) = \{0\}$. Assume $\alpha_n = \frac{1}{n}$, $\beta_n = \frac{n}{2n+2}$, $r_n = 1 - \frac{1}{n}$ and $e_n = \frac{1}{n^2}$. From Theorem 3, they satisfy these conditions. $\{x_n\}$ is generated by (6). So we find that $\{x_n\}$ strongly converges to 0.*

From (6), we have

$$x_{n+1} = \frac{-21n^3 + 51n^2 + 30n}{203n^3 + 210n^2 + 21n - 14}x_n + \frac{n^3 - 3n + 2}{29n^5 + 30n^4 + 3n^3 - 2n^2} \tag{11}$$

Next, let $x_1 = 1$ in (11) and then we obtain the desired results; see Figure 2.

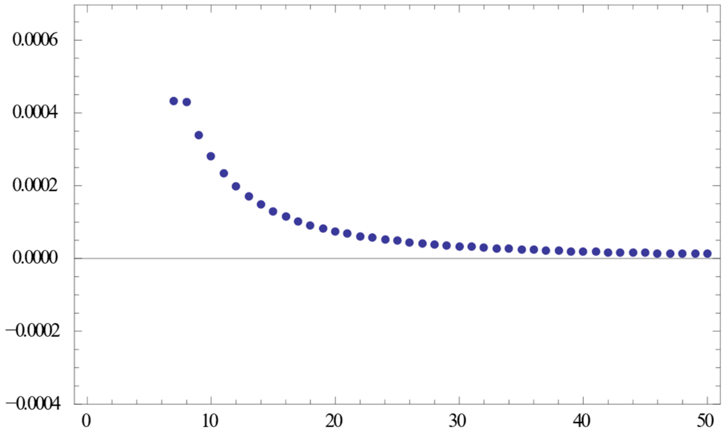

Figure 2. Numerical results.

Example 3. *The inner product* $\langle \cdot, \cdot \rangle : R^3 \times R^3 \to R$ *is defined by*

$$\langle x, y \rangle = x_1 y_1 + x_2 y_2 + x_2 y_3.$$

The usual norm $\|\cdot\| : R^3 \to R$ *is defined by*

$$\|x\| = \sqrt{x_1^2 + x_2^2 + x_3^2}, x = (x_1, x_2, x_3) \in R^3.$$

For any $x \in R^3$, *assume* $J_{r_n} x = \frac{r_n x}{3}$, $f(x) = \frac{x}{7}$ *and* $S(x) = \frac{x}{5}$, *then* $F(S) = \{0\}$. *Assume* $\alpha_n = \frac{1}{n}$, $\beta_n = \frac{n}{2n+2}$ *and* $r_n = 1 - \frac{1}{n}$. *From Theorem 1, they satisfy these conditions.* $\{x_n\}$ *is generated by (2). So we find that* $\{x_n\}$ *strongly converges to 0.*

From (2), we have

$$x_{n+1} = \frac{-21n^3 + 51n^2 + 30n}{203n^3 + 210n^2 + 21n - 14} x_n. \tag{12}$$

Let $x_1 = (1, 2, 3)$ in (12) and then we obtain the desired results; see Figure 3.

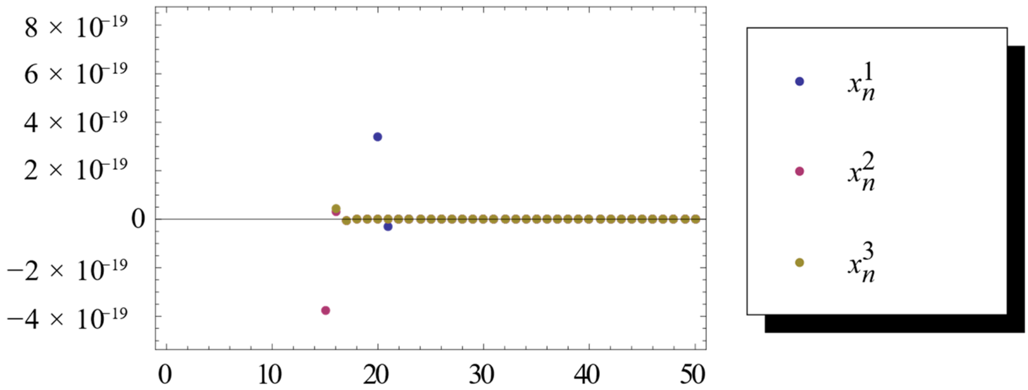

Figure 3. Numerical results.

