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New Generalized Weibull Inverse Gompertz Distribution: Properties and Applications

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Abstract: In parametric statistical modeling, it is essential to create generalizations of current statistical distributions that are more flexible when modeling actual data sets. Therefore, this study introduces a new generalized lifetime model named the odd Weibull Inverse Gompertz distribution (OWIG). The OWIG is derived by combining the odd Weibull family of distributions with the inverse Gompertz distribution. Essential statistical properties are discussed, including reliability functions, moments, Rényi entropy, and order statistics. The proposed OWIG is particularly significant as its hazard rate functions exhibit various monotonic and nonmonotonic shapes. This enables OWIG to model different hazard behaviors more commonly observed in natural phenomena. OWIG's parameters are estimated and its flexibility in predicting unique symmetric and asymmetric patterns is shown by analyzing real-world applications from psychology, environmental, and medical sciences. The results demonstrate that the proposed OWIG is an excellent candidate as it provides the most accurate fits to the data compared with some competing models.

Keywords: odd Weibull inverse Gompertz distribution; odd generalized Weibull generator; inverse Gompertz distribution; Rényi entropy

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1. Introduction

Many real-world problems do not fit the well-known probability models, despite the availability of well-known statistical models. Therefore, it is essential to develop probability models that more accurately capture the behavior of certain real-world phenomena.

Recently, generated families of distributions provide the potential for modeling real data with great flexibility. In addition to improving their applicability to real-life phenomena, adding new parameters to established distributions improves their ability to characterize tail shapes more accurately. Previous studies using novel approaches have generated several distributions and families of distributions. These include the beta-G family by [1], Kumaraswamy-G family by [2], Weibull-G (WG) by [3], exponentiated Weibull-G by [4], and the more general T-X family introduced by [5], among many others.

Here, we concentrate on the WG family in [3], in which the cumulative distribution function (cdf) and the probability density function (pdf) are, respectively, obtained using the T-X method as follows:

$$F_{WG}(x) = \int_0^{\frac{G(x)}{1-G(x)}} \lambda\alpha t^{\alpha-1} e^{-\lambda t}\, dt = 1 - e^{-\lambda\left(\frac{G(x)}{1-G(x)}\right)^{\alpha}}, \quad \alpha, \lambda > 0, \tag{1}$$

$$f_{WG}(x) = \lambda\alpha g(x)\, \frac{G(x)^{\alpha-1}}{[1-G(x)]^{\alpha+1}} e^{-\lambda\left(\frac{G(x)}{1-G(x)}\right)^{\alpha}}, \quad \alpha, \lambda > 0, \tag{2}$$

where G(x) and g(x) are the cdf and pdf of any continuous distribution, respectively. The Weibull generator's extra parameters are sought as a way to generate more flexible distributions. The WG family provides various hazard rate function (hrf) shape characteristics and a variety of lifetime data types can be evaluated using it. Several studies have used the

Weibull generator to introduce new distributions such as the Weibull Rayleigh [6], Weibull Fréchet [7], and odd Weibull inverse Topp–Leone [8].

The Gompertz distribution (GD), named after Benjamin Gompertz, has an exponentially growing failure rate [9,10]. Demographers and actuaries frequently use GD to represent the distribution of adult lifespans [11,12]. Also, GD is used to analyze survival data in several scientific disciplines, including biology, computer programming, marketing, network theory, engineering, behavioral sciences, and gerontology [13–19].

Unfortunately, GD's increasing failure rate decreased its flexibility and ability to describe numerous occurrences in various domains. To compensate for these limits, a modified variant of GD with an upside-down bathtub shape hrf, called the inverse Gompertz distribution (IG), is introduced by [20]. The cdf and pdf of the IG distribution are expressed, respectively, as

$$G_{IG}(x) = \exp\left[-\frac{a}{\beta}\left(e^{\frac{\beta}{x}}-1\right)\right], \quad x>0,\ a,\ \beta>0, \tag{3}$$

$$g_{IG}(x) = \frac{ae^{\frac{\beta}{x}}}{x^2}\exp\left[-\frac{a}{\beta}\left(e^{\frac{\beta}{x}}-1\right)\right], \quad x>0,\ a,\ \beta>0. \tag{4}$$

In recent years, some generalizations of the IG distribution have been introduced to increase its flexibility. For example, the Kumaraswamy inverse Gompertz [21], exponentiated generalized inverted Gompertz [22], extended inverse Gompertz [23], and inverse power Gompertz [24].

The primary goal of this research is to investigate a new lifetime distribution called the Odd Weibull Inverse Gompertz distribution (OWIG), which is based on the Weibull generator and IG distribution. Including additional parameters will help with the IG distribution's inability to fit real-world data that showed non-monotone failure rates. Therefore, the motivation for introducing OWIG distribution arises from the need to

- Increase IG's flexibility by introducing new generalizations.
- Add greater versatility for modeling real-world data in numerous fields.
- Modeling different forms of hrf, which will help provide a "more effective fit" in many practical scenarios.

This article is outlined as follows: Sections 2 and 3 introduce OWIG distribution and drive some of its theoretical features, with a focus on those that could be broadly significant in probability and statistics. To estimate OWIG's parameters, the maximum likelihood (ML) technique is used in Section 4, and the performance of the estimators is examined with simulation studies in Section 5. The effectiveness of the OWIG distribution in comparison with certain competing distributions is demonstrated in Section 6 using real data sets from various fields. Finally, some concluding remarks are presented in Section 7.

