symmetry

MDPI

*Article*

# Improved Fractional Differences with Kernels of Delta Mittag–Leffler and Exponential Functions

**Miguel Vivas-Cortez [1], Pshtiwan Othman Mohammed [2,*], Juan L. G. Guirao [3], Majeed A. Yousif [4], Ibrahim S. Ibrahim [4] and Nejmeddine Chorfi [5]**

[1] Faculty of Exact and Natural Sciences, School of Physical Sciences and Mathematics, Pontifical Catholic University of Ecuador, Av. 12 de Octubre 1076 y Roca, Quito 17-01-2184, Ecuador
[2] Department of Mathematics, College of Education, University of Sulaimani, Sulaimani 46001, Iraq
[3] Department of Applied Mathematics and Statistics, Technical University of Cartagena, Hospital de Marina, 30203 Cartagena, Spain
[4] Department of Mathematics, College of Education, University of Zakho, Zakho 42002, Iraq; majeed.yousif@uoz.edu.krd (M.A.Y.); ibrahim.ibrahim@uoz.edu.krd (I.S.I.)
[5] Department of Mathematics, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia; nchorfi@ksu.edu.sa
* Correspondence: pshtiwansangawi@gmail.com

**Abstract:** Special functions have been widely used in fractional calculus, particularly for addressing the symmetric behavior of the function. This paper provides improved delta Mittag–Leffler and exponential functions to establish new types of fractional difference operators in the setting of Riemann–Liouville and Liouville–Caputo. We give some properties of these discrete functions and use them as the kernel of the new fractional operators. In detail, we propose the construction of the new fractional sums and differences. We also find the Laplace transform of them. Finally, the relationship between the Riemann–Liouville and Liouville–Caputo operators are examined to verify the feasibility and effectiveness of the new fractional operators.

**Keywords:** Mittag–Leffler function; Riemann–Liouville and Liouville–Caputo operators; CF and AB fractional operators

**MSC:** 26A48; 39B62; 33B10; 39A12



**Citation:** Vivas-Cortez, M.; Mohammed, P.O.; Guirao, J.L.G.; Yousif, M.A.; Ibrahim, I.S.; Chorfi, N. Improved Fractional Differences with Kernels of Delta Mittag–Leffler and Exponential Functions. *Symmetry* **2024**, *16*, 1562. https://doi.org/10.3390/sym16121562

Academic Editor: Jaume Giné

Received: 25 October 2024
Revised: 8 November 2024
Accepted: 13 November 2024
Published: 21 November 2024



## 1. Introduction

The discrete fractional operator theory in fractional calculus settings plays an important role in solving difference equations. Fractional difference models are a core issue in mathematical and numerical analysis. It is also widely applied in many fields, including dynamical systems, artificial intelligence, computer communication, compressed sensing and so on, e.g., see [1–3].

In particular, mathematical and physical models of continuous and discrete fractional calculus have become important tools for analyzing differential and difference equations [4–7]. However, many researchers have proposed and studied the existence and uniqueness of discrete fractional models in different systems [8–12].

Various fractional operators have been used worldwide to study mathematical models in the presence of continuous fractional differential equations, for example [13–16], and discrete fractional difference equations [17–20] for nabla and delta types in the sense of Riemann–Liouville (RL), Liouville–Caputo (LC), Caputo–Fabrizio (CF) and Attangana–Baleanu (AB) operators.

Discrete fractional calculus has several correlations with special functions. An important application of discrete fractional calculus in applied mathematics is proving new identities and relationships between special functions. Specifically, enormous efforts have been made to accurately calculate special functions, which can be challenging for researchers who model real-world problems in discrete times. A function with particularly strong ties to discrete fractional calculus is the discrete Mittag–Leffler (ML) function—see, e.g., [21–25]. Furthermore, different authors have developed the problem of discrete fractional calculus in the time scale $\mathbb{Z}$ to the time scale $\mathsf{h}\mathbb{Z}$, see [26–28].

This study proposes discrete $\Delta_{\mathsf{h}}$-CF and $\Delta_{\mathsf{h}}$-AB fractional operators, including delta exponential and ML functions, in their kernels. The key contributions of the proposed research article are summarized as follows:

- A new delta exponential and ML function with some special cases are proposed in the context of $\mathsf{h}$-discrete fractional calculus.
- Some properties of the proposed delta ML functions have been addressed by utilizing Laplace transformations and sum/difference rules.
- The concept of the $\Delta_{\mathsf{h}}$-CF and $\Delta_{\mathsf{h}}$-AB fractional sums and differences are introduced based on the new delta exponential and Mittag–Leffler functions.
- In conclusion, some properties of the $\Delta_{\mathsf{h}}$-CF and $\Delta_{\mathsf{h}}$-AB fractional difference operators are provided based on the Laplace transformation on the left and right sides.

In addition, the article has the following sections: in Section 2, essential definitions in discrete fractional calculus and Laplace transformation with some of its properties are recalled. In Section 3, we formulate the new $\mathsf{h}$-ML functions; we divide this section into two subsections: $\Delta_{\mathsf{h}}$-CF fractional operators in Section 3.1 and $\Delta_{\mathsf{h}}$-AB fractional operators in Section 3.2. Finally, some discussions finish this article in Section 4.

## 2. Preliminaries

Let $\alpha, \mathsf{h} > 0$. Then, we consider the notations $\mathbb{N}_{\xi_0,\mathsf{h}} := \xi_0 + \mathsf{h}\,\mathbb{N}$ and ${}_{\xi,\mathsf{h}}\mathbb{N} := \xi - \mathsf{h}\,\mathbb{N}$, for $\xi_0, \xi \in \mathbb{R}$. Furthermore, let $\mathbb{T}_{\mathsf{h}} := \mathbb{N}^{\mathsf{h}}_{\xi_0;\xi} = \{\xi_0, \xi_0 + \mathsf{h}, \ldots, \xi\}$ such that $\xi = \xi_0 + k\mathsf{h}$, for some $k \in \mathbb{N}$. From Definition 2.25 in [1], we recall the definition of $\Delta_{\mathsf{h}}$ fractional sums as follows:

$$\left({}_{\xi_0}\Delta_{\mathsf{h}}^{-\alpha} v\right)(\tau) = \sum_{\ell=\frac{\xi_0}{\mathsf{h}}}^{\frac{\tau}{\mathsf{h}}-\alpha} \wp_{\alpha-1,\mathsf{h}}(\tau, \sigma_{\mathsf{h}}(\ell)) v(\ell), \quad \tau \in \mathbb{N}_{\xi_0+\alpha\,\mathsf{h},\mathsf{h}}, \tag{1}$$

where $v : \mathbb{N}_{\xi_0,\mathsf{h}} \to \mathbb{R}$, $\sigma_{\mathsf{h}}(\tau) := \tau + \mathsf{h}$ and

$$\wp_{\alpha,\mathsf{h}}(\tau, \ell) = \frac{(\tau-\ell)_{\mathsf{h}}^{(\alpha)}}{\Gamma(\alpha+1)} \quad \text{with} \quad (\tau-\ell)_{\mathsf{h}}^{(\alpha)} = \frac{\Gamma\left(\frac{\tau}{\mathsf{h}} - \ell + 1\right)}{\Gamma\left(\frac{\tau}{\mathsf{h}} - \ell + 1 - \alpha\right)}, \tag{2}$$

whenever $\frac{\Gamma\left(\frac{\tau}{\mathsf{h}}-\ell+1\right)}{\Gamma\left(\frac{\tau}{\mathsf{h}}-\ell+1-\alpha\right)}$ is well-defined for $\tau, \alpha \in \mathbb{R}$.

**Lemma 1** (Theorem 2.50 in [1])**.** *If $v$ is defined on $\mathbb{N}_{\xi_0,\mathsf{h}}$, then for $\alpha > 0$, it can be expressed that*

$$\left({}_{\xi_0}\Delta_{\mathsf{h}}^{-\alpha}\Delta_{\mathsf{h}}\, v\right)(\tau) = \left(\Delta_{\mathsf{h}}\, {}_{\xi_0}\Delta_{\mathsf{h}}^{-\alpha} v\right)(\tau) - \frac{(\tau-\xi_0)_{\mathsf{h}}^{(\alpha-1)}}{\Gamma(\alpha)} v(\xi_0),$$

*for $\tau \in \mathbb{N}_{\xi_0+\alpha\mathsf{h},\mathsf{h}}$.*

**Remark 1.** *For each $\alpha \in \mathbb{R}$ and $\mathsf{h} > 0$, we have*

*(i)* $\Delta_{\mathsf{h}}\left(\tau_{\mathsf{h}}^{(\alpha)}\right) = \alpha\, \tau_{\mathsf{h}}^{(\alpha-1)}$.

