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Geometric Properties of Normalized Galué Type Struve Function

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Abstract: The field of geometric function theory has thoroughly investigated starlike functions concerning symmetric points. The main objective of this work is to derive certain geometric properties, such as the starlikeness of order δ, convexity of order δ, k-starlikeness, k-uniform convexity, lemniscate starlikeness and convexity, exponential starlikeness and convexity, and pre-starlikeness for the Galué type Struve function (GTSF). Furthermore, the conditions for GTSF belonging to the Hardy space are also derived. The results obtained in this work generalize several results available in the literature.

Keywords: analytic function; univalent function; starlike function; k-starlike function; pre-starlike function; convex functions; k-uniformly convex function; lemniscate of Bernoulli; Hardy space; Galué type Struve function

1. Introduction

Let us consider the disk of radius r to be represented as $\mathbb{D}_r = \{z \in \mathbb{C} : |z| < r\}$, and for simplicity, we denote $\mathbb{D}_1$ as $\mathbb{D}$. Suppose the class of analytic functions f defined on $\mathbb{D}$, and normalized by the condition $f(0) = f'(0) - 1 = 0$, can be denoted by $\mathcal{A}$. If a function $f \in \mathcal{A}$ is univalent in $\mathbb{D}$, and $f(\mathbb{D})$ is a starlike domain with respect to the origin, then it is said to be starlike [1]. Analytically,

$$f \in \mathcal{A} \text{ is starlike } \iff \Re\left(\frac{zf'(z)}{f(z)}\right) > 0 \text{ for } z \in \mathbb{D}.$$

For $0 \leq \delta < 1$,

$$f \in \mathcal{A} \text{ is starlike of order } \delta \iff \Re\left(\frac{zf'(z)}{f(z)}\right) > \delta \text{ for } z \in \mathbb{D}.$$

The class of the starlike function of order δ is denoted by $\mathcal{ST}(\delta)$. We simply denote $\mathcal{ST}(0)$ as $\mathcal{ST}$.

Also, if a function $f \in \mathcal{A}$ is univalent in $\mathbb{D}$, and $f(\mathbb{D})$ is a convex domain, then the function f is said to be convex [1]. Analytically,

$$f \in \mathcal{A} \text{ is convex } \iff \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \text{ for } z \in \mathbb{D}.$$

For $0 \leq \delta < 1$, the function

$$f \in \mathcal{A} \text{ is convex of order } \delta \iff \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \delta \text{ for } z \in \mathbb{D}.$$

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We denote the class of convex functions of order δ by $\mathcal{CV}(\delta)$. For $\delta = 0$, the class of the convex function is simply denoted by $\mathcal{CV}$.

Kanas and Wiśniowska in [2] introduced the class k-$\mathcal{UCV}$ of k-uniformly convex functions, defined as the collection of functions $f \in \mathcal{A}$ such that the image of every circular arc contained in $\mathbb{D}$, with center ζ, where $|\zeta| \leq k$, is convex and also provided the one variable characterization. Let $f \in \mathcal{A}$ and $0 \leq k < \infty$, then

$$f \in \mathcal{A} \text{ is } k\text{-uniformly convex} \iff \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > k\left|\frac{zf''(z)}{f'(z)}\right| \text{ for } z \in \mathbb{D}.$$

According to [3], 1-$\mathcal{UCV} = \mathcal{UCV}$ and 0-$\mathcal{UCV} = \mathcal{CV}$.

In [4], Kanas and Wiśniowska had also defined a similar class k-$\mathcal{ST}$, related to the starlike functions, known as the k-starlike function.

$$f \in \mathcal{A} \text{ is } k\text{-starlike} \iff \Re\left(\frac{zf'(z)}{f(z)}\right) > k\left|\frac{zf'(z)}{f(z)} - 1\right| \text{ for } z \in \mathbb{D}.$$

In the case when $k = 0$, we obtain the known class $\mathcal{ST}$ of starlike functions. For $k = 1$, the class 1-$\mathcal{ST}$ coincides with the class $\mathcal{S}_p$, introduced by Rønning [5]. Geometrically, the class k-$\mathcal{ST}$ (k-$\mathcal{UCV}$) can be described as $f \in k$-$\mathcal{ST}$ ($f \in k$-$\mathcal{UCV}$) if the image of $\mathbb{D}$ under the function $\mathcal{SQ}_f(z) = \frac{zf'(z)}{f(z)}$, $\left(\mathcal{CQ}_f(z) = 1 + \frac{zf''(z)}{f'(z)}\right)$ is contained in the conic domain Ω_k, where $1 \in \Omega_k$ and Ω_k is bounded by the curve given by

$$\partial\Omega_k = \{w = x + iy \in \mathbb{C} : x^2 = k^2(x-1)^2 + k^2y^2\},\ 0 \leq k < \infty.$$

Some of the widely known subclasses of starlike functions associated with domains that are symmetric with respect to the real axis are the class of lemniscate starlike functions $\mathcal{S}_L^*$, which was studied by Sokól and Stankiewicz in [6] and the class $\mathcal{S}_e^*$ of starlike functions associated with exponential functions, which was introduced by Mendiratta et al. [7]. These classes are also characterized by the quantities $\mathcal{SQ}_f$ and $\mathcal{CQ}_f$. A function $f \in \mathcal{A}$ is said to be lemniscate starlike (lemniscate convex) on $\mathbb{D}$ if $\left\{\mathcal{SQ}_f(z) : |z| < 1\right\}$ $\left(\left\{\mathcal{CQ}_f(z) : |z| < 1\right\}\right)$ contained in the interior of the region bounded by the right half of the lemniscate of Bernoulli $L = \{w \in \mathbb{C} : \Re(w) > 0, |w^2 - 1| = 1\}$. The classes of lemniscate starlike functions and lemniscate convex functions are denoted by $\mathcal{S}_L^*$ and $\mathcal{C}_L^*$, respectively. The classes $\mathcal{S}_e^*$ and $\mathcal{C}_e^*$ represent the starlike and convex functions associated with exponential functions, which are given by

$$\mathcal{S}_e^* = \left\{f \in \mathcal{A} : \mathcal{SQ}_f(\mathbb{D}) \subset \mathcal{E}\right\} \text{ and } \mathcal{C}_e^* = \left\{f \in \mathcal{A} : \mathcal{CQ}_f(\mathbb{D}) \subset \mathcal{E}\right\},$$

where $\mathcal{E} = \{\exp(z) : z \in \mathbb{D}\}$. It can easily be observed that the domains L and $\mathcal{E}$ are symmetric with respect to the real axis [8]. Geometric function theory shares a close connection with symmetry. For example, both Möbius transformation theory and hyperbolic geometry employ symmetric principles. Furthermore, within function theory, there has been a comprehensive exploration of starlike functions in relation to symmetric points.