2. Odd Weibull Inverse Gompertz Distribution

The cdf of OWIG can be obtained by replacing the $G(x)$ in (1) by (4) as follows:

$$F(x) = 1-\exp\left[-\lambda\left(e^{\frac{a}{\beta}\left(e^{\frac{\beta}{x}}-1\right)}-1\right)^{-\alpha}\right], \quad x>0,\ \alpha,\lambda,a,\beta>0. \tag{5}$$

The corresponding pdf of OWIG is obtained by replacing $G(x)$ and $g(x)$ in (2) by (3) and (4), as

$$f(x) = \frac{a\lambda\alpha e^{\frac{\beta}{x}}}{x^2}\left(e^{-\frac{a}{\beta}\left(e^{\frac{\beta}{x}}-1\right)}\right)^{\alpha}\left(1-e^{-\frac{a}{\beta}\left(e^{\frac{\beta}{x}}-1\right)}\right)^{-\alpha-1}\exp\left[-\lambda\left(e^{\frac{a}{\beta}\left(e^{\frac{\beta}{x}}-1\right)}-1\right)^{-\alpha}\right]. \tag{6}$$

The Survival, $S(x)$, and hrf of OWIG are expressed as

$$S(x) = \exp\left[-\lambda\left(e^{\frac{a}{\beta}\left(e^{\frac{\beta}{x}}-1\right)}-1\right)^{-\alpha}\right], \tag{7}$$

$$h(x) = \frac{a\lambda\alpha}{x^2}e^{\frac{\beta}{x}}\left(e^{-\frac{a}{\beta}\left(e^{\frac{\beta}{x}}-1\right)}\right)^{\alpha}\left(1-e^{-\frac{a}{\beta}\left(e^{\frac{\beta}{x}}-1\right)}\right)^{-\alpha-1}. \tag{8}$$

Figures 1 and 2 show the various forms of OWIG's density and hrf at some values of the parameters. The pdf of OWIG in Figure 1 shows left-skewed, symmetrical, asymmetrical, and J-shaped densities. In addition, OWIG's hrf is attractive, as seen in Figure 2 as it exhibits a wide range of asymmetrical forms, including increasing, bath-tab, upside down bath-tab, decreasing, and reversed J-shapes. As a result, OWIG may be deemed to be a suitable model for fitting a wide range of lifetime data in practical applications.

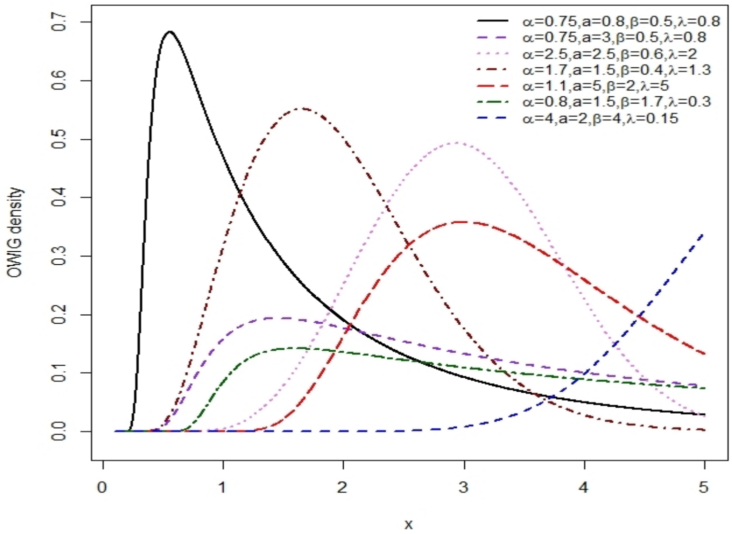

Figure 1. The plots for the OWIG pdf for some certain values.

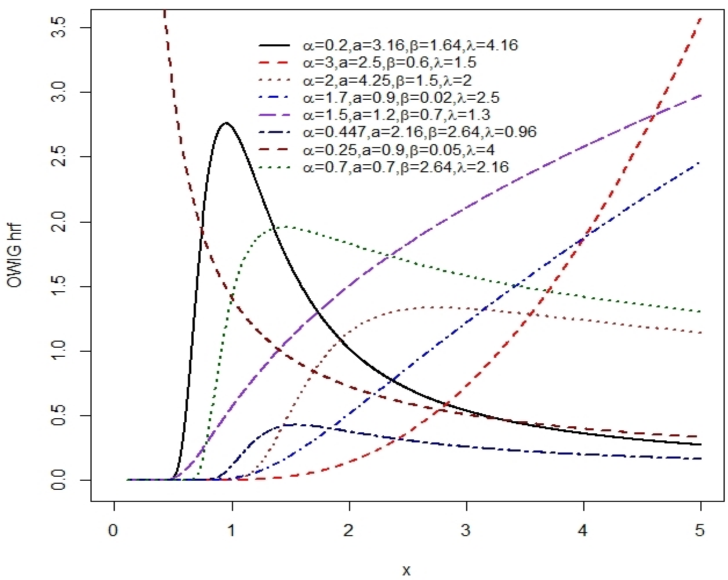

Figure 2. OWIG's hrf plots at some selected values.

3. Statistical Properties of the OWIG

This section investigates some essential statistical properties of OWIG.

3.1. Useful Expansions for OWIG's Density

This subsection provides expansions for OWIG's density provided in Equation (6) by first considering the following expansion, defined as

$$e^{-x} = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} x^i. \tag{9}$$

Then, the pdf of OWIG will become

$$f(x) = \frac{a\alpha e^{\frac{\beta}{x}}}{x^2} \sum_{s_1=0}^{\infty} \frac{(-1)^{s_1}}{s_1!} \lambda^{s_1+1} \left(e^{-\frac{a}{\beta}\left(e^{\frac{\beta}{x}}-1\right)} \right)^{\alpha(1+s_1)} \left(1 - e^{-\frac{a}{\beta}\left(e^{\frac{\beta}{x}}-1\right)} \right)^{-(\alpha s_1+\alpha+1)}. \tag{10}$$

The negative binomial series formula is defined by

$$(1-z)^{-q} = \sum_{s=0}^{\infty} \frac{\Gamma(q+s)}{s!\Gamma(q)} z^s. \tag{11}$$

Employing (11), OWIG's pdf in Equation (10) is rewritten as

$$f(x) = \frac{a\alpha e^{\frac{\beta}{x}}}{x^2} \sum_{s_1,s_2=0}^{\infty} \frac{(-1)^{s_1}\Gamma(\alpha(s_1+1)+s_2+1)}{s_1!s_2!\Gamma(\alpha(s_1+1)+1)} \lambda^{s_1+1} \left(e^{-\frac{a}{\beta}\left(e^{\frac{\beta}{x}}-1\right)} \right)^{\alpha(1+s_1)+s_2}.$$

