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Constructing and Analyzing BiHom-(Pre-)Poisson Conformal Algebras

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Abstract: This study introduces the notions of BiHom-Poisson conformal algebra, BiHom-pre-Poisson conformal algebra, and their related structures. We show that many new BiHom-Poisson conformal algebras can be constructed from a BiHom-Poisson conformal algebra. In particular, the direct product of two BiHom-Poisson conformal algebras is also a BiHom-Poisson conformal algebra. We further describe the conformal bimodule and representation theory of the BiHom-Poisson conformal algebra. In addition, we define BiHom-pre-Poisson conformal algebra as the combination of BiHom-pre-Lie conformal algebra and BiHom-dendriform conformal algebra under some compatibility conditions. We further demonstrate a way to construct BiHom-Poisson conformal algebra from BiHom-pre-Poisson conformal algebra and provide the representation theory for BiHom-pre-Poisson conformal algebra. Finally, a detailed description of $\mathcal{O}$-operators and Rota–Baxter operators on BiHom-Poisson conformal algebra is provided.

Keywords: BiHom-associative conformal algebra; BiHom-Lie conformal algebra; BiHom-Poisson conformal algebra; BiHom-preLie conformal algebra; BiHom-pre-Poisson conformal algebra; conformal representation

JEL Classification: Primary 17B65; 17B10; 15A99; Secondary 16G30

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1. Introduction

Conformal algebra is a growing mathematical structure yielding a variety of important structures such as associative conformal algebra, (pre-)Lie conformal algebra, (tri-)dendriform conformal algebra, (Zinbiel)–Leibniz conformal algebra and (pre-)Poisson conformal algebra. It first arose in the study of vertex algebra in [1], where conformal modules of infinite dimensional Lie algebras were discussed. This study shows that conformal algebra is related to infinite-dimensional algebra that satisfies the locality property. Later on, the classification of irreducible modules over Virasoro conformal algebras was discussed by Wu et al. in [2]. Algebraic structures, such as derivations, representations, and cohomologies of the mentioned conformal algebras, were investigated in [3–6]. Recently, conformal algebras concerning operators such as Rota–Baxter operators and Nijenhuis operators and their cohomology theories were explored in [7,8].

Research on structural map-based algebra has been extensively studied in [9–11]. This domain comprises various algebras, including Hom-associative algebra, Hom-Lie algebra, etc. Makhlouf et al. explored Hom-Lie algebra, Hom-associative algebra, and Hom-dialgebras (see [12]). Numerous results related to structural maps are currently being investigated within a broader framework of rings and algebras, specifically alternative algebras and rings; see [13,14]. Meanwhile, Frieger et al., in [15], provided sufficient conditions for Hom-associative algebra to be associative in detail. However, Armakan et al. provided sufficient conditions in the context of Hom-Lie algebra in [16]. This concept was studied with respect to conformal algebras in [7,17,18]. Later on, it was discovered that an algebra in the presence of two structural maps is called a BiHom-type algebra. Subsequent studies on this subject can be found in [19–21]. In these papers, the researchers explicitly explored the structure theory of BiHom-Lie algebras and BiHom-bialgebras. The study of BiHom-bialgebra includes the study of BiHom-algebra and BiHom-coalgebra.

On the other hand, a Poisson algebra is the combination of an associative algebra and a Lie algebra in the presence of Leibniz's identity law. This is a very important mathematical structure that generalizes three structures and has been widely studied by various researchers. Kosmann-Schwarzbach explored how to obtain Gerstenhaber algebras from Poisson algebras in [22]. Later on, Poisson algebra in a BiHom-setting was studied in [23–26]. An associative algebra yields a dendriform algebra, and a Lie algebra yields a pre-Lie algebra; in the same way, the combination of a dendriform algebra and a pre-Lie algebra under some compatible conditions gives rise to a pre-Poisson algebra. This structure is also very significant and has been studied in Hom- and BiHom-settings in [10,27,28]. Recently, studies on the (pre-)Poisson conformal algebra were conducted in [5,29,30], and many useful results were obtained; i.e., the quantization of Poisson conformal algebras in [5]. Meanwhile, Kolensikov studied the universal enveloping of Poisson conformal algebra in [29]. Being motivated by the above-cited literature, we observed that despite the extensive research in this field, many questions remain unanswered, and these research gaps need to be filled. Therefore, the structural theory of BiHom-Poisson conformal algebra and its related structures is deemed an essential topic. This paper addresses these gaps by presenting many important structures concerning BiHom-Poisson conformal algebras. It introduces the notion of BiHom-(pre-)Poisson conformal algebras and describes the representation theory of BiHom-(pre-)Poisson conformal algebras. Moreover, it highlights the relationship between BiHom-(pre-)Poisson conformal algebras and the $\mathcal{O}$-operators and Rota–Baxter operators.

Collectively, our main goal is to show the transition between BiHom-Poisson conformal algebras and BiHom-pre-Poisson conformal algebras.

1. The first case can be illustrated by the following diagram:

$$\begin{array}{ccccc}
\begin{array}{c}\textit{BiHom-dendriform}\\ \textit{conformal algebra}\end{array} & + & \begin{array}{c}\textit{BiHom-pre-Lie}\\ \textit{conformal algebra}\end{array} & \longrightarrow & \begin{array}{c}\textit{BiHom-pre-Poisson}\\ \textit{conformal algebra}\end{array}\\
\Big\downarrow{\scriptstyle x\prec_\lambda y+x\succ_\lambda y,\ x*_\lambda y-\alpha^{-1}\beta(y)*_{-\partial-\lambda}\alpha\beta^{-1}(x)} & & & & \Big\downarrow{\scriptstyle x\prec_\lambda y+x\succ_\lambda y,\ x*_\lambda y-\alpha^{-1}\beta(y)*_{-\partial-\lambda}\alpha\beta^{-1}(x)}\\
\begin{array}{c}\textit{BiHom-associative}\\ \textit{conformal algebra}\end{array} & + & \begin{array}{c}\textit{BiHom-Lie}\\ \textit{conformal algebra}\end{array} & \longrightarrow & \begin{array}{c}\textit{BiHom-Poisson}\\ \textit{conformal algebra.}\end{array}
\end{array}$$

2. The second case/reverse case can be illustrated in the following diagram:

$$\begin{array}{ccccc}
\begin{array}{c}\textit{BiHom-associative}\\ \textit{conformal algebra}\end{array} & + & \begin{array}{c}\textit{BiHom-Lie}\\ \textit{conformal algebra}\end{array} & \longrightarrow & \begin{array}{c}\textit{BiHom-Poisson}\\ \textit{conformal algebra}\end{array}\\
\Big\downarrow{\scriptstyle R(x)\cdot_\lambda y,x\cdot_\lambda R(y),[R(x)_\lambda y]} & & & & \Big\downarrow{\scriptstyle R(x)\cdot_\lambda y,x\cdot_\lambda R(y),[R(x)_\lambda y]}\\
\begin{array}{c}\textit{BiHom-dendriform}\\ \textit{conformal algebra}\end{array} & + & \begin{array}{c}\textit{BiHom-pre-Lie}\\ \textit{conformal algebra}\end{array} & \longrightarrow & \begin{array}{c}\textit{BiHom-pre-Poisson}\\ \textit{conformal algebra.}\end{array}
\end{array}$$

Aside from this section, this paper is divided into four additional sections. In Section 2, we provide some preliminary definitions and notions about the BiHom-Lie conformal algebras to support our findings in the next sections. In Section 3, we define BiHom-Poisson conformal algebra and construct new structures out of it. We further show that the direct product of two BiHom-Poisson conformal algebras is a BiHom-Poisson conformal algebra. Further, we introduce the representation theory for the BiHom-Poisson conformal algebra. Moving on to Section 4, we introduce the BiHom-pre-Poisson conformal algebra followed by BiHom-pre Lie conformal algebra and provide its representation theory. Finally, in Section 5, we show the relationship between the algebras under consideration and the $\mathcal{O}$-operators and Rota-Baxter operators.

Throughout this paper, $\mathbb{C}$ represents the set of complex numbers. On the other hand, all tensor products and vector spaces are considered over $\mathbb{C}$. Without any uncertainty, we abbreviate $\otimes_{\mathbb{C}}$ by $\otimes$.

2. Preliminaries

In the present section, we recall the notion of BiHom-associative conformal algebra and BiHom-Lie conformal algebra, which are important to define the BiHom-Poisson conformal algebra.

Definition 1. *A BiHom-conformal algebra B is a $\mathbb{C}[\partial]$-module, equipped with the conformal bilinear map $\cdot_\lambda : B \otimes B \to B[\lambda]$, defined by $x \otimes y \mapsto x \cdot_\lambda y$ and two structural maps $\alpha, \beta : B \to B$ satisfying the following identities:*

$$(\partial x) \cdot_\lambda y = -\lambda(x \cdot_\lambda y), \quad x \cdot_\lambda (\partial y) = (\partial + \lambda)(x \cdot_\lambda y), \quad \alpha\partial = \partial\alpha, \quad \beta\partial = \partial\beta,$$

for all $x, y \in B$ and $\lambda, \mu \in \mathbb{C}$. We denote it by $(B, \alpha, \beta, \cdot_\lambda)$.

Definition 2. *A BiHom-conformal algebra $(B, \alpha, \beta, \cdot_\lambda)$ is said to be a BiHom-associative conformal algebra if the following associative condition holds for all $x, y, z \in B$ and $\lambda, \mu \in \mathbb{C}$:*

$$\alpha(x) \cdot_\lambda (y \cdot_\mu z) = (x \cdot_\lambda y) \cdot_{\lambda+\mu} \beta(z).$$

A BiHom-associative conformal algebra is said to be multiplicative if the following condition holds:

$$\alpha(x \cdot_\lambda y) = \alpha(x) \cdot_\lambda \alpha(y), \quad \beta(x \cdot_\lambda y) = \beta(x) \cdot_\lambda \beta(y).$$

Moreover, a BiHom-associative conformal algebra is said to be regular if the twist maps α and β are invertible. A BiHom-associative conformal algebra is called commutative if

$$x \cdot_\lambda y = y \cdot_{-\partial-\lambda} x.$$

Example 1. *Consider that $(B, *_\lambda)$ is an associative conformal algebra, and define two structure maps, $\alpha, \beta : B \to B$, on the associative conformal algebra, such that α and β commute with each other and satisfy $\alpha(x *_\lambda y) = \alpha(x) *_\lambda \alpha(y)$ and $\beta(x *_\lambda y) = \beta(x) *_\lambda \beta(y)$ for all $x, y \in B$ and $\lambda, \mu \in \mathbb{C}$. Then, $(B, \alpha(\cdot *_\lambda \cdot), \beta(\cdot *_\lambda \cdot), \alpha, \beta)$ is a multiplicative BiHom-associative conformal algebra.*

Similar to BiHom-associative conformal algebra, we define BiHom-Lie conformal algebra as follows:

Definition 3. *A BiHom-Lie conformal algebra B is a $\mathbb{C}[\partial]$-module equipped with the $\mathbb{C}$-bilinear map $[\cdot_\lambda \cdot] : B \otimes B \to B[\lambda]$ in such a way that $x \otimes y \mapsto [x_\lambda y]$, with two structural maps α, β satisfying the following identities:*

$$\alpha\partial = \partial\alpha, \ \beta\partial = \partial\beta, \tag{1}$$

$$[(\partial x)_\lambda y] = -\lambda[x_\lambda y], \ [x_\lambda(\partial y)] = (\partial + \lambda)[x_\lambda y], \tag{2}$$

$$[\beta(x)_\lambda \alpha(y)] = -[\beta(y)_{-\partial-\lambda} \alpha(x)], \tag{3}$$

$$[\beta^2(x)_\lambda[\beta(y)_\mu \alpha(z)]] + [\beta^2(y)_\mu[\beta(z)_{-\partial-\lambda} \alpha(x)]] + [\beta^2(z)_{-\partial-\lambda-\mu}[\beta(x)_\lambda \alpha(y)]] = 0. \tag{4}$$

for all $x, y, z \in B$ and $\lambda, \mu \in \mathbb{C}$. Note that the third identity is called conformal BiHom-skew-symmetry, while the fourth one is called a conformal BiHom-Jacobi identity. It can also be written as

$$[\alpha\beta(x)_\lambda[y_\mu z]] = [[\beta(x)_\lambda y]_{\lambda+\mu}\beta(z)] + [\beta(y)_\mu[\alpha(x)_\lambda z]].$$

A finite BiHom-Lie conformal algebra is considered finitely generated as a $\mathbb{C}[\partial]$-module. The rank of a BiHom-Lie conformal algebra, B, is its rank as a $\mathbb{C}[\partial]$-module. A BiHom-Lie conformal algebra $(B, [\cdot_\lambda\cdot], \alpha, \beta)$ is called regular if twist maps α and β are invertible.

Example 2. *Let $(B, [\cdot_\lambda\cdot])$ be a Lie conformal algebra, and define two structure maps, α and β, on B that commute with each other and satisfy the following identities:*

$$\alpha([x_\lambda y]) = [\alpha(x)_\lambda\alpha(y)], \quad \beta([x_\lambda y]) = [\beta(x)_\lambda\beta(y)], \text{ for all } x, y \in B,\ \lambda, \mu \in \mathbb{C}.$$

Then, the tuple $(B, \alpha([\cdot_\lambda\cdot]), \beta([\cdot_\lambda\cdot]), \alpha, \beta)$ is a multiplicative BiHom-Lie conformal algebra.

Let $(B, \alpha, \beta, *_\lambda)$ be a BiHom-associative conformal algebra; then, λ-bracket, defined by

$$[x_\lambda y] = x *_\lambda y - y *_{-\partial-\lambda} x,$$

yields a BiHom-Lie conformal algebra, denoted by $(B, \alpha, \beta, [\cdot_\lambda\cdot])$.

Definition 4. *Let $(B, *_\lambda, \alpha, \beta)$ be a BiHom-associative conformal algebra, and let (M, ϕ, ψ) be a $\mathbb{C}[\partial]$-module equipped with the $\mathbb{C}$-linear commuting maps $\phi, \psi : M \to M$ satisfying $\phi\partial = \partial\phi$, $\psi\partial = \partial\psi$. Let $l, r : B \to gc(M)$ be two $\mathbb{C}$-linear maps. The quadruple (M, l, r, ϕ, ψ) is called a conformal representation of B if the following equations hold:*

$$\begin{aligned}
l(\partial x)_\lambda m &= -\lambda(l(x)_\lambda m),\\
l(x)_\lambda(\partial m) &= (\partial + \lambda)(l(x)_\lambda m),\\
r(\partial x)_\lambda m &= -\lambda(r(x)_\lambda m),\\
r(x)_\lambda(\partial m) &= (\partial + \lambda)(r(x)_\lambda m),\\
\phi(l(x)_\lambda m) &= l(\alpha(x))_\lambda\phi(m),\\
\phi(r(x)_\lambda m) &= r(\alpha(x))_\lambda\phi(m),\\
\psi(l(x)_\lambda m) &= l(\beta(x))_\lambda\psi(m),\\
\psi(r(x)_\lambda m) &= r(\beta(x))_\lambda\psi(m),\\
l(x *_\lambda y)_{\lambda+\mu}\psi(m) &= l(\alpha(x))_\lambda(l(y)_\mu m),\\
r(x *_\lambda y)_{\lambda+\mu}\phi(m) &= r(\beta(y))_\mu(r(x)_\lambda m),\\
l(\alpha(x))_\lambda(r(y)_\mu m) &= r(\beta(y))_\mu(l(x)_\lambda m),
\end{aligned}$$

for all $x, y \in B$, $m \in M$ and $\lambda, \mu \in \mathbb{C}$. Note that $l(x)_\lambda m = x *_\lambda m$ and $r(x)_\lambda m = m *_{-\partial-\lambda} x$.