*(ii)* ${}_{\xi_0+\mu\mathsf{h}}\Delta_{\mathsf{h}}^{-\alpha}\, (\tau-\xi_0)_{\mathsf{h}}^{(\mu)} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+1+\alpha)} (\tau-\xi_0)_{\mathsf{h}}^{(\alpha+\mu)}$, *for $\alpha > 0$, $\mu \geq 0$ and $\tau \in \mathbb{N}_{\xi_0+(\alpha+\mu)\mathsf{h},\mathsf{h}}$.*

**Lemma 2** ([26,27])**.** *Let $\alpha > 0$. Then, for $\tau \in \mathbb{T}_{\mathsf{h}}$, we have*

$$\lim_{\alpha\to 0}\left({}_{\xi_0}\Delta_{\mathsf{h}}^{-\alpha}v\right)(\tau+\alpha\mathsf{h})=v(\tau),\quad \lim_{\alpha\to 0}\left({}_{\mathsf{h}}\Delta_{\xi}^{-\alpha}v\right)(\tau-\alpha\mathsf{h})=v(\tau).$$

The following definitions are given in [26].

**Definition 1.** *For the function $v$ defined on $\mathbb{N}_{\xi_0,\mathsf{h}}$ and ${}_{\xi,\mathsf{h}}\mathbb{N}$, the h-RL fractional difference on the left side is defined by*

$$\left({}_{\xi_0}\Delta_{\mathsf{h}}^{\alpha}v\right)(\tau)=\left(\Delta_{\mathsf{h}}^{\imath}\,{}_{\xi_0}\Delta_{\mathsf{h}}^{-(\imath-\alpha)}v\right)(\tau),\quad \tau\in\mathbb{N}_{\xi_0+(\imath-\alpha)h,\mathsf{h}},$$

*and on the right side by*

$$\left({}_{\mathsf{h}}\Delta_{\xi}^{\alpha}v\right)(\tau)=(-1)^{\imath}\left(\Delta_{\mathsf{h}}^{\imath}\,{}_{\mathsf{h}}\Delta_{\xi}^{-(\imath-\alpha)}v\right)(\tau),\quad \tau\in{}_{\xi-(\imath-\alpha)\mathsf{h},\mathsf{h}}\mathbb{N},$$

*where $\imath=[\alpha]+1$.*

**Definition 2.** *For a function $v$ defined on $\mathbb{N}_{\xi_0,\mathsf{h}}$, the h-LC fractional difference on the left side is defined by*

$$\left({}_{\xi_0}^{LC}\Delta_{\mathsf{h}}^{\alpha}v\right)(\tau)=\left({}_{\xi_0}\Delta_{\mathsf{h}}^{-(\imath-\alpha)}\,\Delta_{\mathsf{h}}^{\imath}v\right)(\tau),\quad \tau\in\mathbb{N}_{\xi_0+(\imath-\alpha)\mathsf{h},\mathsf{h}},$$

*and on the right side by*

$$\left({}_{\mathsf{h}}^{LC}\Delta_{\xi}^{\alpha}v\right)(\tau)=\left({}_{\mathsf{h}}\Delta_{\xi}^{-(\imath-\alpha)}(-1)^{\imath}\Delta_{\mathsf{h}}^{\imath}v\right)(\tau),\quad \tau\in{}_{\xi-(\imath-\alpha)\mathsf{h},\mathsf{h}}\mathbb{N},$$

*where $\imath=[\alpha]+1$.*

The operator of Q is defined by

$$(\mathrm{Q}v)(\tau)=v(\xi_0+\xi-\tau),$$

and was used in [29,30] to obtain the right sides of fractional sum and difference operators by knowing the left sides of these operators without conducting their proofs.

**Lemma 3** ([26])**.** *Let $\alpha,\mathsf{h}>0$ and $v$ be defined on $\mathbb{T}_{\mathsf{h}}$. Then,*

*(i)* $\left({}_{\xi_0}\Delta_{\mathsf{h}}^{-\alpha}\mathrm{Q}v\right)(\tau)=\mathrm{Q}\left({}_{\mathsf{h}}\Delta_{\xi}^{-\alpha}v\right)(\tau)=\left({}_{\mathsf{h}}\Delta_{\xi}^{-\alpha}v\right)v(\xi_0+\xi-\tau).$

*(ii)* $\left({}_{\xi_0}\Delta_{\mathsf{h}}^{\alpha}\mathrm{Q}v\right)(\tau)=\mathrm{Q}\left({}_{\mathsf{h}}\Delta_{\xi}^{\alpha}v\right)(\tau)=\left({}_{\mathsf{h}}\Delta_{\xi}^{\alpha}v\right)v(\xi_0+\xi-\tau).$

*(iii)* $\left({}_{\xi_0}^{LC}\Delta_{\mathsf{h}}^{\alpha}\mathrm{Q}v\right)(\tau)=\mathrm{Q}\left({}_{\mathsf{h}}^{LC}\Delta_{\xi}^{\alpha}v\right)(\tau)=\left({}_{\mathsf{h}}^{LC}\Delta_{\xi}^{\alpha}v\right)v(\xi_0+\xi-\tau).$

According to Theorem 2.2 in the monograph [1] and Definition 14 in [26], we can introduce the following definition of the $\Delta_{\mathsf{h}}$-Laplace transformation.

**Definition 3.** *For the function $v$ defined on $\mathbb{N}_{\xi_0,\mathsf{h}}$, the $\Delta_{\mathsf{h}}$-Laplace transformation of $v$ is defined by*

$$\mathcal{L}_{\xi_0,\mathsf{h}}\{v(\tau)\}(s)=\int_{\xi_0}^{\infty}\frac{{}_{\mathsf{h}}\tilde{e}_{\ominus s}(\tau,\xi_0)}{\mathsf{h}\,s+1}v(\tau)\Delta_{\mathsf{h}}\tau=\mathsf{h}\sum_{k=0}^{\infty}\frac{v(\xi_0+k\mathsf{h})}{(\mathsf{h}\,s+1)^{k+1}},$$

*where ${}_{\mathsf{h}}\tilde{e}_{\ominus s}$ is defined later in Section 3.1.*

**Definition 4.** *The $\Delta_{\mathsf{h}}$-convolution of two functions $v$ and $g$ defined on $\mathbb{N}_{\xi_0,\mathsf{h}}$ is given by*

$$(v * g)(\tau) = \int_{\xi_0}^{\tau} g(\tau - \sigma(s) + \xi_0))v(s)\Delta_{\mathsf{h}} s = \mathsf{h} \sum_{k=0}^{\infty} v(k\mathsf{h})g(\tau - \sigma(k\mathsf{h}) + \xi_0)). \tag{3}$$

*Furthermore, the Laplace convolution of these functions can be expressed by*

$$\mathcal{L}_{\xi_0,\mathsf{h}}\{(v * g)(\tau)\}(s) = \mathcal{L}\{v(\tau)\}(s)\,\mathcal{L}\{g(\tau)\}(s), \tag{4}$$

*for* $s \in \mathbb{R}$ *and* $\tau \in \mathbb{N}_{\xi_0,\mathsf{h}}$.

**Lemma 4.** *For* $v$ *on* $\mathbb{N}_{\xi_0,\mathsf{h}}$, *we have*

$$\mathcal{L}_{\xi_0,\mathsf{h}}\{\Delta_{\mathsf{h}} v(\tau)\}(s) = s\mathcal{L}\{v(\tau)\}(s) - v(\xi_0).$$

**Lemma 5** ([1]). *Let* $\xi_0, \mathsf{h} \geq 0$ $v$ *and* $\alpha \in \mathbb{R} - \{-1, -2, \ldots\}$. *Then, for* $\left|\mathsf{h}\, s + 1\right| < 1$, *it can be expressed that*

$$\mathcal{L}_{\xi_0+\mathsf{h}\alpha,\mathsf{h}}\{\,_{\mathsf{h}}\tilde{H}_{\alpha}(\tau, \xi_0)\}(s) = \frac{(\mathsf{h}\, s + 1)^{\alpha}}{s^{\alpha+1}},$$

*where* $_{\mathsf{h}}\tilde{H}_{\alpha}(\tau, \xi_0) = \frac{(\tau-\xi_0)_{\mathsf{h}}^{(\alpha)}}{\Gamma(\alpha+1)}$.

