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Parametrized Half-Hyperbolic Tangent Function-Activated Complex-Valued Neural Network Approximation

George A. Anastassiou [1,*] and Seda Karateke [2]

[1] Department of Mathematical Sciences, University of Memphis, Memphis, TN 38152, USA
[2] Department of Software Engineering, Faculty of Engineering and Natural Sciences, Istanbul Atlas University, Kagithane, Istanbul 34408, Türkiye; seda.karateke@atlas.edu.tr
* Correspondence: ganastss@memphis.edu

Abstract: In this paper, we create a family of neural network (NN) operators employing a parametrized and deformed half-hyperbolic tangent function as an activation function and a density function produced by the same activation function. Moreover, we consider the univariate quantitative approximations by complex-valued neural network (NN) operators of complex-valued functions on a compact domain. Pointwise and uniform convergence results on Banach spaces are acquired through trigonometric, hyperbolic, and hybrid-type hyperbolic–trigonometric approaches.

Keywords: parametrized half-hyperbolic tangent function; Banach space-valued neural network approximation; Ostrowski- and Opial-type inequalities; complex-valued neural network operators; trigonometric- and hyperbolic-type Taylor formulae; activation function; neural networks

MSC: 26A33; 41A17; 41A25; 41A30; 46B25

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1. Motivation

Under the umbrella of Artificial Intelligence (AI), neural network (NN) operators supply a vigorous structure for solving complex real-life problems in various fields of science such as cybersecurity, healthcare, economics, medicine, psychology, sociology, etc. Thanks to their differentiability properties, they can optimize parameters directly. Moreover, NN operators need correctly selected activation functions to work effectively. When we look at the literature, we encounter a very wide range of activation functions [1]. In this study, we choose $\widehat{h}_{\alpha}$—a parametrized and deformed half-hyperbolic tangent function—as the activation function. $\tan h$-like functions can be more effective in reaching the optimum solution due to their trainability [2]. One of the interesting aspects of this work is that all higher-order approximations are based on trigonometric and hyperbolic-type Taylor's formulae inequalities (see [3–5]). Next, we emphasize the following: as is well known, the human brain has been proven medically to be a non-symmetrical human organ. As a result, NNs try to imitate its operation and are not symmetrical mathematical structures. But in our paper, we build an approximation apparatus that is as close as possible to symmetry. Namely the activation function we initially use is described as reciprocal anti-symmetrical (see Proposition 1). This is the building block for our density function which is used in our approximation NN operators. We prove that this density function is a reciprocal symmetric function (see Equation (3)). This represents our study's interesting connection to the general "symmetry" phenomenon.

The first construction of approximation by NN operators in the sense of the "Cardaliaguet Euvrard" and "Squashing" types was made by G.A. Anastassiou in 1997. As a fruit of this construction, he also brought to the literature the ability to calculate the convergence speed with the help of convergence rates using the modulus of continuity [6].

The mathematical expression of the one-hidden-layer neural network (NN) architecture is presented as

$$N_{\psi,n}(x) = \sum_{i=0}^{n} k_i \psi(\langle \widetilde{\omega}_i \cdot x \rangle + \varpi_i), \quad x \in \mathbb{R}^s, \ s \in \mathbb{N},$$

for $0 \le i \le n$, where $\widetilde{\omega}_i = (\widetilde{\omega}_{i,1}, \widetilde{\omega}_{i,2}, \ldots, \widetilde{\omega}_{i,s}) \in \mathbb{R}^s$ are the connection weights, k_i is the coefficient, $\langle \widetilde{\omega}_i \cdot x \rangle$ is the inner product of $\widetilde{\omega}_i$, x; $\varpi_i \in \mathbb{R}$ is the threshold, and ψ is the activation function. For more knowledge of NNs, readers are recommended to read [7–9].

This paper is organized as follows: in Section 1, we present our motivation to the readers. In Section 2, step by step, we build up our activation function $\widehat{h}_\alpha$. We also include our basic estimations, which form the basis for the main results. We devote Sections 3 and 4 to the construction of density function A_α, and the creation of the $\mathbb{C}-$valued linear NN operators, respectively. In Section 5, the authors perform deformed and parametrized half-hyperbolic tangent function-activated high-order neural network approximations on continuous functions over compact intervals of a real line with complex values. All convergences have rates expressed via the modulus of continuity of the involved functions' high-order derivatives, derived from very tight Jackson-type inequalities. We conclude with Section 6.

2. Activation Function: Parametrized Half-Hyperbolic Tangent Function $\widehat{h}_\alpha$

We consider the following function inspired by [10]

$$\widehat{h}_\alpha(x) := \frac{1 - \alpha e^{-\gamma x}}{1 + \alpha e^{-\gamma x}}, \ \forall\, x \in \mathbb{R}, \tag{1}$$

to be a parametrized half-hyperbolic tangent function for $\alpha, \gamma > 0$. Also, one has $\widehat{h}_\alpha(0) = \frac{1-\alpha}{1+\alpha}$.

Proposition 1. *We observe that*

$$\widehat{h}_\alpha(-x) = -\widehat{h}_{\frac{1}{\alpha}}(x)$$

for $x \in \mathbb{R}; \alpha > 0$.

Proposition 2. *The function $\widehat{h}_\alpha$ is strictly increasing on $\mathbb{R}$. Because*

$$\widehat{h}'_\alpha(x) = \left(\frac{1 - \alpha e^{-\gamma x}}{1 + \alpha e^{-\gamma x}} \right)' = \frac{2\alpha\gamma e^{\gamma x}}{(e^{\gamma x} + \alpha)^2} > 0,$$

for every $x \in \mathbb{R}; \alpha, \gamma > 0$.

Proposition 3. *When we take the second-order derivative of $\widehat{h}_\alpha$, we have*

$$\widehat{h}''_\alpha(x) = \left(\frac{2\alpha\gamma e^{\gamma x}}{(e^{\gamma x} + \alpha)^2} \right)' = 2\alpha\gamma^2 e^{\gamma x} \frac{(\alpha - e^{\gamma x})}{(\alpha + e^{\gamma x})^3} \in C(\mathbb{R}), \forall x \in \mathbb{R}; \alpha, \gamma > 0. \tag{2}$$

According to the above calculation, the following are true:

Case 1: if $x < \frac{\ln \alpha}{\gamma}$, then $\widehat{h}_\alpha$ is strictly concave upward, with $\widehat{h}''_\alpha\left(\frac{\ln \alpha}{\gamma}\right) = 0$.
Case 2: if $x > \frac{\ln \alpha}{\gamma}$, then $\widehat{h}_\alpha$ is strictly concave downward.

3. Construction of Density Function A_α

In this section, we aim to create the density function A_α, and we also present its basic properties, which will be used throughout the paper. It is clear that $1 > -1 \Longrightarrow x+1 > x-1$. So, for every $x \in \mathbb{R}; \alpha, \gamma > 0$, let us consider the following density function:

$$A_\alpha(x) := \frac{1}{4}\Big(\widehat{h}_\alpha(x+1) - \widehat{h}_\alpha(x-1)\Big) > 0.$$

Furthermore, note that

$$\lim_{x\to+\infty} A_\alpha(x) = \lim_{x\to-\infty} A_\alpha(x) = 0,$$

so the x-axis is a horizontal asymptote. One can write that

$$\begin{aligned}
A_\alpha(-x) &= \frac{1}{4}\Big(\widehat{h}_\alpha(-x+1) - \widehat{h}_\alpha(-x-1)\Big) \\
&= \frac{1}{4}\left(-\frac{\frac{1}{\alpha}e^{-\gamma(x+1)} - 1}{\frac{1}{\alpha}e^{-\gamma(x+1)} + 1} + \frac{\frac{1}{\alpha}e^{-\gamma(x-1)} - 1}{\frac{1}{\alpha}e^{-\gamma(x-1)} + 1}\right) \\
&= \frac{1}{4}\Big(\widehat{h}_{\frac{1}{\alpha}}(x+1) - \widehat{h}_{\frac{1}{\alpha}}(x-1)\Big) \\
&= A_{\frac{1}{\alpha}}(x),\ x \in \mathbb{R}; \alpha > 0.
\end{aligned}$$

Thus,

$$A_\alpha(-x) = A_{\frac{1}{\alpha}}(x), \forall x \in \mathbb{R}; \alpha > 0. \tag{3}$$

Remark 1. *We have*

$$A_\alpha'(x) := \frac{1}{4}\Big(\widehat{h}_\alpha'(x+1) - \widehat{h}_\alpha'(x-1)\Big).$$

Then,

(i) Let $x < \frac{\ln\alpha}{\gamma} - 1$, then $x-1 < x+1 < \frac{\ln\alpha}{\gamma}$, and $\widehat{h}_\alpha'(x+1) > \widehat{h}_\alpha'(x-1)$, that is, $A_\alpha'(x) > 0$. Thus, A_α is strictly increasing on $\left(-\infty, \frac{\ln\alpha}{\gamma} - 1\right)$.

(ii) Let $x-1 > \frac{\ln\alpha}{\gamma}$, then $x+1 > x-1 > \frac{\ln\alpha}{\gamma}$, and $\widehat{h}_\alpha'(x+1) < \widehat{h}_\alpha'(x-1)$, namely, $A_\alpha'(x) < 0$. Therefore, A_α is strictly decreasing on $\left(\frac{\ln\alpha}{\gamma} + 1, +\infty\right)$.

Remark 2. *Let $\frac{\ln\alpha}{\gamma} - 1 \le x \le \frac{\ln\alpha}{\gamma} + 1$. Then,*

$$\begin{aligned}
A_\alpha''(x) &= \frac{1}{4}\left(\widehat{h}_\alpha''(x+1) - \widehat{h}_\alpha''(x-1)\right) \\
&= \frac{\alpha\gamma^2}{2}\left(e^{\gamma(x+1)}\frac{\left(\alpha - e^{\gamma(x+1)}\right)}{\left(\alpha + e^{\gamma(x+1)}\right)^3} - e^{\gamma(x-1)}\frac{\left(\alpha - e^{\gamma(x-1)}\right)}{\left(\alpha + e^{\gamma(x-1)}\right)^3}\right)
\end{aligned} \tag{4}$$

Explicitly, according to (4); one determines that $A_\alpha''(x) \le 0$ for $x \in \left[\frac{\ln\alpha}{\gamma} - 1, \frac{\ln\alpha}{\gamma} + 1\right]$, and is strictly concave downward on $\left(\frac{\ln\alpha}{\gamma} - 1, \frac{\ln\alpha}{\gamma} + 1\right)$. Therefore, A_α is a bell-shaped function on $\mathbb{R}$. Moreover, $A_\alpha''\left(\frac{\ln\alpha}{\gamma}\right) < 0$ is satisfied.