Proposition 1. *Let (l, ϕ, ψ, M) be a conformal representation of a BiHom-associative conformal algebra $(B, *_\lambda, \alpha, \beta)$, where M is a $\mathbb{C}[\partial]$-module, and ϕ and ψ are $\mathbb{C}$-linear maps, satisfying $\partial\phi = \phi\partial$ and $\partial\psi = \psi\partial$. Then, the direct sum, $B \oplus M$, of $\mathbb{C}[\partial]$-modules is turned into a BiHom-associative conformal algebra by defining the twist maps $\alpha + \phi$ and $\beta + \psi$ and λ-multiplication $*'_\lambda$ in $B \oplus M$ as follows:*

$$\begin{aligned}
(x_1 + m_1) *'_\lambda (x_2 + m_2) &:= x_1 *_\lambda x_2 + (l(x_1)_\lambda m_2 + r(x_2)_{-\partial-\lambda} m_1),\\
(\alpha \oplus \phi)(x + m) &:= \alpha(x) + \phi(m),\\
(\beta \oplus \psi)(x + m) &:= \beta(x) + \psi(m),
\end{aligned} \tag{5}$$

for all $x, x_1, x_2 \in B$; $m, m_1, m_2 \in M$; and $\lambda, \mu \in \mathbb{C}$.

We denote this conformal algebra by $(B \oplus M, *'_\lambda, \alpha + \phi, \beta + \psi)$, or simply $(B \ltimes_{l,r,\alpha,\beta,\phi,\psi} M)$.

Definition 5. *A conformal representation of a BiHom-Lie conformal algebra* $(B, [\cdot_\lambda \cdot], \alpha, \beta)$ *on a* $\mathbb{C}[\partial]$*-module M equipped with* $\mathbb{C}$*-linear commuting maps* $\phi, \psi : M \to M$ *satisfying* $\phi\partial = \partial\phi$ *and* $\psi\partial = \partial\psi$ *is a* $\mathbb{C}$*-linear map* $\rho : B \to gc(M)$*, such that the following equations hold:*

$$\begin{aligned}
\rho(\partial x)_\lambda m &= -\lambda(\rho(x)_\lambda m),\\
\rho(x)_\lambda(\partial m) &= (\partial + \lambda)(\rho(x)_\lambda m),\\
\phi(\rho(x)_\lambda m) &= \rho(\alpha(x))_\lambda \phi(m),\\
\psi(\rho(x)_\lambda m) &= \rho(\beta(x))_\lambda \psi(m),\\
\rho([\beta(x)_\lambda y])_{\lambda+\mu}\psi(m) &= \rho(\alpha\beta(x))_\lambda \rho(y)_\mu m - \rho(\beta(y))_\mu \rho(\alpha(x))_\lambda m,
\end{aligned}$$

for all $x, y \in B$*,* $m \in M$ *and* $\lambda, \mu \in \mathbb{C}$*.*

The conformal representation of the BiHom-Lie conformal algebra is denoted by (ρ, ϕ, ψ, M).

Proposition 2. *Let* (ρ, ϕ, ψ, M) *be a conformal representation of a BiHom-Lie conformal algebra* $(B, [\cdot_\lambda \cdot], \alpha, \beta)$*, where M is a* $\mathbb{C}[\partial]$*-module, and* ϕ *and* ψ *are* $\mathbb{C}$*-linear maps satisfying* $\partial\phi = \phi\partial$ *and* $\partial\psi = \psi\partial$*. Moreover, assume that* α *and* ψ *are bijective maps. Then, the direct sum,* $B \oplus M$*, of* $\mathbb{C}[\partial]$*-modules is turned into a BiHom-Lie conformal algebra. The* λ*-bracket* $[\cdot_\lambda \cdot]_\rho$ *in* $B \oplus M$ *can be defined as follows:*

$$\begin{aligned}
\left[(x_1 + m_1)_\lambda (x_2 + m_2)\right]_\rho &:= [x_{1\lambda} x_2] + (\rho(x_1)_\lambda m_2 - \rho(\alpha^{-1}\beta(x_2))_{-\partial-\lambda}\phi\psi^{-1}(m_1)),\\
(\alpha \oplus \phi)(x + m) &:= \alpha(x) + \phi(m),\\
(\beta \oplus \psi)(x + m) &:= \beta(x) + \psi(m),
\end{aligned} \tag{6}$$

for all $x, x_1, x_2 \in B$ *and* $m, m_1, m_2 \in M$*.*

We denote this BiHom-Lie conformal algebra by $(B \oplus M, [\cdot_\lambda \cdot]_\rho, \alpha + \phi, \beta + \psi)$, or simply by $(B \ltimes_{\rho,\alpha,\beta,\phi,\psi} M)$.

Example 3. *Consider that* $ad : B \to gc(B)$ *is a linear map defined by* $ad(x)_\lambda y = [x_\lambda y]$ *for all* $x, y \in B$*. Then,* (B, α, β, ad) *is a conformal representation of the BiHom-Lie conformal algebra. It is also known as an adjoint representation of B.*

Next, we define the BiHom-analog of Poisson conformal algebra in the following section.

3. BiHom-Poisson Conformal Algebra and Its Representations

Definition 6. *A BiHom-Poisson conformal algebra* $(B, *_\lambda, \alpha, \beta, [\cdot_\lambda \cdot])$ *is a tuple consisting of a* $\mathbb{C}[\partial]$*-module B, two* $\mathbb{C}$*-bilinear maps* $*_\lambda, [\cdot_\lambda \cdot] : B \otimes B \to B[\lambda]$*, and two* $\mathbb{C}$*-linear maps* $\alpha, \beta : B \to B$*, such that the following identities hold:*

1. $(B, *_\lambda, \alpha, \beta)$ *is a commutative BiHom-associative conformal algebra.*
2. $(B, [\cdot_\lambda \cdot], \alpha, \beta)$ *is a BiHom-Lie conformal algebra.*
3. *The BiHom-Leibniz identity,*

$$[\alpha\beta(x)_\lambda (y *_\mu z)] = ([\beta(x)_\lambda y]) *_{\lambda+\mu} \beta(z) + \beta(y) *_\mu ([\alpha(x)_\lambda z]), \tag{7}$$

holds for all $x, y, z \in B$ *and* $\lambda, \mu \in \mathbb{C}$*.*

In the BiHom-Poisson conformal algebra $(B, [\cdot_\lambda \cdot], *_\lambda, \alpha, \beta)$*, the operations* $*_\lambda$ *and* $[\cdot_\lambda \cdot]$ *are called the conformal product and the BiHom-Poisson conformal bracket, respectively.*

When $\alpha = \beta$, we obtain Hom-Poisson conformal algebra. Additionally, for $\alpha = \beta = id$, we simply obtain a Poisson conformal algebra.

Definition 7. *Let $(B, [\cdot_\lambda \cdot], *_\lambda, \alpha, \beta)$ be a BiHom-Poisson conformal algebra. If $Z(B) = \{x \in B \,|\, [x_\lambda y] = x \cdot_\lambda y = 0,\ \forall x, y \in B\}$, then $Z(B)$ is called the centralizer of B.*

Remark 1. *Let $(B, [\cdot_\lambda \cdot], *_\lambda, \alpha, \beta)$ be a BiHom-Poisson conformal algebra; then,*

1. *$(B, [\cdot_\lambda \cdot], *_\lambda, \alpha, \beta)$ is multiplicative if*

$$\alpha([x_\lambda y]) = [\alpha(x)_\lambda \alpha(y)],\ \ \beta([x_\lambda y]) = [\beta(x)_\lambda \beta(y)] \text{ and}$$
$$\alpha(x *_\lambda y) = \alpha(x) *_\lambda \alpha(y),\ \ \beta(x *_\lambda y) = \beta(x) *_\lambda \beta(y).$$

2. *$(B, [\cdot_\lambda \cdot], *_\lambda, \alpha, \beta)$ is said to be regular if α and β are bijective maps.*
3. *$(B, [\cdot_\lambda \cdot], *_\lambda, \alpha, \beta)$ is said to be involutive if α and β satisfiy $\alpha^2 = id = \beta^2$.*
4. *Let $(B', [\cdot_\lambda \cdot]', *'_\lambda, \alpha', \beta')$ be another BiHom-Poisson conformal algebra. A morphism $f : B \to B'$ is a linear map such that the following conditions are satisfied:*

$$\alpha f = f\alpha', \beta f = f\beta', f([x_\lambda y]) = [f(x)_\lambda f(y)]' \text{ and } f(x *_\lambda y) = f(x) *'_\lambda f(y).$$

Proposition 3. *Let $(B, *_\lambda, \alpha, \beta)$ be a regular BiHom-associative conformal algebra. Then, $B' = (B, [\cdot_\lambda \cdot], *_\lambda, \alpha, \beta)$ is a noncommutative BiHom-Poisson conformal algebra, where $[x_\lambda y] = x *_\lambda y - \alpha^{-1}\beta(y) *_{-\partial-\lambda} \alpha\beta^{-1}(x)$ for all $x, y \in B$ and $\lambda \in \mathbb{C}$.*

Proposition 4. *Let $(B, *_\lambda, [\cdot_\lambda \cdot], \alpha, \beta)$ be a BiHom-Poisson conformal algebra. Then, $(B, *^n_\lambda = \alpha^n \circ *_\lambda, [\cdot_\lambda \cdot]^n = \alpha^n \circ [\cdot_\lambda \cdot], \alpha^{n+1}, \alpha^n\beta)$ is a BiHom-Poisson conformal algebra.*

Proof.

1. Here, we show that $(B, *^n_\lambda = \alpha^n \circ *_\lambda, \alpha^{n+1}, \alpha^n\beta)$ is a BiHom-associative conformal algebra. For this, we consider
   - Conformal sesqui-linearity:

$$\begin{aligned}\partial(x) *^n_\lambda y = \alpha^n(\partial(x) *_\lambda y) &= \alpha^n(-\lambda(x *_\lambda y)) \\ &= -\lambda\alpha^n(x *_\lambda y) = -\lambda(x *^n_\lambda y),\end{aligned}$$

$$\begin{aligned}x *^n_\lambda \partial(y) = \alpha^n(x *_\lambda \partial(y)) &= \alpha^n((\lambda+\partial)(x *_\lambda y)) \\ &= (\lambda+\partial)\alpha^n(x *_\lambda y) = (\lambda+\partial)(x *^n_\lambda y).\end{aligned}$$

   - BiHom-skew symmetry:

$$\begin{aligned}\alpha^n\beta(x) *^n_\lambda \alpha^{n+1}(y) &= \alpha^n(\alpha^n\beta(x) *_\lambda \alpha^{n+1}(y)) = \alpha^{2n}(\beta(x) *_\lambda \alpha(y)) \\ &= -\alpha^{2n}(\beta(y) *_{-\partial-\lambda} \alpha(x)) = -\alpha^n(\alpha^n\beta(y) *_{-\partial-\lambda} \alpha^{n+1}(x)) \\ &= -(\alpha^n\beta(y) *^n_{-\partial-\lambda} \alpha^{n+1}(x)).\end{aligned}$$

   - BiHom-associative conformal identity:

$$\begin{aligned}&\alpha^{n+1}(x) *^n_\lambda (y *^n_\mu z) - (x *^n_\lambda y) *^n_{\lambda+\mu} \alpha^n\beta(z) \\ =&\alpha^n(\alpha^{n+1}(x) *_\lambda \alpha^n(y *_\mu z)) - \alpha^n(\alpha^n(x *_\lambda y) *_{\lambda+\mu} \alpha^n\beta(z)) \\ =&\alpha^{2n+1}(x) *_\lambda \alpha^{2n}(y *_\mu z) - \alpha^{2n}(x *_\lambda y) *_{\lambda+\mu} \alpha^{2n}\beta(z) \\ =&\alpha^{2n}(\alpha(x) *_\lambda (y *_\mu z) - (x *_\lambda y) *_{\lambda+\mu} \beta(z)) \\ =&0.\end{aligned}$$

   Thus, $(B, *^n_\lambda = \alpha^n \circ *_\lambda, \alpha^{n+1}, \alpha^n\beta)$ is a BiHom-associative conformal algebra.
2. Similarly, we can show that $(B, [\cdot_\lambda \cdot]^n = \alpha^n \circ [\cdot_\lambda \cdot], \alpha^{n+1}, \alpha^n\beta)$ is a BiHom-Lie conformal algebra.

3. Now, we are only left to prove the BiHom-Leibniz conformal identity as follows:

$$\begin{aligned}
&([\alpha^n\beta(x)_\lambda y]^n) *^n_{\lambda+\mu} \alpha^n\beta(z) + \alpha^n\beta(y) *^n_\mu ([\alpha^{n+1}(x)_\lambda z]^n)\\
=&\alpha^n((\alpha^n[\alpha^n\beta(x)_\lambda y]) *_{\lambda+\mu} \alpha^n\beta(z)) + \alpha^n(\alpha^n\beta(y) *_\mu \alpha^n([\alpha^{n+1}(x)_\lambda z]))\\
=&\alpha^{2n}([\alpha^n\beta(x)_\lambda y] *_{\lambda+\mu} \beta(z)) + \alpha^{2n}(\beta(y) *_\mu [\alpha^{n+1}(x)_\lambda z])\\
=&\alpha^{2n}([\alpha^n\beta(x)_\lambda y] *_{\lambda+\mu} \beta(z) + \beta(y) *_\mu [\alpha^{n+1}(x)_\lambda z]),\\
=&\alpha^{2n}([\alpha\beta(\alpha^n(x))_\lambda (y *_\mu z)])\\
=&\alpha^{2n}[\alpha^{n+1}\beta(x)_\lambda (y *_\mu z)]\\
=&\alpha^n[\alpha^{2n+1}\beta(x)_\lambda \alpha^n(y *_\mu z)]\\
=&[\alpha^{n+1}\alpha^n\beta(x)_\lambda (y *^n_\mu z)]^n.
\end{aligned}$$

This completes the proof. □

In the following proposition, we show that by defining new multiplications on the BiHom-Poisson conformal algebra with the aid of the Rota–Baxter operator, we can preserve the BiHom-structure.

Proposition 5. *Let $(B, *_\lambda, [\cdot_\lambda\cdot], \alpha, \beta)$ be a BiHom-Poisson conformal algebra and R be a Rota–Baxter operator. Define two new λ-multiplication maps $*'_\lambda$ and $[\cdot_\lambda\cdot]'$ with*

$$x *'_\lambda y = R(x) *_\lambda y + x *_\lambda R(y), \quad [x_\lambda y]' = [R(x)_\lambda y] + [x_\lambda R(y)].$$

Then, $(B, *'_\lambda, [\cdot_\lambda\cdot]', \alpha, \beta)$ is again a BiHom-Poisson conformal algebra.

Proof. Here, we show that $(B, [\cdot_\lambda\cdot]', \alpha, \beta)$ is a BiHom-Lie conformal algebra.

1. For conformal sesqui-linearity:

$$\begin{aligned}
[\partial(x)_\lambda y]' =& [R(\partial(x))_\lambda y] + [\partial(x)_\lambda R(y)] = [\partial(R(x))_\lambda y] + [\partial(x)_\lambda R(y)]\\
&= -\lambda[R(x)_\lambda y] - \lambda[x_\lambda R(y)] = -\lambda([R(x)_\lambda y] + [x_\lambda R(y)])\\
&= -\lambda([x_\lambda y]').
\end{aligned}$$

Similarly, we can show $[x_\lambda \partial(y)]' = (\lambda+\partial)([x_\lambda y]')$.