Example 4. *The inner product* $\langle \cdot, \cdot \rangle : R^3 \times R^3 \to R$ *is defined by*

$$\langle x, y \rangle = x_1 y_1 + x_2 y_2 + x_2 y_3.$$

The usual norm $\|\cdot\| : R^3 \to R$ *is defined by*

$$\|x\| = \sqrt{x_1^2 + x_2^2 + x_3^2}, x = (x_1, x_2, x_3) \in R^3.$$

For any $x \in R^3$, assume $J_{r_n}x = \frac{r_n x}{3}$, $f(x) = \frac{x}{7}$ and $S(x) = \frac{x}{5}$, then $F(S) = \{0\}$. Assume $\alpha_n = \frac{1}{n}$, $\beta_n = \frac{n}{2n+2}$, $r_n = 1 - \frac{1}{n}$ and $e_n = \frac{1}{n^2}$. From Theorem 3, they satisfy these conditions. $\{x_n\}$ is generated by (6). So we find that $\{x_n\}$ strongly converges to 0.

From (6), we have

$$x_{n+1} = \frac{-21n^3 + 51n^2 + 30n}{203n^3 + 210n^2 + 21n - 14} x_n + \frac{n^3 - 3n + 2}{29n^5 + 30n^4 + 3n^3 - 2n^2}. \tag{13}$$

Let $x_1 = (1, 100, 10)$ in (13) and then we obtain the desired results; see Figure 4.

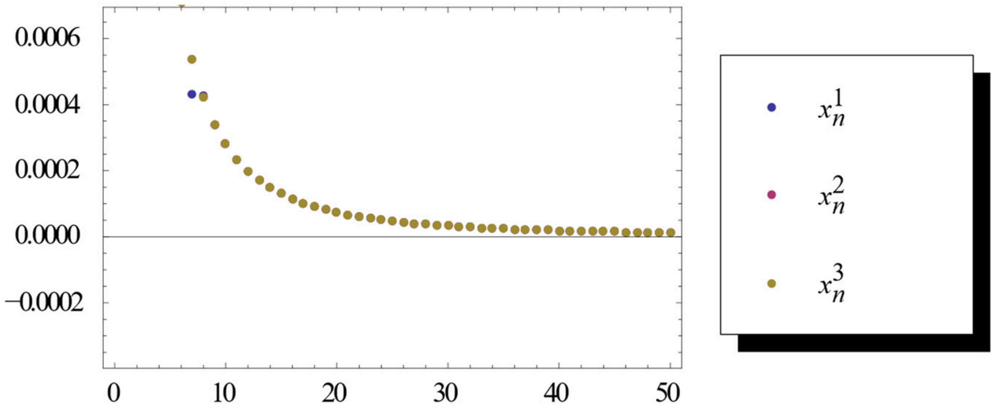

Figure 4. Numerical results.

Example 5. *Let $\{x_n\}$ be generated by Example 1. $\{x_n'\}$ is generated by*

$$\begin{cases} y_n = \beta_n\left(\frac{x_n + x_{n+1}}{2}\right) + (1-\beta_n)J_{r_n}\left(\frac{x_n + x_{n+1}}{2}\right), \\ x_{n+1} = \alpha_n f x_n + (1-\alpha_n)Sy_n. \end{cases}$$

$\{x_n''\}$ is generated by

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T(s_n x_n + (1-s_n)x_{n+1}).$$

And assume most of the conditions of Example 1 are satisfied except $\beta_n = \frac{1}{n}$, $s_n = 1 - \frac{1}{n}$ and $T(x) = \frac{x}{5}$, so we obtain

$$x_{n+1}' = \frac{3 - 9n + 9n^2 + 3n^3 - 14n^4}{-3 + 9n - 9n^2 - 51n^3 + 6n^4} x_n',$$

and

$$x_{n+1}'' = \frac{2 - 4n + 2n^2 + n^3}{6n^2 - 2n^3} x_n'',$$

Let $x_1 = 1$ in $\{x_n\}$, $\{x_n'\}$ and $\{x_n''\}$, then we obtain the desired results; see Figure 5.