Remark 3. *The maximum value of A_α is*

$$A_\alpha\left(\frac{\ln\alpha}{\gamma}\right) = \frac{\widehat{h}_1(1)}{2}.$$

Theorem 1 ([10]). *We determine that*

$$\sum_{j=-\infty}^{\infty} A_\alpha(x-j) = 1,\ \forall\, x \in \mathbb{R}, \forall\, \alpha > 0.$$

Thus

$$\sum_{j=-\infty}^{\infty} A_\alpha(nx-j) = 1, \forall\, n \in \mathbb{N},\ \forall\, x \in \mathbb{R}.$$

In this manner, it holds that

$$\sum_{j=-\infty}^{\infty} A_{\frac{1}{\alpha}}(x-j) = 1, \forall\, x \in \mathbb{R}.$$

However,

$$A_{\frac{1}{\alpha}}(x-j) \overset{(3)}{=} A_\alpha(j-x).$$

So

$$\sum_{j=-\infty}^{\infty} A_\alpha(j-x) = 1, \forall\, x \in \mathbb{R}, \alpha > 0,$$

and

$$\sum_{j=-\infty}^{\infty} A_\alpha(j+x) = 1, \forall\, x \in \mathbb{R}, \alpha > 0.$$

Theorem 2 ([10]). *It holds that*

$$\int_{-\infty}^{\infty} A_\alpha(x)dx = 1;\ \alpha > 0.$$

Thus, this means that A_α is a density function over $\mathbb{R}$ such that $\alpha > 0$.

Next, the following result is needed.

Theorem 3 ([10]). *Let $0 < \lambda < 1$, and $n \in \mathbb{N}$ with $n^{1-\lambda} > 2$; $\alpha, \gamma > 0$. Then,*

$$\sum_{\begin{cases} k = -\infty \\ : |nx-k| \geq n^{1-\lambda} \end{cases}}^{\infty} A_\alpha(nx-k) < 2\max\left\{\alpha, \frac{1}{\alpha}\right\} e^{2\gamma} e^{-\gamma n^{(1-\lambda)}} =: Ce^{-\gamma n^{(1-\lambda)}}, \tag{5}$$

where $C := 2\max\left\{\alpha, \frac{1}{\alpha}\right\} e^{2\gamma}$.

Let $\lceil \cdot \rceil$ and $\lfloor \cdot \rfloor$ be the ceiling of the number and the integral part of the number, respectively.

Let us continue with the following conclusion:

Theorem 4 ([10]). *Let $x \in [a,b] \subset \mathbb{R}$ and $n \in \mathbb{N}$ so that $\lceil na \rceil \leq \lfloor nb \rfloor$. For $\alpha, \gamma > 0$, we consider the number $\gamma_\alpha > \rho_0 > 0$ with $A_\alpha(\rho_0) = A_\alpha(0)$, and $\gamma_\alpha > 1$. Then,*

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} A_\alpha(nx-k)} < max\left\{ \frac{1}{A_\alpha(\gamma_\alpha)}, \frac{1}{A_{\frac{1}{\alpha}}\left(\gamma_{\frac{1}{\alpha}}\right)} \right\} =: \nabla(\alpha). \tag{6}$$

We also mention the following:

Remark 4 ([10])**.** *(i) We also notice that*

$$\lim_{n\to+\infty}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} A_\alpha(nx-k)\neq 1,\ \text{for at least some } x\in[a,b],$$

where $\alpha>0$.

(ii) Let $[a,b]\subset\mathbb{R}$*. For large n, we always have* $\lceil na\rceil\leq\lfloor nb\rfloor$*. Also* $a\leq\frac{k}{n}\leq b$*, iff* $\lceil na\rceil\leq k\leq\lfloor nb\rfloor$*. In general, it holds that*

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} A_\alpha(nx-k)\leq 1. \tag{7}$$

4. Generation of $\mathbb{C}$-Valued Linear NN Operators

Let $(\mathbb{C},|\cdot|)$ be the Banach space of the complex numbers on the reals.

Definition 1. *Let* $f\in C([a,b],\mathbb{C})$ *and* $n\in\mathbb{N}:\lceil na\rceil\leq\lfloor nb\rfloor$*. We introduce and define the* $\mathbb{C}$*-valued linear neural network operators as follows:*

$$\Theta_n(f,x):=\frac{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} f\left(\frac{k}{n}\right)A_\alpha(nx-k)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} A_\alpha(nx-k)},\ x\in[a,b];\ \alpha>0,\alpha\neq 1. \tag{8}$$

For large enough n, we always obtain $\lceil na\rceil\leq\lfloor nb\rfloor$*. Also,* $a\leq\frac{k}{n}\leq b$*, iff* $\lceil na\rceil\leq k\leq\lfloor nb\rfloor$*. The same* Θ_n *may be used for real-valued functions. Here, we study the pointwise and uniform convergence of* $\Theta_n(f,x)$ *to* $f(x)$ *with rates.*

Clearly, here, $\Theta_n(f)\in C([a,b],\mathbb{C})$.

For the sake of usefulness, we can follow the following:

$$\Theta_n^*(f,x):=\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} f\left(\frac{k}{n}\right)A_\alpha(nx-k), \tag{9}$$

that is,

$$\Theta_n(f,x):=\frac{\Theta_n^*(f,x)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} A_\alpha(nx-k)}. \tag{10}$$

so that

$$\Theta_n(f,x)-f(x)=\frac{\Theta_n^*(f,x)-f(x)\left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} A_\alpha(nx-k)\right)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} A_\alpha(nx-k)}. \tag{11}$$

Therefore, we state that

$$|\Theta_n(f,x)-f(x)|\leq\nabla(\alpha)\left|\Theta_n^*(f,x)-f(x)\left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} A_\alpha(nx-k)\right)\right|$$

$$= \nabla(\alpha) \left| \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left(f\left(\frac{k}{n}\right) - f(x) \right) A_\alpha(nx-k) \right|, \tag{12}$$

where $\nabla(\alpha)$ is as in (6).

We will calculate (12) by virtue of the classical first modulus of continuity for $f \in C([a,b],\mathbb{C})$ defined below:

$$\omega_1(f,\delta) := \sup_{\substack{x,y \in [a,b] \\ |x-y| \leq \delta}} |f(x) - f(y)|, \ \ \delta > 0.$$

Moreover, $f \in C([a,b],\mathbb{C})$ is equivalent to $\lim_{\delta \to 0} \omega_1(f,\delta) = 0$ (see [6]).

5. Approximation Results

Now, we are ready to perform $\mathbb{C}$-valued neural network high-order approximations to a function f given with rates. Let us start with a trigonometric approach.

Theorem 5. *Let $f \in C^2([a,b],\mathbb{C})$, $0 < \lambda < 1$, $n \in \mathbb{N} : n^{1-\lambda} > 2$, $x \in [a,b]$. Then,*

(1)

$$\begin{aligned} |\Theta_n(f,x) - f(x)| \leq \nabla(\alpha) \Bigg[|f'(x)| \left(\frac{1}{n^\lambda} + (b-a)Ce^{-\gamma n^{(1-\lambda)}} \right) \\ + \frac{|f''(x)|}{2} \left(\frac{1}{n^{2\lambda}} + (b-a)^2 Ce^{-\gamma n^{(1-\lambda)}} \right) + \\ \left(\frac{\omega_1\left(f''+f, \frac{1}{n^\lambda}\right)}{2n^{2\lambda}} + \|f''+f\|_\infty (b-a)^2 Ce^{-\gamma n^{(1-\lambda)}} \right) \Bigg], \end{aligned}$$

(2) If $f'(x) = f''(x) = 0$, we obtain

$$|\Theta_n(f,x) - f(x)| \leq \nabla(\alpha) \left[\frac{\omega_1\left(f''+f, \frac{1}{n^\lambda}\right)}{2n^{2\lambda}} + \|f''+f\|_\infty (b-a)^2 Ce^{-\gamma n^{(1-\lambda)}} \right],$$

Note here that there is a high rate of convergence at $n^{-3\lambda}$.

(3) In addition, we obtain

$$\begin{aligned} \|\Theta_n f - f\|_\infty \leq \nabla(\alpha) \Bigg[\|f'\|_\infty \left(\frac{1}{n^\lambda} + (b-a)Ce^{-\gamma n^{(1-\lambda)}} \right) \\ + \frac{\|f''\|_\infty}{2} \left(\frac{1}{n^{2\lambda}} + (b-a)^2 Ce^{-\gamma n^{(1-\lambda)}} \right) \\ + \left(\frac{\omega_1\left(f''+f, \frac{1}{n^\lambda}\right)}{2n^{2\lambda}} + \|f''+f\|_\infty (b-a) Ce^{-\gamma n^{(1-\lambda)}} \right) \Bigg], \end{aligned}$$

namely, $\lim_{n \to +\infty} \Theta_n(f) = f$, pointwise and uniformly,

(4) Eventually, it holds that

$$\begin{aligned} \left| \Theta_n(f,x) - f'(x)\Theta_n(\sin(\cdot - x), x) - 2f''(x)\Theta_n\left(\sin^2\left(\frac{\cdot - x}{2}\right), x\right) - f(x) \right| \\ \leq \nabla(\alpha) \left[\frac{\omega_1\left(f''+f, \frac{1}{n^\lambda}\right)}{2n^{2\lambda}} + \|f''+f\|_\infty (b-a)^2 Ce^{-\gamma n^{(1-\lambda)}} \right], \end{aligned}$$

and a high speed of convergence at $n^{-3\lambda}$ is gained.

Proof. Inspired by [3], and applying the trigonometric Taylor's formula for $f \in C^2([a,b],\mathbb{C})$, let $\frac{k}{n}, x \in [a,b]$; then,

$$f\left(\frac{k}{n}\right) = f(x) + f'(x)\sin\left(\frac{k}{n} - x\right) + 2f''(x)\sin^2\left(\frac{\frac{k}{n} - x}{2}\right)$$

$$+ \int_x^{\frac{k}{n}} \left[\left(f''(t) + f(t)\right) - \left(f''(x) + f(x)\right)\right]\sin\left(\frac{k}{n} - t\right)dt.$$

Furthermore, it holds that

$$f\left(\frac{k}{n}\right)A_\alpha(nx-k) = f(x)A_\alpha(nx-k)$$

$$+f'(x)\sin\left(\frac{k}{n} - x\right)A_\alpha(nx-k) + 2f''(x)\sin^2\left(\frac{\frac{k}{n} - x}{2}\right)A_\alpha(nx-k)$$

$$+A_\alpha(nx-k)\left(\int_x^{\frac{k}{n}} \left[\left(f''(t) + f(t)\right) - \left(f''(x) + f(x)\right)\right]\sin\left(\frac{k}{n} - t\right)dt\right).$$

So, we have

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} f\left(\frac{k}{n}\right)A_\alpha(nx-k) - f(x)\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} A_\alpha(nx-k)$$

$$= f'(x)\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} A_\alpha(nx-k)\sin\left(\frac{k}{n} - x\right) + 2f''(x)\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} A_\alpha(nx-k)\sin^2\left(\frac{\frac{k}{n} - x}{2}\right)$$

$$+ \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} A_\alpha(nx-k)\left(\int_x^{\frac{k}{n}} \left[\left(f''(t) + f(t)\right) - \left(f''(x) + f(x)\right)\right]\sin\left(\frac{k}{n} - t\right)dt\right).$$

Hence, we gain

$$\Theta_n^*(f,x) - f(x)\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} A_\alpha(nx-k)$$

$$= f'(x)\Theta_n^*(\sin(\cdot - x), x) + 2f''(x)\Theta_n^*\left(\sin^2\left(\frac{\cdot - x}{2}\right), x\right) + \Gamma_n(x),$$

where

$$\Gamma_n(x) := \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} A_\alpha(nx-k)\left(\int_x^{\frac{k}{n}} \left[\left(f''(t) + f(t)\right) - \left(f''(x) + f(x)\right)\right]\sin\left(\frac{k}{n} - t\right)dt\right).$$

We assume that

$$\widehat{\Gamma}(n) := \int_x^{\frac{k}{n}} \left[\left(f''(t) + f(t)\right) - \left(f''(x) + f(x)\right)\right]\sin\left(\frac{k}{n} - t\right)dt.$$

when $n > \left\lceil (b-a)^{-\frac{1}{\lambda}} \right\rceil$, in other words, when large enough $n \in \mathbb{N}$, $b - a > \frac{1}{n^\lambda}$ is assumed. Thus, this yields $\left|\frac{k}{n} - x\right| \leq \frac{1}{n^\lambda}$ or $\left|\frac{k}{n} - x\right| > \frac{1}{n^\lambda}$.