2. For BiHom-skew-symmetry:

$$\begin{aligned}
[\beta(x)_\lambda \alpha(y)]' &= [R(\beta(x))_\lambda \alpha(y)] + [\beta(x)_\lambda R(\alpha(y))]\\
&= [\beta(R(x))_\lambda \alpha(y)] + [\beta(x)_\lambda \alpha(R(y))]\\
&= -[\beta(y)_{-\partial-\lambda} \alpha(R(x))] - [\beta(R(y))_{-\partial-\lambda} \alpha(x)]\\
&= -([\beta(y)_{-\partial-\lambda} R(\alpha(x))] + [R(\beta(y))_{-\partial-\lambda} \alpha(x)])\\
&= -([\beta(y)_{-\partial-\lambda} \alpha(x)]').
\end{aligned}$$

3. The BiHom-Jacobi identity is given as follows:

$$\begin{aligned}
&[[\beta(x)_\lambda y]'_{\lambda+\mu} \beta(z)]' + [\beta(y)_\mu [\alpha(x)_\lambda z]']'\\
=&[([R(\beta(x))_\lambda y] + [\beta(x)_\lambda R(y)])_{\lambda+\mu} \beta(z)]' + [\beta(y)_\mu ([R(\alpha(x))_\lambda z] + [\alpha(x)_\lambda R(z)])]'\\
=&([R([R(\beta(x))_\lambda y] + [\beta(x)_\lambda R(y)])_{\lambda+\mu} \beta(z)] + [([R(\beta(x))_\lambda y] + [\beta(x)_\lambda R(y)])_{\lambda+\mu} R\beta(z)])\\
&+ ([R\beta(y)_\mu ([R(\alpha(x))_\lambda z] + [\alpha(x)_\lambda R(z)])] + [\beta(y)_\mu R([R(\alpha(x))_\lambda z] + [\alpha(x)_\lambda R(z)])])\\
=&[([R(\beta(x))_\lambda R(y)])_{\lambda+\mu} \beta(z)] + [([R(\beta(x))_\lambda y] + [\beta(x)_\lambda R(y)])_{\lambda+\mu} R\beta(z)]\\
&+ ([R\beta(y)_\mu ([R(\alpha(x))_\lambda z] + [\alpha(x)_\lambda R(z)])] + [\beta(y)_\mu [R(\alpha(x))_\lambda R(z)]])\\
=&[([\beta(R(x))_\lambda R(y)])_{\lambda+\mu} \beta(z)] + [[\beta(R(x))_\lambda y]_{\lambda+\mu} \beta(R(z))] + [[\beta(x)_\lambda R(y)]_{\lambda+\mu} \beta(R(z))]\\
&+ [\beta(R(y))_\mu [\alpha(R(x))_\lambda z]] + [\beta(R(y))_\mu [\alpha(x)_\lambda R(z)]] + [\beta(y)_\mu [\alpha(R(x))_\lambda R(z)]]
\end{aligned}$$

$$=[\alpha\beta(R(x))_\lambda[R(y)_\mu z]] + [\alpha\beta R(x)_\lambda[y_\mu R(z)]] + [\alpha\beta(x)_\lambda[R(y)_\mu R(z)]]$$
$$=[\alpha\beta(R(x))_\lambda[R(y)_\mu z]] + [\alpha\beta(R(x))_\lambda[y_\mu R(z)]] + [\alpha\beta(x)_\lambda R([R(y)_\mu z] + [y_\mu R(z)])]$$
$$=[\alpha\beta(R(x))_\lambda[R(y)_\mu z]] + [\alpha\beta(x)_\lambda R([R(y)_\mu z])] + [\alpha\beta(R(x))_\lambda[y_\mu R(z)]] + [\alpha\beta(x)_\lambda R([y_\mu R(z)])]$$
$$=[R(\alpha\beta(x))_\lambda[R(y)_\mu z]] + [\alpha\beta(x)_\lambda R([R(y)_\mu z])] + [R(\alpha\beta(x))_\lambda[y_\mu R(z)]] + [\alpha\beta(x)_\lambda R([y_\mu R(z)])]$$
$$=[\alpha\beta(x)_\lambda[R(y)_\mu z]]' + [\alpha\beta(x)_\lambda[y_\mu R(z)]]'[\alpha\beta(x)_\lambda[y_\mu z]']'$$
$$=[\alpha\beta(x)_\lambda([R(y)_\mu z] + [y_\mu R(z)])]'$$
$$=[\alpha\beta(x)_\lambda[y_\mu z]']'.$$

Similarly, we can show that $(B, *'_\lambda, \alpha, \beta)$ is BiHom-associative conformal algebra. Thus, we are only left to verify BiHom-Leibniz's conformal identity. This can be seen as

$$[\alpha\beta(x)_\lambda(y *'_\mu z)]'$$
$$=[R(\alpha\beta(x))_\lambda(R(y) *_\mu z + y *_\mu R(z))] + [\alpha\beta(x)_\lambda R(R(y) *_\mu z + y *_\mu R(z))]$$
$$=[R(\alpha\beta(x))_\lambda(R(y) *_\mu z)] + [R(\alpha\beta(x))_\lambda(y *_\mu R(z))] + [\alpha\beta(x)_\lambda(R(y) *_\mu R(z))]$$
$$=[R(\beta(x))_\lambda R(y)] *_{\lambda+\mu} \beta(z) + R(\beta(y)) *_\mu [R(\alpha(x))_\lambda z] + [R\beta(x)_\lambda y] *_{\lambda+\mu} R\beta(z)$$
$$+ \beta(y) *_\mu [R\alpha(x)_\lambda R(z)] + [\beta(x)_\lambda R(y)] *_{\lambda+\mu} R\beta(z) + R\beta(y) *_\mu [\alpha(x)_\lambda R(z)]$$
$$=([\beta(Rx)_\lambda Ry]) *_{\lambda+\mu} \beta(z) + [\beta(Rx)_\lambda y] *_{\lambda+\mu} \beta(Rz) + [\beta(x)_\lambda Ry] *_{\lambda+\mu} \beta(Rz)$$
$$+ \beta(Ry) *_\mu [\alpha(Rx)_\lambda z] + \beta(Ry) *_\mu [\alpha(x)_\lambda Rz] + \beta(y) *_\mu ([\alpha(Rx)_\lambda Rz])$$
$$=R([\beta(Rx)_\lambda y] + [\beta(x)_\lambda Ry]) *_{\lambda+\mu} \beta(z) + ([\beta(Rx)_\lambda y] + [\beta(x)_\lambda Ry]) *_{\lambda+\mu} \beta(Rz)$$
$$+ \beta(Ry) *_\mu ([\alpha(Rx)_\lambda z] + [\alpha(x)_\lambda Rz]) + \beta(y) *_\mu R([\alpha(Rx)_\lambda z] + [\alpha(x)_\lambda Rz])$$
$$=([\beta(x)_\lambda y]') *'_{\lambda+\mu} \beta(z) + \beta(y) *'_\mu ([\alpha(x)_\lambda z]').$$

This completes the proof. □

Now, we define the tensor product of BiHom-Poisson conformal algebra as follows:

Lemma 1. *Let $(B_1, *^1_\lambda, [\cdot_\lambda\cdot]_1, \alpha_1, \beta_1, \partial_1)$ and $(B_2, *^2_\lambda, [\cdot_\lambda\cdot]_2, \alpha_2, \beta_2, \partial_2)$ be two (noncommutative) BiHom-Poisson conformal algebras. Define two linear maps, $\alpha, \beta : B_1 \oplus B_2 \to B_1 \oplus B_2$, such that*

$$\alpha(x, p) = (\alpha_1(x), \alpha_2(p)) \text{ and } \beta(x, p) = (\beta_1(x), \beta_2(p))$$

and two λ-multiplication maps, $*_{\oplus\lambda}, [\cdot_{\oplus\lambda}\cdot] : (B_1 \oplus B_2) \otimes (B_1 \oplus B_2) \to (B_1 \oplus B_2)[\lambda]$, such that the following conditions hold for all $x, y \in B_1$, $p, q \in B_2$ and $\lambda \in \mathbb{C}$:

$$[(x+p)_{\oplus\lambda}(y+q)] = [x_\lambda y]_1 + [p_\lambda q]_2,$$
$$(x+p)_{\oplus\lambda}(y+q) = (x *^1_\lambda y) + (p *^2_\lambda q).$$

Then, $(B_1 \oplus B_2, *_{\oplus\lambda}, [\cdot_{\oplus\lambda}\cdot], \alpha, \beta)$ is a BiHom-Poisson conformal algebra.

Proof. The following is used to show that $(B_1 \oplus B_2, \cdot_{\oplus\lambda}, [\cdot_{\oplus\lambda}\cdot], \alpha, \beta, \partial_\oplus)$ is a BiHom-Poisson conformal algebra:

1. We first show that $(B_1 \oplus B_2, [\cdot_{\oplus\lambda}\cdot], \alpha, \beta, \partial_\oplus)$ is a BiHom-Poisson conformal algebra. For this, we need to satisfy the following identities:
   (a) Conformal sesqui-linearity:

$$\begin{aligned}[\partial_\oplus(x+p)_{\oplus\lambda}(y+q)] &= [\partial_1(x) + \partial_2(p)_\lambda(y+q)] \\ &= [\partial_1(x)_\lambda(y)] + [\partial_2(p)_\lambda(q)] \\ &= -\lambda[x_\lambda(y)]_1 - \lambda[p_\lambda(q)]_2 \\ &= -\lambda([x_\lambda(y)]_1 + [p_\lambda(q)]_2) \\ &= -\lambda([(x+p)_{\oplus\lambda}(y+q)]).\end{aligned}$$

Similarly, we can show that

$$[(x+p)_{\oplus\lambda}\partial(y+q)] = (\lambda+\partial)([(x+p)_{\oplus\lambda}(y+q)]).$$

(b) For BiHom-skew symmetry:

$$\begin{aligned}[\beta(x+p)_{\lambda\oplus}\alpha(y+q)] &= [\beta_1(x)+\beta_2(p)_{\lambda\oplus}\alpha_1(y)+\alpha_2(q)]\\ &= [\beta_1(x)_\lambda\alpha_1(y)]_1 + [\beta_2(p)_\lambda\alpha_2(q)]_2\\ &= -[\beta_1(y)_{-\partial-\lambda}\alpha_1(x)]_1 - [\beta_2(q)_{-\partial-\lambda}\alpha_2(p)]_2\\ &= -[\beta_1(y)+\beta_2(q)_{\oplus(-\partial-\lambda)}\alpha_1(x)+\alpha_2(p)]\\ &= -[\beta(y+q)_{\oplus(-\partial-\lambda)}\alpha(x+p)].\end{aligned}$$

Here, we have used the fact that B_1 and B_2 are Lie conformal algebras and satisfy the skew-symmetric identity.

(c) For the BiHom-Jacobi identity, we compute that for $x,y,z\in B_1$ and $p,q,r\in B_2$,

$$\begin{aligned}&\circlearrowleft_{x,y,z}^{p,q,r} [\beta^2(x+p)_{\oplus\lambda}[\beta(y+q)_{\oplus\mu}\alpha(z+r)]]\\ &=\circlearrowleft_{x,y,z}^{p,q,r} [\beta_1^2(x)+\beta_2^2(p)_{\oplus\lambda}[\beta_1(y)+\beta_2(q)_{\oplus\mu}\alpha_1(z)+\alpha_2(r)]]\\ &=\circlearrowleft_{x,y,z}^{p,q,r} [\beta_1^2(x)_\lambda[\beta_1(y)_\mu\alpha_1(z)]_1]_1 + [\beta_2^2(p)_\lambda[\beta_2(q)_\mu\alpha_2(r)]_2]_2\\ &= 0.\end{aligned}$$

2. It is easy to show that $(B_1\oplus B_2, *_{\oplus\lambda}, \alpha, \beta)$ is a BiHom-associative conformal algebra.
3. Now, we show that the BiHom-Leibniz conformal identity holds for this considering that

$$\begin{aligned}&([\beta(x+p)_{\oplus\lambda}(y+q)]) *_{\oplus(\lambda+\mu)} \beta(z+r)\\ &+ \beta(y+q) *_{\oplus\mu} ([\alpha(x+p)_{\oplus\lambda}(z+r)])\\ &= ([(\beta_1(x)+\beta_2(p))_{\oplus\lambda}(y+q)]) *_{\oplus(\lambda+\mu)} (\beta_1(z)+\beta_2(r))\\ &+ (\beta_1(y)+\beta_2(q)) *_{\oplus\mu} ([(\alpha_1(x)+\alpha_2(p))_{\oplus\lambda} z+r])\\ &= ([\beta_1(x)_\lambda y]_1 + [\beta_2(p)_\lambda q]_2) *_{\oplus(\lambda+\mu)} (\beta_1(z)+\beta_2(r))\\ &+ (\beta_1(y)+\beta_2(q)) *_{\oplus\mu} ([\alpha_1(x)_\lambda z]_1 + [\alpha_1(p)_\lambda r]_2)\\ &= ([\beta_1(x)_\lambda y]_1) *_{1(\lambda+\mu)} \beta_1(z) + ([\beta_2(p)_\lambda q]_2) *_{2(\lambda+\mu)} \beta_2(r)\\ &+ \beta_1(y) *_{1\mu} ([\alpha_1(x)_\lambda z]_1) + \beta_2(q) *_{2\mu} ([\alpha_2(p)_\lambda r]_2)\\ &= ([\beta_1(x)_\lambda y]_1) *_{1(\lambda+\mu)} \beta_1(z) + \beta_1(y) *_{1\mu} ([\alpha_1(x)_\lambda z]_1)\\ &+ ([\beta_2(p)_\lambda q]_2) *_{2(\lambda+\mu)} \beta_2(r) + \beta_2(q) *_{2\mu} ([\alpha_2(p)_\lambda r]_2)\\ &= [\alpha_1\beta_1(x)_\lambda(y *_{1\mu} z)]_1 + [\alpha_2\beta_2(p)_\lambda(q *_{2\mu} r)]_2\\ &= [\alpha\beta(x+p)_{\oplus\lambda}((y+q) *_{\oplus\mu} (z+r))].\end{aligned}$$

This completes the proof. □

Theorem 1. *Consider that $(B_1, *_\lambda^1, [\cdot_\lambda\cdot]_1, \alpha_1, \beta_1, \partial_1)$ is a BiHom-Poisson conformal algebra, where α_1 and β_1 are structural maps. Assume that there exist $\phi_1, \psi_1 \in gc(B_1)$ such that $\alpha_1, \beta_1, \phi_1$, and ψ_1 commute. In this case, $B_1' = (B_1, *_\lambda'^1 = (\phi\otimes\psi)\circ *_\lambda^1, [\cdot_\lambda\cdot]_1' = (\phi\otimes\psi)\circ[\cdot_\lambda\cdot]_1', \alpha_1\phi, \beta_1\psi, \partial_1)$ is a BiHom-Poisson conformal algebra.*

Now consider that $(B_2, *_\lambda^2, [\cdot_\lambda\cdot]_2, \alpha_2, \beta_2, \partial_2)$ is another BiHom-Poisson conformal algebra and there exist ϕ_2 and ψ_2 such that $\alpha_2, \beta_2, \phi_2$ and ψ_2 commute.
Let f be the morphism of these BiHom-Poisson conformal algebras, given by

$$f : (B_1, *_\lambda^1, [\cdot_\lambda\cdot]_1, \alpha_1, \beta_1, \partial_1) \to (B_2, *_\lambda^2, [\cdot_\lambda\cdot]_2, \alpha_2, \beta_2, \partial_2)$$

such that $f\phi = \phi' f$ and $f\psi = \psi' f$. In this way, $f : B_1' \to B_2'$ is also a morphism.