Moreover, applying (9), then

$$f(x) = \sum_{s_1,s_2,s_3=0}^{\infty} \frac{(-1)^{s_1+s_3}\Gamma(\alpha(s_1+1)+s_2+1)\alpha a^{s_3+1}}{s_1!s_2!s_3!\Gamma(\alpha(s_1+1)+1)\beta^{s_3}x^2} \left(e^{\frac{\beta}{x}}\right)^{s_3+1} \lambda^{s_1+1}(\alpha(1+s_1)+s_2)^{s_3}\left(1-e^{-\frac{\beta}{x}}\right)^{s_3}.$$

Additionally, by employing both (9) and (11), the pdf of OWIG is reduced to

$$f(x) = \sum_{s_5=0}^{s_3} \eta_{s_5} \frac{\beta^{s_4-s_3}}{x^{s_4+2}} e^{-\frac{s_5\beta}{x}}, \tag{12}$$

where

$$\eta_{s_5} = \sum_{s_1,s_2,s_3,s_4=0}^{\infty} \frac{(-1)^{s_1+s_3+s_5}\Gamma(\alpha(s_1+1)+s_2+1)\alpha\lambda a^{s_1+1}a^{s_3+1}}{s_1!s_2!s_4!s_5!(s_3-s_5)!\Gamma(\alpha(s_1+1)+1)}(s_3+1)^{s_4}(\alpha(1+s_1)+s_2)^{s_3}. \tag{13}$$

3.2. Quantile Function

The quantile function , $QN(p)$, of OWIG is expressed as

$$QN(p) = \frac{\beta}{\log\left[1+\frac{\beta}{a}\log\left(1+\left(\frac{-\log(1-p)}{\lambda}\right)^{-1/\alpha}\right)\right]}, \qquad 0 < p < 1. \tag{14}$$

Therefore, OWIG's median can be obtained as

$$Median = x(0.50) = \frac{\beta}{\log\left[1+\frac{\beta}{a}\log\left(1+\left(\frac{-\log(0.5)}{\lambda}\right)^{-1/\alpha}\right)\right]}.$$

Hence, the 25th and 75th percentiles of OWIG are obtained by replacing p by $(0.25, 0.75)$, respectively, in (14) .

3.3. The Galton Skewness and Moors Kurtosis

The Galton skewness (GS) by [25] and the Moors kurtosis (MK) by [26] measures can be obtained for OWIG using (14) as follows:

$$GS = \frac{QN(\frac{6}{8}) - 2QN(\frac{4}{8}) + QN(\frac{2}{8})}{QN(\frac{6}{8}) - QN(\frac{2}{8})}, \tag{15}$$

$$MK = \frac{(QN(\frac{7}{8}) - QN(\frac{5}{8})) + (QN(\frac{3}{8}) - QN(\frac{1}{8}))}{(QN(\frac{6}{8}) - QN(\frac{2}{8}))}. \tag{16}$$

3.4. Moments

If X has an OWIG with density (12), then the rth moment of X is provided by

$$E(x^r) = \int_0^\infty x^r f(x) dx = \sum_{s_5=0}^{s_3} \eta_{s_5} \beta^{s_4 - s_3} \int_0^\infty x^{r-s_4-2}\, e^{-\frac{s_5\beta}{x}}\, dx, \tag{17}$$

where η_{s_5} is provided by (13). Taking $u = \frac{s_5\beta}{x}$, limits change from ∞ to 0, then simplifying, we obtain

$$E(x^r) = \sum_{s_5=0}^{s_3} \eta_{s_5} s_5^{r-s_4-1} \beta^{r-s_3-1} \int_0^\infty u^{s_4-r} e^{-u}\, du.$$

Thus, the rth moment is expressed as

$$\mu_r = E(x^r) = \sum_{s_5=0}^{s_3} \eta_{s_5} s_5^{r-s_4-1} \beta^{r-s_3-1}\, \Gamma\left(s_4 - r + 1\right), \quad r < s_4 + 1. \tag{18}$$

Therefore, the mean of OWIG is provided by

$$\mu = E(x) = \sum_{s_5=0}^{s_3} \frac{\eta_{s_5}}{s_5^{s_4} \beta^{s_3}}\, \Gamma\left(s_4\right). \tag{19}$$

3.5. Moment Generating Function

The OWIG's moment-generating function (MGF) is obtained as

$$MGF(t) = E(e^{tx}) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu_r = \alpha\beta \sum_{r=0}^{\infty} \sum_{s_5=0}^{s_3} \frac{t^r}{r!} \eta_{s_5} s_5^{r-s_4-1} \beta^{r-s_3-1}\, \Gamma\left(s_4 - r + 1\right), \tag{20}$$

where η_{s_5} is given by (13).

3.6. Characteristic Function

OWIG's characteristic function is obtained as follows:

$$\phi_x(t) = E(e^{itx}) = \alpha\beta \sum_{r=0}^{\infty} \sum_{s_5=0}^{s_3} \frac{(it)^r}{r!} \eta_{s_5} s_5^{r-s_4-1} \beta^{r-s_3-1}\, \Gamma\left(s_4 - r + 1\right), \tag{21}$$

where η_{s_5} is provided by (13).

3.7. Rényi Entropy

Entropies are measures of the variability or uncertainty of a R.V.X. The Rényi entropy, denoted by $I_\nu(X)$, is formulated as

$$I_\nu(X) = \frac{1}{1-\nu} \log\left(\int_0^\infty f(x)^\nu dx\right); \quad \nu > 0, \nu \neq 1.$$

Therefore, using the pdf of OWIG in (6), $f(x)^\nu$ is expressed as

$$f^\nu(x) = \left(\frac{a\lambda\alpha e^{\frac{\beta}{x}}}{x^2}\right)^\nu \left(e^{-\frac{a}{\beta}\left(e^{\frac{\beta}{x}}-1\right)}\right)^{\alpha\nu} \left[e^{\left[-\lambda\left(\frac{e^{-\frac{a}{\beta}\left(e^{\frac{\beta}{x}}-1\right)}}{1-e^{-\frac{a}{\beta}\left(e^{\frac{\beta}{x}}-1\right)}}\right)^{\alpha}\right]}\right]^\nu \left(1-e^{-\frac{a}{\beta}\left(e^{\frac{\beta}{x}}-1\right)}\right)^{\nu(-\alpha-1)}.$$