One of the special functions that has captured the interest of numerous researchers is the Struve function, along with its generalizations that arise in the field of mathematical physics and engineering. The generalized Struve function [9] is defined as a particular solution of the second-order inhomogeneous differential equation

$$z^2w''(z) + bzw'(z) + [cz^2 - p^2 + (1-b)p]w(z) = \frac{4(z/2)^{p+1}}{\sqrt{\pi}\Gamma(p + \frac{1}{2})}, \tag{1}$$

such that

$$w_{p,b,c}(z) = \sum_{n=0}^{\infty} \frac{(-c)^n (z/2)^{2n+p+1}}{\Gamma(n + \frac{3}{2})\Gamma(p + n + \frac{b+2}{2})}, \tag{2}$$

where $b, c, p \in \mathbb{C}$. This function generalizes the Struve function H_p of order p for $b = 1, c = 1$ and the modified Struve function L_p of order p for $b = 1, c = -1$. In [10], Nisar et al. introduced another generalization of the Struve function named the Galué type Struve function (GTSF), defined as

$$ {}_{\alpha}\mathcal{W}^{\lambda,\mu}_{p,b,c,\xi}(z) = \sum_{n=0}^{\infty} \frac{(-c)^n (z/2)^{2n+p+1}}{\Gamma(\lambda n + \mu)\Gamma(\alpha n + \frac{p}{\xi} + \frac{b+2}{2})}. \tag{3} $$

It can be easily seen that the Function (3) is a generalization of (2) and various other special functions frequently used in several branches of mathematics. For example:

(i) If we put $\alpha = \xi = \lambda = 1$ and $\mu = \frac{3}{2}$, we obtain

$$ {}_{1}\mathcal{W}^{1,\frac{3}{2}}_{p,b,c,1}(z) = w_{p,b,c}(z), $$

where $w_{p,b,c}(z)$ is the generalized Struve function defined as (2).

(ii) If we put $\alpha = c = \xi = \lambda = \mu = 1$, $p = n - 1$, and $b = 2$, we obtain

$$ {}_{1}\mathcal{W}^{1,1}_{n-1,2,1,1}(z) = J_n(z), $$

where $J_n(z)$ is the Bessel function [11].

(iii) If we put $\alpha = b = c = \xi = \lambda = 1$ and $\mu = \frac{3}{2}$, we obtain

$$ {}_{1}\mathcal{W}^{1,\frac{3}{2}}_{p,b,c,1}(z) = H_p(z), $$

where $H_p(z)$ is the Struve function of order p.

(iv) If we put $\alpha = \xi = 1$, $b = 2$, and $p = c = -1$, we obtain

$$ {}_{1}\mathcal{W}^{\lambda,\mu}_{-1,2,-1,1}(2\sqrt{z}) = \phi(\lambda, \mu; z), $$

where $\phi(\lambda, \mu, z)$ is the Wright function [12].

(v) ${}_{\alpha}\mathcal{W}^{\lambda,\mu}_{p,b,c,\xi}(z)$ have a connection with the Fox–Wright function ${}_{r}\Psi_{s}[z]$:

$$ {}_{\alpha}\mathcal{W}^{\lambda,\mu}_{p,b,c,\xi}(2\sqrt{z}) = \left(\frac{z}{p}\right)^{p+1} {}_{1}\Psi_{2}\left[\begin{matrix} (1,1) \\ (\lambda,\mu), (\frac{p}{\xi} + \frac{b+1}{2}, \alpha) \end{matrix} ; \frac{-cz^2}{4} \right]. $$

(vi) Setting $\delta = \mu = 1$, $\lambda = b = 0$, and $c = -1$, we obtain generalized the Mittag-Leffler function [13]:

$$ {}_{\alpha}\mathcal{W}^{\lambda,\mu}_{p-1,0,-1,1}(2\sqrt{z}) = (z)^{\frac{p}{2}} {}_{\alpha}E_p(z), \quad (p, \alpha > 0). $$

The GTSF in (3) does not belong to the class $\mathcal{A}$; thus, we use the following normalization for our study:

$$ {}_{\alpha}\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}(z) := 2^{p+1}\Gamma(\mu)\Gamma\left(\frac{p}{\xi} + \frac{b+2}{2}\right) z^{\frac{1-p}{2}} \, {}_{\alpha}\mathcal{W}^{\lambda,\mu}_{p,b,c,\xi}(\sqrt{z}). \tag{4} $$

It is noteworthy that research on special functions, such as the Struve functions, is a broad and ongoing field. Scholars subsequent to Struve have persisted in investigating diverse aspects, uses, and expansions of these functions. Further studies have broadened our knowledge and application of Struve functions in a variety of domains, such as engineering, physics, and mathematical analysis. The GTSF also plays a significant role in mathematical analysis, including applications in fractional calculus and integral transformations (see [10,14,15]).

Several researchers have investigated the geometric properties such as univalency, starlikeness, convexity, close to convexity, exponential starlikeness, exponential convexity, and the Hardy space of Struve functions and its generalizations, e.g., [9,16–21]. In [22], the strong starlikeness, strong convexity, and uniform convexity properties of GTSF were obtained. Motivated by these developments, we aim, here, to obtain the geometrical properties related to various kinds of starlikeness and convexity for normalized GTSF.