Proof. To show that $B_1' = (B_1, *_\lambda'^1 = (\phi \otimes \psi) \circ *_\lambda^1, [\cdot_\lambda \cdot]_1' = (\phi \otimes \psi) \circ [\cdot_\lambda \cdot]_1', \alpha_1\phi, \beta_1\psi, \partial_1)$ is a BiHom-Poisson conformal algebra, we only show the associative conformal algebra case; other cases can be proved similarly. Note the following definitions:

1. Conformal sesqui-linearity:

$$\begin{aligned}\partial(x) *_\lambda'^1 y &= (\phi \otimes \psi)(\partial(x) *_\lambda^1 y) = (\partial(\phi(x)) *_\lambda^1 \psi(y))\\ &= -\lambda(\phi(x) *_\lambda^1 \psi(y)) = -\lambda(\phi \otimes \psi)(x *_\lambda^1 y) = -\lambda(x *_\lambda'^1 y),\end{aligned}$$

$$\begin{aligned}x *_\lambda'^1 \partial(y) =&(\phi \otimes \psi)(x *_\lambda^1 \partial(y)) = \phi(x) *_\lambda^1 \partial(\psi(y))\\ &= (\lambda + \partial)\phi(x) *_\lambda^1 \psi(y) = (\lambda + \partial)(x *_\lambda'^1 y).\end{aligned}$$

2. BiHom-skew symmetry:

$$\begin{aligned}\beta_1(\psi(x)) *_\lambda'^1 \alpha_1(\phi(y)) &= \beta_1(\phi\psi(x)) *_\lambda^1 \alpha_1(\phi\psi(y))\\ &= \beta_1(\phi\psi(y)) *_{-\partial-\lambda}^1 \alpha_1(\phi\psi(x)) = \beta_1\psi(y) *_{-\partial-\lambda}'^1 \alpha_1\phi(x).\end{aligned}$$

3. BiHom-associative conformal identity:

$$\begin{aligned}\alpha\phi(x) *_\lambda'^1 (y *_\mu'^1 z) &= \alpha\phi\phi(x) *_\lambda^1 \psi(\phi(y) *_\mu^1 \psi(z)) = \alpha\phi^2(x) *_\lambda^1 (\psi\phi(y) *_\mu^1 \psi^2(z))\\ &= (\phi^2(x) *_\lambda^1 \phi\psi(y)) *_{\lambda+\mu}^1 \beta\psi^2(z) = \phi(\phi(x) *_\lambda^1 \psi(y)) *_{\lambda+\mu}^1 \psi\beta\psi(z)\\ &= (x *_\lambda'^1 y) *_{\lambda+\mu}'^1 \beta\psi(z).\end{aligned}$$

Thus, $B_1' = (B_1, *_\lambda'^1 = (\phi \otimes \psi) \circ *_\lambda^1, \alpha_1\phi, \beta_1\psi, \partial_1)$ is an associative conformal algebra.

Now, we show that there is an algebra morphism from $f : B_1' \to B_2'$. Assume that $x, y \in B_1'$ and $\lambda, \mu \in \mathbb{C}$, in this case

$$\begin{aligned}f[x_\lambda y]_1' &= f[\phi(x)_\lambda \psi(y)]_1 = [f(\phi(x))_\lambda f(\psi(y))]_2\\ &= [\phi' f(x)_\lambda \psi' f(y)]_2 = (\phi' \otimes \psi') \circ [f(x)_\lambda f(y)]_2\\ &= [f(x)_\lambda f(y)]_2'.\end{aligned}$$

Similarly, we obtain

$$f(x *_\lambda'^1 y) = (f(x) *_\lambda'^2 f(y)).$$

This completes the proof. □

From Theorem 1, we obtain the following two results:

Corollary 1. *Let $(B, *_\lambda, [\cdot_\lambda \cdot], \alpha, \beta)$ be a BiHom-Poisson conformal algebra. Then, $B^k = (B, *_\lambda^k = (\alpha^k \otimes \beta^k) \circ (\cdot *_\lambda \cdot), [\cdot_\lambda \cdot]^k = (\alpha^k \otimes \beta^k) \circ [\cdot_\lambda \cdot], \alpha^{k+1}, \beta^{k+1})$ is also a BiHom-Poisson conformal algebra.*

Proof. Here, the proof can be completed by using Theorem 1 and replacing ϕ and ψ with α^k and β^k, respectively. □

Corollary 2. *Assume that $(P, *_\lambda, [\cdot_\lambda \cdot])$ is a Poisson conformal algebra. Let α and β be $\mathbb{C}$-linear endomorphisms of P; then, $B' = (P, *_\lambda' = (\alpha \otimes \beta) \circ *_\lambda, [\cdot_\lambda \cdot]' = (\alpha \otimes \beta) \circ [\cdot_\lambda \cdot], \alpha, \beta)$ is an associated BiHom-Poisson conformal algebra.*

Proof. The proof is followed by considering $\alpha = id$ in Theorem 1. □

Now, we introduce the conformal representations of BiHom-Poisson conformal algebra as follows:

Definition 8. *A conformal representation of a BiHom-Poisson conformal algebra* $(B, *_\lambda, [\cdot_\lambda\cdot], \alpha, \beta)$ *is defined on a* $\mathbb{C}[\partial]$*-module M endowed with the* $\mathbb{C}$*-linear maps* $\phi, \psi : M \to M$ *such that* $\phi\partial = \partial\phi$ *and* $\partial\psi = \psi\partial$ *form a tuple* $(M, l, r, \rho, \phi, \psi)$*, where* (M, l, r, ϕ, ψ) *is a conformal representation of the BiHom-associative conformal algebra* $(B, *_\lambda, \alpha, \beta)$ *and* (M, ρ, ϕ, ψ) *is a conformal representation of the BiHom-Lie conformal algebra* $(B, [\cdot_\lambda\cdot], \alpha, \beta)$*, such that for all* $x, y \in B$*,* $m \in M$ *and* $\lambda, \mu \in \mathbb{C}$*, the following equations hold:*

$$l([\beta(x)_\lambda y])_{\lambda+\mu}\psi(m) = \rho(\alpha\beta(x))_\lambda(l(y)_\mu m) - l(\beta(y))_\mu\rho(\alpha(x))_\lambda m, \quad (8)$$

$$r([\alpha(x)_\lambda y])_{\lambda+\mu}\psi(m) = \rho(\alpha\beta(x))_\lambda(r(y)_\mu m) - r(\beta(y))_\mu\rho(\beta(x))_\lambda m, \quad (9)$$

$$\rho(x *_\lambda y)_{\lambda+\mu}\phi\psi(m) = l(\alpha(x))_\lambda(\rho(y)_\mu\phi(m)) - l(\alpha(y))_\mu(\rho(x)_\lambda\psi(m)). \quad (10)$$

Proposition 6. *Let* $(l, r, \rho, \phi, \psi, M)$ *be a conformal representation of a BiHom-Poisson conformal algebra* $(B, *_\lambda, [\cdot_\lambda\cdot], \alpha, \beta)$*, where M is a* $\mathbb{C}[\partial]$*-module, and* ϕ *and* ψ *are* $\mathbb{C}$*-linear maps. Moreover, assume that* α *and* ψ *are bijective maps. Then, the semi-direct product,* $B \oplus M$*, of the* $\mathbb{C}[\partial]$*-modules is turned into a BiHom-Poisson conformal algebra* $(B \oplus M, *'_\lambda, [\cdot_\lambda\cdot]', \alpha, \beta)$ *by the* λ*-multiplication in* $B \oplus M$*, defined as follows:*

$$\begin{aligned}
(x_1 + m_1) *'_\lambda (x_2 + m_2) &:= x_1 *_\lambda x_2 + (l(x_1)_\lambda m_2 + r(x_2)_\lambda m_1), \\
[(x_1 + m_1)_\lambda(x_2 + m_2)]' &:= [x_{1\lambda} x_2] + (\rho(x_1)_\lambda m_2 - \rho(\alpha^{-1}\beta(x_2))_{-\partial-\lambda}\phi\psi^{-1}(m_1)), \\
(\alpha \oplus \phi)(x_1 + m_1) &:= \alpha(x_1) + \phi(m_1), \\
(\beta \oplus \psi)(x_1 + m_1) &:= \beta(x_1) + \psi(m_1),
\end{aligned}$$

for all $x_1, x_2 \in B, m_1, m_2 \in M$ *and* $\lambda, \mu \in \mathbb{C}$.

Proof. To show that $B \oplus M$ is a space of BiHom-Poisson conformal algebra, we need to satisfy the axioms of Definition 6. The first axiom is straightforward to show, and the second axiom can be seen as similar to Proposition 3.1 of [31]. Now, we are only left to prove the BiHom-Leibniz conformal identity. For all $x_1, x_2, x_3 \in B$, $m_1, m_2, m_3 \in M$ and $\lambda, \mu \in \mathbb{C}$, we have

$$\begin{aligned}
&[(\alpha\beta + \phi\psi)(x_1 + m_1)_\lambda((x_2 + m_2) *'_\mu (x_3 + m_3))]' \\
&- [(\beta + \psi)(x_1 + m_1)_\lambda(x_2 + m_2)]' *'_{\lambda+\mu}(\beta + \psi)(x_3 + m_3) \\
&- (\beta + \psi)(x_2 + m_2) *'_\mu[(\alpha + \phi)(x_1 + m_1)_\lambda(x_3 + m_3)]' \\
=&[(\alpha\beta(x_1) + \phi\psi(m_1))_\lambda(x_2 *_\mu x_3 + l(x_2)_\mu m_3 + r(x_3)_\mu m_2)]' \\
&- [(\beta(x_1) + \psi(m_1))_\lambda(x_2 + m_2)]' *'_{\lambda+\mu}(\beta(x_3) + \psi(m_3)) \\
&- (\beta(x_2) + \psi(m_2)) *'_\mu[(\alpha(x_1) + \phi(m_1))_\lambda(x_3 + m_3)]' \\
=&[(\alpha\beta(x_1) + \phi\psi(m_1))_\lambda(x_2 *_\mu x_3 + l(x_2)_\mu m_3 + r(x_3)_\mu m_2)]' \\
&- ([\beta(x_1)_\lambda x_2] + \rho(\beta(x_1))_\lambda m_2 - \rho(\alpha^{-1}\beta(x_2))_{-\partial-\lambda}\phi(m_1)) *'_{\lambda+\mu}(\beta(x_3) + \psi(m_3)) \\
&- (\beta(x_2) + \psi(m_2)) *'_\mu[\alpha(x_1)_\lambda(x_3)] + \rho(\alpha(x_1))_\lambda m_3 - \rho(\alpha^{-1}\beta(x_3))_{-\partial-\lambda}\phi^2\psi^{-1}(m_1) \\
=&[\alpha\beta(x_1)_\lambda(x_2 *_\mu x_3)] + \rho(\alpha\beta(x_1))_\lambda l(x_2)_\mu m_3 + \rho(\alpha\beta(x_1))_\lambda(r(x_3)_\mu m_2) \\
&- \rho(\alpha^{-1}\beta(x_{2\mu}x_3))_{-\partial-\lambda}\phi^2(m_1) \\
&- [\beta(x_1)_\lambda x_2] *_{\lambda+\mu} \beta(x_3) - l([\beta(x_1)_\lambda x_2])_{\lambda+\mu}\psi(m_3) \\
&- r(\beta(x_3))_{-\partial-\mu}\rho(\beta(x_1))_\lambda m_2 + r(\beta(x_3))_{-\partial-\mu}\rho(\alpha^{-1}\beta(x_2))_{-\partial-\lambda}\phi(m_1) \\
&- \beta(x_2) *_\mu [\alpha(x_1)_\lambda x_3] - l(\beta(x_2))_\mu(\rho(\alpha(x_1))_\lambda m_3) \\
&+ l(\beta(x_2))_\mu(\rho(\alpha^{-1}\beta(x_3))_{-\partial-\lambda}\phi^2\psi^{-1}(m_1)) + r([\alpha(x_1)_\lambda x_3])_{-\partial-\mu}\psi(m_2)
\end{aligned}$$

$$
\begin{aligned}
=&([\alpha\beta(x_1)_\lambda(x_2 *_\mu x_3)] - [\beta(x_1)_\lambda x_2] *_{\lambda+\mu} \beta(x_3) - \beta(x_2) *_\mu [\alpha(x_1)_\lambda x_3])\\
&+ (\rho(\alpha\beta(x_1))_\lambda l(x_2)_\mu(m_3) - l([\beta(x_1)_\lambda x_2])\psi(m_3) - l(\beta(x_2))_\mu\rho(\alpha(x_1))_\lambda(m_3))\\
&+ (\rho(\alpha\beta(x_1))_\lambda r(x_3)_\mu(m_2) - r(\beta(x_3))_{-\partial-\mu}\rho(\beta(x_1))_\lambda(m_2) - r([\alpha(x_1)_\lambda x_3])_{-\partial-\mu}\psi(m_2))\\
&- (\rho(\alpha^{-1}\beta(x_2 *_\mu x_3))_{-\partial-\lambda}\phi^2(m_1) + r(\beta(x_3))_{-\partial-\mu}\rho(\alpha^{-1}\beta(x_2))_{-\partial-\lambda}\phi(m_1)\\
&+ l(\beta(x_2))_\mu\rho(\alpha^{-1}\beta(x_3))_{-\partial-\lambda}\phi^2\psi^{-1}(m_1))\\
=&0.
\end{aligned}
$$

This finishes the proof. □

Example 4. *Let $(B, *_\lambda, [\cdot_\lambda\cdot], \alpha, \beta)$ be a BiHom-Poisson conformal algebra. Then, $(l, r, ad, \alpha, \beta, B)$ is a regular representation of B, where $l(x)_\lambda y = x *_\lambda y, r(x)_\lambda y = y *_{-\partial-\lambda} x$ and $ad(x)_\lambda y = [x_\lambda y]$ for all $x, y \in B$ and $\lambda \in \mathbb{C}$.*

4. BiHom-Pre-Poisson Conformal Algebra and Its Conformal Bimodule

In this section, we first introduce the notion of the conformal representation of the BiHom-pre-Lie conformal algebra, which leads us to describe a (noncommutative) BiHom-pre-Poisson conformal algebra. We also discuss the conformal bimodule structure of it.

Definition 9. *A BiHom-pre-Lie conformal algebra $(B, *_\lambda, \alpha, \beta)$ is a tuple consisting of a $\mathbb{C}[\partial]$-module, B; a $\mathbb{C}$-bilinear map, $*_\lambda : B \otimes B \to B[\lambda]$; and two commutative multiplicative linear maps, $\alpha, \beta : B \to B$, such that $\partial\alpha = \alpha\partial$ and $\beta\partial = \partial\beta$, satisfying the following equation for all $x, y, z \in B$ and $\lambda, \mu \in \mathbb{C}$:*

$$
\begin{aligned}
(\beta(x) *_\lambda \alpha(y)) *_{\lambda+\mu} \beta(z) &- \alpha\beta(x) *_\lambda (\alpha(y) *_\mu z)\\
&= (\beta(y) *_\mu \alpha(x)) *_{\lambda+\mu} \beta(z) - \alpha\beta(y) *_\mu (\alpha(x) *_\lambda z).
\end{aligned}
$$

If B is finitely generated, then a BiHom-pre-Lie conformal algebra is called finite.

Proposition 7. *Let $(B, *_\lambda, \alpha, \beta)$ be a regular BiHom-pre-Lie conformal algebra with bijective structure maps α and β. Then, $(B, [\cdot_\lambda\cdot], \alpha, \beta)$ is called a BiHom-Lie conformal algebra with the λ-bracket given by*

$$[x_\lambda y] = x *_\lambda y - \alpha^{-1}\beta(y) *_{-\partial-\lambda} \alpha\beta^{-1}(x),$$

for all $x, y \in B$. We call this algebra, $B^c = (B, [\cdot_\lambda\cdot], \alpha, \beta)$, a subadjacent BiHom-Lie conformal algebra of $(B, *_\lambda, \alpha, \beta)$.