**Lemma 6** ([1]). *Let* $M \in \mathbb{N}_0$ *and* $v : \mathbb{N}_{\xi_0 - M\mathsf{h},\mathsf{h}} \to \mathbb{R}$ *and* $g : \mathbb{N}_{\xi_0,\mathsf{h}} \to \mathbb{R}$ *be two functions of exponential order* $\ell > 0$. *Then, for* $\left|\mathsf{h}\, s + 1\right| > \ell$, *it can be expressed that*

$$\mathcal{L}_{\xi_0 - M\mathsf{h}\alpha,\mathsf{h}}\{v(\tau)\}(s) = \frac{1}{(\mathsf{h}\, s + 1)^M}\mathcal{L}_{\xi_0,\mathsf{h}}\{v(\tau)\}(s) + \mathsf{h}\sum_{j=0}^{M-1} \frac{v(\xi_0 + (j - M)\mathsf{h})}{(\mathsf{h}\, s + 1)^{j+1}}, \tag{5}$$

*and*

$$\mathcal{L}_{\xi_0 + M\mathsf{h}\alpha,\mathsf{h}}\{g(\tau)\}(s) = (\mathsf{h}\, s + 1)^M \mathcal{L}_{\xi_0,\mathsf{h}}\{g(\tau)\}(s) - \mathsf{h}\sum_{j=0}^{M-1} (\mathsf{h}\, s + 1)^{M-j-1} g(\xi_0 + j\mathsf{h}). \tag{6}$$

## 3. Delta ML Functions and Related Operators

In this part of our article, we state the delta h-ML function and derive the CF and AB fractional differences and sums including delta h-ML function in their kernel.

**Definition 5.** *For any complex numbers* $\alpha, \mu\, \gamma, \lambda$ *with* $() > 0$ *and* $\left|\lambda \mathsf{h}^{\alpha}\right| < 1$, *we express the delta* h-ML *function as follows:*

$$_{\mathsf{h}}E^{\gamma}_{(\alpha,\mu)}(\lambda, \tau) = \sum_{\imath=0}^{\infty} \lambda^{\imath} \frac{(\tau + \imath(\alpha - 1) + (\mu - 1)\mathsf{h})_{\mathsf{h}}^{(\imath\alpha+\mu-1)}(\gamma)_{\imath}}{\Gamma(\alpha\,\imath + \mu)}. \tag{7}$$

*From* (7), *the following special cases can be observed:*

- *If* $\gamma = 1$, *then* (7) *becomes*

$$_{\mathsf{h}}E_{(\alpha,\mu)}(\lambda, \tau) = \sum_{\imath=0}^{\infty} \lambda^{\imath} \frac{(\tau + \imath(\alpha - 1)\mathsf{h} + (\mu - 1)\mathsf{h})_{\mathsf{h}}^{(\imath\alpha+\mu-1)}}{\Gamma(\alpha\,\tau + \mu)}. \tag{8}$$

- *If* $\gamma = \mu = 1$, *then* (7) *becomes*

$$_{\mathsf{h}}E_{(\alpha)}(\lambda, \tau) = \sum_{\imath=0}^{\infty} \lambda^{\imath} \frac{(\tau + \imath(\alpha - 1)\mathsf{h})_{\mathsf{h}}^{(\imath\alpha)}}{\Gamma(\alpha\,\imath + 1)}. \tag{9}$$

For the main concepts about the discrete Mittag–Liffler functions and their properties, we refer the reader to [21–27].

**Remark 2.** *One of the motivations make us to work on* $\mathsf{h}\mathbb{Z}$ *is that a smaller* $\mathsf{h} \in (0,1)$ *allows us to use a larger interval of* $\lambda$*. For example, if* $\mathsf{h} = 1$*, we need* $0 < \alpha < \frac{1}{2}$ *to guarantee convergence as* $\lambda = -\frac{1}{1-\alpha}$*. However, as* $\mathsf{h} \to 0$*, we can attain that the series will be convergent for* $0 < \alpha < 1$*.*

### *3.1. Delta Fractional Differences with Exponential Kernels*

Let us denote the $\Delta_{\mathsf{h}}$ exponential kernel by

$$ {}_{\mathsf{h}}\tilde{e}_{\lambda}(\tau,\sigma(\ell)) = (1+\lambda\mathsf{h})^{\frac{\tau-\sigma(\ell)}{\mathsf{h}}} = \left(\frac{1-\alpha-\alpha\mathsf{h}}{1-\alpha}\right)^{\frac{\tau-\sigma(r)}{\mathsf{h}}}, $$

where $\lambda = -\frac{\alpha}{1-\alpha}$ with $|\lambda\mathsf{h}^{\alpha}| < 1$. In addition, for $\ominus s = \frac{1+\mathsf{h}\,s}{s}$, it follows that

$$ {}_{\mathsf{h}}\tilde{e}_{\ominus s}(\tau,\xi_0) = \left(\frac{1}{1+\mathsf{h}\,s}\right)^{\frac{\tau-\xi_0}{\mathsf{h}}}, \quad \tau \in \mathbb{N}_{\xi_0,\mathsf{h}}. \tag{10} $$

**Definition 6.** *For* $0 < \alpha < 1$ *and with* $v$ *defined on* $\mathbb{N}_{\xi_0,\mathsf{h}}$ *in the left case and* ${}_{\xi,\mathsf{h}}\mathbb{N}$ *in the right case. Let* $\mathscr{D}(\alpha)$ *be the normalization, as defined in [26]. Then, we define the following:*

*(a)* *The* $\Delta_{\mathsf{h}}$-CF *of Liouville–Caputo type on the left side by*

$$ \begin{aligned} \left({}^{CFLC}_{\xi_0}\Delta^{\alpha}_{\mathsf{h}}v\right)(\tau) &= \frac{\mathscr{D}(\alpha)}{1-\alpha}\sum_{\ell=\frac{\xi_0}{\mathsf{h}}}^{\frac{\tau}{\mathsf{h}}-1}\mathsf{h}(\Delta_{\mathsf{h}}v)(\ell\mathsf{h})(1+\lambda\mathsf{h})^{\frac{\tau-\sigma(r)}{\mathsf{h}}} \\ &= \frac{\mathscr{D}(\alpha)}{1-\alpha}\sum_{\ell=\frac{\xi_0}{\mathsf{h}}}^{\frac{\tau}{\mathsf{h}}-1}\mathsf{h}(\Delta_{\mathsf{h}}v)(\ell\mathsf{h})\left(\frac{1-\alpha-\alpha\mathsf{h}}{1-\alpha}\right)^{\frac{\tau-\sigma(r)}{\mathsf{h}}}, \end{aligned} \tag{11} $$

*for* $\tau \in \mathbb{N}_{\xi_0+\mathsf{h},\mathsf{h}}$*.*

*(b)* *The* $\Delta_{\mathsf{h}}$-CF *of Liouville–Caputo type on the right side by*

$$ \begin{aligned} \left({}^{CFLC}_{\mathsf{h}}\Delta^{\alpha}_{\xi}v\right)(\tau) &= \frac{\mathscr{D}(\alpha)}{1-\alpha}\sum_{\ell=\frac{\tau}{\mathsf{h}}+1}^{\frac{\xi}{\mathsf{h}}}\mathsf{h}(-\nabla_{\mathsf{h}}v)(\ell\mathsf{h})(1+\lambda\mathsf{h})^{\frac{\ell\mathsf{h}-\sigma(\tau)}{\mathsf{h}}} \\ &= \frac{\mathscr{D}(\alpha)}{1-\alpha}\sum_{\ell=\frac{\tau}{\mathsf{h}}+1}^{\frac{\xi}{\mathsf{h}}}\mathsf{h}(-\nabla_{\mathsf{h}}v)(\ell\mathsf{h})\left(\frac{1-\alpha-\alpha\mathsf{h}}{1-\alpha}\right)^{\frac{\ell\mathsf{h}-\sigma(\tau)}{\mathsf{h}}}, \end{aligned} \tag{12} $$