Outline

The rest of this paper is organized as follows. The Lemmas that are used to prove the main results are listed in Section 2. Section 3 contains the results related to the starlikeness and convexity of order δ and the geometrical properties on $\mathbb{D}_{\frac{1}{2}}$ for ${}_{\alpha}\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}(z)$. The k-starlikeness and k-uniform convexity properties are given in Section 4. In Section 5, the starlikeness and convexity associated with the exponential function and lemniscate of Bernoulli for ${}_{\alpha}\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}(z)$ are obtained. The conditions under which the function ${}_{\alpha}\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}(z)$ belongs to the class $\mathcal{L}[\rho,\delta]$ are provided in Section 6. In Section 7, the results associated with the Hardy space for ${}_{\alpha}\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}(z)$ are presented.

2. Useful Lemmas

This section contains some Lemmas that will be useful in proving the main results.

Lemma 1 ([23])**.** *For any real number $s > 1$, the digamma function $\psi(s) = \frac{\Gamma'(s)}{\Gamma(s)}$ satisfies the following inequality:*

$$\log(s) - \gamma \leq \psi(s) \leq \log(s),$$

where γ is the Euler–Mascheroni constant.

Lemma 2 ([24])**.** *Let $f \in \mathcal{A}$ and $\left|\frac{f(z)}{z} - 1\right| < 1$ for each $z \in \mathbb{D}$, then f is univalent and starlike in $\mathbb{D}_{\frac{1}{2}} = \{z : |z| < \frac{1}{2}\}$.*

Lemma 3 ([25])**.** *Let $f \in \mathcal{A}$ and $|f'(z) - 1| < 1$ for each $z \in \mathbb{D}$, then f is convex in $\mathbb{D}_{\frac{1}{2}} = \{z : |z| < \frac{1}{2}\}$.*

Lemma 4 ([4])**.** *Assume that $f \in \mathcal{A}$ with $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. If*

$$\sum_{n=2}^{\infty} (n + k(n+1))|a_n| < 1, \text{ for some } 0 \leq k < \infty,$$

then $f \in k$-$\mathcal{ST}$.

Lemma 5 ([2])**.** *Assume that $f \in \mathcal{A}$ with $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. If*

$$\sum_{n=2}^{\infty} n(n+1)|a_n| < \frac{1}{k+2}, \text{ for some } 0 \leq k < \infty,$$

then $f \in k$-$\mathcal{UCV}$.

Lemma 6 ([26])**.** *If the function f, convex of order δ $(0 \leq \delta < 1)$, is not of the following form:*

$$f(z) = \begin{cases} m + d \cdot z(1 - ze^{i\eta})^{2\delta - 1} & \delta \neq \dfrac{1}{2}, \\ m + d \cdot \log(1 - ze^{i\eta}) & \delta = \dfrac{1}{2}, \end{cases}$$

for $d, m \in \mathbb{C}$, $\eta \in \mathbb{R}$, then the following statements hold true:

(i) *If $0 \leq \delta < \frac{1}{2}$, then $\exists\, \sigma > 0$, such that $f \in \mathcal{H}^{\sigma + \frac{1}{1-2\delta}}$.*
(ii) *If $\delta \geq \frac{1}{2}$, then $f \in \mathcal{H}^{\infty}$.*

(iii) $\exists\, \rho > 0$, *such that* $f' \in \mathcal{H}^{\rho+\frac{1}{2(1-\delta)}}$.

Here, the notation $\mathcal{H}^p(0 < p \leq \infty)$ is associated with the Hardy space described in Section 7.

3. Starlikeness and Convexity of GTSF

In this section, we establish various properties related to starlikeness and convexity for the normalized GTSF. Additionally, some corollaries and examples for particular cases of the GTSF are provided. Initially, we derive conditions for the starlikeness and convexity of order δ of ${}_{\alpha}\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}(z)$.

Theorem 1. *Assume that* $\alpha \in \mathbb{N},\ \xi, \lambda, \mu > 0,\ |c| < 4$, *such that* $2p + \xi(b+2) > 0$. *If the following holds true:*

(i) $(2\xi)^{\alpha} e^{1+\gamma(\lambda+\alpha)} < (\mu+\lambda)^{\lambda}(2p+\xi(b+2)+2\xi\alpha)^{\alpha}$;

(ii) $|c|(2-\delta)\Gamma(\mu)\Gamma\left(\frac{p}{\xi}+\frac{b+2}{2}\right) < (4-|c|)(1-\delta)\Gamma(\mu+\lambda)\Gamma\left(\frac{p}{\xi}+\frac{b+2}{2}+\alpha\right)$,

then ${}_{\alpha}\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi} \in \mathcal{ST}(\delta)$.

Proof. To establish the required result, it suffices to show that

$$\left|\frac{z\ {}_{\alpha}\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}{}'(z)}{{}_{\alpha}\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}(z)} - 1\right| = \left|\frac{{}_{\alpha}\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}{}'(z) - \frac{{}_{\alpha}\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}(z)}{z}}{\frac{{}_{\alpha}\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}(z)}{z}}\right| < 1-\delta, \qquad (\forall z \in \mathbb{D}).$$

Now, by a calculation, we have

$$\begin{aligned}\left|{}_{\alpha}\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}{}'(z) - \frac{{}_{\alpha}\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}(z)}{z}\right| &= \left|\sum_{n=0}^{\infty} \frac{\Gamma(\mu)\Gamma\left(\frac{p}{\xi}+\frac{b+2}{2}\right) n \left(\frac{c}{4}\right)^n z^n}{\Gamma(\mu+\lambda n)\Gamma\left(\frac{p}{\xi}+\frac{b+2}{2}+\alpha n\right)}\right| \\ &< \Gamma(\mu)\Gamma\left(\frac{p}{\xi}+\frac{b+2}{2}\right)\sum_{n=1}^{\infty} d_n(\alpha,p,b,c,\xi,\lambda,\mu)\left|\frac{c}{4}\right|^n, \quad (\forall z \in \mathbb{D}),\end{aligned} \tag{5}$$

where

$$d_n = d_n(\alpha,p,b,c,\xi,\lambda,\mu) = \frac{n}{\Gamma(\mu+\lambda n)\Gamma\left(\frac{p}{\xi}+\frac{b+2}{2}+\alpha n\right)}, \quad n \in \mathbb{N}.$$