Now, we introduce the conformal representation of a BiHom-pre-Lie conformal algebra in the following definition.

Definition 10. *Let $(B, *_\lambda, \alpha, \beta)$ be a BiHom-pre-Lie conformal algebra and (M, ϕ, ψ) be a BiHom-conformal module. Let $l_*, r_* : B \to gc(M)$ be two $\mathbb{C}$-linear maps. Then, the tuple $(l_*, r_*, \phi, \psi, M)$ is considered a conformal representation of a BiHom-pre-Lie conformal algebra B if the following equations hold:*

$$
\begin{aligned}
l_*(\partial x)_\lambda m &= -\lambda(l_*(x)_\lambda m),\\
l_*(x)_\lambda(\partial m) &= (\partial + \lambda)(l_*(x)_\lambda m),\\
r_*(\partial x)_\lambda m &= -\lambda(r_*(x)_\lambda m),\\
r_*(x)_\lambda(\partial m) &= (\partial + \lambda)(r_*(x)_\lambda m),\\
\phi(l_*(x)_\lambda m) &= l_*(\alpha(x))_\lambda\phi(m),\\
\phi(r_*(x)_\lambda m) &= r_*(\alpha(x))_\lambda\phi(m),\\
\psi(l_*(x)_\lambda m) &= l_*(\beta(x))_\lambda\psi(m),
\end{aligned}
$$

$$\psi(r_*(x)_\lambda m) = r_*(\beta(x))_\lambda \psi(m),$$
$$l_*([\beta(x)_\lambda \alpha(y)])_{\lambda+\mu}\psi(m) = l_*(\alpha\beta(x))_\lambda l_*(\alpha(y))_\mu m - l_*(\alpha\beta(y))_\mu l_*(\alpha(x))_\lambda m,$$
$$r_*(\beta(y))_\mu \rho(\beta(x))_\lambda \phi(m) = l_*(\alpha\beta(x))_\lambda r_*(y)_\mu \phi(m) - r_*(\alpha(x) *_\lambda y)_{\lambda+\mu}\phi\psi(m),$$

for all $x, y \in B, m \in M$ and $\lambda, \mu \in \mathbb{C}$. Additionally, $[\beta(x)_\lambda \alpha(y)] = \beta(x) *_\lambda \alpha(y) - \beta(y) *_{-\lambda-\partial} \alpha(x)$ and $(\rho \circ \beta)\phi = (l_* \circ \beta)\phi - (r_* \circ \alpha)\psi$.

Proposition 8. *Let $(B, *_\lambda, \alpha, \beta)$ be a BiHom-pre-Lie conformal algebra and $(l_*, r_*, \phi, \psi, M)$ be its conformal representation, where M is a $\mathbb{C}[\partial]$-module, and ϕ and ψ are $\mathbb{C}$-linear maps, satisfying $\partial\phi = \phi\partial$ and $\partial\psi = \psi\partial$. Then, the direct sum, $B \oplus M$, of the $\mathbb{C}[\partial]$-modules is turned into a BiHom-pre-Lie conformal algebra by defining the λ-multiplication $*'_\lambda$ on $B \oplus M$ as follows:*

$$\begin{aligned}(x_1 + m_1) *'_\lambda (x_2 + m_2) &:= x_1 *_\lambda x_2 + (l_*(x_1)_\lambda m_2 + r_*(x_2)_{-\partial-\lambda} m_1),\\ (\alpha \oplus \phi)(x + m) &:= \alpha(x) + \phi(m),\\ (\beta \oplus \psi)(x + m) &:= \beta(x) + \psi(m),\end{aligned} \tag{11}$$

for all $x, x_1, x_2 \in B, m, m_1, m_2 \in M$ and $\lambda \in \mathbb{C}$.

We denote this BiHom-pre-Lie conformal algebra by $(B \oplus M, *'_\lambda, \alpha + \beta, \phi + \psi)$, or simply $(B \ltimes_{l_*, r_*, \alpha, \beta, \phi, \psi} M)$.

Proposition 9. *Consider a regular BiHom-pre-Lie conformal algebra $(B, *_\lambda, \alpha, \beta)$ and let $(l_*, r_*, \phi, \psi, M)$ be a conformal representation of it in such a way that ϕ is bijective. Let $(B, [\cdot_\lambda \cdot], \alpha, \beta)$ be the subadjacent BiHom-Lie conformal algebra of $(B, *_\lambda, \alpha, \beta)$. Then, $(l_* - (r_* \circ \alpha\beta^{-1})\phi^{-1}\psi, \phi, \psi, M)$ is a conformal representation of the BiHom-Lie conformal algebra $(B, [\cdot_\lambda \cdot], \alpha, \beta)$.*

Proof. To show that $(l_* - (r_* \circ \alpha\beta^{-1})\phi^{-1}\psi, \phi, \psi, M)$ is a conformal representation of a BiHom-Lie conformal algebra $(B, [\cdot_\lambda \cdot], \alpha, \beta)$, we need to satisfy the axioms of Definition 5. Let us check them one by one

1. First, we show that

$$\begin{aligned}(l_* - (r_* \circ \alpha\beta^{-1})\phi^{-1}\psi)(\partial(x))_\lambda m &= l_*(\partial(x))_\lambda m - (r_* \circ \alpha\beta^{-1}(\partial(x)))_\lambda \phi^{-1}\psi(m)\\ &= -\lambda(l_*(x)_\lambda m) + \lambda(r_* \circ \alpha\beta^{-1}(x))_\lambda \phi^{-1}\psi(m)\\ &= -\lambda((l_* - (r_* \circ \alpha\beta^{-1})_\lambda \phi^{-1}\psi)(x))_\lambda m.\end{aligned}$$

Similarly, we can show that

$$(l_* - (r_* \circ \alpha\beta^{-1})\phi^{-1}\psi)(x)_\lambda \partial(m) = (\partial + \lambda)((l_* - (r_* \circ \alpha\beta^{-1})\phi^{-1}\psi)(x))_\lambda m.$$

2. Next, we show that

$$\begin{aligned}\phi(l_*(x) - r_*(\alpha\beta^{-1}(x))\phi^{-1}\psi)_\lambda m &= \phi(l_*(x))_\lambda m - \phi(r_*(\alpha\beta^{-1}(x))_\lambda \phi^{-1}\psi(m))\\ &= (l_*(\alpha(x)))_\lambda \phi(m) - (r_*(\alpha\beta^{-1}\alpha(x))_\lambda \phi\phi^{-1}\psi(m))\\ &= (l_*(\alpha(x)))_\lambda \phi(m) - (r_*(\alpha\beta^{-1}\alpha(x))_\lambda \psi(m))\\ &= (l_* - (r_* \circ \alpha\beta^{-1})\phi\psi^{-1})(\alpha(x))_\lambda \phi(m).\end{aligned}$$

Similarly, we can show that

$$\psi(l_*(x) - r_*(\alpha\beta^{-1}(x))\phi^{-1}\psi)_\lambda m = (l_* - (r_* \circ \alpha\beta^{-1})\phi\psi^{-1})(\beta(x))_\lambda \psi(m).$$

3. Finally, we show that

$$
\begin{aligned}
&(l^*(\alpha\beta(x)) - (r^*\alpha\beta^{-1}(\alpha\beta(x))\phi^{-1}\psi))_\lambda(l^*(y) - (r^*\circ\alpha\beta^{-1}(y))\phi^{-1}\psi)_\mu(m) \\
&\quad - (l^*(\beta(y)) - (r^*\alpha\beta^{-1}(\beta(y))\circ\phi^{-1}\psi))_\mu(l^*(\alpha(x)) - (r^*\circ\alpha\beta^{-1}(\alpha(x))\phi^{-1}\psi)_\lambda(m)) \\
&= l^*(\alpha\beta(x))_\lambda(l^*(y)_\mu m) - (r^*\alpha^2(x))_\lambda\phi^{-1}\psi(l^*(y)_\mu m) \\
&\quad - l^*(\alpha\beta(x))_\lambda(r^*\alpha\beta^{-1}(y))_\mu\phi^{-1}\psi(m) + (r^*\alpha^2(x))_\lambda\phi^{-1}\psi(r^*\alpha\beta^{-1}(y))_\mu\phi^{-1}\psi(m) \\
&\quad - l^*(\beta(y))_\mu l^*(\alpha(x))_\lambda m + l^*(\beta(y))_\mu(r^*\circ\alpha^2\beta^{-1}(x))_\lambda\phi^{-1}\psi(m) \\
&\quad + (r^*\circ\alpha(y))_\mu\phi^{-1}\psi\circ l^*(\alpha(x))_\lambda m - (r^*\circ\alpha(y))_\mu\phi^{-1}\psi\circ(r^*\circ\alpha^2\beta^{-1}(x))_\lambda\phi^{-1}\psi(m) \\
&= l^*(\alpha\beta(x))_\lambda(l^*(y)_\mu m) - (r^*\circ\alpha^2(x))_\lambda(l^*(\alpha^{-1}\beta(y))_\mu\phi^{-1}\psi(m)) \\
&\quad - l^*(\alpha\beta(x))_\lambda((r^*\circ\alpha\beta^{-1}(y))_\mu\phi^{-1}\psi(m)) + (r^*\circ\alpha^2(x))_\lambda((r^*(y))_\mu\phi^{-2}\psi^2(m)) \\
&\quad - l^*(\beta(y))_\mu(l^*(\alpha(x))_\lambda m) + l^*(\beta(y))_\mu((r^*\circ\alpha^2(x))_\lambda\phi^{-1}\psi(m)) \\
&\quad + (r^*\circ\alpha(y))_\mu(l^*(\beta(x))_\lambda\phi^{-1}\psi(m)) - (r^*\circ\alpha(y))_\mu((r^*\circ\alpha(x))_\lambda\phi^{-2}\psi^2(m)) \\
&= \{l^*(\alpha\beta(x))_\lambda(l^*(y)_\mu m) - l^*(\beta(y))_\mu(l^*(\alpha(x))_\lambda m)\} \\
&\quad + \{-l^*(\alpha\beta(x))_\lambda((r^*\circ\alpha\beta^{-1}(y))_\mu\phi^{-1}\psi(m)) \quad + (r^*\circ\alpha(y))_\mu(l^*(\beta(x))_\lambda\phi^{-1}\psi(m)) \\
&\quad - (r^*\circ\alpha(y))_\mu((r^*\circ\alpha(x))_\lambda\phi^{-2}\psi^2(m))\} \\
&\quad - \{(r^*\circ\alpha^2(x))_\lambda(l^*(\alpha^{-1}\beta(y))_\mu\phi^{-1}\psi(m)) \quad - (r^*\circ\alpha^2(x))_\lambda((r^*(y))_\mu\phi^{-2}\psi^2(m)) \\
&\quad - l^*(\beta(y))_\mu((r^*\circ\alpha^2\beta^{-1}(x))_\lambda\phi^{-1}\psi(m))\} \\
&= l_*([\beta(x)_\lambda y])_{\lambda+\mu}\psi(m) - r_*(\alpha(x)*_\lambda\alpha\beta^{-1}(y))_{\lambda+\mu}\phi^{-1}\psi^2(m) \\
&\quad + r_*(y*_{-\partial-\lambda}\alpha^2\beta^{-1}(x))_{\lambda+\mu}\phi^{-1}\psi^2(m) \\
&= l_*([\beta(x)_\lambda y])_{\lambda+\mu}\psi(m) - r_*(\alpha(x)*_\lambda\alpha\beta^{-1}(y) - y*_{-\partial-\lambda}\alpha^2\beta^{-1}(x))_{\lambda+\mu}\phi^{-1}\psi^2(m) \\
&= l_*([\beta(x)_\lambda y])_{\lambda+\mu}\psi(m) - r_*([\alpha(x)_\lambda\alpha\beta^{-1}(y)])_{\lambda+\mu}\phi^{-1}\psi^2(m) \\
&= \rho([\beta(x)_\lambda y])_{\lambda+\mu}\psi(m).
\end{aligned}
$$

This completes the proof. □

Definition 11. *[32] A 5-tuple $(B, \prec_\lambda, \succ_\lambda, \alpha, \beta)$ equipping a $\mathbb{C}[\partial]$-module B, bilinear multiplication maps $\prec_\lambda, \succ_\lambda$: $B \otimes B \to B[\lambda]$ and commuting $\mathbb{C}$-linear maps $\alpha, B \to B$ is said to be a BiHom-dendriform conformal algebra if the following conditions hold:*

$$(\partial x) \succ_\lambda y = -\lambda(x \succ_\lambda y), \quad x \succ_\lambda (\partial y) = (\lambda+\partial)(x \succ_\lambda y), \tag{12}$$
$$(\partial x) \prec_\lambda y = -\lambda(x \prec_\lambda y), \quad x \prec_\lambda (\partial y) = (\lambda+\partial)(x \prec_\lambda y), \tag{13}$$
$$\alpha(x \prec_\lambda y) = \alpha(x) \prec_\lambda \alpha(y), \quad \alpha(x \succ_\lambda y) = \alpha(x) \succ_\lambda \alpha(y), \tag{14}$$
$$\beta(x \prec_\lambda y) = \beta(x) \prec_\lambda \beta(y), \quad \beta(x \succ_\lambda y) = \beta(x) \succ_\lambda \beta(y), \tag{15}$$
$$(x \prec_\lambda y) \prec_{\lambda+\mu} \beta(z) = \alpha(x) \prec_\lambda (y \prec_\mu z + y \succ_\mu z), \tag{16}$$
$$(x \succ_\lambda y) \prec_{\lambda+\mu} \beta(z) = \alpha(x) \succ_\lambda (y \prec_\mu r), \tag{17}$$
$$\alpha(x) \succ_\lambda (y \succ_\mu z) = (x \prec_\lambda y + x \succ_\lambda y) \succ_{\lambda+\mu} \beta(z), \tag{18}$$

for all $x, y, z \in B$ and $\lambda, \mu \in \mathbb{C}$.

Lemma 2. *The tuple $(B, \cdot_\lambda = \prec_\lambda + \succ_\lambda, \alpha, \beta)$ is a BiHom-associative conformal algebra provided that $(B, \prec_\lambda, \succ_\lambda, \alpha, \beta)$ is a BiHom-dendriform conformal algebra.*

Now, we introduce the conformal representation of BiHom-dendriform conformal algebra.