Applying similar concepts in Section 3.1 and using both (11) and (9), then

$$f^\nu(x) = \sum_{s_5=0}^{s_3} \eta^*_{s_5} \frac{\beta^{s_4-s_3}}{x^{s_4+2\nu}} e^{-\frac{s_5\beta}{x}},$$

where

$$\eta^*_{s_5} = \alpha^\nu \sum_{s_1,s_2,s_3,s_4=0}^{\infty} \frac{(-1)^{s_1+s_3+s_5}\Gamma(\alpha(s_1+\nu)+s_2+\nu)\lambda^{s_1+\nu}a^{s_3+\nu}\nu^{s_1}}{s_1!s_2!s_4!s_5!(s_3-s_5)!\Gamma(\alpha(s_1+\nu)+\nu)}(s_3+\nu)^{s_4}(\alpha(\nu+s_1)+s_2)^{s_3}.$$

By setting $u = \frac{s_5\beta}{x}$, the Rényi entropy of the OWIG, is provided by

$$I_\nu(X) = \frac{1}{1-\nu} \log\left[\sum_{s_5=0}^{s_3} \eta^*_{s_5} s_5^{1-s_4-2\nu} \beta^{1-s_3-2\nu}\,\Gamma\left(s_4+2\nu-1\right)\right].$$

3.8. Order Statistics

Suppose $X_1, X_2, \ldots, X_n$ is a random sample (R.S.) from OWIG and $X_{i:n}$ is the ith order statistics. Therefore, the pdf, $f_{i:n}(x)$, of the ith order statistics is

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} f(x)[F(x)]^{i-1}[1-F(x)]^{n-i}. \tag{22}$$

Applying the expansion (11) to (22), then

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} f(x) \sum_{k=0}^{\infty}(-1)^k \binom{n-i}{k} [F(x)]^{k+i-1}. \tag{23}$$

Substituting (5) into (23), $f_{i:n}(x)$ will be

$$f_{i:n}(x) = \frac{n!f(x)}{(i-1)!(n-i)!} \sum_{k=0}^{\infty}(-1)^k \binom{n-i}{k} \left[1-\exp\left(-\lambda\left(e^{\frac{a}{\beta}\left(e^{\frac{\beta}{x}}-1\right)}-1\right)^{-\alpha}\right)\right]^{k+i-1},$$

where f(x) is the OWIG's pdf, provided by (6).

4. ML Estimation

Let $x_1, x_2, \ldots, x_n$ be am R.S. from OWIG. The log-likelihood (ℓ) for $\Theta = (\alpha, \lambda, a, \beta)$, can be written as follows

$$\begin{aligned}\ell(\Theta) =& n\ln(\alpha\lambda a) - 2\sum_{i=1}^{n}\ln(x_i) + \beta\sum_{i=1}^{n}\frac{1}{x_i} - \frac{\alpha a}{\beta}\sum_{i=1}^{n}\left(e^{\frac{\beta}{x_i}} - 1\right) \\ & - \lambda\sum_{i=1}^{n}\left(\frac{e^{-\frac{a}{\beta}\left(e^{\frac{\beta}{x_i}}-1\right)}}{1-e^{-\frac{a}{\beta}\left(e^{\frac{\beta}{x_i}}-1\right)}}\right)^{\alpha} - (\alpha+1)\sum_{i=1}^{n}\ln\left(1-e^{-\frac{a}{\beta}\left(e^{\frac{\beta}{x_i}}-1\right)}\right).\end{aligned} \tag{24}$$

Then, the parameters' ML estimates (MLEs) can be obtained via maximizing (24). That is, finding the partial derivatives of (24) with respect to α, λ, a, and β, respectively, will result in the following

$$\frac{\partial\ell}{\partial\lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n}\left(\frac{e^{-\frac{a}{\beta}\left(e^{\frac{\beta}{x_i}}-1\right)}}{1-e^{-\frac{a}{\beta}\left(e^{\frac{\beta}{x_i}}-1\right)}}\right)^{\alpha}, \tag{25}$$

$$\frac{\partial\ell}{\partial\alpha} = \frac{n}{\alpha} - \frac{a}{\beta}\sum_{i=1}^{n}\left(\mathrm{e}^{\frac{\beta}{x_i}} - 1\right) + \frac{a\lambda}{\beta}\sum_{i=1}^{n}\left(\frac{\left(\mathrm{e}^{\frac{\beta}{x_i}}-1\right)\mathrm{e}^{-\frac{a\alpha\left(\mathrm{e}^{\frac{\beta}{x_i}}-1\right)}{\beta}}}{\left(1-\mathrm{e}^{-\frac{a}{\beta}\left(\mathrm{e}^{\frac{\beta}{x_i}}-1\right)}\right)^{\alpha}}\right) \tag{26}$$

$$+\lambda\sum_{i=1}^{n}\left(\frac{\ln\left(1-\mathrm{e}^{-\frac{a}{\beta}\left(\mathrm{e}^{\frac{\beta}{x_i}}-1\right)}\right)\mathrm{e}^{-\frac{a\alpha\left(\mathrm{e}^{\frac{\beta}{x_i}}-1\right)}{\beta}}}{\left(1-\mathrm{e}^{-\frac{a}{\beta}\left(\mathrm{e}^{\frac{\beta}{x_i}}-1\right)}\right)^{\alpha}}\right) - \sum_{i=1}^{n}\ln\left(1-\mathrm{e}^{-\frac{a}{\beta}\left(\mathrm{e}^{\frac{\beta}{x_i}}-1\right)}\right), \tag{27}$$