Next, consider the function $\mathscr{D}_1 : [1,\infty) \to \mathbb{R}$ as

$$\mathscr{D}_1(s) := \frac{s}{\Gamma(\mu+\lambda s)\Gamma\left(\frac{p}{\xi}+\frac{b+2}{2}+\alpha s\right)}, \quad s \in [1,\infty). \tag{6}$$

Therefore,

$$\mathscr{D}_1'(s) = \mathscr{D}_1(s)\mathscr{D}_2(s), \tag{7}$$

where $\mathscr{D}_2$ is given by

$$\mathscr{D}_2(s) = \frac{1}{s} - \lambda\psi(\mu+\lambda s) - \alpha\psi\left(\frac{p}{\xi}+\frac{b+2}{2}+\alpha s\right), \quad s \in [1,\infty). \tag{8}$$

From Lemma 1, we obtain

$$\mathscr{D}_2(s) \leq \frac{1}{s} - \lambda\log(\mu+\lambda s) - \alpha\log\left(\frac{p}{\xi}+\frac{b+2}{2}+\alpha s\right) + \gamma(\lambda+\alpha) = \mathscr{D}_3(s), \quad s \in [1,\infty).$$

This leads to

$$\mathscr{D}_3'(s) = \frac{-1}{s^2} - \frac{\lambda^2}{\mu + \lambda s} - \frac{\alpha^2}{\frac{p}{\xi} + \frac{b+2}{2} + \alpha s} < 0, \quad s \in [1, \infty).$$

This implies that $\mathscr{D}_3(s)$ is decreasing on $[1, \infty)$. Also, under the given hypothesis (i), $\mathscr{D}_3(1) < 0$ and, thus, $\mathscr{D}_1'(s) < 0$ for $s \in [1, \infty)$. Consequently, $\{d_n\}_{n \geq 1}$ is a decreasing sequence. Now, from (5), we have

$$\begin{aligned}\left| {}_{\alpha}\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}{}'(z) - \frac{{}_{\alpha}\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}(z)}{z} \right| &< \Gamma(\mu)\Gamma\left(\frac{p}{\xi} + \frac{b+2}{2}\right) \sum_{n=1}^{\infty} d_1(\alpha, p, b, c, \xi, \lambda, \mu) \left|\frac{c}{4}\right|^n, \\ &= \Gamma(\mu)\Gamma\left(\frac{p}{\xi} + \frac{b+2}{2}\right) d_1(\alpha, p, b, c, \xi, \lambda, \mu) \frac{|c|}{4 - |c|}, \quad (\forall z \in \mathbb{D}). \end{aligned} \tag{9}$$

Now,

$$\left| \frac{{}_{\alpha}\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}(z)}{z} \right| \geq 1 - \sum_{n=1}^{\infty} r_n(\alpha, p, b, c, \xi, \lambda, \mu) \left(\frac{|c|}{4}\right)^n, \ (\forall z \in \mathbb{D}), \tag{10}$$

where

$$r_n = r_n(\alpha, p, b, c, \xi, \lambda, \mu) = \frac{\Gamma(\mu)\Gamma\left(\frac{p}{\xi} + \frac{b+2}{2}\right)}{\Gamma(\mu + \lambda n)\Gamma\left(\frac{p}{\xi} + \frac{b+2}{2} + \alpha n\right)}.$$

Similarly, it can be shown that $\{r_n\}_{n \geq 1}$ is a decreasing sequence. Therefore, using (10), we have

$$\begin{aligned}\left| \frac{{}_{\alpha}\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}(z)}{z} \right| &\geq 1 - \sum_{n=1}^{\infty} r_1(\alpha, p, b, c, \xi, \lambda, \mu) \left(\frac{|c|}{4}\right)^n \\ &= 1 - \frac{r_1(\alpha, p, b, c, \xi, \lambda, \mu)|c|}{4 - |c|}, \quad (\forall z \in \mathbb{D}). \end{aligned} \tag{11}$$

Combining (9) and (11), we have

$$\begin{aligned}&\left| \frac{{}_{\alpha}\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}{}'(z) - \frac{{}_{\alpha}\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}(z)}{z}}{\frac{{}_{\alpha}\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}(z)}{z}} \right| \\ &\quad < \frac{\Gamma(\mu)\Gamma\left(\frac{p}{\xi} + \frac{b+2}{2}\right)|c|}{(4 - |c|)\Gamma(\mu + \lambda)\Gamma\left(\frac{p}{\xi} + \frac{b+2}{2} + \alpha\right) - |c|\Gamma(\mu)\Gamma\left(\frac{p}{\xi} + \frac{b+2}{2}\right)}. \end{aligned} \tag{12}$$

From the condition *(ii)*, the following holds:

$$\frac{\Gamma(\mu)\Gamma\left(\frac{p}{\xi} + \frac{b+2}{2}\right)|c|}{(4 - |c|)\Gamma(\mu + \lambda)\Gamma\left(\frac{p}{\xi} + \frac{b+2}{2} + \alpha\right) - |c|\Gamma(\mu)\Gamma\left(\frac{p}{\xi} + \frac{b+2}{2}\right)} < 1 - \delta, \ (\forall z \in \mathbb{D}).$$

Hence, the theorem is proved. □

Corollary 1. *Following are special cases from Theorem 1 when $\alpha = \xi = \lambda = 1$, and μ respectively $3/2$ and 1:*

(i) *If* $p + \dfrac{b+2}{2} > \max\left\{\dfrac{2e^{1+2\gamma}}{5} - 1, \dfrac{4|c|}{3(4-|c|)}\right\}$, *then the function* ${}_1\mathcal{G}^{1,\frac{3}{2}}_{p,b,c,1}$ *is starlike in* $\mathbb{D}$.

(ii) *If* $p + \dfrac{b+2}{2} > \max\left\{\dfrac{e^{1+2\gamma}}{2} - 1, \dfrac{3|c|}{(4-|c|)}\right\}$, *then the function* ${}_1\mathcal{G}^{1,1}_{p,b,c,1}$ *is starlike in* $\mathbb{D}$.