Definition 12. *Let $(B, \prec_\lambda, \succ_\lambda, \alpha, \beta)$ be a BiHom-dendriform conformal algebra and M be a $\mathbb{C}[\partial]$-module. Let $l_\prec, r_\prec, l_\succ, r_\succ : B \to gc(M)$ and $\phi, \psi : M \to M$ be six $\mathbb{C}$-linear maps. Then, the tuple*

$(l_\prec, r_\prec, l_\succ, r_\succ, \phi, \psi, M)$ *is called a conformal representation of B if the following equations hold for any* $x, y \in B$, $m \in M$ *and* $\lambda, \mu \in \mathbb{C}$:

$$\begin{aligned}
l_\prec(x \prec_\lambda y)_{\lambda+\mu}\phi(m) &= l_\prec(\alpha(x))_\lambda(l_\cdot(y)_\mu m),\\
r_\prec(\beta(x))_\lambda(l_\prec(y)_\mu m) &= l_\prec(\alpha(y))_\mu(r_\cdot(x)_\lambda m),\\
r_\prec(\beta(x))_\lambda(r_\prec(y)_\mu m) &= r_\prec(x \cdot_\lambda y)_{\lambda+\mu}\phi(m),\\
l_\prec(x \succ_\lambda y)_{\lambda+\mu}\psi(m) &= l_\succ(\alpha(x))_\lambda(l_\prec(y)_\mu m),\\
r_\prec(\beta(x))_\lambda(l_\succ(y)_\mu m) &= l_\succ(\alpha(y))_\mu(r_\prec(x)_\lambda m),\\
r_\prec(\beta(x))_\lambda(r_\succ(y)_\mu m) &= r_\succ(y \prec_{\mu=-\partial-\lambda} x)_{\lambda+\mu}\phi(m),\\
l_\succ(x \cdot_\lambda y)_{\lambda+\mu}\psi(m) &= l_\succ(\alpha(x))_\lambda(l_\succ(y)_\mu m),\\
r_\succ(\beta(x))_\lambda(l_\cdot(y)_\mu m) &= l_\succ(\alpha(y))_\mu(r_\succ(x)_\lambda m),\\
r_\succ(\beta(x))_\lambda(r_\cdot(y)_\mu m) &= r_\succ(y \succ_\mu x)_{\lambda+\mu}\phi(m),\\
\phi(l_\prec(x)_\lambda m) &= l_\prec(\alpha(x))_\lambda\phi(m),\\
\phi(r_\prec(x)_\lambda m) &= r_\prec(\alpha(x))_\lambda\phi(m),\\
\psi(l_\prec(x)_\lambda m) &= l_\prec(\beta(x))_\lambda\psi(m),\\
\psi(r_\prec(x)_\lambda m) &= r_\prec(\beta(x))_\lambda\psi(m),\\
\phi(l_\succ(x)_\lambda m) &= l_\succ(\alpha(x))_\lambda\phi(m),\\
\phi(r_\succ(x)_\lambda m) &= r_\succ(\alpha(x))_\lambda\phi(m),\\
\psi(l_\succ(x)_\lambda m) &= l_\succ(\beta(x))_\lambda\psi(m),\\
\psi(r_\succ(x)_\lambda m) &= r_\succ(\beta(x))_\lambda\psi(m),
\end{aligned}$$

where $x \cdot_\lambda y = x \prec_\lambda y + x \succ_\lambda y, l_\cdot = l_\prec + l_\succ$ *and* $r_\cdot = r_\prec + r_\succ$.

Proposition 10. *Let* $(l_\prec, r_\prec, l_\succ, r_\succ, \phi, \psi, M)$ *be a conformal representation of a BiHom-dendriform conformal algebra* $(B, \prec_\lambda, \succ_\lambda, \alpha, \beta)$, *where M is a* $\mathbb{C}[\partial]$*- module, and* ϕ *and* ψ *are* $\mathbb{C}$*-linear maps, satisfying* $\partial\phi = \phi\partial$ *and* $\partial\psi = \psi\partial$. *Then, the direct sum,* $B \oplus M$, *of the* $\mathbb{C}[\partial]$*-modules is turned into a BiHom-dendriform conformal algebra by defining* λ*-multiplication operators* $\prec'_\lambda$ *and* $\succ'_\lambda$ *on* $B \oplus M$ *as follows:*

$$\begin{aligned}
(x_1 + m_1) \prec'_\lambda (x_2 + m_2) &:= x_1 \prec_\lambda x_2 + (l_\prec(x_1)_\lambda m_2 + r_\prec(x_2)_{-\partial-\lambda} m_1),\\
(x_1 + m_1) \succ'_\lambda (x_2 + m_2) &:= x_1 \succ_\lambda x_2 + (l_\succ(x_1)_\lambda m_2 + r_\succ(x_2)_{-\partial-\lambda} m_1),\\
(\alpha \oplus \phi)(x + m) &:= \alpha(x) + \phi(m),\\
(\beta \oplus \psi)(x + m) &:= \beta(x) + \psi(m),
\end{aligned} \tag{19}$$

for all $x, x_1, x_2 \in B; m, m_1, m_2 \in M;$ *and* $\lambda \in \mathbb{C}$.

We denote this BiHom-dendriform conformal algebra by $(B \oplus M, \prec'_\lambda, \succ'_\lambda, \alpha + \phi, \beta + \psi)$, or simply $(B \ltimes_{l_\prec, r_\prec, l_\succ, r_\succ, \alpha, \beta, \phi, \psi} M)$.

Proposition 11. *Let* $(l_\prec, r_\prec, l_\succ, r_\succ, \phi, \psi, M)$ *be a conformal bimodule of BiHom-dendriform conformal algebra* $(B, \prec_\lambda, \succ_\lambda, \alpha, \beta)$. *Let* $(B, \cdot_\lambda = \prec_\lambda + \succ_\lambda, \alpha, \beta)$ *be a BiHom-associative conformal algebra. Then,* $(l_\prec + l_\succ, r_\prec + r_\succ, \phi, \psi, M)$ *is a conformal bimodule of* $(B, \cdot_\lambda = \prec_\lambda + \succ_\lambda, \alpha, \beta)$.

Proof. Let us prove the 10th identity in Definition 4; other cases can be proved similarly:

$$\begin{aligned}
&(l_\prec + l_\succ)(x \cdot_\lambda y)_{\lambda+\mu}\psi(m)\\
&= (l_\prec + l_\succ)(x \prec_\lambda y + x \succ_\lambda y)_{\lambda+\mu}\psi(m)\\
&= l_\prec(x \prec_\lambda y)_{\lambda+\mu}\psi(m) + l_\prec(x \succ_\lambda y)_{\lambda+\mu}\psi(m) + l_\succ(x \cdot_\lambda y)_{\lambda+\mu}\psi(m)\\
&= l_\prec(\alpha(x))_\lambda(l_\prec + l_\succ)(y)_\mu m + l_\succ(\alpha(x))_\lambda l_\prec(y)_\mu m + l_\succ(\alpha(x))_\lambda l_\succ(y)_\mu m
\end{aligned}$$

$$= l_{\prec}(\alpha(x))_{\lambda}((l_{\prec}+l_{\succ})(y)_{\mu}m) + l_{\succ}(\alpha(x))_{\lambda}((l_{\prec}+l_{\succ})(y)_{\mu}m)$$
$$= (l_{\prec}+l_{\succ})(\alpha(x))_{\lambda}((l_{\prec}+l_{\succ})(y)_{\mu})m.$$

□

Next, we introduce the notion of a BiHom-pre-Poisson conformal algebra and give some important results.

Definition 13. *A noncommutative BiHom-pre-Poisson conformal algebra is a 6-tuple $(B, \prec_\lambda, \succ_\lambda, *_\lambda, \alpha, \beta)$, such that $(B, \prec_\lambda, \succ_\lambda, \alpha, \beta)$ is a BiHom-dendriform conformal algebra and $(B, *_\lambda, \alpha, \beta)$ is a BiHom-pre-Lie conformal algebra satisfying the following compatibility conditions:*

$$\begin{aligned}
(\beta(x) *_\lambda \alpha(y) - \beta(y) *_{-\partial-\lambda} \alpha(x)) \prec_{\lambda+\mu} \beta(z) &= \alpha\beta(x) *_\lambda (\alpha(y) \prec_\mu z) - \alpha\beta(y) \prec_\mu (\alpha(x) *_\lambda z), \\
\beta(x) \succ_\lambda (\alpha\beta(y) *_\mu \alpha(z) - \beta(z) *_{-\partial-\mu} \alpha^2(y)) &= \alpha\beta^2(y) *_\mu (x \succ_\lambda \alpha(z)) - (\beta^2(y) *_\mu x) \succ_{\lambda+\mu} \alpha\beta(z), \\
(\beta(x) \prec_\lambda \alpha(y) + \beta(x) \succ_\lambda \alpha(y)) *_{\lambda+\mu} \beta(z) &= (\beta(x) *_\lambda \alpha(z)) \succ_{-\partial-\mu} \beta(y) + \alpha\beta(x) \prec_\lambda (\alpha(y) *_\mu z).
\end{aligned} \tag{20}$$

Theorem 2. *Let $(B, \prec_\lambda, \succ_\lambda, *_\lambda)$ be a pre-Poisson conformal algebra and $\alpha, \beta \in End(B)$ be two commuting $\mathbb{C}$-linear morphisms of B. Then, $B_{\alpha,\beta} := (B, \prec_\lambda^{\alpha,\beta} = \prec_\lambda \circ(\alpha \otimes \beta), \succ_\lambda^{\alpha,\beta} = \succ_\lambda \circ(\alpha \otimes \beta), *_\lambda^{\alpha,\beta} = *_\lambda \circ (\alpha \otimes \beta), \alpha, \beta)$ is a BiHom-pre-Poisson conformal algebra, known as the Yau-twist of B. Moreover, assume that there is another BiHom-pre-Poisson conformal algebra $B'_{\alpha',\beta'}$ generated from the pre-Poisson conformal algebra $(B', \prec'_\lambda, \succ'_\lambda, *'_\lambda)$ in the presence of structure maps α' and β'. Assume that $f : B \to B'$ is a BiHom-pre-Poisson conformal algebra morphism that satisfies $f \circ \alpha' = \alpha \circ f$, $f \circ \beta' = \beta \circ f$. Then, $f : B_{\alpha,\beta} \to B'_{\alpha',\beta'}$ is a BiHom-pre-Poisson conformal algebra morphism.*

Proof. We shall only prove the first relation in Equation (20); the other conditions can be proved analogously. Then, for any $x, y, z \in B$ and $\lambda, \mu \in \mathbb{C}$,

$$\begin{aligned}
&(\beta(x) *_\lambda^{\alpha,\beta} \alpha(y) - \beta(y) *_{-\partial-\lambda}^{\alpha,\beta} \alpha(x)) \prec_{\lambda+\mu}^{\alpha,\beta} \beta(z) \\
&= (\alpha\beta(x) *_\lambda \alpha\beta(y) - \alpha\beta(y) *_{-\partial-\lambda} \alpha\beta(x)) \prec_{\lambda+\mu}^{\alpha,\beta} \beta(z) \\
&= (\alpha^2\beta(x) *_\lambda \alpha^2\beta(y) - \alpha^2\beta(y) *_{-\partial-\lambda} \alpha^2\beta(x)) \prec_{\lambda+\mu} \beta^2(z) \\
&= \alpha^2\beta(x) *_\lambda (\alpha^2\beta(y) \prec_\mu \beta^2(z)) - \alpha^2\beta(y) \prec_\mu (\alpha^2\beta(x) *_\lambda \alpha^2(z)) \\
&= \alpha\beta(x) *_\lambda^{\alpha,\beta} (\alpha(y) \prec_\mu^{\alpha,\beta} z) - \alpha\beta(y) \prec_\mu^{\alpha,\beta} (\alpha(x) *_\lambda^{\alpha,\beta} z) \\
&= \alpha\beta(x) *_\lambda^{\alpha,\beta} (\alpha(y) \prec_\mu^{\alpha,\beta} z) - \alpha\beta(y) \prec_\mu^{\alpha,\beta} (\alpha(x) *_\lambda^{\alpha,\beta} z).
\end{aligned}$$

For the second assertion, we have

$$\begin{aligned}
f(x \prec_\lambda^{\alpha,\beta} y) &= f(\alpha(x) \prec_\lambda \beta(y)) \\
&= f(\alpha(x)) \prec'_\lambda f(\beta(y)) \\
&= \alpha' f(x) \prec'_\lambda \beta' f(y) \\
&= f(x) \prec_\lambda^{'\alpha',\beta'} f(y).
\end{aligned}$$

Similarly, we have $f(x \prec_\lambda^{\alpha,\beta} y) = f(x) \prec_\lambda^{'\alpha',\beta'} f(y)$ and $f(x *_\lambda^{'\alpha',\beta'} y) = f(x) *_\lambda^{'\alpha',\beta'} f(y)$. This completes the proof. □

Proposition 12. *Let $(B, \prec_\lambda, \succ_\lambda, *_\lambda, \alpha, \beta)$ be a commutative BiHom-pre-Poisson conformal algebra and $\alpha', \beta' : B \to B$ be two noncommutative BiHom-pre-Poisson conformal algebra morphisms such that any two of the maps α, β, α' and β' commute. Then, $(B, \prec_\lambda^{\alpha',\beta'}, \succ_\lambda^{\alpha',\beta'}, *_\lambda^{\alpha',\beta'}, \alpha \circ \alpha', \beta \circ \beta')$ is a noncommutative BiHom-pre-Poisson conformal algebra.*

Corollary 3. *Let $(B, \prec_\lambda, \succ_\lambda, *_\lambda, \alpha, \beta)$ be a noncommutative BiHom-pre-Poisson conformal algebra and $m \in \mathbb{N}/\{0\}$; then, two types of mth-derived noncommutative BiHom-pre-Poisson conformal algebras are defined by*

1. $B_1^m = (B, \prec_\lambda^{(m)} = \prec_\lambda \circ(\alpha^m \otimes \beta^m), \succ_\lambda^{(m)} = \succ_\lambda \circ(\alpha^m \otimes \beta^m), *_\lambda^{(m)} = *_\lambda \circ (\alpha^m \otimes \beta^m), \alpha^{m+1}, \beta^{m+1})$.
2. $B_2^m = (B, \prec_\lambda^{(2m-1)} = \prec_\lambda \circ(\alpha^m \otimes \beta^m), \succ_\lambda^{(2m-1)} = \succ_\lambda \circ(\alpha^{2m-1} \otimes \beta^{2m-1}), *_\lambda^{(2m-1)} = *_\lambda \circ (\alpha^m \otimes \beta^m), \alpha^{2m}, \beta^{2m})$.