For the case $\left|\frac{k}{n} - x\right| \leq \frac{1}{n^\lambda}$, the following are obtained:

(i) If $\frac{k}{n} \geq x$, then

$$\left|\widehat{\Gamma}(n)\right| = \left|\int_x^{\frac{k}{n}} \left[\left(f''(t) + f(t)\right) - \left(f''(x) + f(x)\right)\right] \sin\left(\frac{k}{n} - t\right) dt\right|$$
$$\leq \int_x^{\frac{k}{n}} \omega_1(f'' + f, t - x) \left|\sin\left(\frac{k}{n} - t\right)\right| dt$$

(employing $|\sin x| \leq |x|, \forall\, x \in \mathbb{R}$)

$$\leq \int_x^{\frac{k}{n}} \omega_1(f'' + f, t - x)\left(\frac{k}{n} - t\right) dt \leq \omega_1\left(f'' + f, \frac{k}{n} - x\right) \frac{\left(\frac{k}{n} - x\right)^2}{2}$$
$$\leq \frac{\omega_1\left(f'' + f, \frac{1}{n^\lambda}\right)}{2n^{2\lambda}},$$

namely,

$$\left|\widehat{\Gamma}(n)\right| \leq \frac{\omega_1\left(f'' + f, \frac{1}{n^\lambda}\right)}{2n^{2\lambda}}.$$

(ii) If $\frac{k}{n} < x$, then

$$\left|\widehat{\Gamma}(n)\right| = \left|\int_x^{\frac{k}{n}} \left[\left(f''(t) + f(t)\right) - \left(f''(x) + f(x)\right)\right] \sin\left(\frac{k}{n} - t\right) dt\right|$$
$$= \left|\int_{\frac{k}{n}}^{x} \left[\left(f''(t) + f(t)\right) - \left(f''(x) + f(x)\right)\right] \sin\left(\frac{k}{n} - t\right) dt\right|$$
$$\leq \int_{\frac{k}{n}}^{x} \left|\left(f''(t) + f(t)\right) - \left(f''(x) + f(x)\right)\right| \left|\sin\left(\frac{k}{n} - t\right)\right| dt$$
$$\leq \int_{\frac{k}{n}}^{x} \omega_1\left(f'' + f, x - \frac{k}{n}\right)\left(t - \frac{k}{n}\right) dt$$
$$\leq \omega_1\left(f'' + f, x - \frac{k}{n}\right) \frac{\left(x - \frac{k}{n}\right)^2}{2}$$
$$\leq \frac{\omega_1\left(f'' + f, \frac{1}{n^\lambda}\right)}{2n^{2\lambda}}.$$

Therefore,

$$\left|\widehat{\Gamma}(n)\right| \leq \frac{\omega_1\left(f'' + f, \frac{1}{n^\lambda}\right)}{2n^{2\lambda}}.$$

So, we have proven that when $\left|\frac{k}{n} - x\right| \leq \frac{1}{n^\lambda}$, it is always true that

$$\left|\widehat{\Gamma}(n)\right| \leq \frac{\omega_1\left(f'' + f, \frac{1}{n^\lambda}\right)}{2n^{2\lambda}}.$$

(a) Again let $\frac{k}{n} \geq x$:

$$\left|\widehat{\Gamma}(n)\right| = \left|\int_x^{\frac{k}{n}} \left[\left(f''(t) + f(t)\right) - \left(f''(x) + f(x)\right)\right] \sin\left(\frac{k}{n} - t\right) dt\right|$$
$$\leq \int_x^{\frac{k}{n}} \left|\left(f''(t) + f(t)\right) - \left(f''(x) + f(x)\right)\right| \left|\sin\left(\frac{k}{n} - t\right)\right| dt$$

(employing $|\sin x| \leq |x|, \forall\, x \in \mathbb{R}$)

$$\leq 2\|f'' + f\|_\infty \left(\int_x^{\frac{k}{n}} \left(\frac{k}{n} - t \right) \right) dt$$

$$= 2\|f'' + f\|_\infty \frac{\left(\frac{k}{n} - x \right)^2}{2} \leq \|f'' + f\|_\infty (b-a)^2.$$

Thus, we obtain

$$\left| \widehat{\Gamma}(n) \right| \leq \|f'' + f\|_\infty (b-a)^2.$$

(b) One more time, let $\frac{k}{n} < x$,

$$\begin{aligned}
\left| \widehat{\Gamma}(n) \right| &= \left| \int_x^{\frac{k}{n}} \left[\left(f''(t) + f(t) \right) - \left(f''(x) + f(x) \right) \right] \sin\left(\frac{k}{n} - t \right) dt \right| \\
&= \left| \int_{\frac{k}{n}}^x \left[\left(f''(x) + f(x) \right) - \left(f''(t) + f(t) \right) \right] \sin\left(\frac{k}{n} - t \right) dt \right| \\
&\leq \int_{\frac{k}{n}}^x \left| \left(f''(x) + f(x) \right) - \left(f''(t) + f(t) \right) \right| \left| \sin\left(\frac{k}{n} - t \right) \right| dt \\
&\leq 2\|f'' + f\|_\infty \int_{\frac{k}{n}}^x \left| \sin\left(\frac{k}{n} - t \right) \right| dt \\
&\leq 2\|f'' + f\|_\infty \int_{\frac{k}{n}}^x \left(t - \frac{k}{n} \right) dt = \|f'' + f\|_\infty \left(x - \frac{k}{n} \right)^2 \\
&\leq \|f'' + f\|_\infty (b-a)^2.
\end{aligned}$$

So, we gain

$$\left| \widehat{\Gamma}(n) \right| \leq \|f'' + f\|_\infty (b-a)^2.$$

Also, we have

$$\Gamma_n(x) = \sum_{\substack{k = \lceil na \rceil \\ : \left| \frac{k}{n} - x \right| \leq \frac{1}{n^\lambda}}}^{\lfloor nb \rfloor} A_\alpha(nx-k) \widehat{\Gamma}(n) + \sum_{\substack{k = \lceil na \rceil \\ : \left| \frac{k}{n} - x \right| > \frac{1}{n^\lambda}}}^{\lfloor nb \rfloor} A_\alpha(nx-k) \widehat{\Gamma}(n).$$

Thus, it follows that

$$|\Gamma_n(x)| \leq \sum_{\substack{k = \lceil na \rceil \\ : \left| \frac{k}{n} - x \right| \leq \frac{1}{n^\lambda}}}^{\lfloor nb \rfloor} A_\alpha(nx-k) \left| \widehat{\Gamma}(n) \right| + \sum_{\substack{k = \lceil na \rceil \\ : \left| \frac{k}{n} - x \right| > \frac{1}{n^\lambda}}}^{\lfloor nb \rfloor} A_\alpha(nx-k) \left| \widehat{\Gamma}(n) \right|$$

$$\leq \left(\sum_{\begin{cases} k = \lceil na \rceil \\ : \left| \frac{k}{n} - x \right| \leq \frac{1}{n^{\lambda}} \end{cases}}^{\lfloor nb \rfloor} A_{\alpha}(nx-k) \right) \frac{\omega_1\left(f'' + f, \frac{1}{n^{\lambda}}\right)}{2n^{2\lambda}}$$

$$+ \sum_{\begin{cases} k = \lceil na \rceil \\ : \left| \frac{k}{n} - x \right| > \frac{1}{n^{\lambda}} \end{cases}}^{\lfloor nb \rfloor} A_{\alpha}(nx-k) \left\| f'' + f \right\|_{\infty} (b-a)^2 \overset{\text{(by (7))}}{\leq}$$

$$\frac{\omega_1\left(f'' + f, \frac{1}{n^{\lambda}}\right)}{2n^{2\lambda}} + \left\| f'' + f \right\|_{\infty} (b-a)^2$$

$$\left(\sum_{\begin{cases} k = \lceil na \rceil \\ : \left| \frac{k}{n} - x \right| > \frac{1}{n^{\lambda}} \end{cases}}^{\lfloor nb \rfloor} A_{\alpha}(nx-k) \right) \overset{\text{(by Theorem 3)}}{\leq}$$

$$\frac{\omega_1\left(f'' + f, \frac{1}{n^{\lambda}}\right)}{2n^{2\lambda}} + \left\| f'' + f \right\|_{\infty} (b-a)^2 C e^{-\gamma n^{(1-\lambda)}}.$$

As a result, we derive

$$|\Gamma_n(x)| \leq \frac{\omega_1\left(f'' + f, \frac{1}{n^{\lambda}}\right)}{2n^{2\lambda}} + \left\| f'' + f \right\|_{\infty} (b-a)^2 C e^{-\gamma n^{(1-\lambda)}}. \tag{13}$$

Again, we apply $|\sin x| \leq |x|$, $\forall\, x \in \mathbb{R}$.
We obtain

$$\Theta_n^*(\sin(\cdot - x), x) = \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} A_{\alpha}(nx-k) \sin\left(\frac{k}{n} - x\right),$$

and

$$|\Theta_n^*(\sin(\cdot - x), x)| \leq \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} A_{\alpha}(nx-k) \left| \sin\left(\frac{k}{n} - x\right) \right|$$

$$= \sum_{\begin{cases} k = \lceil na \rceil \\ : \left| \frac{k}{n} - x \right| \leq \frac{1}{n^{\lambda}} \end{cases}}^{\lfloor nb \rfloor} A_{\alpha}(nx-k) \left| \sin\left(\frac{k}{n} - x\right) \right|$$

$$+ \sum_{\begin{cases} k = \lceil na \rceil \\ : \left| \frac{k}{n} - x \right| > \frac{1}{n^{\lambda}} \end{cases}}^{\lfloor nb \rfloor} A_{\alpha}(nx-k) \left| \sin\left(\frac{k}{n} - x\right) \right|$$

$$\leq \sum_{\begin{cases} k = \lceil na \rceil \\ : \left| \frac{k}{n} - x \right| \leq \frac{1}{n^{\alpha}} \end{cases}}^{\lfloor nb \rfloor} A_{\alpha}(nx-k) \left| \frac{k}{n} - x \right|$$

$$+ \sum_{\begin{cases} k = \lceil na \rceil \\ : \left| \frac{k}{n} - x \right| > \frac{1}{n^{\lambda}} \end{cases}}^{\lfloor nb \rfloor} A_{\alpha}(nx-k) \left| \frac{k}{n} - x \right|$$

$$\leq \frac{1}{n^{\lambda}} + (b-a) \left(\sum_{\begin{cases} k = \lceil na \rceil \\ : \left| \frac{k}{n} - x \right| > \frac{1}{n^{\lambda}} \end{cases}}^{\lfloor nb \rfloor} A_{\alpha}(nx-k) \right)$$

$$\overset{(\text{by } (7))}{\leq} \frac{1}{n^{\lambda}} + (b-a) C e^{-\gamma n^{(1-\lambda)}}.$$