$$\begin{aligned}\frac{\partial\ell}{\partial a} =& \frac{n}{a} - \frac{\alpha}{\beta}\sum_{i=1}^{n}\left(\mathrm{e}^{\frac{\beta}{x_i}} - 1\right) + \frac{\alpha\lambda}{\beta}\sum_{i=1}^{n}\left(\left(\mathrm{e}^{\frac{\beta}{x_i}}-1\right)\left(1-\mathrm{e}^{-\frac{a}{\beta}\left(\mathrm{e}^{\frac{\beta}{x_i}}-1\right)}\right)^{-\alpha-1}\mathrm{e}^{-\frac{a(\alpha+1)}{\beta}\left(\mathrm{e}^{\frac{\beta}{x_i}}-1\right)}\right) \\ & + \frac{\alpha\lambda}{\beta}\sum_{i=1}^{n}\frac{\left(\mathrm{e}^{\frac{\beta}{x_i}}-1\right)\mathrm{e}^{-\frac{a\alpha}{\beta}\left(\mathrm{e}^{\frac{\beta}{x_i}}-1\right)}}{\left(1-\mathrm{e}^{-\frac{a}{\beta}\left(\mathrm{e}^{\frac{\beta}{x_i}}-1\right)}\right)^{\alpha}} + \frac{(\alpha+1)}{\beta}\sum_{i=1}^{n}\frac{\left(\mathrm{e}^{\frac{\beta}{x_i}}-1\right)\mathrm{e}^{\frac{-a}{\beta}\left(\mathrm{e}^{\frac{\beta}{x_i}}-1\right)}}{\left(1-\mathrm{e}^{-\frac{a}{\beta}\left(\mathrm{e}^{\frac{\beta}{x_i}}-1\right)}\right)},\end{aligned} \tag{28}$$

$$\frac{\partial \ell}{\partial \beta}=\sum_{i=1}^{n}\frac{1}{x_i}-\frac{a\alpha}{\beta}\sum_{i=1}^{n}\frac{e^{\frac{\beta}{x_i}}}{x_i}+\frac{a\alpha}{\beta^2}\sum_{i=1}^{n}\left(e^{\frac{\beta}{x_i}}-1\right)-\lambda\sum_{i=1}^{n}\left(\frac{\left(\frac{a\alpha}{\beta^2}\left(e^{\frac{b}{x_i}}-1\right)-\frac{a\alpha}{\beta x_i}e^{\frac{\beta}{x_i}}\right)e^{-\frac{a\alpha}{\beta}\left(e^{\frac{\beta}{x_i}}-1\right)}}{\left(1-e^{-\frac{a}{\beta}\left(e^{\frac{\beta}{x_i}}-1\right)}\right)^{\alpha}}\right)$$
$$-\alpha\lambda\sum_{i=1}^{n}\left(\frac{a}{\beta^2}\left(e^{\frac{\beta}{x_i}}-1\right)-\frac{a}{\beta x_i}e^{\frac{\beta}{x_i}}\right)e^{-\frac{a(\alpha+1)}{\beta}\left(e^{\frac{\beta}{x_i}}-1\right)}\left(1-e^{-\frac{a}{\beta}\left(e^{\frac{\beta}{x_i}}-1\right)}\right)^{-\alpha-1} \tag{29}$$
$$-(\alpha+1)\sum_{i=1}^{n}\frac{\left(\frac{a}{\beta^2}\left(e^{\frac{\beta}{x_i}}-1\right)-\frac{a}{\beta x_i}e^{\frac{\beta}{x_i}}\right)e^{-\frac{a}{\beta}\left(e^{\frac{\beta}{x_i}}-1\right)}}{\left(1-e^{-\frac{a}{\beta}\left(e^{\frac{\beta}{x_i}}-1\right)}\right)}.$$

The MLE for each parameter cannot be calculated directly from Equations (25)–(29). Therefore, optimization techniques such as the Newton–Rapshon algorithm in "R software version 4.3.2" can be utilized to obtain $\hat{\alpha}, \hat{\lambda}, \hat{a}, \hat{\beta}$.

5. Simulation Studies

A Monte Carlo simulation is conducted to illustrate the performance of MLE of the OWIG parameters based on their mean square error (MSE) and root mean square error (RMSE) using the following expression:

$$MSE = \text{var}(\hat{\theta}) + [\text{Bias}(\hat{\theta})]^2 = \frac{1}{N}\sum_{i=1}^{n}(\hat{\theta}-\theta_{tr})^2$$

where, $Bias = \frac{1}{N}\sum_{i=1}^{n}(\hat{\theta}-\theta_{tr})$

$$RMSE = \sqrt{MSE} = \sqrt{\frac{1}{N}\sum_{i=1}^{n}(\hat{\theta}-\theta_{tr})^2}$$

The simulation results are conducted via R program by generating 1000 samples from OWIG using the quantile function provided in Equation (14). Also, by using different sample sizes when $n = 50, 100, 200$, and 500 and several values of true parameters are obtained, as follows:

Case I: $\alpha = 0.7, a = 0.5, \beta = 0.2, \lambda = 0.2$
Case II: $\alpha = 0.9, a = 1.1, \beta = 0.35, \lambda = 1.1$
Case III: $\alpha = 2.0, a = 1.5, \beta = 4.0, \lambda = 0.1$
Case IV: $\alpha = 2.0, a = 0.5, \beta = 1.2, \lambda = 0.1$

It can be observed from Tables 1 and 2 that MSE and RMSE decrease when the sample size n increases. In addition, when the sample size n increases, the estimates approach the true values of the parameters.

Table 1. Simulation study: Parameter estimates, MSE, and RMSE for case I and case II.

Sample Size	Parameter	Case I			Case II		
		Estimate	MSE	RMSE	Estimate	MSE	RMSE
n = 50	α	1.9142	0.3997	0.6322	0.8690	0.0586	0.2421
	a	0.4209	0.6975	0.8352	1.0842	3.5263	1.8778
	β	0.6130	0.3843	0.6199	0.9822	1.0892	1.0437
	λ	0.1735	0.0621	0.2493	1.0883	2.8870	1.6991

Table 1. *Cont.*

Sample Size	Parameter	Case I			Case II		
		Estimate	MSE	RMSE	Estimate	MSE	RMSE
n = 100	α	0.6880	0.0049	0.0702	0.8745	0.0298	0.1728
	a	0.4195	0.3496	0.5913	1.0149	2.2070	1.4856
	β	0.4756	0.1924	0.4387	0.8286	0.6433	0.8021
	λ	0.1698	0.0360	0.1896	1.0241	1.7828	1.3352
n = 200	α	0.6923	0.0024	0.0494	0.8889	0.0156	0.1248
	a	0.4258	0.2052	0.4530	0.9865	0.9415	0.9703
	β	0.4011	0.1276	0.3572	0.6753	0.4045	0.6360
	λ	0.1714	0.0231	0.1521	1.0205	1.0337	1.0167
n = 500	α	0.6976	0.0009	0.0307	0.8956	0.0076	0.0872
	a	0.4216	0.1029	0.3208	1.0151	0.4463	0.6680
	β	0.3288	0.0720	0.2684	0.5313	0.2020	0.4494
	λ	0.1704	0.0118	0.1088	1.0395	0.5193	0.7206

Table 2. Simulation study: Parameter estimates, MSE, and RMSE for case III and case IV.