Example 1. *Following examples can be construct from Theorem 1*

(i) ${}_1\mathcal{G}^{1,\frac{3}{2}}_{-1.05,1,1,1}$ *is starlike in* $\mathbb{D}$.

(ii) ${}_1\mathcal{G}^{1,1}_{-1.3,2,1,1}$ *is starlike in* $\mathbb{D}$.

Now, in this following theorem, the conditions for the convexity of order δ are derived for GTSF.

Theorem 2. *Assume that* $\alpha \in \mathbb{N}$, $\xi, \lambda, \mu > 0$, $|c| < 4$, *such that* $2p + \xi(b+2) > 0$. *If the following holds true:*

(i) $(2\xi)^\alpha e^{\frac{3}{2}+\gamma(\lambda+\alpha)} < (\mu+\lambda)^\lambda (2p + \xi(b+2) + 2\xi\alpha)^\alpha$;

(ii) $2|c|(2-\delta)\Gamma(\mu)\Gamma\left(\dfrac{p}{\xi} + \dfrac{b+2}{2}\right) < (4-|c|)(1-\delta)\Gamma(\mu+\lambda)\Gamma\left(\dfrac{p}{\xi} + \dfrac{b+2}{2} + \alpha\right)$,

then ${}_\alpha\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi} \in \mathcal{CV}(\delta)$.

Proof. Clearly, we are finished if we can show that

$$\left|\frac{z\, {}_\alpha\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}{}''(z)}{{}_\alpha\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}{}'(z)}\right| < 1-\delta, \qquad (\forall z \in \mathbb{D}).$$

Now,

$$\begin{aligned}\left|z{}_\alpha\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}{}''(z)\right| &= \left|\sum_{n=0}^{\infty} \frac{\Gamma(\mu)\Gamma\left(\frac{p}{\xi} + \frac{b+2}{2}\right) n(n+1)\left(\frac{c}{4}\right)^n z^n}{\Gamma(\mu+\lambda n)\Gamma\left(\frac{p}{\xi} + \frac{b+2}{2} + \alpha n\right)}\right| \\ &< \Gamma(\mu)\Gamma\left(\frac{p}{\xi} + \frac{b+2}{2}\right) \sum_{n=1}^{\infty} h_n(\alpha, p, b, c, \xi, \lambda, \mu)\left|\frac{c}{4}\right|^n, \quad (\forall z \in \mathbb{D}),\end{aligned} \tag{13}$$

where

$$h_n = h_n(\alpha, p, b, c, \xi, \lambda, \mu) = \frac{n(n+1)}{\Gamma(\mu+\lambda n)\Gamma\left(\frac{p}{\xi} + \frac{b+2}{2} + \alpha n\right)}, \quad n \in \mathbb{N}.$$

Now, consider the function $\mathscr{H}_1(s)$ as

$$\mathscr{H}_1(s) := \frac{s(s+1)}{\Gamma(\mu+\lambda s)\Gamma\left(\frac{p}{\xi} + \frac{b+2}{2} + \alpha s\right)}, \quad s \in [1, \infty). \tag{14}$$

Therefore,

$$\mathscr{H}_1'(s) = \mathscr{H}_1(s)\mathscr{H}_2(s), \tag{15}$$

where $\mathscr{H}_2$ is given by

$$\mathscr{H}_2(s) = \frac{1}{s} + \frac{1}{s+1} - \lambda\psi(\mu+\lambda s) - \alpha\psi\left(\frac{p}{\xi} + \frac{b+2}{2} + \alpha s\right), \quad s \in [1, \infty). \tag{16}$$

From Lemma 1, we obtain

$$\mathscr{H}_2(s) \leq \frac{1}{s} + \frac{1}{s+1} - \lambda \log(\mu + \lambda s) - \alpha \log\left(\frac{p}{\xi} + \frac{b+2}{2} + \alpha s\right) \\ + \gamma(\lambda + \alpha) = \mathscr{H}_3(s), \;\; s \in [1, \infty).$$

This leads to

$$\mathscr{H}_3'(s) = -\frac{1}{s^2} - \frac{1}{(s+1)^2} - \frac{\lambda^2}{\mu + \lambda s} - \frac{\alpha^2}{\frac{p}{\xi} + \frac{b+2}{2} + \alpha s} < 0, \;\; s \in [1, \infty).$$

This implies that $\mathscr{H}_3(s)$ is decreasing on $[1, \infty)$. Also, under the given hypothesis (i) $\mathscr{H}_3(1) < 0$ and, thus, $\mathscr{H}_1'(s) < 0$ for $s \in [1, \infty)$. Consequently, $\{h_n\}_{n\geq 1}$ is decreasing sequence. Now, from (13), we have

$$\begin{aligned}\left|z_\alpha \mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}{}''(z)\right| &< \Gamma(\mu)\Gamma\left(\frac{p}{\xi} + \frac{b+2}{2}\right) \sum_{n=1}^{\infty} h_1(\alpha, p, b, c, \xi, \lambda, \mu)\left|\frac{c}{4}\right|^n, \\ &= \Gamma(\mu)\Gamma\left(\frac{p}{\xi} + \frac{b+2}{2}\right) h_1(\alpha, p, b, c, \xi, \lambda, \mu) \frac{|c|}{4 - |c|}, \;\; (\forall z \in \mathbb{D}). \end{aligned} \tag{17}$$

Now,

$$\left|{}_\alpha\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}{}'(z)\right| \geq 1 - \sum_{n=1}^{\infty} j_n(\alpha, p, b, c, \xi, \lambda, \mu)\left(\frac{|c|}{4}\right)^n, \;(\forall z \in \mathbb{D}), \tag{18}$$

where

$$j_n = j_n(\alpha, p, b, c, \xi, \lambda, \mu) = \frac{\Gamma(\mu)\Gamma\left(\frac{p}{\xi} + \frac{b+2}{2}\right)(n+1)}{\Gamma(\mu + \lambda n)\Gamma\left(\frac{p}{\xi} + \frac{b+2}{2} + \alpha n\right)}.$$