We determine that

$$\left| \Theta_n^*(\sin(\cdot - x), x) \right| \leq \frac{1}{n^{\lambda}} + (b-a) C e^{-\gamma n^{(1-\lambda)}}.$$

Then, we calculate

$$\Theta_n^*\left(\sin^2\left(\frac{\cdot - x}{2} \right), x \right) = \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} A_{\alpha}(nx-k) \sin^2\left(\frac{\frac{k}{n} - x}{2} \right),$$

using $|\sin x| \leq |x|$, $\forall\, x \in \mathbb{R}$, and calculate

$$\Theta_n^*\left(\sin^2\left(\frac{\cdot - x}{2} \right), x \right) = \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} A_{\alpha}(nx-k) \left| \sin\left(\frac{\frac{k}{n} - x}{2} \right) \right|^2$$

$$\leq \frac{1}{4} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} A_{\alpha}(nx-k) \left| \frac{k}{n} - x \right|^2$$

$$= \frac{1}{4} \left[\sum_{\begin{cases} k = \lceil na \rceil \\ : \left| \frac{k}{n} - x \right| \leq \frac{1}{n^{\lambda}} \end{cases}}^{\lfloor nb \rfloor} A_{\alpha}(nx-k) \left| \frac{k}{n} - x \right|^2 \right.$$

$$\left. + \sum_{\begin{cases} k = \lceil na \rceil \\ : \left| \frac{k}{n} - x \right| > \frac{1}{n^{\lambda}} \end{cases}}^{\lfloor nb \rfloor} A_{\alpha}(nx-k) \left| \frac{k}{n} - x \right|^2 \right]$$

$$\leq \frac{1}{4} \left[\frac{1}{n^{2\lambda}} + (b-a)^2 C e^{-\gamma n^{(1-\lambda)}} \right],$$

i.e.,

$$\Theta_n^*\left(\sin^2\left(\frac{\cdot - x}{2} \right), x \right) \leq \frac{1}{4} \left[\frac{1}{n^{2\lambda}} + (b-a)^2 C e^{-\gamma n^{(1-\lambda)}} \right].$$

As a consequence, we obtain the following:

(1)

$$|\Theta_n(f,x) - f(x)| \leq \nabla(\alpha)\left[|f'(x)|\left(\frac{1}{n^{\lambda}} + (b-a)Ce^{-\gamma n^{(1-\lambda)}}\right) + \frac{|f''(x)|}{2}\left(\frac{1}{n^{2\lambda}} + (b-a)^2 Ce^{-\gamma n^{(1-\lambda)}}\right) + \left(\frac{\omega_1\left(f''+f, \frac{1}{n^{\lambda}}\right)}{2n^{2\lambda}} + \|f''+f\|_{\infty}(b-a)^2 Ce^{-\gamma n^{(1-\lambda)}}\right)\right]. \quad (14)$$

(2) If $f'(x) = f''(x) = 0$, according to (14), we have

$$|\Theta_n(f,x) - f(x)| \leq \nabla(\alpha)\left[\frac{\omega_1\left(f''+f, \frac{1}{n^{\lambda}}\right)}{2n^{2\lambda}} + \|f''+f\|_{\infty}(b-a)^2 Ce^{-\gamma n^{(1-\lambda)}}\right].$$

Here, we keep in mind that $n^{-3\lambda}$ has a high convergence rate.

(3) In addition, according to (14), we have

$$\|\Theta_n f - f\|_{\infty} \leq \nabla(\alpha)\left[\|f'\|_{\infty}\left(\frac{1}{n^{\lambda}} + (b-a)Ce^{-\gamma n^{(1-\lambda)}}\right) + \frac{\|f''\|_{\infty}}{2}\left(\frac{1}{n^{2\lambda}} + (b-a)^2 Ce^{-\gamma n^{(1-\lambda)}}\right) + \left(\frac{\omega_1\left(f''+f, \frac{1}{n^{\lambda}}\right)}{2n^{2\lambda}} + \|f''+f\|_{\infty}(b-a)^2 Ce^{-\gamma n^{(1-\lambda)}}\right)\right].$$

We state that convergence is pointwise and uniform such that $\lim_{n\to+\infty}\Theta_n(f) = f$.

We consider that

$$\Theta_n(f,x) - f'(x)\Theta_n(\sin(\cdot - x), x) - 2f''(x)\Theta_n\left(\sin^2\left(\frac{\cdot - x}{2}\right), x\right) - f(x)$$

$$= \frac{\Theta_n^*(f,x)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} A_\alpha(nx-k)} - f'(x)\frac{\Theta_n^*(\sin(\cdot - x), x)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} A_\alpha(nx-k)} - 2f''(x)\frac{\Theta_n^*\left(\sin^2\left(\frac{\cdot - x}{2}\right), x\right)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} A_\alpha(nx-k)} - f(x)\left(\frac{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} A_\alpha(nx-k)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} A_\alpha(nx-k)}\right)$$

$$= \frac{1}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} A_\alpha(nx-k)}\left[\Theta_n^*(f,x) - f'(x)\Theta_n^*(\sin(\cdot - x), x) - 2f''(x)\Theta_n^*\left(\sin^2\left(\frac{\cdot - x}{2}\right), x\right) - f(x)\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} A_\alpha(nx-k)\right]$$

$$= \frac{1}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} A_\alpha(nx-k)}(\Gamma_n(x)).$$

Lastly, we obtain ($\forall\, x \in [a,b]$, $n \in \mathbb{N}$):

(4)

$$\left|\Theta_n(f,x) - f'(x)\Theta_n(\sin(\cdot - x), x) - 2f''(x)\Theta_n\left(\sin^2\left(\frac{\cdot - x}{2}\right), x\right) - f(x)\right|$$
$$\overset{(6)}{\leq} \nabla(\alpha)|\Gamma_n(x)|$$
$$\overset{(13)}{\leq} \nabla(\alpha)\left[\frac{\omega_1\left(f''+f, \frac{1}{n^\lambda}\right)}{2n^{2\lambda}} + \|f''+f\|_\infty (b-a)^2 C e^{-\gamma n^{(1-\lambda)}}\right].$$

The theorem is proved. □

We move ahead with a hyperbolic high-order neural network approximation.

Theorem 6. *Let $f \in C^2([a,b],\mathbb{C})$, $0 < \lambda < 1$, $n \in \mathbb{N} : n^{1-\lambda} > 2$, $x \in [a,b]$. Then,*

(1)

$$|\Theta_n(f,x) - f(x)| \leq \nabla(\alpha)\cosh(b-a)\left[|f'(x)|\left(\frac{1}{n^\lambda} + (b-a)Ce^{-\gamma n^{(1-\lambda)}}\right)\right.$$
$$\frac{|f''(x)|}{2}\left(\frac{1}{n^{2\lambda}} + (b-a)^2 Ce^{-\gamma n^{(1-\lambda)}}\right)$$
$$\left. + \left(\frac{\omega_1\left(f''-f, \frac{1}{n^\lambda}\right)}{2n^{2\lambda}} + \|f''-f\|_\infty (b-a)^2 C e^{-\gamma n^{(1-\lambda)}}\right)\right],$$

(2) If $f'(x) = f''(x) = 0$, we obtain

$$|\Theta_n(f,x) - f(x)| \leq \nabla(\alpha)\cosh(b-a)$$
$$\left[\frac{\omega_1\left(f''-f, \frac{1}{n^\lambda}\right)}{2n^{2\lambda}} + \|f''-f\|_\infty (b-a)^2 C e^{-\gamma n^{(1-\lambda)}}\right],$$

considering the high rate of convergence at $n^{-3\lambda}$.

(3) In addition, we obtain

$$\|\Theta_n f - f\|_\infty \leq \nabla(\alpha)\cosh(b-a)\left[\|f'\|_\infty\left(\frac{1}{n^\lambda} + (b-a)Ce^{-\gamma n^{(1-\lambda)}}\right)\right.$$
$$\frac{\|f''\|_\infty}{2}\left(\frac{1}{n^{2\lambda}} + (b-a)^2 Ce^{-\gamma n^{(1-\lambda)}}\right)$$
$$\left. + \left(\frac{\omega_1\left(f''-f, \frac{1}{n^\lambda}\right)}{2n^{2\lambda}} + \|f''-f\|_\infty (b-a)^2 C e^{-\gamma n^{(1-\lambda)}}\right)\right],$$

it yields that $\lim_{n\to+\infty} \Theta_n(f) = f$, pointwise and uniformly, and

(4)

$$\left|\Theta_n(f,x) - f'(x)\Theta_n(\sinh(\cdot - x), x) - 2f''(x)\Theta_n\left(\sinh^2\left(\frac{\cdot - x}{2}\right), x\right) - f(x)\right|$$
$$\leq \nabla(\alpha)\cosh(b-a)\left[\frac{\omega_1\left(f''-f, \frac{1}{n^\lambda}\right)}{2n^{2\lambda}} + \|f''-f\|_\infty (b-a)^2 C e^{-\gamma n^{(1-\lambda)}}\right],$$

Here, again, $n^{-3\lambda}$ is our high convergence speed.

Proof. Using the mean value theorem, we write

$$\sinh x = \sinh x - \sinh 0 = (\cosh \xi)(x - 0),$$

for some ξ in $(0, x)$, for any $x \in \mathbb{R}$.

Thus,

$$|\sinh x| \leq \|\cosh\|_{\infty,[-(b-a),b-a]}|x|, \ \forall\, x \in [-(b-a), b-a].$$

In other words, there exists $M \geq 1$ such that

$$|\sinh x| \leq M|x|, \ \forall\, x \in [-(b-a), b-a], \tag{15}$$

where $M := \|\cosh\|_{\infty,[-(b-a),b-a]} = \cosh(b-a)$.