Sample Size	Parameter	Case III			Case IV		
		Estimate	MSE	RMSE	Estimate	MSE	RMSE
n = 50	α	1.9452	0.4048	0.6363	1.8989	0.3381	0.5814
	a	1.5665	5.2237	2.2855	0.4679	0.2520	0.5020
	β	5.8516	27.9653	5.2882	1.9143	3.2040	1.7900
	λ	0.3557	3.1456	1.7736	0.2428	1.0497	1.0245
n = 100	α	1.9801	0.1765	0.4202	1.9668	0.1699	0.4122
	a	1.4780	2.7060	1.6450	0.5088	0.2192	0.4682
	β	4.9706	11.2436	3.3532	1.5100	1.1033	1.0504
	λ	0.2225	1.2119	1.1009	0.2230	0.5908	0.7686
n = 200	α	1.9557	0.0829	0.2879	1.9545	0.0855	0.2925
	a	1.5413	1.1824	1.0874	0.5213	0.1734	0.4164
	β	4.6192	5.2342	2.2878	1.4016	0.5594	0.7479
	λ	0.1880	0.2274	0.4768	0.2164	0.4497	0.6706
n = 500	α	1.9712	0.0323	0.1796	1.9688	0.0326	0.1805
	a	1.5281	0.7382	0.8592	0.5113	0.0892	0.2987
	β	4.3456	2.2659	1.5053	1.3180	0.2617	0.5115
	λ	0.1547	0.0946	0.3076	0.1546	0.0738	0.2717

6. Applications

The efficacy of OWIG is investigated by examining three data sets from different disciplines. The data are listed as follows:

Data 1: Pre-schoolers data

The following are the General Rating of Affective Symptoms for Preschoolers (GRASP) scores, which indicate how children's emotional and behavioral issues are measured (frequency in parentheses) [27]:

19 (16)	20 (15)	21 (14)	22 (9)	23 (12)	24 (10)	25 (6)	26 (9)	27 (8)	28 (5)	29 (6)
30 (4)	31 (3)	32 (4)	33	34	35 (4)	36 (2)	37 (2)	39	42	44

Data 2: Precipitation data

The following is the precipitation data, which represents the annual maximum precipitation (inches) in Fort Collins, Colorado, for one rain gauge (1900–1999) [28]:

239	232	434	85	302	174	170	121	193	168	148	116	132
132	144	183	223	96	298	97	116	146	84	230	138	170
117	115	132	125	156	124	189	193	71	176	105	93	354
60	151	160	219	142	117	87	223	215	108	354	213	306
169	184	71	98	96	218	176	121	161	321	102	269	98
271	95	212	151	136	240	162	71	110	285	215	103	443
185	199	115	134	297	187	203	146	94	129	162	112	348
95	249	103	181	152	135	463	183	241				

Data 3: Survival times of cancer patients

The survival rates for 44 people with head and neck cancer are listed below. Chemotherapy and radiation (RT+CT) are used to treat patients [28]:

12.20	23.56	23.74	25.87	31.98	37	41.35	47.38	55.46	58.36	63.47	68.46
78.26	74.47	81.43	84	92	94	110	112	119	127	130	133
140	146	155	159	173	179	194	195	209	249	281	319
339	432	469	519	633	725	817	1776				

The appropriateness of the three data sets for OWIG is evaluated by comparing its fit to the following distributions:

- The Weibull Inverse Gompertz (WIG) using the Weibull-G family in [5], where the G is represented by the IG distribution

$$F(x) = 1 - \exp\left[-\lambda^{-\alpha}\left(-log\left(1 - \exp\left(-\frac{a}{\beta}\left(e^{\frac{\beta}{x}} - 1\right)\right)\right)\right)^{\alpha}\right].$$

- The generalized inverse Gompertz (GIG) by [23]

$$F(x) = 1 - \left[1 - \exp\left(-\frac{a}{\beta}\left(e^{\frac{\beta}{x}} - 1\right)\right)\right]^{\alpha}.$$

- The inverse power Gompertz (PIG) by [24]

$$F(x) = \exp\left(-\frac{a}{\beta}\left(e^{\frac{\beta}{x^{\alpha}}} - 1\right)\right).$$

The performance of OWIG is assessed using the goodness of fit criteria (GoF), which include the $-\ell$, Akaike information criterion (AIC), corrected AIC (CAIC), Bayesian information criterion (BIC), Kramér-von Mises (W*), Anderson–Darling (AD*), and Kolmogorov–Smirnov (KS) test statistics with its corresponding p-value. In general, the model with the lowest AIC, CAIC, BIC, and KS values with the highest p-value will provide a better fit for the data.

Tables 3–5 present the MLEs, as well as the GoF criteria of OWIG and the competing distributions for all datasets. Furthermore, the estimated pdf and cdf of OWIG and rival distributions are shown in Figures 3–5.

Table 3. MLEs and GoF measures for GRASP data.