*for* $\tau \in {}_{\xi-\mathsf{h},\mathsf{h}}\mathbb{N}$*.*

*(c)* *The* $\Delta_{\mathsf{h}}$-CF *of Riemann–Liouville type on the left side by*

$$ \begin{aligned} \left({}^{CFRL}_{\xi_0}\Delta^{\alpha}_{\mathsf{h}}v\right)(\tau) &= \frac{\mathscr{D}(\alpha)}{1-\alpha}\Delta_{\mathsf{h}}\sum_{\ell=\frac{\xi_0}{\mathsf{h}}}^{\frac{\tau}{\mathsf{h}}-1}\mathsf{h}v(\ell\mathsf{h})(1+\lambda\mathsf{h})^{\frac{\ell\mathsf{h}-\sigma(\tau)}{\mathsf{h}}} \\ &= \frac{\mathscr{D}(\alpha)}{1-\alpha}\Delta_{\mathsf{h}}\sum_{\ell=\frac{\xi_0}{\mathsf{h}}}^{\frac{\tau}{\mathsf{h}}-1}\mathsf{h}v(\ell\mathsf{h})\left(\frac{1-\alpha-\alpha\mathsf{h}}{1-\alpha}\right)^{\frac{\ell\mathsf{h}-\sigma(\tau)}{\mathsf{h}}}, \end{aligned} \tag{13} $$

*for* $\tau \in \mathbb{N}_{\xi_0+\mathsf{h},\mathsf{h}}$*.*

*(d)* *The* $\Delta_{\mathsf{h}}$-CF *of Riemann–Liouville type on the right side as follows:*

$$
\begin{aligned}
\left({}_{\mathsf{h}}^{CFRL}\Delta_{\xi}^{\alpha}v\right)(\tau) &= \frac{\mathscr{D}(\alpha)}{1-\alpha}(-\nabla_{\mathsf{h}})\sum_{\ell=\frac{\tau}{\mathsf{h}}+1}^{\frac{\xi}{\mathsf{h}}}\mathsf{h}v(\ell\mathsf{h})(1+\lambda\mathsf{h})^{\frac{\ell\mathsf{h}-\sigma(\tau)}{\mathsf{h}}} \\
&= \frac{\mathscr{D}(\alpha)}{1-\alpha}(-\nabla_{\mathsf{h}})\sum_{\ell=\frac{\tau}{\mathsf{h}}+1}^{\frac{\xi}{\mathsf{h}}}\mathsf{h}v(\ell\mathsf{h})\left(\frac{1-\alpha-\alpha\mathsf{h}}{1-\alpha}\right)^{\frac{\ell\mathsf{h}-\sigma(\tau)}{\mathsf{h}}}, \qquad (14)
\end{aligned}
$$

*for* $\tau \in {}_{\xi-\mathsf{h},\mathsf{h}}\mathbb{N}$.

**Remark 3.** *Considering the above definition, we can note the following:*

*(i)* *As* $\alpha \to 0$, *we have*

$$
\lim_{\alpha\to 0}\left({}_{\xi_0}^{CFLC}\Delta_{\mathsf{h}}^{\alpha}v\right)(\tau) \to v(\tau)-v(\xi_0), \quad \textit{and} \quad \lim_{\alpha\to 0}\left({}_{\mathsf{h}}^{CFLC}\Delta_{\xi}^{\alpha}v\right)(\tau) \to v(\tau)-v(\xi),
$$
$$
\lim_{\alpha\to 0}\left({}_{\xi_0}^{CFRL}\Delta_{\mathsf{h}}^{\alpha}v\right)(\tau) \to v(\tau), \quad \textit{and} \quad \lim_{\alpha\to 0}\left({}_{\mathsf{h}}^{CFRL}\Delta_{\xi}^{\alpha}v\right)(\tau) \to v(\tau).
$$

*(ii)* *As* $\alpha \to 1$, *we have*

$$
\lim_{\alpha\to 1}\left({}_{\xi_0}^{CFLC}\Delta_{\mathsf{h}}^{\alpha}v\right)(\tau) \to \Delta v(\tau), \quad \textit{and} \quad \lim_{\alpha\to 1}\left({}_{\mathsf{h}}^{CFLC}\Delta_{\xi}^{\alpha}v\right)(\tau) \to -\nabla v(\tau),
$$
$$
\lim_{\alpha\to 1}\left({}_{\xi_0}^{CFRL}\Delta_{\mathsf{h}}^{\alpha}v\right)(\tau) \to \Delta v(\tau), \quad \textit{and} \quad \lim_{\alpha\to 1}\left({}_{\mathsf{h}}^{CFRL}\Delta_{\xi}^{\alpha}v\right)(\tau) \to -\nabla v(\tau).
$$

**Remark 4.** *The expression of* Q *can be applied to the above operators to have*

*(i)* $\left(\mathrm{Q}\,{}_{\xi_0}^{CFLC}\Delta_{\mathsf{h}}^{\alpha}v\right)(\tau) = \left({}_{\mathsf{h}}^{CFLC}\Delta_{\xi}^{\alpha}\mathrm{Q}v\right)(\tau)$,

*(ii)* $\left(\mathrm{Q}\,{}_{\xi_0}^{CFRL}\Delta_{\mathsf{h}}^{\alpha}v\right)(\tau) = \left({}_{\mathsf{h}}^{CFRL}\Delta_{\xi}^{\alpha}\mathrm{Q}v\right)(\tau)$.

Consider the difference equation

$$
\left({}_{\xi_0}^{CFRL}\Delta_{\mathsf{h}}^{\alpha}v\right)(\tau) = g(\tau),
$$

or, equivalently,

$$
\frac{\mathscr{D}(\alpha)}{1-\alpha}\Delta_{\mathsf{h}}\left(v * {}_{\mathsf{h}}\tilde{e}_{\lambda}(\tau,\xi_0)\right) = g(\tau). \qquad (15)
$$

By applying $\mathcal{L}_{\xi_0,\mathsf{h}}$ on both sides of (15) and considering (4), Lemma 4 and

$$
\mathcal{L}_{\xi_0,\mathsf{h}}\{{}_{\mathsf{h}}\tilde{e}_{\lambda}(\tau,\xi_0)\}(s) = \frac{1}{s-\lambda} = \frac{(1-\alpha)}{(1-\alpha)s+\alpha},
$$

we obtain

$$
\frac{s\,\mathscr{D}(\alpha)}{(1-\alpha)s+\alpha}V(s) = G(s), \qquad (16)
$$

where $G(s) = \mathcal{L}_{\xi_0,\mathsf{h}}\{g(\tau)\}(s)$ and $V(s) = \mathcal{L}_{\xi_0,\mathsf{h}}\{v(\tau)\}$. This leads to

$$
V(s) = \frac{1-\alpha}{\mathscr{D}(\alpha)}G(s) + \frac{\alpha}{\mathscr{D}(\alpha)}\frac{1}{s}G(s).
$$

Apply $\mathcal{L}_{\xi_0,\mathsf{h}}^{-1}$ on both sides to reach

$$
v(\tau) = \frac{1-\alpha}{\mathscr{D}(\alpha)}g(\tau) + \frac{\alpha}{\mathscr{D}(\alpha)}\sum_{k=\frac{\xi_0}{\mathsf{h}}}^{\frac{\tau}{\mathsf{h}}-1}\mathsf{h}g(k\mathsf{h}).
$$

This helps us introduce the following CF fractional definitions as follows:

**Definition 7.** *For $\alpha \in (0,1)$ and $v$ defined on $\mathbb{T}_{\mathsf{h}}$, we express the $\Delta_{\mathsf{h}}$-CF fractional sum in the left side case by*

$$\left({}^{CF}_{\xi_0}\Delta_{\mathsf{h}}^{-\alpha}v\right)(\tau) = \frac{1-\alpha}{\mathscr{D}(\alpha)}v(\tau) + \frac{\alpha}{\mathscr{D}(\alpha)}\sum_{k=\frac{\xi_0}{\mathsf{h}}}^{\frac{\tau}{\mathsf{h}}-1}\mathsf{h}v(k\mathsf{h}),$$

*and in the right side case by*

$$\left({}^{CF}_{\mathsf{h}}\Delta_{\xi}^{-\alpha}v\right)(\tau) = \frac{1-\alpha}{\mathscr{D}(\alpha)}v(\tau) + \frac{\alpha}{\mathscr{D}(\alpha)}\sum_{k=\frac{\tau}{\mathsf{h}}+1}^{\frac{\xi}{\mathsf{h}}}\mathsf{h}v(k\mathsf{h}).$$

It can be easily shown that $\left({}^{CF}_{\xi_0}\Delta_{\mathsf{h}}^{-\alpha}\mathrm{Q}v\right)(\tau) = \mathrm{Q}\left({}^{CF}_{\mathsf{h}}\Delta_{\xi}^{-\alpha}v\right)(\tau)$. Furthermore, we have $\left({}^{CF}_{\mathsf{h}}\Delta_{\xi}^{-\alpha}\,{}^{CFRL}_{\mathsf{h}}\Delta_{\xi}^{\alpha}v\right)(\tau) = v(\tau)$.