By similar arguments, we see that $\{j_n\}_{n\geq 1}$ is a decreasing sequence. Therefore, using (18), we have

$$\begin{aligned}\left|{}_\alpha\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}{}'(z)\right| &\geq 1 - \sum_{n=1}^{\infty} j_1(\alpha, p, b, c, \xi, \lambda, \mu)\left(\frac{|c|}{4}\right)^n \\ &= 1 - \frac{j_1(\alpha, p, b, c, \xi, \lambda, \mu)|c|}{4 - |c|}, \;\; (\forall z \in \mathbb{D}). \end{aligned} \tag{19}$$

Combining (17) and (19), we have

$$\left|\frac{z_\alpha \mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}{}''(z)}{{}_\alpha\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}{}'(z)}\right| < \frac{2\Gamma(\mu)\Gamma\left(\frac{p}{\xi} + \frac{b+2}{2}\right)|c|}{(4 - |c|)\Gamma(\mu + \lambda)\Gamma\left(\frac{p}{\xi} + \frac{b+2}{2} + \alpha\right) - 2|c|\Gamma(\mu)\Gamma\left(\frac{p}{\xi} + \frac{b+2}{2}\right)}. \tag{20}$$

Finally, using the given hypothesis (ii), the desired result can be established. □

Corollary 2. *If* $p + \frac{b+2}{2} > \max\left\{\frac{2e^{\frac{3}{2}+2\gamma}}{5} - 1, \frac{8|c|}{3(4-|c|)}\right\}$, *then the function* ${}_1\mathcal{G}^{1,\frac{3}{2}}_{p,b,c,1}$ *is convex in* $\mathbb{D}$.

Example 2. *Following examples can be construct from Theorem 2*

(i) ${}_1\mathcal{G}^{1,\frac{3}{2}}_{-0.6,1,1,1}$ *is convex in* $\mathbb{D}$.

(ii) ${}_1\mathcal{G}^{1,1}_{-0.65,2,1,1}$ *is convex in* $\mathbb{D}$.

Next, we will obtain the starlikeness and convexity conditions over $\mathbb{D}_{\frac{1}{2}}$.

Theorem 3. *Assume that* $\alpha \in \mathbb{N}$, $\xi, \lambda, \mu > 0$, $|c| < 4$, *such that* $2p + \xi(b+2) > 0$. *If the following holds true:*

(i) $(2\xi)^{\alpha} e^{\gamma(\lambda+\alpha)} < (\mu+\lambda)^{\lambda}(2p + \xi(b+2) + 2\xi\alpha)^{\alpha}$;

(ii) $|c|\Gamma(\mu)\Gamma(\frac{p}{\xi} + \frac{b+2}{2}) < (4-|c|)\Gamma(\mu+\lambda)\Gamma(\frac{p}{\xi} + \frac{b+2}{2} + \alpha)$,

then ${}_{\alpha}\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}(z)$ *is starlike in* $\mathbb{D}_{\frac{1}{2}}$.

Proof. A simple computation gives

$$\begin{aligned} \left| \frac{{}_{\alpha}\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}(z)}{z} - 1 \right| &= \left| \sum_{n=0}^{\infty} \frac{\Gamma(\mu)\Gamma\left(\frac{p}{\xi} + \frac{b+2}{2}\right)\left(\frac{c}{4}\right)^n z^n}{\Gamma(\mu+\lambda n)\Gamma\left(\frac{p}{\xi} + \frac{b+2}{2} + \alpha n\right)} \right| \\ &< \Gamma(\mu)\Gamma\left(\frac{p}{\xi} + \frac{b+2}{2}\right) \sum_{n=1}^{\infty} k_n(\alpha,p,b,c,\xi,\lambda,\mu)\left|\frac{c}{4}\right|^n, \quad (\forall z \in \mathbb{D}), \end{aligned} \tag{21}$$

where

$$k_n = k_n(\alpha,p,b,c,\xi,\lambda,\mu) = \frac{1}{\Gamma(\mu+\lambda n)\Gamma\left(\frac{p}{\xi} + \frac{b+2}{2} + \alpha n\right)}, \quad n \in \mathbb{N}.$$

Now, we define the function $\mathscr{K}_1(s)$ as

$$\mathscr{K}_1(s) := \frac{1}{\Gamma(\mu+\lambda s)\Gamma\left(\frac{p}{\xi} + \frac{b+2}{2} + \alpha s\right)}, \quad s \in [1,\infty). \tag{22}$$

Therefore,

$$\mathscr{K}_1'(s) = \mathscr{K}_1(s)\mathscr{K}_2(s), \tag{23}$$

where

$$\mathscr{K}_2(s) = -\lambda\psi(\mu+\lambda s) - \alpha\psi\left(\frac{p}{\xi} + \frac{b+2}{2} + \alpha s\right), \quad s \in [1,\infty).$$

Again, applying Lemma 1, we obtain

$$\mathscr{K}_2(s) \le -\lambda\log(\mu+\lambda s) - \alpha\log\left(\frac{p}{\xi} + \frac{b+2}{2} + \alpha s\right) + \gamma(\lambda+\alpha) = \mathscr{K}_3(s), \quad s \in [1,\infty). \tag{24}$$

Therefore,

$$\mathscr{K}_3'(s) = -\frac{\lambda^2}{\mu+\lambda s} - \frac{\alpha^2}{\frac{p}{\xi} + \frac{b+2}{2} + \alpha s} < 0, \quad s \in [1,\infty).$$

This implies that the function $\mathscr{K}_3(s)$ is decreasing on $[1,\infty)$ and by hypothesis (*i*), $\mathscr{K}_3(1) < 0$. So, $\mathscr{K}_3(s) < 0$ for all $s \ge 1$. Consequently, the function $\mathscr{K}_1(s)$ is decreasing with the aid of (23) and (24). Hence, the sequence $\{k_k\}_{n \ge \infty}$ is decreasing. Therefore, using (21), we obtain

$$\begin{aligned} \left| \frac{{}_{\alpha}\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}(z)}{z} - 1 \right| &< \Gamma(\mu)\Gamma\left(\frac{p}{\xi} + \frac{b+2}{2}\right) \sum_{n=1}^{\infty} k_1(\alpha,p,b,c,\xi,\lambda,\mu)\left|\frac{c}{4}\right|^n \\ &\le \frac{\Gamma(\mu)\Gamma\left(\frac{p}{\xi} + \frac{b+2}{2}\right)|c|}{(4-|c|)\Gamma(\mu+\lambda)\Gamma\left(\frac{p}{\xi} + \frac{b+2}{2} + \alpha\right)}, \quad (\forall z \in \mathbb{D}). \end{aligned} \tag{25}$$