Distributions	OWIG	WIG	GIG	PIG	IG
Estimates	$\hat{\lambda} = 4.5623$	$\hat{\lambda} = 0.1231$			
	$\hat{\alpha} = 0.4747$	$\hat{\alpha} = 0.9684$	$\hat{\alpha} = 1.5841$	$\hat{\alpha} = 5.2337$	
	$\hat{a} = 0.0062$	$\hat{a} = 0.9731$	$\hat{a} = 0.0532$	$\hat{a} = 3.9800$	$\hat{a} = 0.0101$
	$\hat{b} = 10.9913$	$\hat{b} = 3.3355$	$\hat{b} = 7.2699$	$\hat{b} = 2.8420$	$\hat{b} = 9.4218$

Table 3. *Cont.*

Distributions	OWIG	WIG	GIG	PIG	IG
$-\ell$	−15.4156	−22.5233	−21.6541	−18.5220	−21.8509
AIC	38.8313	53.0465	49.3083	43.0441	47.7019
CAIC	44.6270	58.8422	53.6551	47.3908	50.5998
BIC	50.4227	64.6379	58.0019	51.7376	53.4976
W*	0.1231	0.2307	0.2092	0.2141	0.2177
AD*	1.0045	1.9120	1.5296	1.5319	1.6097
KS	0.0886	0.0891	0.0991	0.0945	0.0973
p-value	0.2428	0.2384	0.1385	0.1821	0.1581

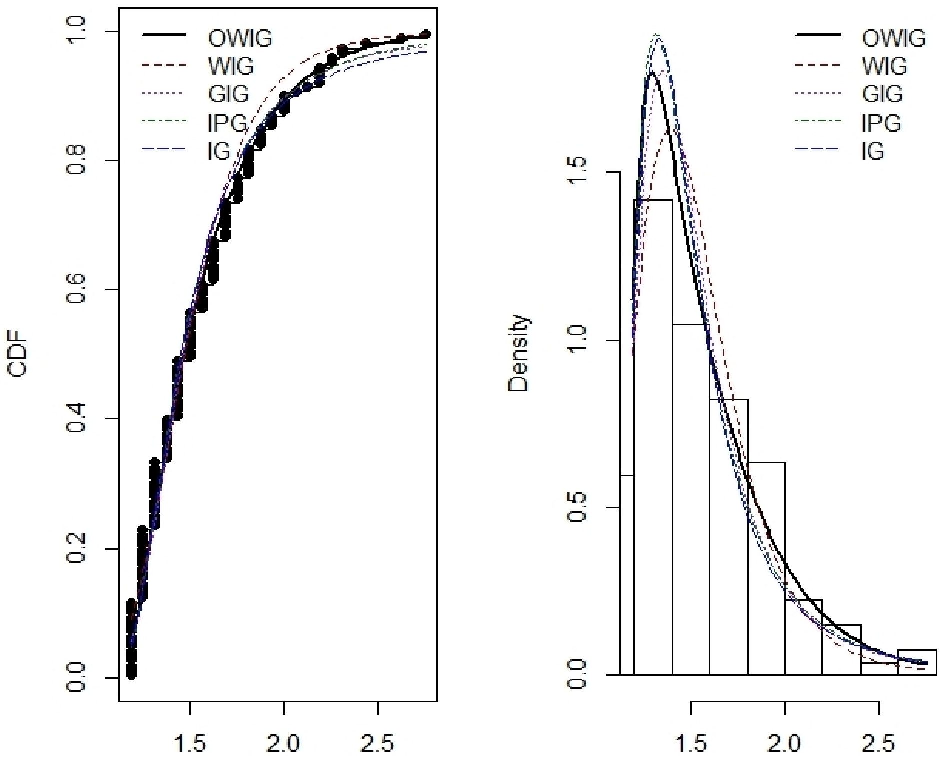

Figure 3. Estimated cdfs and pdfs for GRASP data.

Table 4. MLEs and GoF measures for precipitation data.

Distributions	OWIG	WIG	GIG	PIG	IG
Estimates	$\hat{\lambda} = 0.9241$	$\hat{\lambda} = 1.9389$			
	$\hat{\alpha} = 1.0991$	$\hat{\alpha} = 4.3919$	$\hat{\alpha} = 0.8167$	$\hat{\alpha} = 0.2891$	
	$\hat{a} = 79.9890$	$\hat{a} = 26.8193$	$\hat{a} = 54.4072$	$\hat{a} = 0.0071$	$\hat{a} = 40.7834$
	$\hat{b} = 141.2530$	$\hat{b} = 31.4120$	$\hat{b} = 16.0385$	$\hat{b} = 34.3842$	$\hat{b} = 238.2838$
$-\ell$	−564.6826	−564.8341	−641.5151	−572.47	−579.5334
AIC	1137.365	1139.994	1289.030	1150.940	1163.067
CAIC	1142.575	1147.745	1292.938	1154.848	1165.672
BIC	1147.786	1152.955	1296.846	1158.755	1168.277
W*	0.0156	0.0465	4.3555	0.1391	0.3240
AD*	0.1524	0.3555	21.4890	1.1192	2.4481
KS	0.0407	0.0551	0.3884	0.0764	0.1215
p-value	0.9964	0.8714	1.559×10^{-13}	0.6022	0.1041