Conversely, consider the difference equation

$$\left({}^{CF}_{\xi_0}\Delta_{\mathsf{h}}^{-\alpha}v\right)(\tau) = \frac{1-\alpha}{\mathscr{D}(\alpha)}v(\tau) + \frac{\alpha}{\mathscr{D}(\alpha)}\sum_{k=\frac{\xi_0}{\mathsf{h}}}^{\frac{\tau}{\mathsf{h}}-1}\mathsf{h}v(k\mathsf{h}) = g(\tau).$$

Applying $\mathcal{L}_{\xi_0,\mathsf{h}}$ on both sides to obtain

$$\frac{1-\alpha}{\mathscr{D}(\alpha)}V(s) + \frac{\alpha}{s\,\mathscr{D}(\alpha)}V(s) = G(s).$$

leads to

$$\begin{aligned} V(s) &= \frac{s\,\mathscr{D}(\alpha)}{(1-\alpha)s+\alpha}G(s) \\ &\overset{by}{\underset{(16)}{=}} \mathcal{L}_{\xi_0,\mathsf{h}}\left\{\left({}^{CFRL}_{\xi_0}\Delta_{\mathsf{h}}^{\alpha}g\right)(\tau)\right\}(s) \\ &= \mathcal{L}_{\xi_0,\mathsf{h}}\left\{\left({}^{CFRL}_{\xi_0}\Delta_{\mathsf{h}}^{\alpha}\,{}^{CF}_{\xi_0}\Delta_{\mathsf{h}}^{-\alpha}v\right)(\tau)\right\}(s). \end{aligned}$$

It follows that

$$\left({}^{CFRL}_{\xi_0}\Delta_{\mathsf{h}}^{\alpha}\,{}^{CF}_{\xi_0}\Delta_{\mathsf{h}}^{-\alpha}v\right)(\tau) = v(\tau). \tag{17}$$

By the same method, or by applying the operator of Q, we can show that

$$\left({}^{CF}_{\mathsf{h}}\Delta_{\xi}^{-\alpha}\,{}^{CFRL}_{\mathsf{h}}\Delta_{\xi}^{\alpha}v\right)(\tau) = v(\tau). \tag{18}$$

It is always of interest to make a connection between $\Delta_{\mathsf{h}}$-CF fractional differences of RL and LC types. For this reason, the following theorem will be raised.

**Theorem 1.** *For the function $v$ on $\mathbb{N}_{\xi_0,\mathsf{h}}$, we have the following relationships:*

*(i)* $\left({}^{CFLC}_{\xi_0}\Delta_{\mathsf{h}}^{\alpha}v\right)(\tau) = \left({}^{CFRL}_{\xi_0}\Delta_{\mathsf{h}}^{\alpha}v\right)(\tau) - \frac{\mathscr{D}(\alpha)}{1-\alpha}v(\xi_0)\left(\frac{1-\alpha-\alpha\mathsf{h}}{1-\alpha}\right)^{\frac{\tau-\xi_0}{\mathsf{h}}}$,

*(ii)* $\left({}^{CFLC}_{\mathsf{h}}\Delta_{\xi}^{\alpha}v\right)(\tau) = \left({}^{CFRL}_{\mathsf{h}}\Delta_{\xi}^{\alpha}v\right)(\tau) - \frac{\mathscr{D}(\alpha)}{1-\alpha}v(\xi)\left(\frac{1-\alpha-\alpha\mathsf{h}}{1-\alpha}\right)^{\frac{\xi-\tau}{\mathsf{h}}}$.

**Proof.** Having (16), we see that

$$\mathcal{L}_{\xi_0,\mathsf{h}}\left\{{}^{CFRL}_{\xi_0}\Delta_{\mathsf{h}}^{\alpha}v(\tau)\right\}(s) = \frac{s\,\mathscr{D}(\alpha)}{(1-\alpha)s+\alpha}V(s). \tag{19}$$

By applying $\mathcal{L}_{\xi_0,\mathrm{h}}$ on (11) with the use of (19), we obtain

$$\begin{aligned}
\mathcal{L}_{\xi_0,\mathrm{h}}\Big\{ {}^{CFLC}_{\xi_0}\Delta^{\alpha}_{\mathrm{h}} v(\tau) \Big\}(s) &= \frac{\mathscr{D}(\alpha)}{1-\alpha}\mathcal{L}_{\xi_0,\mathrm{h}}\Big\{ (\Delta v(\tau)) * {}_{\mathrm{h}}\tilde{e}_{\lambda}(\tau,\xi_0) \Big\}(s) \\
&= \frac{\mathscr{D}(\alpha)}{1-\alpha}\mathcal{L}_{\xi_0,\mathrm{h}}\{\Delta v(\tau)\}(s)\cdot \mathcal{L}_{\xi_0,\mathrm{h}}\{ {}_{\mathrm{h}}\tilde{e}_{\lambda}(\tau,\xi_0)\}(s) \\
&= \frac{\mathscr{D}(\alpha)}{1-\alpha}\frac{sV(s)}{s-\alpha} - \frac{\mathscr{D}(\alpha)}{1-\alpha}\frac{(1-\alpha)}{(1-\alpha)s+\alpha}v(\xi_0) \\
&= \mathcal{L}_{\xi_0,\mathrm{h}}\Big\{ \Big( {}^{CFRL}_{\xi_0}\Delta^{\alpha}_{\mathrm{h}} v\Big)(\tau) \Big\}(s) - \frac{\mathscr{D}(\alpha)}{1-\alpha}\frac{(1-\alpha)}{(1-\alpha)s+\alpha}v(\xi_0).
\end{aligned} \tag{20}$$

Applying $\mathcal{L}^{-1}_{\xi_0,\mathrm{h}}$ on both sides of (20), we obtain (i) as desired. The proof of (ii) will be obtained by applying the operator of Q on the first item (i). □

### *3.2. Delta Fractional Differences with ML Kernels*

This subsection is dedicated to introduce the $\Delta_{\mathrm{h}}$-AB fractional differences and sums, and some of their properties. We start by introducing the $\Delta_{\mathrm{h}}$-AB fractional difference operators.