Thus, the condition (*ii*) completes the proof. □

Corollary 3. *Following are special cases from Theorem 3 when* $\alpha = \xi = \lambda = 1$, *and* μ *respectively* 1 *and* $3/2$*:*

(i) *If* $p + \dfrac{b+2}{2} > \max\left\{\dfrac{2e^{2\gamma}}{5} - 1, \dfrac{2|c|}{3(4-|c|)}\right\}$, *then the function* ${}_1\mathcal{G}^{1,\frac{3}{2}}_{p,b,c,1}$ *is starlike in* $\mathbb{D}_{\frac{1}{2}}$.

(ii) *If* $p + \dfrac{b+2}{2} > \max\left\{\dfrac{e^{1+2\gamma}}{2} - 1, \dfrac{3|c|}{(4-|c|)}\right\}$, *then the function* ${}_1\mathcal{G}^{1,1}_{p,b,c,1}$ *is starlike in* $\mathbb{D}_{\frac{1}{2}}$.

Example 3. *Following examples can be construct from Theorem 3*

(i) ${}_1\mathcal{G}^{1,\frac{3}{2}}_{-1.2,1,1,1}$ *is starlike in* $\mathbb{D}_{\frac{1}{2}}$.

(ii) ${}_1\mathcal{G}^{1,1}_{0.1,2,1,1}$ *is starlike in* $\mathbb{D}_{\frac{1}{2}}$.

Theorem 4. *Assume that* $\alpha \in \mathbb{N}$, $\xi, \lambda, \mu > 0$, $|c| < 4$, *such that* $2p + \xi(b+2) > 0$. *If the following holds true:*

(i) $(2\xi)^{\alpha} e^{\frac{1}{2}+\gamma(\lambda+\alpha)} < (\mu+\lambda)^{\lambda}(2p + \xi(b+2) + 2\xi\alpha)^{\alpha}$;

(ii) $|c|\Gamma(\mu)\Gamma(\frac{p}{\xi} + \frac{b+2}{2}) < (4-|c|)\Gamma(\mu+\lambda)\Gamma(\frac{p}{\xi} + \frac{b+2}{2} + \alpha)$,

then ${}_\alpha\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}(z)$ *is convex in* $\mathbb{D}_{\frac{1}{2}}$.

Proof. A direct computation gives

$$\left|{}_\alpha\mathcal{G}^{\lambda,\mu\,\prime}_{p,b,c,\xi}(z) - 1\right| = \left|\sum_{n=1}^{\infty} \frac{\Gamma(\mu)\Gamma\left(\frac{p}{\xi} + \frac{b+2}{2}\right)(n+1)\left(\frac{c}{4}\right)^n z^n}{\Gamma(\mu+\lambda n)\Gamma\left(\frac{p}{\xi} + \frac{b+2}{2} + \alpha n\right)}\right|$$
$$< \Gamma(\mu)\Gamma\left(\frac{p}{\xi} + \frac{b+2}{2}\right)\sum_{n=1}^{\infty} x_n(\alpha, p, b, c, \xi, \lambda, \mu)\left|\frac{c}{4}\right|^n, \quad (\forall z \in \mathbb{D}), \tag{26}$$

where

$$x_n = x_n(\alpha, p, b, c, \xi, \lambda, \mu) = \frac{n+1}{\Gamma(\mu+\lambda n)\Gamma\left(\frac{p}{\xi} + \frac{b+2}{2} + \alpha n\right)}, \quad n \in \mathbb{N}.$$

Now, we define the function $\mathscr{X}_1(s)$ as

$$\mathscr{X}_1(s) := \frac{s+1}{\Gamma(\mu+\lambda s)\Gamma\left(\frac{p}{\xi} + \frac{b+2}{2} + \alpha s\right)}, \quad s \in [1, \infty). \tag{27}$$

Differentiation gives

$$\mathscr{X}_1'(s) = \mathscr{X}_1(s)\mathscr{X}_2(s), \tag{28}$$

where

$$\mathscr{X}_2(s) = \frac{1}{s+1} - \lambda\psi(\mu+\lambda s) - \alpha\psi\left(\frac{p}{\xi} + \frac{b+2}{2} + \alpha s\right), \quad s \in [1, \infty).$$

Using Lemma 1, we obtain

$$\mathscr{X}_2(s) \leq \frac{1}{s+1} - \lambda\log(\mu+\lambda s) - \alpha\log\left(\frac{p}{\xi} + \frac{b+2}{2} + \alpha s\right)$$
$$+ \gamma(\lambda+\alpha) = \mathscr{X}_3(s), \quad s \in [1, \infty). \tag{29}$$

Thus, we have

$$\mathscr{X}_3'(s) = -\frac{1}{(s+1)^2} - \frac{\lambda^2}{\mu+\lambda s} - \frac{\alpha^2}{\frac{p}{\xi} + \frac{b+2}{2} + \alpha s} < 0, \quad s \in [1, \infty).$$

We observe that the function $\mathscr{X}_3(s)$ is decreasing on $[1, \infty)$ and also by hypothesis (i), $\mathscr{X}_3(1) < 0$. So, $\mathscr{X}_3(s) < 0$ for all $s \geq 1$. Now, with the aid of (28) and (29), the function

$\mathscr{X}_1(s)$ is decreasing. Hence, the sequence $\{x_n\}_{n\geq 1}$ is decreasing. Therefore, using (26), we obtain