Inspired by [3,4], and applying the hyperbolic Taylor's formula for $f \in C^2([a,b], \mathbb{C})$, when $\frac{k}{n}, x \in [a,b]$, then

$$f\left(\frac{k}{n}\right) = f(x) + f'(x)\sinh\left(\frac{k}{n} - x\right) + 2f''(x)\sinh^2\left(\frac{\frac{k}{n} - x}{2}\right)$$
$$+ \int_x^{\frac{k}{n}} \left[\left(f''(t) - f(t)\right) - \left(f''(x) - f(x)\right)\right] \sinh\left(\frac{k}{n} - t\right) dt.$$

So, it holds that

$$f\left(\frac{k}{n}\right) A_\alpha(nx-k) = f(x) A_\alpha(nx-k)$$
$$+ f'(x)\sinh\left(\frac{k}{n} - x\right) A_\alpha(nx-k) + 2f''(x)\sinh^2\left(\frac{\frac{k}{n} - x}{2}\right) A_\alpha(nx-k)$$
$$+ A_\alpha(nx-k)\left(\int_x^{\frac{k}{n}} \left[\left(f''(t) - f(t)\right) - \left(f''(x) - f(x)\right)\right] \sinh\left(\frac{k}{n} - t\right) dt\right).$$

Accordingly, we obtain

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) A_\alpha(nx-k) - f(x) \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} A_\alpha(nx-k)$$
$$= f'(x) \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} A_\alpha(nx-k)\sinh\left(\frac{k}{n} - x\right) + 2f''(x) \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} A_\alpha(nx-k)\sinh^2\left(\frac{\frac{k}{n} - x}{2}\right)$$
$$+ \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} A_\alpha(nx-k)\left(\int_x^{\frac{k}{n}} \left[\left(f''(t) - f(t)\right) - \left(f''(x) - f(x)\right)\right] \sinh\left(\frac{k}{n} - t\right) dt\right).$$

So, we obtain

$$\Theta_n^*(f, x) - f(x) \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} A_\alpha(nx-k)$$
$$= f'(x)\Theta_n^*(\sinh(\cdot - x), x) + 2f''(x)\Theta_n^*\left(\sinh^2\left(\frac{\cdot - x}{2}\right), x\right) + \Gamma_n(x), \tag{16}$$

where

$$\Gamma_n(x) := \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} A_\alpha(nx-k)\left(\int_x^{\frac{k}{n}} \left[\left(f''(t) - f(t)\right) - \left(f''(x) - f(x)\right)\right] \sinh\left(\frac{k}{n} - t\right) dt\right).$$

We assume that

$$\widehat{\Gamma}(n) := \int_x^{\frac{k}{n}} \left[\left(f''(t) - f(t) \right) - \left(f''(x) - f(x) \right) \right] \sinh\left(\frac{k}{n} - t \right) dt.$$

For large enough $n \in \mathbb{N}$, let $b - a > \frac{1}{n^\lambda}$, that is, when $n > \left\lceil (b-a)^{-\frac{1}{\lambda}} \right\rceil$.
Hence, $\left| \frac{k}{n} - x \right| \leq \frac{1}{n^\lambda}$ or $\left| \frac{k}{n} - x \right| > \frac{1}{n^\lambda}$.
For $\left| \frac{k}{n} - x \right| \leq \frac{1}{n^\lambda}$, we have the following:

(i) If $\frac{k}{n} \geq x$, then

$$\begin{aligned} \left| \widehat{\Gamma}(n) \right| &= \left| \int_x^{\frac{k}{n}} \left[\left(f''(t) - f(t) \right) - \left(f''(x) - f(x) \right) \right] \sinh\left(\frac{k}{n} - t \right) dt \right| \\ &\leq \int_x^{\frac{k}{n}} \omega_1(f'' - f, t - x) \left| \sinh\left(\frac{k}{n} - t \right) \right| dt \end{aligned}$$

$$\overset{(15)}{\leq} \int_x^{\frac{k}{n}} \omega_1(f'' - f, t - x) M \left(\frac{k}{n} - t \right) dt \leq M\omega_1\left(f'' - f, \frac{k}{n} - x \right) \int_x^{\frac{k}{n}} \left(\frac{k}{n} - t \right) dt$$

$$= M\omega_1\left(f'' - f, \frac{k}{n} - x \right) \frac{\left(\frac{k}{n} - x \right)^2}{2} \leq \frac{M\omega_1\left(f'' - f, \frac{1}{n^\lambda} \right)}{2n^{2\lambda}},$$

that is,

$$\left| \widehat{\Gamma}(n) \right| \leq \frac{M\omega_1\left(f'' - f, \frac{1}{n^\lambda} \right)}{2n^{2\lambda}}.$$

(ii) If $\frac{k}{n} < x$, then

$$\begin{aligned} \left| \widehat{\Gamma}(n) \right| &= \left| \int_x^{\frac{k}{n}} \left[\left(f''(t) - f(t) \right) - \left(f''(x) - f(x) \right) \right] \sinh\left(\frac{k}{n} - t \right) dt \right| \\ &= \left| \int_{\frac{k}{n}}^{x} \left[\left(f''(t) - f(t) \right) - \left(f''(x) - f(x) \right) \right] \sinh\left(\frac{k}{n} - t \right) dt \right| \\ &\leq \int_{\frac{k}{n}}^{x} \left| \left(f''(t) - f(t) \right) - \left(f''(x) - f(x) \right) \right| \left| \sinh\left(\frac{k}{n} - t \right) \right| dt \\ &\leq M\omega_1\left(f'' - f, x - \frac{k}{n} \right) \int_{\frac{k}{n}}^{x} \left(t - \frac{k}{n} \right) dt = M\omega_1\left(f'' - f, x - \frac{k}{n} \right) \frac{\left(x - \frac{k}{n} \right)^2}{2} \\ &\leq \frac{M\omega_1\left(f'' - f, \frac{1}{n^\lambda} \right)}{2n^{2\lambda}}, \end{aligned}$$

i.e.,

$$\left| \widehat{\Gamma}(n) \right| \leq \frac{M\omega_1\left(f'' - f, \frac{1}{n^\lambda} \right)}{2n^{2\lambda}}.$$

So, we have verified that when $\left| \frac{k}{n} - x \right| \leq \frac{1}{n^\alpha}$, it gives

$$\left| \widehat{\Gamma}(n) \right| \leq \frac{M\omega_1\left(f'' - f, \frac{1}{n^\lambda} \right)}{2n^{2\lambda}}.$$

Now, let us again assume that $\frac{k}{n} \geq x$; then,

$$\begin{aligned}
\left|\widehat{\Gamma}(n)\right| &= \left|\int_x^{\frac{k}{n}} \left[\left(f''(t) - f(t)\right) - \left(f''(x) - f(x)\right)\right] \sinh\left(\frac{k}{n} - t\right) dt\right| \\
&\leq \int_x^{\frac{k}{n}} \left|\left(f''(t) - f(t)\right) - \left(f''(x) - f(x)\right)\right| \left|\sinh\left(\frac{k}{n} - t\right)\right| dt \\
&\leq 2M\|f'' - f\|_\infty \int_x^{\frac{k}{n}} \left(\frac{k}{n} - t\right) dt \\
&= 2M\|f'' - f\|_\infty \frac{\left(\frac{k}{n} - x\right)^2}{2} \leq M\|f'' - f\|_\infty (b-a)^2.
\end{aligned}$$

Thus,

$$\left|\widehat{\Gamma}(n)\right| \leq M\|f'' - f\|_\infty (b-a)^2.$$

If $\frac{k}{n} < x$, then

$$\begin{aligned}
\left|\widehat{\Gamma}(n)\right| &= \left|\int_{\frac{k}{n}}^{x} \left[\left(f''(t) - f(t)\right) - \left(f''(x) - f(x)\right)\right] \sinh\left(\frac{k}{n} - t\right) dt\right| \\
&\leq \int_{\frac{k}{n}}^{x} \left|\left(f''(t) - f(t)\right) - \left(f''(x) - f(x)\right)\right| \left|\sinh\left(\frac{k}{n} - t\right)\right| dt \\
&\leq 2M\|f'' - f\|_\infty \int_{\frac{k}{n}}^{x} \left(t - \frac{k}{n}\right) dt \\
&= 2M\|f'' - f\|_\infty \frac{\left(x - \frac{k}{n}\right)^2}{2} \leq M\|f'' - f\|_\infty (b-a)^2.
\end{aligned}$$

Hence, it verifies that

$$\left|\widehat{\Gamma}(n)\right| \leq M\|f'' - f\|_\infty (b-a)^2.$$

Also, we have

$$\begin{aligned}
\Gamma_n(x) = &\sum_{\substack{k = \lceil na \rceil \\ : \left|\frac{k}{n} - x\right| \leq \frac{1}{n^\lambda}}}^{\lfloor nb \rfloor} A_\alpha(nx - k)\widehat{\Gamma}(n) \\
&+ \sum_{\substack{k = \lceil na \rceil \\ : \left|\frac{k}{n} - x\right| > \frac{1}{n^\lambda}}}^{\lfloor nb \rfloor} A_\alpha(nx - k)\widehat{\Gamma}(n).
\end{aligned}$$

Thus, it yields that

$$\begin{aligned}
|\Gamma_n(x)| \leq &\sum_{\substack{k = \lceil na \rceil \\ : \left|\frac{k}{n} - x\right| \leq \frac{1}{n^\lambda}}}^{\lfloor nb \rfloor} A_\alpha(nx - k)\left|\widehat{\Gamma}(n)\right| \\
&+ \sum_{\substack{k = \lceil na \rceil \\ : \left|\frac{k}{n} - x\right| > \frac{1}{n^\lambda}}}^{\lfloor nb \rfloor} A_\alpha(nx - k)\left|\widehat{\Gamma}(n)\right|
\end{aligned}$$

$$\leq \left(\sum_{\substack{k=\lceil na\rceil \\ :\left|\frac{k}{n}-x\right|\leq\frac{1}{n^{\lambda}}}}^{\lfloor nb\rfloor} A_\alpha(nx-k) \right) \frac{\omega_1\left(f''-f,\frac{1}{n^{\lambda}}\right)M}{2n^{2\lambda}}$$

$$+ \left(\sum_{\substack{k=\lceil na\rceil \\ :\left|\frac{k}{n}-x\right|>\frac{1}{n^{\lambda}}}}^{\lfloor nb\rfloor} A_\alpha(nx-k) \right) M\|f''-f\|_\infty (b-a)^2$$

$$\overset{(\text{by } (5))}{\leq} \frac{M\omega_1\left(f''-f,\frac{1}{n^{\lambda}}\right)}{2n^{2\lambda}} + M\|f''-f\|_\infty (b-a)^2 \left(\sum_{\substack{k=\lceil na\rceil \\ :\left|\frac{k}{n}-x\right|>\frac{1}{n^{\lambda}}}}^{\lfloor nb\rfloor} A_\alpha(nx-k) \right)$$

$$\overset{(\text{by Theorem 3})}{\leq} M\frac{\omega_1\left(f''-f,\frac{1}{n^{\lambda}}\right)}{2n^{2\lambda}} + M\|f''-f\|_\infty (b-a)^2 Ce^{-\gamma n^{(1-\lambda)}}.$$

Therefore, we determine that

$$|\Gamma_n(x)| \leq M\left[\frac{\omega_1\left(f''-f,\frac{1}{n^{\lambda}}\right)}{2n^{2\lambda}} + \|f''-f\|_\infty (b-a)^2 Ce^{-\gamma n^{(1-\lambda)}}\right]. \tag{17}$$