Proof. Apply Proposition 12 with $\alpha' = \alpha^m$ and $\beta' = \beta^m$ and $\alpha' = \alpha^{2m-1}$ and $\beta' = \beta^{2m-1}$, respectively. □

Theorem 3. *Let $(B, \prec_\lambda, \succ_\lambda, *_\lambda, \alpha, \beta)$ be a regular noncommutative BiHom-pre-Poisson conformal algebra. Then, $(B, [\cdot_\lambda \cdot], \cdot_\lambda, \alpha, \beta)$ is a noncommutative BiHom-Poisson conformal algebra with $x \cdot_\lambda y = x \prec_\lambda y + x \succ_\lambda y$ and $[x_\lambda y] = x *_\lambda y - \alpha^{-1}\beta(y) *_{-\partial-\lambda} \alpha\beta^{-1}(x)$ for any $x, y \in B$, $\lambda \in \mathbb{C}$. We say that $(B, [\cdot_\lambda \cdot], \cdot_\lambda, \alpha, \beta)$ is the subadjacent noncommutative BiHom-Poisson conformal algebra of $(B, \prec_\lambda, \succ_\lambda, *_\lambda, \alpha, \beta)$ and is denoted by B^c.*

Proof. From Proposition 7 and Lemma 2, we deduce that $(B, \cdot_\lambda, \alpha, \beta)$ is a BiHom-associative conformal algebra and $(B, [\cdot_\lambda \cdot], \alpha, \beta)$ is a BiHom-Lie conformal algebra. Now, we are only left to show the BiHom-Leibniz conformal identity:

$$
\begin{aligned}
&[\alpha\beta(x)_\lambda(y \cdot_\mu z)] - [\beta(x)_\lambda y] \cdot_{\lambda+\mu} \beta(z) - \beta(y) \cdot_\mu [\alpha(x)_\lambda z] \\
=&[\alpha\beta(x)_\lambda(y \prec_\mu z + y \succ_\mu z)] - [\beta(x)_\lambda y] \prec_{\lambda+\mu} \beta(z) - [\beta(x)_\lambda y] \succ_{\lambda+\mu} \beta(z) \\
&- \beta(y) \prec_\mu [\alpha(x)_\lambda z] - \beta(y) \succ_\mu [\alpha(x)_\lambda z] \\
=&\alpha\beta(x) *_\lambda (y \prec_\mu z) - \alpha^{-1}\beta(y \prec_\mu z) *_{-\partial-\lambda} \alpha^2(x) \\
&+ \alpha\beta(x) *_\lambda (y \succ_\mu z) - \alpha^{-1}\beta(y \succ_\mu z) *_{-\partial-\lambda} \alpha^2(x) \\
&- (\beta(x) *_\lambda y) \prec_{\lambda+\mu} \beta(z) + (\alpha^{-1}\beta(y) *_{-\partial-\lambda} \alpha(x)) \prec_{\lambda+\mu} \beta(z) \\
&- (\beta(x) *_\lambda y) \succ_{\lambda+\mu} \beta(z) + (\alpha^{-1}\beta(y) *_{-\partial-\lambda} \alpha(x)) \succ_{\lambda+\mu} \beta(z) \\
&- \beta(y) \prec_\mu (\alpha(x) *_\lambda z) + \beta(y) \prec_\mu (\alpha^{-1}\beta(z) *_{-\partial-\lambda} \alpha^2\beta^{-1}(x)) \\
&- \beta(y) \succ_\mu (\alpha(x) *_\lambda z) + \beta(y) \succ_\mu (\alpha^{-1}\beta(z) *_{-\partial-\lambda} \alpha^2\beta^{-1}(x)) \\
=&\{^1\alpha\beta(x) *_\lambda (y \prec_\mu z) -^5 (\beta(x) *_\lambda y) \prec_{\lambda+\mu} \beta(z) \\
&-^9 \beta(y) \prec_\mu (\alpha(x) *_\lambda z) +^{10} \beta(y) \prec_\mu (\alpha^{-1}\beta(z) *_{-\partial-\lambda} \alpha^2\beta^{-1}(x))\} \\
&+ \{^2-\alpha^{-1}\beta(y \prec_\mu z) *_{-\partial-\lambda} \alpha^2(x) -^4 \alpha^{-1}\beta(y \succ_\mu z) *_{-\partial-\lambda} \alpha^2(x) \\
&+^6 (\alpha^{-1}\beta(y) *_{-\partial-\lambda} \alpha(x)) \prec_{\lambda+\mu} \beta(z) +^8 (\alpha^{-1}\beta(y) *_{-\partial-\lambda} \alpha(x)) \succ_{\lambda+\mu} \beta(z)\} \\
&+ \{^3+\alpha\beta(x) *_\lambda (y \succ_\mu z) -^7 (\beta(x) *_\lambda y) \succ_{\lambda+\mu} \beta(z) \\
&-^{11} \beta(y) \succ_\mu (\alpha(x) *_\lambda z) +^{12} \beta(y) \succ_\mu (\alpha^{-1}\beta(z) *_{-\partial-\lambda} \alpha^2\beta^{-1}(x))\} \\
=&0 + 0 + 0 \\
=&0.
\end{aligned}
$$

The above result is obtained by using Equation (20). □

In the following, we introduce the conformal bimodule of noncommutative BiHom-pre-Poisson conformal algebras. Additionally, some relevant properties are also given.

Definition 14. *Let $(B, \prec_\lambda, \succ_\lambda, *_\lambda, \alpha, \beta)$ be a BiHom-pre-Poisson conformal algebra. A conformal bimodule of B is a 9-tuple $(l_\prec, r_\succ, l_\succ, r_\prec, l_*, r_*, \phi, \psi, M)$ such that $(l_*, r_*, \phi, \psi, M)$ is a conformal bimodule of the BiHom-pre-Lie conformal algebra $(B, *_\lambda, \alpha, \beta)$ and $(l_\prec, r_\prec, l_\succ, r_\succ, \phi, \psi, M)$ is a conformal bimodule of the BiHom-dendriform conformal algebra $(B, \prec_\lambda, \succ_\lambda, \alpha, \beta)$ satisfying*

$$l_{\prec}([\beta(x)_\lambda \alpha(y)])_{\lambda+\mu}\psi(m) = l_*(\alpha\beta(x))_\lambda(l_{\prec}(\alpha(y))_\mu m) - l_{\prec}(\alpha\beta(y))_\mu(l_*(\alpha(x))_\lambda m), \tag{21}$$

$$r_{\prec}(\beta(x))_\lambda(\rho(\beta(y))_\mu\phi(m)) = l_*(\alpha\beta(y))_\mu(r_{\prec}(x)_\lambda\phi(m)) - r_{\prec}(\alpha(y) *_\mu x)_{\lambda+\mu}\phi\psi(m), \tag{22}$$

$$-r_{\prec}(\beta(x))_\lambda(\rho(\beta(y))_\mu\phi(m)) = r_*(\alpha(y) \prec_\mu x)_{\lambda+\mu}\phi\psi(m) - l_{\prec}(\alpha\beta(y))_\mu(r_*(x)_\lambda(\phi(m))), \tag{23}$$

$$l_{\succ}(\beta(x))_\lambda(\rho(\alpha\beta(y))_\mu\phi(m)) = l_*(\alpha\beta^2(y))_\mu(l_{\succ}(x)_\lambda\phi(m)) - l_{\succ}(\beta^2(y) *_\mu x)_{\lambda+\mu}\phi\psi(m), \tag{24}$$

$$r_{\succ}([\alpha\beta(x)_\lambda\alpha(y)])_{\lambda+\mu}\psi(m) = l_*(\alpha\beta^2(x))_\lambda(r_{\succ}(\alpha(y))_\mu m) - r_{\succ}(\alpha\beta(y))_\mu(l_*(\beta^2(x))_\lambda m), \tag{25}$$

$$-l_{\succ}(\beta(x))_\lambda(\rho(\beta(y))_\mu\phi^2(m)) = r_*(x \succ_\lambda \alpha(y))_{\lambda+\mu}\phi\psi^2(m) - r_{\succ}(\alpha\beta(y))_\mu(r_*(x)_\lambda\psi^2(m)), \tag{26}$$

$$l_*(\beta(x) \cdot_\lambda \alpha(y))_{\lambda+\mu}\psi(m) = r_{\succ}(\beta(y))_\mu l_*(\beta(x))_\lambda\phi(m) + l_{\prec}(\alpha\beta(x))_\lambda l_*(\alpha(y))_\mu m, \tag{27}$$

$$r_*(\beta(x))_\lambda(l_\cdot(\beta(y))_\mu\phi(m)) = l_{\succ}(\beta(y) *_\mu \alpha(x))_{\lambda+\mu}\phi(m) + l_{\prec}(\alpha\beta(y))_\mu(r_*(x)_\lambda\phi(m)), \tag{28}$$

$$r_*(\beta(x))_\lambda r_\cdot(\alpha(y))_\mu\psi(m) = r_{\succ}(\beta(y))_\mu r_*(\alpha(x))_\lambda\psi(m) + r_{\prec}(\alpha(y) *_\mu x)_{\lambda+\mu}\phi\psi(m), \tag{29}$$

for all $x, y \in B$ and $m \in M$. Here,

$$x \cdot_\lambda y = x \prec_\lambda y + x \succ_\lambda y, l_\cdot = l_{\prec} + l_{\succ}, r_\cdot = r_{\prec} + r_{\succ},$$

$$[\beta(x)_\lambda\alpha(y)] = \beta(x) *_\lambda \alpha(y) - \beta(y) *_{-\partial-\lambda} \alpha(x),$$

$$(\rho \circ \beta)\phi = (l_* \circ \beta)\phi - (r_* \circ \alpha)\psi.$$

Proposition 13. *Let $(l_{\prec}, r_{\prec}, l_{\succ}, r_{\succ}, l_*, r_*, \phi, \psi, M)$ be a conformal bimodule of a BiHom-pre-Poisson conformal algebra $(B, \prec_\lambda, \succ_\lambda, *_\lambda, \alpha, \beta)$, where M is a $\mathbb{C}[\partial]$-module, and ϕ and ψ are $\mathbb{C}$-linear maps, satisfying $\partial\phi = \phi\partial$ and $\partial\psi = \psi\partial$. Then, the direct sum $B \oplus M$ is found to be a BiHom-pre-Poisson conformal algebra by defining the λ-multiplications $\prec'_\lambda$ $\succ'_\lambda$ and $*'_\lambda$ in $B \oplus M$ as follows:*

$$\begin{aligned}
(x_1 + m_1) \prec'_\lambda (x_2 + m_2) &:= x_1 \prec_\lambda x_2 + (l_{\prec}(x_1)_\lambda m_2 + r_{\prec}(x_2)_{-\partial-\lambda} m_1), \\
(x_1 + m_1) \succ'_\lambda (x_2 + m_2) &:= x_1 \succ_\lambda x_2 + (l_{\succ}(x_1)_\lambda m_2 + r_{\succ}(x_2)_{-\partial-\lambda} m_1), \\
(x_1 + m_1) *'_\lambda (x_2 + m_2) &:= x_1 *_\lambda x_2 + (l_*(x_1)_\lambda m_2 + r_*(x_2)_{-\partial-\lambda} m_1), \\
(\alpha \oplus \phi)(x + m) &:= \alpha(x) + \phi(m), \\
(\beta \oplus \psi)(x + m) &:= \beta(x) + \psi(m),
\end{aligned} \tag{30}$$

for all $x, x_1, x_2 \in B$; $m, m_1, m_2 \in M$; and $\lambda, \mu \in \mathbb{C}$.

We denote this BiHom-pre-Poisson conformal algebra by $(B \oplus M, \prec'_\lambda, \succ'_\lambda, *'_\lambda, \alpha + \phi, \beta + \psi)$, or simply $(B \ltimes_{l_{\prec}, r_{\prec} l_*, r_*, l_{\succ}, r_{\succ}, \alpha, \beta, \phi, \psi} M)$.

Proof. To show that $(B \oplus M, \prec'_\lambda, \succ'_\lambda, *'_\lambda, \alpha + \phi, \beta + \psi)$ is a BiHom-pre-Poisson conformal algebra, we need to satisfy the axioms given in Equation (20); for convenience, we only give the proof of the first axiom; the other can be proved likewise. For any $x, y, z \in B$, $m_1, m_2, m_3 \in M$ and $\lambda, \mu \in \mathbb{C}$, we have

$$\begin{aligned}
&((\beta + \psi)(x + m_1) *'_\lambda (\alpha + \beta)(y + m_2)) \prec'_{\lambda+\mu} (\beta + \psi)(z + m_3) \\
&- ((\beta + \psi)(y + m_2) *'_{-\partial-\lambda} (\alpha + \beta)(x + m_1)) \prec'_{\lambda+\mu} (\beta + \psi)(z + m_3) \\
=&((\beta(x) *_\lambda \alpha(y)) + l_*(\beta(x))_\lambda\phi(m_2) + r_*(\alpha(y)_{-\partial-\lambda}\psi(m_1))) \prec'_{\lambda+\mu} (\beta + \psi)(z + m_3) \\
&- (\beta(y) *_{-\partial-\lambda} \alpha(x) + l_*(\beta(y)_{-\partial-\lambda}\phi(m_1) + r_*(\alpha(x))_\lambda\psi(m_2)) \prec'_{\lambda+\mu} (\beta + \psi)(z + m_3) \\
=&(\beta(x) *_\lambda \alpha(y)) \prec'_{\lambda+\mu} (\beta(z) + \psi(m_3)) + (l_*(\beta(x))_\lambda\phi(m_2)) \prec'_{\lambda+\mu} (\beta(z) + \psi(m_3)) \\
&+ (r_*(\alpha(y)_{-\partial-\lambda}\psi(m_1))) \prec'_{\lambda+\mu} (\beta(z) + \psi(m_3)) \\
&- (\beta(y) *_{-\partial-\lambda} \alpha(x)) \prec'_{\lambda+\mu} (\beta(z) + \psi(m_3)) + (l_*(\beta(y)_{-\partial-\lambda}\phi(m_1)) \prec'_{\lambda+\mu} (\beta(z) + \psi(m_3)) \\
&+ (r_*(\alpha(x))_\lambda\psi(m_2))) \prec'_{\lambda+\mu} (\beta(z) + \psi(m_3))
\end{aligned}$$

$$=[\beta(x)_\lambda\alpha(y)]_{\lambda+\mu}\psi(m_3)+l_\prec([\beta(x)_\lambda\alpha(y)])_{\lambda+\mu}\psi(m_3)+r_\prec(\beta(z))_{-\partial-\lambda-\mu}(\rho(\beta(y))_\lambda\phi(m_1))$$
$$+r_\prec(\beta(z))_{-\partial-\lambda-\mu}(\rho(\alpha(x))_{-\partial-\lambda}\psi(m_2)).$$

On the other hand,

$$\begin{aligned}
&(\alpha\beta(x)+\phi\psi(m_1))*'_\lambda((\alpha(y)+\phi(m_2))\prec_\mu(z+m_3))\\
&-(\alpha\beta(y)+\phi\psi(m_2))\prec'_\mu((\alpha(x)+\phi(m_1))*'_\lambda(z+m_3))\\
=&(\alpha\beta(x)+\phi\psi(m_1))*'_\lambda(\alpha(y)\prec_\mu z+l_\prec(\alpha(y))_\mu m_3+r_\prec(z)_{-\partial-\mu}\phi(m_2)\\
&-(\alpha\beta(y)+\phi\psi(m_2))\prec'_\mu(\alpha(x)*_\lambda z+l_*(\alpha(x))_\lambda(m_3)+r_*(z)_{-\partial-\lambda}\phi(m_1))\\
=&\alpha\beta(x)*_\lambda(\alpha(y)\prec_\mu z)+l_*(\alpha\beta(x))_\lambda(l_\prec(\alpha(y))_\mu m_3)\\
&+l_*(\alpha\beta(x))_\lambda(r_\prec(z)_{-\partial-\mu}\phi(m_2))+r_*(\alpha(y)\prec_\mu z)_{-\partial-\lambda}\phi\psi(m_1)\\
&-\alpha\beta(y)\prec_\mu(\alpha(x)*_\lambda z)-l_\prec(\alpha\beta(y))_\mu(l_*(\alpha(x))_\lambda m_3)\\
&-l_\prec(\alpha\beta(y))_\mu(r_*(z)_{-\partial-\lambda}\phi(m_1))-r_\prec(\alpha(x)*_\lambda z)_{-\partial-\mu}\phi\psi(m_2).
\end{aligned}$$

Using the first three equations of Definition 14, Equation (20) and conformal sesqui-linearity, the proof is clear. However, the first, second, third and fourth terms of the second-to-last equality are equated to the pairs $\{1,5\}$, $\{2,6\}$, $\{3,8\}$ and $\{4,7\}$ in the last equality. □

Example 5. *Let $(B,\prec_\lambda,\succ_\lambda,*_\lambda,\alpha,\beta)$ be a noncommutative BiHom-pre-Poisson conformal algebra. A regular conformal bimodule of B is defined as the tuple $(L_\prec,R_\prec,L_\succ,R_\succ,L_*,R_*,\alpha,\beta,B)$, where $L_\prec(x)_\lambda y=x\prec_\lambda y$, $R_\prec(x)_\lambda y=y\prec_{-\partial-\lambda}x$, $L_\succ(x)_\lambda y=x\succ_\lambda y$, $R_\succ(x)_\lambda y=y\succ_{-\partial-\lambda}x$, $L_*(x)_\lambda y=x*_\lambda y$ and $R_*(x)_\lambda y=y*_{-\partial-\lambda}x$, for all $x,y\in B$, $\lambda\in\mathbb{C}$.*