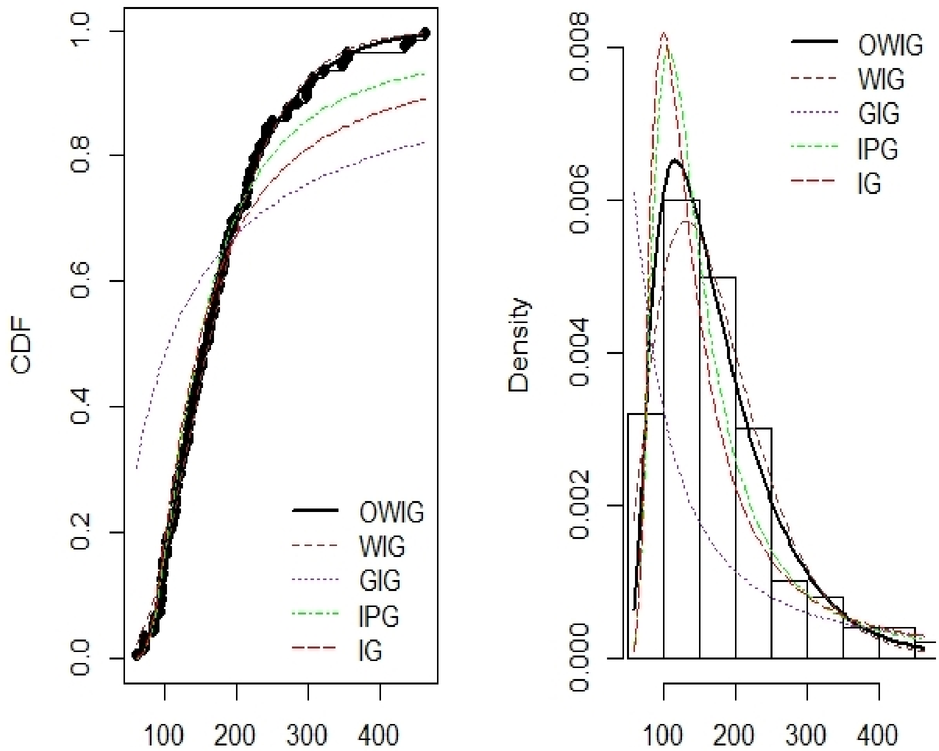

Figure 4. Estimated cdfs and pdfs for precipitation data.

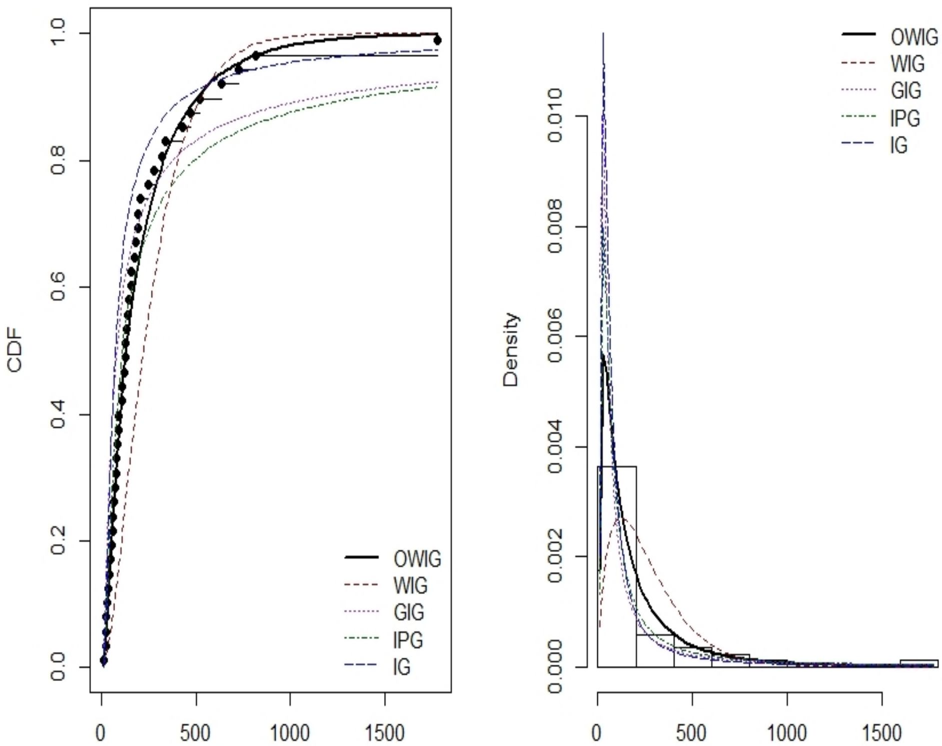

Figure 5. Estimated cdfs and pdfs for cancer data.

Table 5. MLEs and GoF measures for cancer data.

Distributions	OWIG	WIG	GIG	PIG	IG
Estimates	$\hat{\lambda} = 0.0055$	$\hat{\lambda} = 10.5667$			
	$\hat{\alpha} = 0.7541$	$\hat{\alpha} = 14.9225$	$\hat{\alpha} = 0.6323$	$\hat{\alpha} = 0.6977$	
	$\hat{a} = 0.1596$	$\hat{a} = 0.0073$	$\hat{a} = 30.5926$	$\hat{a} = 15.8997$	$\hat{a} = 46.3294$
	$\hat{b} = 80.7622$	$\hat{b} = 3.4091$	$\hat{b} = 8.4041$	$\hat{b} = 8.2868$	$\hat{b} = 9.8322$
$-\ell$	−277.836	−289.9746	−286.0049	−282.6561	−283.3979
AIC	563.6720	587.9493	578.0097	571.3122	570.7958
CAIC	567.2403	591.5176	580.686	573.9885	572.5800
BIC	570.8087	595.0860	583.3623	576.6648	574.3642
W*	0.0432	1.4577	0.6158	0.1885	0.8875
AD*	0.2655	7.9456	3.1290	1.2207	4.0556
KS	0.0908	0.2938	0.2156	0.1195	0.2415
p-value	0.8284	0.0007	0.0282	0.5167	0.0096

Concerning Tables 3–5, OWIG has the largest p-value and the lowest values of GOF criteria. This suggests that OWIG provides better fits for all three applications when compared with the rival distributions. Figures 3–5 also clearly show that OWIG matches the histogram more closely than the other competitive distributions.

7. Conclusions

This work introduced the OWIG distribution derived by considering the Weibull generator with the IG distribution. This is regarded as a new generalization of the IG distribution. The proposed OWIG is more versatile as its density and hrf present attractive shapes. The OWIG's hrf includes increasing, bath-tab, upside down bath-tab, decreasing, reversed J-shape, and unimodal shapes, which are suitable to fit an extensive variety of real data behaviors. Some essential statistical properties of OWIG are obtained. The well-known ML approach is utilized to estimate the parameters of OWIG, and the performance of the ML estimators is examined using Monte Carlo simulation studies. The simulation results confirmed that the ML estimation approach functioned effectively for estimating the OWIG parameters. Three real data sets from psychology, environmental, and medical sciences are analyzed to determine the OWIG's modeling capability and efficiency, demonstrating that it can fit data more accurately than WIG, GIG, PIG, and IG. OWIG has the highest p-value of KS statistics and the lowest GoF criterion for the three data. Compared with many lifetime models, the proposed OWIG will be capable of providing a superior fit for many lifetime and reliability applications.

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