**Definition 8.** *For the function $v$ on $\mathbb{T}_{\mathrm{h}}$ and $0 < \alpha \le 1$ with $|\lambda \mathrm{h}^{\alpha}| < 1$, we express the $\Delta_{\mathrm{h}}$-AB fractional difference of Liouville–Caputo type on the left side by*

$$\begin{aligned}
\Big( {}^{ABLC}_{\xi_0}\Delta^{\alpha}_{\mathrm{h}} v\Big)(\tau) &= \frac{\mathscr{D}(\alpha)}{1-\alpha}\sum_{\ell=\frac{\xi_0}{\mathrm{h}}}^{\frac{\tau}{\mathrm{h}}-1} \mathrm{h}(\Delta_{\mathrm{h}} v)(\ell \mathrm{h})\, {}_{\mathrm{h}}E_{(\alpha)}(\lambda, \tau - \sigma(\ell \mathrm{h})) \\
&= \frac{\mathscr{D}(\alpha)}{1-\alpha}\Big[ \Delta_{\mathrm{h}} v(\tau) * {}_{\mathrm{h}}E_{(\alpha)}(\lambda, \tau-\xi_0)\Big],
\end{aligned} \tag{21}$$

*and of Riemann–Liouville type on the left side by*

$$\begin{aligned}
\Big( {}^{ABRL}_{\xi_0}\Delta^{\alpha}_{\mathrm{h}} v\Big)(\tau) &= \frac{\mathscr{D}(\alpha)}{1-\alpha}\Delta_{\mathrm{h}}\sum_{\ell=\frac{\xi_0}{\mathrm{h}}}^{\frac{\tau}{\mathrm{h}}-1} \mathrm{h} v(\ell \mathrm{h})\, {}_{\mathrm{h}}E_{(\alpha)}(\lambda, \tau - \sigma(\ell \mathrm{h})) \\
&= \frac{\mathscr{D}(\alpha)}{1-\alpha}\Delta_{\mathrm{h}}\Big[ v(\tau) * {}_{\mathrm{h}}E_{(\alpha)}(\lambda, \tau-\xi_0)\Big].
\end{aligned} \tag{22}$$

**Definition 9.** *For the function $v$ on $\mathbb{T}_{\mathrm{h}}$ and $0 < \alpha \le 1$ with $|\lambda \mathrm{h}^{\alpha}| < 1$, we express the $\Delta_{\mathrm{h}}$-AB fractional difference of Liouville–Caputo type on the right side by*

$$\Big( {}^{ABLC}_{\mathrm{h}}\Delta^{\alpha}_{\xi} v\Big)(\tau) = \frac{\mathscr{D}(\alpha)}{1-\alpha}\sum_{\ell=\frac{\tau}{\mathrm{h}}+1}^{\frac{\xi}{\mathrm{h}}} \mathrm{h}(-\nabla_{\mathrm{h}} v)(\ell \mathrm{h})\, {}_{\mathrm{h}}E_{(\alpha)}(\lambda, \sigma(\ell \mathrm{h}) - \tau), \tag{23}$$

*and of Riemann–Liouville type on the right side by*

$$\Big( {}^{ABRL}_{\mathrm{h}}\Delta^{\alpha}_{\xi} v\Big)(\tau) = \frac{\mathscr{D}(\alpha)}{1-\alpha}(-\nabla_{\mathrm{h}})\sum_{\ell=\frac{\tau}{\mathrm{h}}+1}^{\frac{\xi}{\mathrm{h}}} \mathrm{h} v(\ell \mathrm{h})\, {}_{\mathrm{h}}E_{(\alpha)}(\lambda, \sigma(\ell \mathrm{h}) - \tau). \tag{24}$$

To find the Laplace of $\big({}_{\xi_0}\Delta^{-\alpha}_{\mathrm{h}} v\big)(\tau + (\alpha-1)\mathrm{h})$, we need the following lemma on h-ML functions.

**Lemma 7.** *For $0 < \alpha \le 1$, we have*

$$\mathcal{L}_{\xi_0,\mathrm{h}}\Big\{ {}_{\mathrm{h}}E_{(\alpha)}(\lambda, \tau-\xi_0)\Big\}(s) = \frac{s^{\alpha-1}(\mathrm{h}\,s+1)^{1-\alpha}}{s^{\alpha}(\mathrm{h}\,s+1)^{1-\alpha}-\lambda},$$

*provided that* $\left|\lambda(\mathsf{h}\,s+1)^{\alpha-1}\right| < s^{\alpha}$.

**Proof.** From Definitions 3 and 5, and making use of Lemmas 6 and 7, we see that

$$\begin{aligned}
\mathcal{L}_{\xi_0,\mathsf{h}}\left\{ {}_{\mathsf{h}}E_{(\alpha)}(\lambda,\tau-\xi_0)\right\}(s) &= \sum_{\imath=0}^{\infty}\lambda^{\imath}\mathcal{L}_{\xi_0,\mathsf{h}}\left\{\frac{(\tau-\xi_0+\imath(\alpha-1)\mathsf{h})_{\mathsf{h}}^{(\alpha\imath)}}{\Gamma(\alpha\imath+1)}\right\}(s) \\
&= \sum_{\imath=0}^{\infty}\lambda^{\imath}\Bigg[\frac{1}{(\mathsf{h}\,s+1)^{\imath}}\mathcal{L}_{\xi_0+\imath\mathsf{h},\mathsf{h}}\left\{\frac{(\tau-\xi_0+\imath(\alpha-1)\mathsf{h})_{\mathsf{h}}^{(\alpha\imath)}}{\Gamma(\alpha\imath+1)}\right\}(s) \\
&+ \frac{\mathsf{h}}{(\mathsf{h}\,s+1)^{\imath}\Gamma(\alpha\imath+1)}\sum_{j=0}^{\imath-1}(\mathsf{h}\,s+1)^{\imath-j-1}((j+\imath(\alpha-1))\mathsf{h})_{\mathsf{h}}^{(\alpha\imath)}\Bigg] \\
&= \sum_{\imath=0}^{\infty}\lambda^{\imath}\Bigg[\frac{1}{(\mathsf{h}\,s+1)^{\imath}}\frac{(\mathsf{h}\,s+1)^{\alpha\imath}}{s^{\alpha\imath+1}} \\
&+ \mathsf{h}^{\alpha\imath+1}\sum_{j=0}^{\imath-1}(\mathsf{h}\,s+1)^{-j-1}\frac{\Gamma(j+\imath(\alpha-1)+1)}{\Gamma(j-\imath+1)}\Bigg] \\
&= \sum_{\imath=0}^{\infty}\lambda^{\imath}\left[\frac{1}{(\mathsf{h}\,s+1)^{\imath}}\frac{(\mathsf{h}\,s+1)^{\alpha\imath}}{s^{\alpha\imath+1}}+0\right] = \frac{1}{s}\sum_{\imath=0}^{\infty}\left(\frac{\lambda(\mathsf{h}\,s+1)^{\alpha-1}}{s^{\alpha}}\right)^{\imath}.
\end{aligned}$$

This will be convergent if $\left|\lambda(\mathsf{h}\,s+1)^{\alpha-1}\right| < s^{\alpha}$. Therefore, by the test of geometric series, we obtain

$$\begin{aligned}
\mathcal{L}_{\xi_0,\mathsf{h}}\left\{ {}_{\mathsf{h}}E_{(\alpha)}(\lambda,\tau-\xi_0)\right\}(s) &= \frac{1}{s}\frac{1}{1-\frac{\lambda(\mathsf{h}\,s+1)^{\alpha-1}}{s^{\alpha}}} \\
&= \frac{s^{\alpha-1}(\mathsf{h}\,s+1)^{1-\alpha}}{s^{\alpha}(\mathsf{h}\,s+1)^{1-\alpha}-\lambda},
\end{aligned}$$

which completes the proof. □

**Lemma 8.** *For each* $\alpha > 0$, *it can be*

$$\mathcal{L}_{\xi_0,\mathsf{h}}\left\{\left({}_{\xi_0}\Delta_{\mathsf{h}}^{-\alpha}v\right)(\tau+(\alpha-1)\mathsf{h})\right\}(s) = \frac{(\mathsf{h}\,s+1)^{\alpha-1}}{s^{\alpha}}V(s),$$

*where* $V(s) = \mathcal{L}_{\xi_0,\mathsf{h}}\{v\}(s)$.