Proposition 14. *Let $(B_1,\prec^1_\lambda,\succ^1_\lambda,\star^1_\lambda,\alpha,\beta)$ and $(B_2,\prec^2_\lambda,\succ^2_\lambda,\star^2_\lambda,\phi,\psi)$ be two noncommutative BiHom-pre-Poisson conformal algebras and f be the morphism between them. We observe that, by using f, we can form a conformal bimodule of B_1, represented as $(l^1_\prec,r^1_\prec,l^1_\succ,r^1_\succ,l^1_\star,r^1_\star,\phi,\psi,B_2)$ and defined by $l^1_\prec(x)_\lambda y=f(x)\prec^2_\lambda y$, $r^1_\prec(x)_\lambda y=y\prec^2_{-\partial-\lambda}f(x)$, $l^1_\succ(x)_\lambda y=f(x)\succ^2_\lambda y$, $r^1_\succ(x)_\lambda y=y\succ^2_{-\partial-\lambda}f(x)$ and $l^1_\star(x)_\lambda y=f(x)\star^2_\lambda y$, $r^1_\star(x)_\lambda y=y\star^2_{-\partial-\lambda}f(x)$ for all $(x,y)\in B_1\times B_2$ and $\lambda,\mu\in\mathbb{C}$.*

Proof. We need to show the axioms given in Definition 14. Here, we only prove the seventh axiom; the other axioms can be proved similarly. For any $x,y\in B_1$, $z\in B_2$ and $\lambda,\mu\in\mathbb{C}$, we have

$$\begin{aligned}
&l^1_*(\beta(x)\cdot^1_\lambda\alpha(y))_{\lambda+\mu}\psi(z)\\
&=f(\beta(x)\cdot^1_\lambda\alpha(y))*^2_{\lambda+\mu}\psi(z)\\
&=(\psi f(x)\cdot^2_\lambda\phi f(y))*^2_{\lambda+\mu}\psi(z)\\
&=(\psi f(x)\succ^2_\lambda\phi f(y)+\psi f(x)\prec^2_\lambda\phi f(y))*^2_{\lambda+\mu}\psi(z)\\
&=(\psi f(x)\succ^2_\lambda\phi f(y)-\psi f(y)\prec^2_{-\partial-\lambda}\phi f(x))*^2_{\lambda+\mu}\psi(z)\\
&=([\psi f(x)_\lambda\phi f(y)]_2)*^2_{\lambda+\mu}\psi(z)\\
&=f([\beta(x)_\lambda\alpha(y)]_2)*^2_{\lambda+\mu}\psi(z)\\
&=(\psi f(x)*^2_\lambda\phi(z))\succ^2_{\lambda+\mu}\psi f(y)+\phi\psi f(x)\prec^2_\lambda(\phi f(y)*^2_\mu z)\ \text{(by Equation (20))}\\
&=(f(\beta(x))*^2_\lambda\phi(z))\succ^2_{\lambda+\mu}f(\beta(y))+f(\alpha\beta(x))\prec^2_\lambda(f(\alpha(y))*^2_\mu z)\\
&=r^1_\succ(\beta(y))_\mu(f(\beta(x))*^2_\lambda\phi(z))+l^1_\prec(\alpha\beta(x))(f(\alpha(y))*^2_\mu z)\\
&=r^1_\succ(\beta(y))_\mu l^*_1(\beta(x))_\lambda\phi(z)+l^1_\prec(\alpha\beta(x))_\lambda l^1_*(\alpha(y))_\mu z.
\end{aligned}$$

This completes the proof. □

5. BiHom-Poisson Conformal Algebra and $\mathcal{O}$-Operators

In this section, we introduce the notion of an $\mathcal{O}$-operator acting on BiHom-Poisson conformal algebras, and we give some related properties. For this, we first recall the notion of an $\mathcal{O}$-operator acting on the BiHom-associative conformal algebra and the BiHom-Lie conformal algebra as follows.

Definition 15. *Consider that we have a BiHom-associative conformal algebra $(B, *_\lambda, \alpha, \beta)$ and a conformal bimodule (l, r, ϕ, ψ, M) over B. An $\mathcal{O}$-operator is a $\mathbb{C}[\partial]$-module homomorphism, $T : M \to B$, associated with (l, r, ϕ, ψ, M) if it satisfies the following axioms for all $m_1, m_2 \in M$ and $\lambda \in \mathbb{C}$:*

$$\begin{aligned} &\alpha T = T\phi, \quad \beta T = T\psi, \\ &T(m_1)_\lambda T(m_2) = T(l(T(m_1))_\lambda m_2 + r(T(m_2))_{-\partial-\lambda} m_1). \end{aligned}$$

Lemma 3. *If we have an $\mathcal{O}$-operator $T : M \to B$ on a BiHom-associative conformal algebra $(B, *_\lambda, \alpha, \beta)$, we can establish a BiHom-dendriform conformal algebra on the conformal bimodule (l, r, ϕ, ψ, M) given by*

$$m_1 \succ_\lambda m_2 = l(T(m_1))_\lambda m_2, \quad m_1 \prec_\lambda m_2 = r(T(m_2))_{-\partial-\lambda} m_1, \quad \textit{for all } m_1, m_2 \in M, \lambda \in \mathbb{C}.$$

Now, we review the notion of an $\mathcal{O}$-operator acting on a BiHom-Lie conformal algebra that is linked to the conformal representation. Note that these $\mathcal{O}$-operators are the generalization of Rota–Baxter operators of 0 weight.

Definition 16. *Let $(B, [\cdot_\lambda \cdot], \alpha, \beta)$ be a BiHom-Lie conformal algebra and (ρ, ϕ, ψ, M) be its conformal representation. In this context, an $\mathcal{O}$-operator associated with (ρ, ϕ, ψ, M) is a $\mathbb{C}[\partial]$-module mapping $T : M \to B$ that incorporates the following conditions for all $m_1, m_2 \in M, \lambda \in \mathbb{C}$:*

$$\begin{aligned} &\alpha T = T\phi, \ \beta T = T\psi, \\ &[T(m_1)_\lambda T(m_2)] = T(\rho(T(m_1))_\lambda m_2 + \rho(T(\phi^{-1}\psi(m_2)))_{-\partial-\lambda} \phi\psi^{-1}(m_1)). \end{aligned}$$

Lemma 4. *If we have an $\mathcal{O}$-operator $T : M \to B$ acting on a BiHom-Lie conformal algebra concerning the conformal representation (ρ, ϕ, ψ, M), we can generate a BiHom-pre-Lie conformal algebra through the following conformal multiplication, $*_\lambda : M \otimes M \to M[\lambda]$, defined by*

$$m_1 *_\lambda m_2 = \rho(T(m_1))_\lambda m_2, \quad \forall\, m_1, m_2 \in M, \lambda \in \mathbb{C}.$$

We denote this BiHom-pre-Lie conformal algebra as $(M, *_\lambda, \alpha, \beta)$.

Definition 17. *A $\mathbb{C}[\partial]$-module homomorophism $T : M \to B$ is called an $\mathcal{O}$-operator acting on a BiHom-Poisson conformal algebra $(B, *_\lambda, [\cdot_\lambda \cdot], \alpha, \beta)$ with respect to the conformal representation $(l, r, \rho, \phi, \psi, M)$ if T is an $\mathcal{O}$-operator acting on both $(B, *_\lambda, \alpha, \beta)$, the BiHom-associative conformal algebra, and $(B, [\cdot_\lambda \cdot], \alpha, \beta)$, the BiHom-Lie conformal algebra.*

Example 6. *A Rota–Baxter operator acting on a noncommutative BiHom-Poisson conformal algebra $(B, *_\lambda, [\cdot_\lambda \cdot], \alpha, \beta)$ with respect to the regular representation is defined as an $\mathcal{O}$-operator acting on B.*

Theorem 4. *Consider a BiHom-Poisson conformal algebra $(B, *_\lambda, [\cdot_\lambda \cdot], \alpha, \beta)$ and an $\mathcal{O}$-operator $T : M \to B$ acting on B with respect to the conformal representation $(l, r, \rho, \phi, \psi, M)$. Note that $(M, \prec_\lambda, \succ_\lambda, *_\lambda, \alpha, \beta)$ becomes a BiHom-pre-Poisson conformal algebra by defining the new operations $\succ_\lambda$, $\prec_\lambda$ and $*_\lambda$ acting on M given by*

$$m_1 *_\lambda m_2 = \rho(T(m_1))_\lambda m_2, \quad m_1 \succ_\lambda m_2 = r(T(m_2))_{-\partial-\lambda} m_1, \quad m_1 \prec_\lambda m_2 = l(T(m_1))_\lambda m_2.$$

Furthermore, we have that $T(M) = \{T(m); \quad m \in M\} \subset B$ *forms a subalgebra of B. And there exists an induced BiHom-pre-Poisson conformal algebra structure on* $T(M)$*, given by*

$$\begin{aligned} T(m_1) *_\lambda T(m_2) &= T(m_1 *_\lambda m_2), \\ T(m_1) \prec_\lambda T(m_2) &= T(m_1 \prec_\lambda m_2, \\ T(m_1) \succ_\lambda T(m_2) &= T(m_1 \succ_\lambda m_2), \end{aligned}$$

for all $m_1, m_2 \in M$ *and* $\lambda \in \mathbb{C}$ *.*

Proof. Both Lemma 3 and Lemma 4 imply that $(B, \prec_\lambda, \succ_\lambda, \alpha, \beta)$ is a BiHom-dendriform conformal algebra and $(B, *_\lambda, \alpha, \beta)$ is a BiHom-pre-Lie conformal algebra. In this proof, we focus on demonstrating the first axiom of Equation (20) while noting that the remaining axioms can be proven similarly. Let us consider $x, y, z \in M$, $\lambda, \mu \in \mathbb{C}$.

$$\begin{aligned} &(\psi(x) *_\lambda \phi(y) - \psi(y) *_{-\partial-\lambda} \phi(x)) \prec_{\lambda+\mu} \psi(z) \\ &- \phi\psi(x) *_\lambda (\phi(y) \prec_\mu z) + \phi\psi(y) \prec_\mu (\phi(x) *_\lambda z) \\ =&(\rho(T(\psi(x))_\lambda \phi(y)) - \rho(T(\psi(y))_{-\partial-\lambda}\phi(x))) \prec_{\lambda+\mu} \psi(z) \\ &- \rho(T(\phi\psi(x)))_\lambda(\phi(y) \prec_\mu z) + l(T(\phi\psi(y)))_\mu(\phi(x) *_\lambda z) \\ =&l(T(\rho(T(\psi(x)))_\lambda \phi(y) - \rho(T(\psi(y)))_{-\partial-\lambda}\phi(x)))_{\lambda+\mu}\psi(z) \\ &- \rho(T(\phi\psi(x)))_\lambda l(T(\phi(y)))_\mu z + l(T(\phi\psi(y)))_\mu(\rho(\phi(x))_\lambda z) \\ =&l([T(\psi(x))_\lambda T(\phi(y))])_{\lambda+\mu}\psi(z) - \rho(T(\phi\psi(x)))_\lambda l(T(\phi(y)))_\mu z \\ &+ l(T(\phi\psi(y)))_\mu(\rho(\phi(x))_\lambda z) \\ =&0. \end{aligned}$$

The above expressions are obtained by using Equation (8). Therefore, $(M, \prec_\lambda, \succ_\lambda, *_\lambda, \alpha, \beta)$ is a BiHom-pre-Poisson conformal algebra. The remaining part of this proof is fairly intuitive. □

Corollary 4. *Consider a BiHom-Poisson conformal algebra* $(B, *_\lambda, [\cdot_\lambda \cdot], \alpha, \beta)$*. In this case, there exists a BiHom-pre-Poisson conformal algebra structure on B in such a way that its underlying BiHom-Poisson conformal algebra is exactly a BiHom-Poisson conformal algebra* $(B, *_\lambda, [\cdot_\lambda \cdot], \alpha, \beta)$ *if there exists an invertible* $\mathcal{O}$*-operator acting on* $(B, *_\lambda, [\cdot_\lambda \cdot], \alpha, \beta)$*.*

Proof. Suppose there exists a bijective $\mathcal{O}$-operator $T : M \to B$ associated with the conformal representation $(l, r, \rho, \phi, \psi, M)$. Then, for all $x, y \in B$, the compatible BiHom-pre-Poisson conformal algebra structure on B is defined as follows:

$$\begin{aligned} x \prec_\lambda y &= T(l(x)_\lambda T^{-1}(y)), \\ x \succ_\lambda y &= T(r(y)_{-\partial-\lambda} T^{-1}(x)), \\ x *_\lambda y &= T(\rho(x)_\lambda T^{-1}(y)). \end{aligned}$$

Conversely, if $(B, \prec_\lambda, \succ_\lambda, *_\lambda, \alpha, \beta)$ is a BiHom-pre-Poisson conformal algebra and $(B, *_\lambda, [\cdot_\lambda \cdot], \alpha, \beta)$ is the underlying BiHom-Poisson conformal algebra, then the identity map *id* is an $\mathcal{O}$-operator acting on B with respect to the regular conformal representation $(L_\prec, R_\succ, ad, \alpha, \beta, B)$. □

Example 7. *Consider a noncommutative BiHom-Poisson conformal algebra* $(B, *_\lambda, [\cdot_\lambda \cdot], \alpha, \beta)$ *and a Rota–Baxter operator* $R : B \to B$ *acting on it. By defining the new operations* $\prec_\lambda, \succ_\lambda$*, and* $\cdot_\lambda$ *acting on B, we obtain a BiHom-pre-Poisson conformal algebra* $(B, \prec_\lambda, \succ_\lambda, \cdot_\lambda, \alpha, \beta)$ *defined by*

$$x \cdot_\lambda y = [R(x)_\lambda y], x \prec_\lambda y = R(x) *_\lambda y, x \succ_\lambda y = x *_\lambda R(y).$$

In this case, R acts as a homomorphism between the subadjacent BiHom-Poisson conformal algebra $(B, \cdot'_\lambda, [\cdot_\lambda \cdot]', \alpha, \beta)$ *and the BiHom-Poisson conformal algebra* $(B, \cdot_\lambda, [\cdot_\lambda \cdot], \alpha, \beta)$*, where*

$$[x_\lambda y]' = x *_\lambda y - \alpha^{-1}\beta(y) *_{-\partial-\lambda} \alpha\beta^{-1}(x) \quad \text{and} \quad x \cdot'_\lambda y = x \prec_\lambda y + x \succ_\lambda y.$$

6. Conclusions

In the present study, we have established a broad framework for BiHom-Poisson conformal algebras and BiHom-pre-Poisson conformal algebras, highlighting their structural properties and interrelationships. We showcased that numerous new BiHom-Poisson conformal algebras can be established from existing ones, specifically highlighting that the direct product of two BiHom-Poisson conformal algebras preserves the BiHom-Poisson structure. Furthermore, we thoroughly investigated the representation theory of associated algebras, depicting their applicability in broader mathematical contexts.

The introduction of the BiHom-pre-Poisson conformal algebra, defined through the compatibility of BiHom-pre-Lie conformal algebras and BiHom-dendriform conformal algebras, enhances the framework of conformal algebra theory. Our findings reveal a clear pathway for developing BiHom-Poisson conformal algebras from their pre-Poisson counterparts, thereby enhancing existing theories with new connections and insights. Additionally, we have examined the role of $\mathcal{O}$-operators and Rota–Baxter operators within the context of BiHom-Poisson conformal algebras, revealing the significance of these operators in understanding the algebraic structures involved. By addressing existing gaps in the literature and providing a detailed analysis of these algebras, our work contributes to the ongoing dialogue in conformal algebra research and opens avenues for future exploration. Moreover, knowledge of these structures is important to further study the cohomology and deformation of these algebras.

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