Also, we have that

$$\Theta_n^*(\sinh(\cdot-x),x) = \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} A_\alpha(nx-k)\sinh\left(\frac{k}{n}-x\right),$$

and

$$|\Theta_n^*(\sinh(\cdot-x),x)| \leq \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} A_\alpha(nx-k)\left|\sinh\left(\frac{k}{n}-x\right)\right|$$

$$= \sum_{\substack{k=\lceil na\rceil \\ :\left|\frac{k}{n}-x\right|\leq\frac{1}{n^{\lambda}}}}^{\lfloor nb\rfloor} A_\alpha(nx-k)\left|\sinh\left(\frac{k}{n}-x\right)\right|$$

$$+ \sum_{\substack{k=\lceil na\rceil \\ :\left|\frac{k}{n}-x\right|>\frac{1}{n^{\lambda}}}}^{\lfloor nb\rfloor} A_\alpha(nx-k)\left|\sinh\left(\frac{k}{n}-x\right)\right|$$

$$\leq M\left[\sum_{\substack{k=\lceil na\rceil \\ :\left|\frac{k}{n}-x\right|\leq\frac{1}{n^{\lambda}}}}^{\lfloor nb\rfloor} A_{\alpha}(nx-k)\left|\frac{k}{n}-x\right| + \sum_{\substack{k=\lceil na\rceil \\ :\left|\frac{k}{n}-x\right|>\frac{1}{n^{\lambda}}}}^{\lfloor nb\rfloor} A_{\alpha}(nx-k)\left|\frac{k}{n}-x\right|\right]$$

$$\leq M\left[\frac{1}{n^{\lambda}}+(b-a)\left(\sum_{\substack{k=\lceil na\rceil \\ :\left|\frac{k}{n}-x\right|>\frac{1}{n^{\lambda}}}}^{\lfloor nb\rfloor} A_{\alpha}(nx-k)\right)\right] \overset{\text{(by (5))}}{\leq} M\left[\frac{1}{n^{\alpha}}+(b-a)Ce^{-\gamma n^{(1-\lambda)}}\right].$$

We obtain that

$$|\Theta_n^*(\sinh(\cdot-x),x)| \leq M\left[\frac{1}{n^{\lambda}}+(b-a)Ce^{-\gamma n^{(1-\lambda)}}\right]. \tag{18}$$

Then, we calculate

$$\Theta_n^*\left(\sinh^2\left(\frac{\cdot-x}{2}\right),x\right) = \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} A_{\alpha}(nx-k)\sinh^2\left(\frac{\frac{k}{n}-x}{2}\right).$$

One determines that

$$\Theta_n^*\left(\sinh^2\left(\frac{\cdot-x}{2}\right),x\right) = \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} A_{\alpha}(nx-k)\left(\sinh\left(\frac{\frac{k}{n}-x}{2}\right)\right)^2$$

$$\leq \frac{M}{4}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} A_{\alpha}(nx-k)\left(\frac{k}{n}-x\right)^2$$

$$= \frac{M}{4}\left[\sum_{\substack{k=\lceil na\rceil \\ :\left|\frac{k}{n}-x\right|\leq\frac{1}{n^{\lambda}}}}^{\lfloor nb\rfloor} A_{\alpha}(nx-k)\left(\frac{k}{n}-x\right)^2 + \sum_{\substack{k=\lceil na\rceil \\ :\left|\frac{k}{n}-x\right|>\frac{1}{n^{\lambda}}}}^{\lfloor nb\rfloor} A_{\alpha}(nx-k)\left(\frac{k}{n}-x\right)^2\right]$$

$$\leq \frac{M}{4}\left[\frac{1}{n^{2\lambda}} + (b-a)^2 Ce^{-\gamma n^{(1-\lambda)}}\right].$$

That is,

$$\Theta_n^*\left(\sinh^2\left(\frac{\cdot - x}{2}\right), x\right) \leq \frac{M}{4}\left[\frac{1}{n^{2\lambda}} + (b-a)^2 Ce^{-\gamma n^{(1-\lambda)}}\right]. \tag{19}$$

According to (12), and based on (16)–(19), we acquire the following:

(1)

$$|\Theta_n(f,x) - f(x)| \leq \nabla(\alpha) M\Bigg[|f'(x)|\left(\frac{1}{n^{\lambda}} + (b-a)Ce^{-\gamma n^{(1-\lambda)}}\right) + \frac{|f''(x)|}{2}\left(\frac{1}{n^{2\lambda}} + (b-a)^2 Ce^{-\gamma n^{(1-\lambda)}}\right) + \left(\frac{\omega_1\left(f''-f, \frac{1}{n^{\lambda}}\right)}{2n^{2\lambda}} + \|f''-f\|_\infty (b-a)^2 Ce^{-\gamma n^{(1-\lambda)}}\right)\Bigg]. \tag{20}$$

(2) If $f'(x) = f''(x) = 0$, according to (20), we achieve that

$$|\Theta_n(f,x) - f(x)| \leq \nabla(\alpha) M\left[\frac{\omega_1\left(f''-f, \frac{1}{n^{\lambda}}\right)}{2n^{2\lambda}} + \|f''-f\|_\infty (b-a)^2 Ce^{-\gamma n^{(1-\lambda)}}\right],$$

with the high rate of convergence at $n^{-3\lambda}$.

(3) Moreover, according to (20), we gain

$$\|\Theta_n f - f\|_\infty \leq \nabla(\alpha) M\Bigg[\|f'\|_\infty\left(\frac{1}{n^{\lambda}} + (b-a)Ce^{-\gamma n^{(1-\lambda)}}\right) + \frac{\|f''\|_\infty}{2}\left(\frac{1}{n^{2\lambda}} + (b-a)^2 Ce^{-\gamma n^{(1-\lambda)}}\right) + \left(\frac{\omega_1\left(f''-f, \frac{1}{n^{\lambda}}\right)}{2n^{2\lambda}} + \|f''-f\|_\infty (b-a)^2 Ce^{-\gamma n^{(1-\lambda)}}\right)\Bigg].$$

It yields a pointwise and uniform convergence such that $\lim_{n\to+\infty} \Theta_n(f) = f$.

We note that

$$\Theta_n(f,x) - f'(x)\Theta_n(\sinh(\cdot - x), x) - 2f''(x)\Theta_n\left(\sinh^2\left(\frac{\cdot - x}{2}\right), x\right) - f(x)$$

$$= \frac{\Theta_n^*(f,x)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} A_\alpha(nx-k)} - f'(x)\frac{\Theta_n^*(\sinh(\cdot - x), x)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} A_\alpha(nx-k)} - 2f''(x)\frac{\Theta_n^*\left(\sinh^2\left(\frac{\cdot - x}{2}\right), x\right)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} A_\alpha(nx-k)} - f(x)\left(\frac{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} A_\alpha(nx-k)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} A_\alpha(nx-k)}\right)$$

$$= \frac{1}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} A_\alpha(nx-k)}\left[\Theta_n^*(f,x) - f'(x)\Theta_n^*(\sinh(\cdot - x), x)\right.$$

$$-2f''(x)\Theta_n^*\left(\sinh^2\left(\frac{\cdot - x}{2}\right), x\right) - f(x)\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} A_\alpha(nx-k)\Bigg]$$

$$= \frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} A_\alpha(nx-k)}(\Gamma_n(x)).$$

Consequently, we gain ($\forall\ x \in [a,b]$, $n \in \mathbb{N}$):

(4)

$$\left|\Theta_n(f,x) - f'(x)\Theta_n(\sinh(\cdot - x), x) - 2f''(x)\Theta_n\left(\sinh^2\left(\frac{\cdot - x}{2}\right), x\right) - f(x)\right|$$
$$\overset{(6)}{\leq} \nabla(\alpha)|\Gamma_n(x)|$$
$$\overset{(17)}{\leq} \nabla(\alpha)M\left[\frac{\omega_1\left(f''-f, \frac{1}{n^\lambda}\right)}{2n^{2\lambda}} + \|f''-f\|_\infty (b-a)^2 Ce^{-\gamma n^{(1-\lambda)}}\right].$$

The theorem is accomplished. □

Now, we go further with a hybrid-type, i.e., hyperbolic–trigonometric high-order NN approximation.

Theorem 7. *Let* $f \in C^4([a,b],\mathbb{C})$, $0 < \lambda < 1$, $n \in \mathbb{N} : n^{1-\lambda} > 2$, $x \in [a,b]$. *Then,*

(1)

$$\left|\Theta_n(f,x) - f(x) - \frac{f'(x)}{2}\Theta_n((\sinh(\cdot - x) + \sin(\cdot - x)), x)\right.$$
$$-\frac{f''(x)}{2}\Theta_n((\cosh(\cdot - x) - \cos(\cdot - x)), x)$$
$$-\frac{f^{(3)}(x)}{2}\Theta_n((\sinh(\cdot - x) - \sin(\cdot - x)), x)$$
$$\left.-f^{(4)}(x)\Theta_n\left(\left(\sinh^2\left(\frac{\cdot - x}{2}\right) - \sin^2\left(\frac{\cdot - x}{2}\right)\right), x\right)\right|$$
$$\leq \frac{\nabla(\alpha)(\cosh(b-a)+1)}{2}$$
$$\left[\frac{\omega_1\left(f^{(4)}-f, \frac{1}{n^\lambda}\right)}{2n^{2\lambda}} + \left\|f^{(4)}-f\right\|_\infty (b-a)^2 Ce^{-\gamma n^{(1-\lambda)}}\right],$$

(2) *If* $f^{(i)}(x) = 0$, $i = 1,2,3,4$, *we obtain*

$$|\Theta_n(f,x) - f(x)| \leq \frac{\nabla(\alpha)(\cosh(b-a)+1)}{2}$$
$$\left[\frac{\omega_1\left(f^{(4)}-f, \frac{1}{n^\lambda}\right)}{2n^{2\lambda}} + \left\|f^{(4)}-f\right\|_\infty (b-a)^2 Ce^{-\gamma n^{(1-\lambda)}}\right], \quad (21)$$

and in (21), $n^{-3\lambda}$*appears as the highest speed.*

Proof. Inspired by [4], we employ the hyperbolic–trigonometric Taylor's formula for $f \in C^4([a,b],\mathbb{C})$:

When $\frac{k}{n}, x \in [a,b]$, then

$$f\left(\frac{k}{n}\right) - f(x) - f'(x)\left(\frac{\sinh\left(\frac{k}{n}-x\right)+\sin\left(\frac{k}{n}-x\right)}{2}\right)$$
$$-f''(x)\left(\frac{\cosh\left(\frac{k}{n}-x\right)-\cos\left(\frac{k}{n}-x\right)}{2}\right)$$
$$-f^{(3)}(x)\left(\frac{\sinh\left(\frac{k}{n}-x\right)-\sin\left(\frac{k}{n}-x\right)}{2}\right)$$
$$-f^{(4)}(x)\left(\sinh^2\left(\frac{\frac{k}{n}-x}{2}\right)-\sin^2\left(\frac{\frac{k}{n}-x}{2}\right)\right)$$

$$= \int_x^{\frac{k}{n}} \left[\left(f^{(4)}(t)-f(t)\right)-\left(f^{(4)}(x)-f(x)\right)\right]\left(\frac{\sinh\left(\frac{k}{n}-t\right)-\sin\left(\frac{k}{n}-t\right)}{2}\right)dt$$
$$=: \Phi\left(\frac{k}{n}, x\right).$$