**Proof.** In view of (1) and (3), we have

$$\begin{aligned}
\left({}_{\xi_0}\Delta_{\mathsf{h}}^{-\alpha}v\right)(\tau+(\alpha-1)\mathsf{h}) &= \frac{1}{\Gamma(\alpha)}\sum_{k=\frac{\xi_0}{\mathsf{h}}}^{\frac{\tau}{\mathsf{h}}-1}(\tau-\sigma(k\mathsf{h}))_{\mathsf{h}}^{(\alpha-1)}v(k\mathsf{h})\mathsf{h} \\
&= \sum_{k=\frac{\xi_0}{\mathsf{h}}}^{\frac{\tau}{\mathsf{h}}-1}{}_{\mathsf{h}}\tilde{H}_{\alpha-1}(\tau+(\alpha-1)\mathsf{h},\sigma(k\mathsf{h}))v(k\mathsf{h})\mathsf{h} \\
&= \mathsf{h}\sum_{k=\frac{\xi_0}{\mathsf{h}}}^{\frac{\tau}{\mathsf{h}}-1}v(k\mathsf{h})\,{}_{\mathsf{h}}\tilde{H}_{\alpha-1}(\tau+(\alpha-1)\mathsf{h}-\sigma(k\mathsf{h})+\xi_0,\xi_0) \\
&= \left(v*{}_{\mathsf{h}}\tilde{H}_{\alpha-1}(\tau,\xi_0)\right)(\tau+(\alpha-1)\mathsf{h}),
\end{aligned}$$

where ${}_{\mathsf{h}}\tilde{H}_{\alpha-1}(\tau,\xi_0)$ is as defined in Lemma 5. Therefore, by considering Lemma 5, we obtain

$$\mathcal{L}_{\xi_0,\mathsf{h}}\left\{\left({}_{\xi_0}\Delta_{\mathsf{h}}^{-\alpha}v\right)(\tau+(\alpha-1)\mathsf{h})\right\}(s)=\mathcal{L}_{\xi_0,\mathsf{h}}\{v(\tau)\}(s)\cdot\mathcal{L}_{\xi_0,\mathsf{h}}\left\{{}_{\mathsf{h}}\tilde{H}_{\alpha-1}(\tau+(\alpha-1)\mathsf{h},\xi_0)\right\}(s)$$
$$=V(s)\frac{(\mathsf{h}s+1)^{\alpha-1}}{s^{\alpha}},$$

which ends the proof. □

In association to $\left({}^{ABRL}_{\xi_0}\Delta_{\mathsf{h}}^{\alpha}u\right)(\tau)=v(\tau)$, we can formulate the $\Delta_{\mathsf{h}}$-AB fractional sum by considering the difference equation

$$\left({}^{ABRL}_{\xi_0}\Delta_{\mathsf{h}}^{\alpha}g\right)(\tau)=v(\tau). \tag{25}$$

To solve (25), we can apply $\mathcal{L}_{\xi_0,\mathsf{h}}$ on both sides and apply (22) and Lemma 7 to obtain

$$\frac{\mathscr{D}(\alpha)}{1-\alpha}\frac{s^{\alpha-1}(\mathsf{h}s+1)^{1-\alpha}}{s^{\alpha}(\mathsf{h}s+1)^{1-\alpha}-\lambda}G(s)=V(s). \tag{26}$$

From this, we conclude that

$$G(s)=\frac{1-\alpha}{\mathscr{D}(\alpha)}V(s)+\frac{\alpha}{\mathscr{D}(\alpha)}\frac{1}{s^{\alpha}(\mathsf{h}s+1)^{1-\alpha}}V(s).$$

Then, we apply $\mathcal{L}^{-1}_{\xi_0,\mathsf{h}}$ and use Lemma 8 to deduce that

$$g(\tau)=\frac{1-\alpha}{\mathscr{D}(\alpha)}v(\tau)+\frac{\alpha}{\mathscr{D}(\alpha)}\left({}_{\xi_0}\Delta_{\mathsf{h}}^{-\alpha}v\right)(\tau+(\alpha-1)\mathsf{h}). \tag{27}$$

Conversely, if we solve $\left({}^{AB}_{\xi_0}\Delta_{\mathsf{h}}^{-\alpha}g\right)(\tau)=v(\tau)$, we find out that $g(\tau)=\left({}^{ABRL}_{\xi_0}\Delta_{\mathsf{h}}^{\alpha}v\right)(\tau)$.

Depending on (27), we can define the $\Delta_{\mathsf{h}}$-AB fractional sums on the left and right sides as follows.

**Definition 10.** *Let $h>0$ and $0<\alpha<1$. We express the $\Delta_{\mathsf{h}}$-AB fractional sum on the left side by*

$$\left({}^{AB}_{\xi_0}\Delta_{\mathsf{h}}^{-\alpha}v\right)(\tau)=\frac{1-\alpha}{\mathscr{D}(\alpha)}v(\tau)+\left({}_{\xi_0}\Delta_{\mathsf{h}}^{-\alpha}v\right)(\tau+(\alpha-1)\mathsf{h}), \tag{28}$$

*and on the right side by*

$$\left({}^{AB}_{\mathsf{h}}\Delta_{\xi}^{-\alpha}v\right)(\tau)=\frac{1-\alpha}{\mathscr{D}(\alpha)}v(\tau)+\left({}_{\mathsf{h}}\Delta_{\xi}^{-\alpha}v\right)(\tau-(\alpha-1)\mathsf{h}). \tag{29}$$

It is worth noting that $\left({}^{AB}_{\xi_0}\Delta_{\mathsf{h}}^{-\alpha}\mathrm{Q}v\right)(\tau)=\mathrm{Q}\left({}^{AB}_{\mathsf{h}}\Delta_{\xi}^{-\alpha}v\right)(\tau)$. In addition, by considering the operator of Q, it can be proved that $\left({}^{AB}_{\mathsf{h}}\Delta_{\xi}^{-\alpha}\,{}^{ABRL}_{\mathsf{h}}\Delta_{\xi}^{\alpha}v\right)(\tau)=v(\tau)$.

In the last stage of this section, it is worth making a connection between $\Delta_{\mathsf{h}}$-AB fractional differences of RL and LC types. For this reason, we have the following example.

**Example 1.** *Suppose that $v$ is defined on $\mathbb{N}_{\xi_0,\mathsf{h}}$. Then, in the conclusion of* (21), *Lemmas* 4 *and* 7, *we have*

$$\mathcal{L}_{\xi_0,\mathsf{h}}\left\{\left({}^{ABLC}_{\xi_0}\Delta_{\mathsf{h}}^{\alpha}v\right)(\tau)\right\}(s)=\frac{\mathscr{D}(\alpha)}{1-\alpha}\left[(sV(s)-v(\xi_0))\frac{s^{\alpha-1}(\mathsf{h}s+1)^{1-\alpha}}{s^{\alpha}(\mathsf{h}s+1)^{1-\alpha}-\lambda}\right]$$
$$=\frac{\mathscr{D}(\alpha)}{1-\alpha}\frac{s^{\alpha}(\mathsf{h}s+1)^{1-\alpha}}{s^{\alpha}(\mathsf{h}s+1)^{1-\alpha}-\lambda}V(s)-\frac{\mathscr{D}(\alpha)}{1-\alpha}\frac{s^{\alpha-1}(\mathsf{h}s+1)^{1-\alpha}}{s^{\alpha}(\mathsf{h}s+1)^{1-\alpha}-\lambda}v(\xi_0).$$

*By applying $\mathcal{L}^{-1}_{\xi_0,\mathsf{h}}$ on both sides and considering* (26), *we can conclude that we have the following relationships:*

$$\left({}_{\xi_0}^{ABLC}\Delta_{\mathsf{h}}^{\alpha}v\right)(\tau) = \left({}_{\xi_0}^{ABRL}\Delta_{\mathsf{h}}^{\alpha}v\right)(\tau) - \frac{\mathscr{D}(\alpha)}{1-\alpha}v(\xi_0)\,{}_{\mathsf{h}}E_{(\alpha)}(\lambda, \tau - \xi_0). \quad (30)$$

*By applying the operator of* Q *on* (30)*, we can similarly deduce that*

$$\left({}_{\mathsf{h}}^{ABLC}\Delta_{\xi}^{\alpha}v\right)(\tau) = \left({}_{\mathsf{h}}^{ABRL}\Delta_{\xi}^{\alpha}v\right)(\tau) - \frac{\mathscr{D}(\alpha)}{1-\alpha}v(\xi)\,{}_{\mathsf{h}}E_{(\alpha)}(\lambda, \xi - \tau).$$

**Remark 5.** *The application of our theoretical results could not be prepared easily in a simple example, which is why we have not prepared any application in this paper. For this reason, we refer the reader to see the recently accepted paper [31], in which the application of our new findings has been examined in the context of monotonic analysis.*

## 4. Conclusions

This work emphasizes the development of novel special functions named as delta h-exponential and h-ML functions to address the $\Delta_{\mathsf{h}}$-CF and $\Delta_{\mathsf{h}}$-AB fractional difference operators including these functions in their kernels. We aim to reconstruct discrete fractional operators in the delta background sense. Afterwards, Laplace transformation is performed on the $\Delta_{\mathsf{h}}$-CF and $\Delta_{\mathsf{h}}$-AB fractional difference operators to obtain their corresponding $\Delta_{\mathsf{h}}$-CF and $\Delta_{\mathsf{h}}$-AB fractional sum operators. Relationships between the RL and LC cases are considered for the $\Delta_{\mathsf{h}}$-CF and $\Delta_{\mathsf{h}}$-AB fractional difference operators.