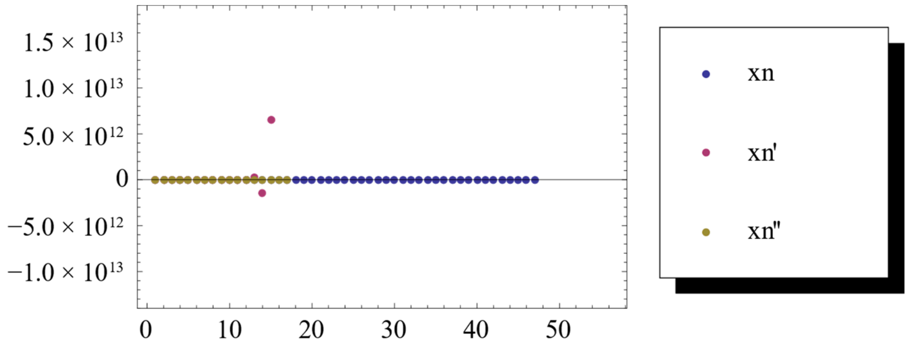

Figure 5. Numerical results.

Figure 5 shows that iterative algorithm (2) has a faster convergence speed than the algorithm of Zhang [20,21]. The stability and effectiveness of iterative algorithm (2) are also better than the algorithm of Zhang [20,21].

5. Conclusions

In the paper, we propose a new iterative algorithm by using the generalized viscosity implicit midpoint rule and Banach space, which is also a symmetric space. Under some conditions, we find that the sequence strongly converges to a common point of the fixed point set of nonexpansive mapping and the zero point set of the accretive operator. Our work extends the results of Xu [17], Luo [18], Ke [19], Zhang [20] and Zhang [21]. In the end, we give five numerical examples and show that our algorithm can achieve faster convergence speed, stability and effectiveness. This work further extends and enriches the relevant theory of symmetric space. In this paper, we considered extending nonexpansive mapping to more general mappings, and will continue to research this issue to find better iterative algorithms.

Funding: This research received no external funding.

Data Availability Statement: Data are contained within the article.

Conflicts of Interest: The author declares no conflicts of interest.

References

1. Chang, S.S.; Lee, H.W.J.; Chan, C.K. Strong convergence theorems by viscosity approximation methods for accretive mappings and nonexpansive mappings. *J. Appl. Math. Inform.* **2009**, *27*, 59–68.
2. Jung, J.S.; Cho, Y.J.; Zhou, H.Y. Iterative processes with mixed errors for nonlinear equations with perturbed m-accretive operators in Banach spaces. *Appl. Math. Comput.* **2002**, *133*, 389–406. [CrossRef]
3. Moudafi, A. Viscosity approximation methods for fixed points problems. *J. Math. Anal. Appl.* **2000**, *241*, 46–55. [CrossRef]
4. Qin, X.L.; Cho, S.Y.; Wang, L. Iterative algorithms with errors for zero points of m-accretive Operators. *Fixed Point Theory Appl.* **2013**, *2013*, 148. [CrossRef]
5. Reich, S. Approximating zeros of accretive operators. *Proc. Am. Math. Soc.* **1975**, *51*, 381–384. [CrossRef]
6. Reich, S. On fixed point theorems obtained from existence theorems for differential equations. *J. Math. Anal. Appl.* **1976**, *54*, 26–36. [CrossRef]
7. Auzinger, W.; Frank, R. Asymptotic error expansions for stiff equations: An analysis for the implicit midpoint and trapezoidal rules in the strongly stiff case. *Numer. Math.* **1989**, *56*, 469–499. [CrossRef]
8. Bader, G.; Deuflhard, P. A semi-implicit mid-point rule for stiff systems of ordinary differential equations. *Numer. Math.* **1983**, *41*, 373–398. [CrossRef]
9. Deuflhard, P. Recent progress in extrapolation methods for ordinary differential equations. *SIAM Rev.* **1985**, *27*, 505–535. [CrossRef]
10. Schneider, C. Analysis of the linearly implicit mid-point rule for differential-algebra equations. *Electron. Trans. Numer. Anal.* **1993**, *1*, 1–10.
11. Somalia, S. Implicit midpoint rule to the nonlinear degenerate boundary value problems. *Int. J. Comput. Math.* **2002**, *79*, 327–332. [CrossRef]
12. Van Veldhuxzen, M. Asymptotic expansions of the global error for the implicit midpoint rule (stiff case). *Computing* **1984**, *33*, 185–192. [CrossRef]