From Theorems 5 and 6, we determine that

$$\Theta_n^*(f,x) - f(x)\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} A_\alpha(nx-k)$$
$$-\frac{f'(x)}{2}\Theta_n^*((\sinh(\cdot - x)+\sin(\cdot - x)),x)$$
$$-\frac{f''(x)}{2}\Theta_n^*((\cosh(\cdot - x)-\cos(\cdot - x)),x)$$
$$-\frac{f^{(3)}(x)}{2}\Theta_n^*((\sinh(\cdot - x)-\sin(\cdot - x)),x)$$
$$-\frac{f^{(4)}(x)}{2}\Theta_n^*\left(\left(\sinh^2\left(\frac{\cdot - x}{2}\right)-\sin^2\left(\frac{\cdot - x}{2}\right)\right),x\right) = \widehat{\Phi}_n(x),$$

where

$$\widehat{\Phi}_n(x) := \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} A_\alpha(nx-k)\Phi_n\left(\frac{k}{n}, x\right).$$

Without loss of generality, let us consider that $n > \left\lceil (b-a)^{-\frac{1}{\lambda}} \right\rceil$.
Hence, $\left|\frac{k}{n}-x\right| \le \frac{1}{n^\lambda}$ or $\left|\frac{k}{n}-x\right| > \frac{1}{n^\lambda}$.
For $\left|\frac{k}{n}-x\right| \le \frac{1}{n^\lambda}$, we gain the following cases:

(i) If $\frac{k}{n} \ge x$, then

$$\left|\Phi\left(\frac{k}{n}, x\right)\right|$$
$$= \left|\frac{1}{2}\int_x^{\frac{k}{n}} \left[\left(f^{(4)}(t)-f(t)\right)-\left(f^{(4)}(x)-f(x)\right)\right]\left(\sinh\left(\frac{k}{n}-t\right)-\sin\left(\frac{k}{n}-t\right)\right)dt\right|$$

$$
\begin{aligned}
&\leq \frac{1}{2}\int_x^{\frac{k}{n}}\left|\left(f^{(4)}(t)-f(t)\right)-\left(f^{(4)}(x)-f(x)\right)\right|\left|\sinh\left(\frac{k}{n}-t\right)-\sin\left(\frac{k}{n}-t\right)\right|dt \\
&\leq \frac{1}{2}\int_x^{\frac{k}{n}}\omega_1\left(f^{(4)}-f,t-x\right)\left(\left|\sinh\left(\frac{k}{n}-t\right)\right|+\left|\sin\left(\frac{k}{n}-t\right)\right|\right)dt \\
&\leq \frac{\omega_1\left(f^{(4)}-f,\frac{k}{n}-x\right)}{2}\int_x^{\frac{k}{n}}\left(\cosh(b-a)\left(\frac{k}{n}-t\right)+\left(\frac{k}{n}-t\right)\right)dt \\
&= \frac{(\cosh(b-a)+1)\omega_1\left(f^{(4)}-f,\frac{k}{n}-x\right)}{2}\int_x^{\frac{k}{n}}\left(\frac{k}{n}-t\right)dt \\
&= \frac{(\cosh(b-a)+1)\omega_1\left(f^{(4)}-f,\frac{k}{n}-x\right)}{4}\left(\frac{k}{n}-x\right)^2 \\
&\leq \frac{(\cosh(b-a)+1)\omega_1\left(f^{(4)}-f,\frac{1}{n^{\lambda}}\right)}{4n^{2\lambda}}.
\end{aligned}
$$

Namely, if $\frac{k}{n} \geq x$, then

$$
\left|\Phi\left(\frac{k}{n},x\right)\right| \leq \frac{(\cosh(b-a)+1)\omega_1\left(f^{(4)}-f,\frac{1}{n^{\lambda}}\right)}{4n^{2\lambda}}.
$$

(ii) When $\frac{k}{n} < x$, then

$$
\begin{aligned}
&\left|\Phi\left(\frac{k}{n},x\right)\right| \\
&= \left|\frac{1}{2}\int_{\frac{k}{n}}^{x}\left[\left(f^{(4)}(t)-f(t)\right)-\left(f^{(4)}(x)-f(x)\right)\right]\left(\sinh\left(\frac{k}{n}-t\right)-\sin\left(\frac{k}{n}-t\right)\right)dt\right| \\
&\leq \frac{\omega_1\left(f^{(4)}-f,x-\frac{k}{n}\right)}{2}\int_{\frac{k}{n}}^{x}\left(\cosh(b-a)\left(t-\frac{k}{n}\right)+\left(t-\frac{k}{n}\right)\right)dt \\
&= \frac{(\cosh(b-a)+1)\omega_1\left(f^{(4)}-f,x-\frac{k}{n}\right)}{2}\int_{\frac{k}{n}}^{x}\left(t-\frac{k}{n}\right)dt \\
&= \frac{(\cosh(b-a)+1)\omega_1\left(f^{(4)}-f,x-\frac{k}{n}\right)}{4}\left(x-\frac{k}{n}\right)^2 \\
&\leq \frac{(\cosh(b-a)+1)\omega_1\left(f^{(4)}-f,\frac{1}{n^{\lambda}}\right)}{4n^{2\lambda}}.
\end{aligned}
$$

Finally, when $\left|\frac{k}{n}-x\right| \leq \frac{1}{n^{\lambda}}$, we always determine that

$$
\left|\Phi\left(\frac{k}{n},x\right)\right| \leq \frac{(\cosh(b-a)+1)\omega_1\left(f^{(4)}-f,\frac{1}{n^{\lambda}}\right)}{4n^{2\lambda}}.
$$

Again, let $\frac{k}{n} \geq x$; then,

$$
\begin{aligned}
&\left|\Phi\left(\frac{k}{n},x\right)\right| \\
&\leq \frac{1}{2}\int_x^{\frac{k}{n}}\left|\left(f^{(4)}(t)-f(t)\right)-\left(f^{(4)}(x)-f(x)\right)\right|\left|\sinh\left(\frac{k}{n}-t\right)-\sin\left(\frac{k}{n}-t\right)\right|dt \\
&\leq \left\|f^{(4)}-f\right\|_\infty \int_x^{\frac{k}{n}}\left[\cosh(b-a)\left(\frac{k}{n}-t\right)+\left(\frac{k}{n}-t\right)\right]dt \\
&= \left\|f^{(4)}-f\right\|_\infty (\cosh(b-a)+1)\int_x^{\frac{k}{n}}\left(\frac{k}{n}-t\right)dt \\
&= \frac{\left\|f^{(4)}-f\right\|_\infty (\cosh(b-a)+1)}{2}\left(\frac{k}{n}-x\right)^2 \\
&\leq \frac{\left\|f^{(4)}-f\right\|_\infty (\cosh(b-a)+1)(b-a)^2}{2}.
\end{aligned}
$$

Thus,

$$
\left|\Phi\left(\frac{k}{n},x\right)\right| \leq \frac{\left\|f^{(4)}-f\right\|_\infty (\cosh(b-a)+1)(b-a)^2}{2}.
$$

If we let $\frac{k}{n} < x$, we obtain

$$
\begin{aligned}
&\left|\Phi\left(\frac{k}{n},x\right)\right| \\
&\leq \frac{1}{2}\int_{\frac{k}{n}}^{x}\left|\left(f^{(4)}(t)-f(t)\right)-\left(f^{(4)}(x)-f(x)\right)\right|\left|\sinh\left(\frac{k}{n}-t\right)-\sin\left(\frac{k}{n}-t\right)\right|dt \\
&\leq \left\|f^{(4)}-f\right\|_\infty \int_{\frac{k}{n}}^{x}\left[\cosh(b-a)\left(t-\frac{k}{n}\right)+\left(t-\frac{k}{n}\right)\right]dt \\
&= \left\|f^{(4)}-f\right\|_\infty (\cosh(b-a)+1)\int_{\frac{k}{n}}^{x}\left(t-\frac{k}{n}\right)dt \\
&= \frac{\left\|f^{(4)}-f\right\|_\infty (\cosh(b-a)+1)}{2}\left(x-\frac{k}{n}\right)^2 \\
&\leq \frac{\left\|f^{(4)}-f\right\|_\infty (\cosh(b-a)+1)(b-a)^2}{2}.
\end{aligned}
$$

which is why there exists that

$$
\left|\Phi\left(\frac{k}{n},x\right)\right| \leq \frac{\left\|f^{(4)}-f\right\|_\infty (\cosh(b-a)+1)(b-a)^2}{2}.
$$

So,

$$
\begin{aligned}
\left|\widehat{\Phi}_n(x)\right| &\leq \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} A_\alpha(nx-k)\left|\Phi\left(\frac{k}{n},x\right)\right| \\
&= \sum_{\substack{k=\lceil na\rceil \\ :\left|\frac{k}{n}-x\right|\leq\frac{1}{n^\lambda}}}^{\lfloor nb\rfloor} A_\alpha(nx-k)\left|\Phi\left(\frac{k}{n},x\right)\right|
\end{aligned}
$$

$$+\sum_{\left\{\begin{array}{c} k=\lceil na\rceil \\ :\left|\frac{k}{n}-x\right|>\frac{1}{n^{\lambda}} \end{array}\right.}^{\lfloor nb\rfloor} A_{\alpha}(nx-k)\left|\Phi\left(\frac{k}{n},x\right)\right|$$

$$\leq\left(\sum_{\left\{\begin{array}{c} k=\lceil na\rceil \\ :\left|\frac{k}{n}-x\right|\leq\frac{1}{n^{\lambda}} \end{array}\right.}^{\lfloor nb\rfloor} A_{\alpha}(nx-k)\right)\frac{(\cosh(b-a)+1)\omega_1\left(f^{(4)}-f,\frac{1}{n^{\lambda}}\right)}{4n^{2\lambda}}$$

$$+\left(\sum_{\left\{\begin{array}{c} k=\lceil na\rceil \\ :\left|\frac{k}{n}-x\right|>\frac{1}{n^{\lambda}} \end{array}\right.}^{\lfloor nb\rfloor} A_{\alpha}(nx-k)\right)$$

$$\left(\frac{\left\|f^{(4)}-f\right\|_{\infty}(\cosh(b-a)+1)(b-a)^2}{2}\right)$$

$$\leq\frac{(\cosh(b-a)+1)\omega_1\left(f^{(4)}-f,\frac{1}{n^{\lambda}}\right)}{4n^{2\lambda}} \tag{22}$$

$$+\frac{\left\|f^{(4)}-f\right\|_{\infty}(\cosh(b-a)+1)(b-a)^2}{2}Ce^{-\gamma n^{(1-\lambda)}}.$$

We try to prove that

$$\left|\widehat{\Phi}_n(x)\right|\leq\frac{(\cosh(b-a)+1)}{2}$$

$$\left[\frac{\omega_1\left(f^{(4)}-f,\frac{1}{n^{\lambda}}\right)}{2n^{2\lambda}}+\left\|f^{(4)}-f\right\|_{\infty}(b-a)^2Ce^{-\gamma n^{(1-\lambda)}}\right]. \tag{23}$$