It can be noted that there may be some limitations in our study. This study only introduces one variate delta Mittag–Leffler function. Bivariate discrete Mittag–Leffler functions [32], higher-order differential equations [33] and monotonicity analysis are also needed to generalize the above new operators and address future work.

**Author Contributions:** Conceptualization, J.L.G.G.; Data curation, I.S.I.; Funding acquisition, M.V.-C.; Investigation, P.O.M.; Methodology, M.A.Y.; Project administration, J.L.G.G.; Resources, M.V.-C.; Software, I.S.I.; Supervision, N.C.; Visualization, N.C.; Writing—review and editing, P.O.M. and M.A.Y. All authors have read and agreed to the published version of the manuscript.

**Funding:** This research received no external funding.

**Data Availability Statement:** Data are contained within the article.

**Acknowledgments:** Researchers Supporting Project number (RSP2024R153), King Saud University, Riyadh, Saudi Arabia.

**Conflicts of Interest:** The authors declare no conflicts of interest.

## References


1. Goodrich, C.; Peterson, A.C. *Discrete Fractional Calculus*; Springer: New York, NY, USA, 2015.
2. Cabada, A.; Dimitrov, N. Nontrivial solutions of non-autonomous Dirichlet fractional discrete problems. *Fract. Calc. Appl. Anal.* **2020**, *23*, 980–995. [CrossRef]
3. Zhang, X.-L.; Yu, Y.; Wang, H.; Feng, J. Stability analysis of discrete-time tempered fractional-order neural networks with time delays. *Fract. Calc. Appl. Anal.* **2024**, *27*, 1972–1993. [CrossRef]
4. Atici, F.M.; Atici, M.; Belcher, M.; Marshall, D. A new approach for modeling with discrete fractional equations. *Fund. Inform.* **2017**, *151*, 313–324.
5. You, X.; Song, Q.; Zhao, Z. Existence and finite-time stability of discrete fractional-order complex-valued neural networks with time delays. *Neural Netw.* **2020**, *123*, 248–260. [CrossRef]
6. Wei, Y.; Chen, Y.; Wei, Y.; Zhao, X. Lyapunov stability analysis for nonlinear nabla tempered fractional order systems. *Asian J. Control* **2023**, *25*, 3057–3066. [CrossRef]
7. Ortigueira, M. Discrete-time fractional difference calculus: Origins, evolutions, and new formalisms. *Fractal Fract.* **2023**, *7*, 502. [CrossRef]
8. Jia, B.; Du, F.; Erbe, L.; Peterson, A. Asymptotic behavior of nabla half order h-difference equations. *J. Appl. Anal. Comput.* **2018**, *8*, 1707–1726.
9. Liu, X.; Yu, Y. Synchronization analysis for discrete fractional-order complex-valued neural networks with time delays. *Neural Comput. Appl.* **2021**, *33*, 10503–10514. [CrossRef]

10. Mohammed, P.O.; Srivastava, H.M.; Baleanu, D.; Al-Sarairah, E.; Yousif, M.A.; Chorfi, N. Analytical and approximate monotone solutions of the mixed order fractional nabla operators subject to bounded conditions. *Math. Comp. Model. Dyn.* **2024**, *30*, 626–639. [CrossRef]
11. Mohammed, P.O. An analysis of exponential kernel fractional difference operator for delta positivity. *Nonlinear Eng.* **2024**, *13*, 20220377. [CrossRef]
12. Chen, C.R.; Bohner, M.; Jia, B.G. Ulam-Hyers stability of Caputo fractional difference equations. *Math. Meth. Appl. Sci.* **2019**, *42*, 7461–7470. [CrossRef]
13. Miller, K.S.; Ross, B. *An Introduction to the Fractional Calculus and Fractional Differential Equations*; Wiley-Interscience Publisher: New York, NY, USA, 1993.
14. Cesarano, C. Generalized special functions in the description of fractional diffusive equations. *Commun. Appl. Ind. Math.* **2019**, *10*, 31–40. [CrossRef]
15. Jajarmi, A.; Baleanu, D.; Sajjadi, S.S.; Nieto, J.J. Analysis and some applications of a regularized Ψ-Hilfer fractional derivative. *J. Comput. Appl. Math.* **2022**, *415*, 114476. [CrossRef]
16. Baleanu, D.; Jajarmi, A.; Hajipour M. On the nonlinear dynamical systems within the generalized fractional derivatives with Mittag–Leffler Kernel. *Nonlinear Dyn.* **2018**, *94*, 397–414. [CrossRef]
17. Abdeljawad, T. On Riemann and Caputo fractional differences. *Comput. Math. Appl.* **2011**, *62*, 1602–1611. [CrossRef]
18. Abdeljawad, T.; Baleanu, D. Discrete fractional differences with nonsingular discrete Mittag–Leffler kernels. *Adv. Differ. Equ.* **2016**, *2016*, 232. [CrossRef]
19. Abdeljawad, T.; Al-Mdallal, Q.M.; Hajji, M.A. Arbitrary order fractional difference operators with discrete exponential kernels and applications. *Discret. Dyn. Nat. Soc.* **2017**, *2017*, 4149320. [CrossRef]
20. Mohammed, P.O.; Abdeljawad, T.; Hamasalh, F.K. On Riemann–Liouville and Caputo fractional forward difference monotonicity analysis. *Mathematics* **2021**, *9*, 1303. [CrossRef]
21. Nagai, A. Discrete Mittag–Leffler function and its applications. *Publ. Res. Inst. Math. Sci. Kyoto Univ.* **2003**, *1302*, 1–20.
22. Wu, G.C.; Baleanu, D.; Zeng, S.D.; Luo, W.H. Mittag–Leffler function for discrete fractional modelling. *J. King Saud Univ. Sci.* **2016**, *28*, 99–102. [CrossRef]
23. Atici, F.M.; Chang, S.; Jonnalagadda, J.M. Mittag–Leffler functions in discrete time. *Fractal Fract.* **2023**, *7*, 254. [CrossRef]
24. Mohammed, P.O.; Abdeljawad, T.; Hamasalh, F.K. Discrete Prabhakar fractional difference and sum operators. *Chaos Soliton. Fract.* **2021**, *150*, 111182. [CrossRef]
25. Saenko, V.V. The calculation of the Mittag–Leffler function. *Int. J. Comput. Math.* **2022**, *99*, 1367–1394. [CrossRef]
26. Abdeljawad, T. Different type kernel $h$–fractional differences and their fractional $h$–sums. *Chaos Solit. Fract.* **2018**, *116*, 146–156. [CrossRef]
27. Mohammed, P.O.; Abdeljawad, T. Discrete generalized fractional operators defined using h-discrete Mittag–Leffler kernels and applications to AB fractional difference systems. *Math. Meth. Appl. Sci.* **2020**, *46*, 7688–7713. [CrossRef]
28. Fereira R.; Torres, D. Fractional $h$–difference equations arising from the calculus of variations. *Appl. Anal. Discret. Math.* **2011**, *5*, 110–121. [CrossRef]
29. Abdeljawad, T. Dual identities in fractional difference calculus within Riemann. *Adv. Differ. Equ.* **2013**, *2013*, 36. [CrossRef]
30. Abdeljawad, T. On delta and nabla caputo fractional differences and dual identities. *Discr. Dynam. Nat. Soc.* **2013**, *2013*, 12. [CrossRef]
31. Mohammed, P.O. On the delta Mittag–Leffler functions and its application in monotonic analysis. *J. Comput. Appl. Math.* 2025, *accepted*.
32. Mohammed, P.O.; Kürt, C.; Abdeljawad, T. Bivariate discrete Mittag–Leffler functions with associated discrete fractional operators. *Chaos Solit. Fractals* **2022**, *165*, 112848. [CrossRef]
33. Zhang, X.; Feng, Y.; Luo, Z.; Liu, J. A spatial sixth-order numerical scheme for solving fractional partial differential equation. *Appl. Math. Lett.* **2025**, *159*, 109265. [CrossRef]