13. Hammad, H.A.; ur Rehman, H.; De la Sen, M. Accelerated modified inertial Mann and viscosity algorithms to find a fixed point of α—Inverse strongly monotone operators. *AIMS Math.* **2021**, *6*, 9000–9019. [CrossRef]
14. Hammad, H.A.; ur Rehman, H.; Kattan, D.A. Strong convergence for split variational inclusion problems under hybrid algorithms with applications. *Alex. Eng. J.* **2024**, *87*, 350–364. [CrossRef]
15. Jung, J.S. Strong convergence of an iterative algorithm for accretive operators and nonexpansive mappings. *J. Nonlinear Sci. Appl.* **2016**, *9*, 2394–2409. [CrossRef]
16. Li, D.F. On nonexpansive and accretive operators in Banach spaces. *J. Nonlinear Sci. Appl.* **2017**, *10*, 3437–3446. [CrossRef]
17. Xu, H.K.; Alghamdi, M.A.; Shahzad, N. The viscosity technique for the implicit midpoint rule of nonexpansive mappings in Hilbert spaces. *Fixed Point Theory Appl.* **2015**, *2015*, 41. [CrossRef]
18. Luo, P.; Cai, G.; Shehu, Y. The viscosity iterative algorithms for the implicit midpoint rule of nonexpansive mappings in uniformly smooth Banach spaces. *J. Inequal. Appl.* **2017**, *2017*, 154. [CrossRef] [PubMed]
19. Ke, Y.F.; Ma, C.F. The generalized viscosity implicit rules of nonexpansive mappings in Hilbert spaces. *Fixed Point Theory Appl.* **2015**, *2015*, 190. [CrossRef]
20. Zhang, H.C.; Qu, Y.H.; Su, Y.F. Strong convergence theorems for fixed point problems for nonexpansive mappings and zero point problems for accretive operators using viscosity implicit midpoint rules in Banach spaces. *Mathematics* **2018**, *6*, 257. [CrossRef]
21. Zhang, H.C.; Qu, Y.H.; Su, Y.F. The generalized viscosity implicit midpoint rule for nonexpansive mappings in Banach space. *Mathematics* **2019**, *7*, 512. [CrossRef]
22. Filali, D.; Ali, F.; Akram, M.; Dilshad, M. A Novel Fixed-Point Iteration Approach for Solving Troesch's Problem. *Symmetry* **2024**, *16*, 856. [CrossRef]
23. Reich, S. On the asymptotic behavior of nonlinear semigroups and the range of accretive operators. *J. Math. Anal. Appl.* **1981**, *79*, 113–126. [CrossRef]
24. Barbu, V. *Nonlinear Semigroups and Differential Equations in Banach Spaces*; Editura Academiei Republicii Socialiste Romania; Springer: Dordrecht, The Netherlands, 1976; Volume 2, pp. 1–6.
25. Liu, L.S. Ishikawa and Mann iterative process with errors for nonlinear strongly accretive mappings in Banach spaces. *J. Math. Anal. Appl.* **1995**, *194*, 114–125. [CrossRef]
26. Reich, S. Strong convergence theorems for resolvents of accretive operators in Banach spaces. *J. Math. Anal. Appl.* **1980**, *75*, 287–292. [CrossRef]