We examine that

$$\Theta_n(f,x)-f(x)-\frac{f'(x)}{2}\Theta_n((\sinh(\cdot-x)+\sin(\cdot-x)),x)$$

$$-\frac{f''(x)}{2}\Theta_n((\cosh(\cdot-x)-\cos(\cdot-x)),x)$$

$$-\frac{f^{(3)}(x)}{2}\Theta_n((\sinh(\cdot-x)-\sin(\cdot-x)),x)$$

$$-f^{(4)}(x)\Theta_n\left(\left(\sinh^2\left(\frac{\cdot-x}{2}\right)-\sin^2\left(\frac{\cdot-x}{2}\right)\right),x\right)$$

$$=\left[\Theta_n^*(f,x)-f(x)\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}A_{\alpha}(nx-k)-\frac{f'(x)}{2}\Theta_n^*((\sinh(\cdot-x)+\sin(\cdot-x)),x)\right.$$

$$-\frac{f''(x)}{2}\Theta_n^*((\cosh(\cdot-x)-\cos(\cdot-x)),x)$$

$$-\frac{f^{(3)}(x)}{2}\Theta_n^*((\sinh(\cdot-x)-\sin(\cdot-x)),x)$$

$$-f^{(4)}(x)\Theta_n^*\left(\left(\sinh^2\left(\frac{\cdot - x}{2}\right) - \sin^2\left(\frac{\cdot - x}{2}\right)\right), x\right)\right] \frac{1}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} A_\alpha(nx-k)}$$

$$= \frac{\widehat{\Phi}_n(x)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} A_\alpha(nx-k)}.$$

As a result, we obtain ($\forall\, x \in [a,b]$, $n \in \mathbb{N}$):

$$\left|\Theta_n(f,x) - f(x) - \frac{f'(x)}{2}\Theta_n((\sinh(\cdot - x) + \sin(\cdot - x)), x)\right.$$

$$-\frac{f''(x)}{2}\Theta_n((\cosh(\cdot - x) - \cos(\cdot - x)), x)$$

$$-\frac{f^{(3)}(x)}{2}\Theta_n((\sinh(\cdot - x) - \sin(\cdot - x)), x)$$

$$\left.-f^{(4)}(x)\Theta_n\left(\left(\sinh^2\left(\frac{\cdot - x}{2}\right) - \sin^2\left(\frac{\cdot - x}{2}\right)\right), x\right)\right|$$

$$= \frac{\left|\widehat{\Phi}_n(x)\right|}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} A_\alpha(nx-k)} \leq \nabla(\alpha)\left|\widehat{\Phi}_n(x)\right| \overset{\text{(by (23))}}{\leq} \frac{\nabla(\alpha)(\cosh(b-a)+1)}{2}$$

$$\left[\frac{\omega_1\left(f^{(4)} - f, \frac{1}{n^\lambda}\right)}{2n^{2\lambda}} + \left\|f^{(4)} - f\right\|_\infty (b-a)^2 C e^{-\gamma n^{(1-\lambda)}}\right].$$

The theorem is established. □

Now, a general trigonometric result will be considered.

Theorem 8. *Let $f \in C^4([a,b],\mathbb{C})$, $0 < \lambda < 1$, $n \in \mathbb{N} : n^{1-\lambda} > 2$, $x \in [a,b]$, and $\widetilde{\xi}, \overline{\xi} \in \mathbb{R}$ such that $\widetilde{\xi}\,\overline{\xi}\left(\widetilde{\xi}^2 - \overline{\xi}^2\right) \neq 0$. Then,*

(1)

$$\left|\Theta_n(f,x) - f(x) - \frac{f'(x)}{\widetilde{\xi}\,\overline{\xi}\left(\overline{\xi}^2 - \widetilde{\xi}^2\right)}\Theta_n\left(\left(\overline{\xi}^3 \sin\left(\widetilde{\xi}(\cdot - x)\right) - \widetilde{\xi}^3 \sin(\overline{\xi}(\cdot - x))\right), x\right)\right.$$

$$-\frac{f''(x)}{\left(\overline{\xi}^2 - \widetilde{\xi}^2\right)}\Theta_n\left(\left(\cos\left(\widetilde{\xi}(\cdot - x)\right) - \cos(\overline{\xi}(\cdot - x))\right), x\right)$$

$$-\frac{f^{(3)}(x)}{\widetilde{\xi}\,\overline{\xi}\left(\overline{\xi}^2 - \widetilde{\xi}^2\right)}\Theta_n\left(\left(\overline{\xi}\sin\left(\widetilde{\xi}(\cdot - x)\right) - \widetilde{\xi}\sin(\overline{\xi}(\cdot - x))\right), x\right)$$

$$-\left(\frac{2f^{(4)}(x) + \left(\widetilde{\xi}^2 + \overline{\xi}^2\right)f''(x)}{\left(\widetilde{\xi}\,\overline{\xi}\right)^2\left(\overline{\xi}^2 - \widetilde{\xi}^2\right)}\right)$$

$$\left.\Theta_n\left(\left(\overline{\xi}^2 \sin^2\left(\frac{\widetilde{\xi}(\cdot - x)}{2}\right) - \widetilde{\xi}^2 \sin^2\left(\frac{\overline{\xi}(\cdot - x)}{2}\right)\right), x\right)\right|$$

$$\leq \frac{\nabla(\alpha)}{\left|\overline{\xi}^2 - \widetilde{\xi}^2\right|}\left[\frac{\omega_1\left(\left(f^{(4)} + \left(\widetilde{\xi}^2 + \overline{\xi}^2\right)f'' + \widetilde{\xi}^2\overline{\xi}^2 f\right), \frac{1}{n^\lambda}\right)}{n^{2\lambda}}\right.$$

$$\left.+2\left\|f^{(4)} + \left(\widetilde{\xi}^2 + \overline{\xi}^2\right)f'' + \widetilde{\xi}^2\overline{\xi}^2 f\right\|_\infty (b-a)^2 C e^{-\gamma n^{(1-\lambda)}}\right],$$

(2) *If* $f^{(i)}(x) = 0$, $i = 1,2,3,4$, *we obtain*

$$|\Theta_n(f,x) - f(x)| \leq \frac{\nabla(\alpha)}{\left|\overline{\xi}^2 - \widetilde{\xi}^2\right|} \left[\frac{\omega_1\left(\left(f^{(4)} + \left(\widetilde{\xi}^2 + \overline{\xi}^2\right)f'' + \widetilde{\xi}^2\overline{\xi}^2 f\right), \frac{1}{n^\lambda}\right)}{n^{2\lambda}} + 2\left\|f^{(4)} + \left(\widetilde{\xi}^2 + \overline{\xi}^2\right)f'' + \widetilde{\xi}^2\overline{\xi}^2 f\right\|_\infty (b-a)^2 C e^{-\gamma n^{(1-\lambda)}}\right].$$

$n^{-3\lambda}$ *is the high convergence speed in both (1) and (2).*

Proof. This proof is inspired by [4] (Chapter 3, Theorem 3.13, pp. 84–89), and also the proof of Theorem 7. □

We finalize with a general hyperbolic result.

Theorem 9. *Let* $f \in C^4([a,b],\mathbb{C})$, $0 < \lambda < 1$, $n \in \mathbb{N} : n^{1-\lambda} > 2$, $x \in [a,b]$, *and let* $\widetilde{\xi}, \overline{\xi} \in \mathbb{R}$ *with* $\widetilde{\xi}\,\overline{\xi}\left(\widetilde{\xi}^2 - \overline{\xi}^2\right) \neq 0$. *Then,*

(1)

$$\Bigg|\Theta_n(f,x) - f(x) - \frac{f'(x)}{\widetilde{\xi}\,\overline{\xi}\left(\overline{\xi}^2 - \widetilde{\xi}^2\right)}\Theta_n\left(\left(\overline{\xi}^3 \sinh\left(\widetilde{\xi}(\cdot - x)\right) - \widetilde{\xi}^3 \sinh\left(\overline{\xi}(\cdot - x)\right)\right), x\right)$$
$$- \frac{f''(x)}{\overline{\xi}^2 - \widetilde{\xi}^2}\Theta_n\left(\left(\cosh\left(\overline{\xi}(\cdot - x)\right) - \cosh\left(\widetilde{\xi}(\cdot - x)\right)\right), x\right)$$
$$- \frac{f'''(x)}{\widetilde{\xi}\,\overline{\xi}\left(\overline{\xi}^2 - \widetilde{\xi}^2\right)}\Theta_n\left(\left(\widetilde{\xi}\sinh\left(\overline{\xi}(\cdot - x)\right) - \overline{\xi}\sinh\left(\widetilde{\xi}(\cdot - x)\right)\right), x\right)$$
$$- \left(\frac{2(f^{(4)}(x) - \left(\widetilde{\xi}^2 + \overline{\xi}^2\right)f''(x))}{\left(\widetilde{\xi}\,\overline{\xi}\right)^2\left(\overline{\xi}^2 - \widetilde{\xi}^2\right)}\right)$$
$$\Theta_n\left(\left(\widetilde{\xi}^2 \sinh^2\left(\frac{\overline{\xi}(\cdot - x)}{2}\right) - \overline{\xi}^2 \sinh^2\left(\frac{\widetilde{\xi}(\cdot - x)}{2}\right)\right), x\right)\Bigg|$$
$$\leq \frac{\nabla(\alpha)\cosh(b-a)}{\left|\overline{\xi}^2 - \widetilde{\xi}^2\right|}\left[\frac{\omega_1\left(\left(f^{(4)} - \left(\widetilde{\xi}^2 + \overline{\xi}^2\right)f'' + \widetilde{\xi}^2\overline{\xi}^2 f\right), \frac{1}{n^\lambda}\right)}{n^{2\lambda}} + 2\left\|f^{(4)} - \left(\widetilde{\xi}^2 + \overline{\xi}^2\right)f'' + \widetilde{\xi}^2\overline{\xi}^2 f\right\|_\infty (b-a)^2 C e^{-\gamma n^{(1-\lambda)}}\right],$$

(2) *If* $f^{(i)}(x) = 0$, $i = 1,2,3,4$, *we obtain*

$$|\Theta_n(f,x) - f(x)| \leq \frac{\nabla(\alpha)\cosh(b-a)}{\left|\overline{\xi}^2 - \widetilde{\xi}^2\right|} \left[\frac{\omega_1\left(\left(f^{(4)} - \left(\widetilde{\xi}^2 + \overline{\xi}^2\right)f'' + \widetilde{\xi}^2\overline{\xi}^2 f\right), \frac{1}{n^\lambda}\right)}{n^{2\lambda}} + 2\left\|f^{(4)} - \left(\widetilde{\xi}^2 + \overline{\xi}^2\right)f'' + \widetilde{\xi}^2\overline{\xi}^2 f\right\|_\infty (b-a)^2 C e^{-\gamma n^{(1-\lambda)}}\right].$$

Again, $n^{-3\lambda}$ *is the high convergence speed in both (1) and (2).*

Proof. This proof is inspired by [4], and also the proof of Theorem 7. □

6. Conclusions

As we have seen, the foundations of neural operators are based on a rich tapestry of branches such as approximation theory and computational analysis. The sophisticated architecture of single-layer neural networks offers effective and faster solutions to many problems in engineering and science. Our study stands out in terms of revealing complex-valued hyperbolic, trigonometric, and hybrid convergence cases with the help of an activation function that is relatively easy to train. Moreover, the types of convergence that reach a promising speed of approximation are our key findings. Finally, we would like to point out that we worked with Ostrowski- and Opial-type inequalities, norms, and also trigonometric- and hyperbolic-type Taylor formulae to reach this convergence rate result.

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