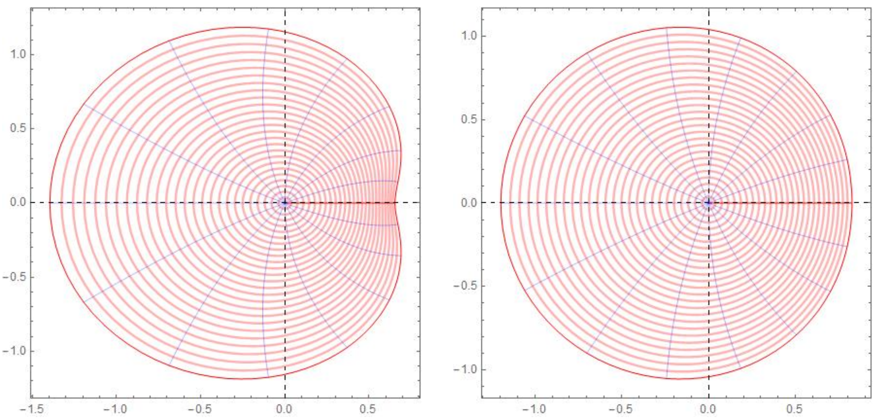
$$\begin{aligned}\left|{}_{\alpha}\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}{}'(z)-1\right| &< \Gamma(\mu)\Gamma\left(\frac{p}{\xi}+\frac{b+2}{2}\right)\sum_{n=1}^{\infty}x_1(\alpha,p,b,c,\xi,\lambda,\mu)\left|\frac{c}{4}\right|^n \\ &\leq \frac{\Gamma(\mu)\Gamma\left(\frac{p}{\xi}+\frac{b+2}{2}\right)|c|}{(4-|c|)\Gamma(\mu+\lambda)\Gamma\left(\frac{p}{\xi}+\frac{b+2}{2}+\alpha\right)}, \quad (\forall z\in\mathbb{D}).\end{aligned} \tag{30}$$

In view of condition (*ii*), the proof of this theorem is completed. □

Corollary 4. *If* $p+\dfrac{b+2}{2}>\max\left\{\dfrac{e^{2\gamma+\frac{1}{2}}}{2}-1,\dfrac{2|c|}{3(4-|c|)}\right\}$*, then the function* ${}_1\mathcal{G}^{1,\frac{3}{2}}_{p,b,c,1}$ *is convex in* $\mathbb{D}_{\frac{1}{2}}$.

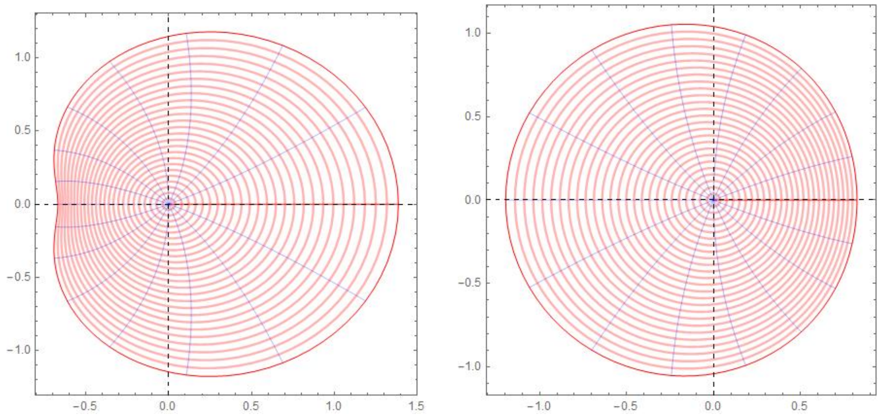
Remark 1. *The significance of Figures 1–3 are illustrate below:*

(i) *Figure 1a and 1b illustrate the starlikeness and convexity of the Struve function, respectively.*
(ii) *The starlikeness and convexity of the Bessel function are depicted in Figure 2a and 2b, respectively.*
(iii) *Figure 3a,b visually represent the starlikeness and convexity properties within the domain* $\mathbb{D}_{\frac{1}{2}}$ *of* ${}_{\alpha}\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}(z)$.



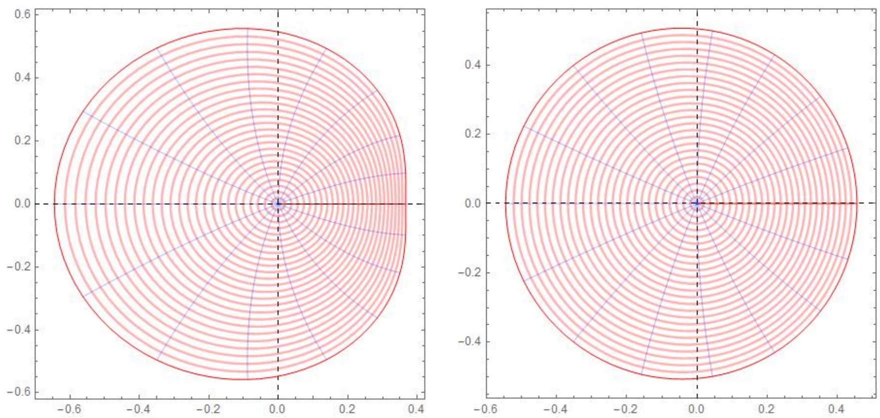
(**a**) (**b**)

Figure 1. Starlikeness and convexity of Struve function of order p. (**a**) Image of $\mathbb{D}$ under ${}_{\alpha}\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}(z)$ for $\alpha=1, p=-1.05, b=1, c=1, \xi=1; \lambda=1; \mu=1.5$; (**b**) image of $\mathbb{D}$ under ${}_{\alpha}\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}(z)$ for $\alpha=1, p=-0.6, b=1, c=1, \xi=1; \lambda=1; \mu=1.5$.



(**a**) (**b**)

Figure 2. Starlikeness and convexity of Bessel function of order $p-1$. (**a**) Image of $\mathbb{D}$ under ${}_{\alpha}\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}(z)$ for $\alpha = 1, p = -1.3, b = 2, c = -1, \xi = 1; \lambda = 1; \mu = 1$; (**b**) image of $\mathbb{D}$ under ${}_{\alpha}\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}(z)$ for $\alpha = 1, p = -0.65, b = 2, c = 1, \xi = 1; \lambda = 1; \mu = 1$.



(**a**) (**b**)

Figure 3. Starlikeness and Convexity in $\mathbb{D}_{\frac{1}{2}}$ of ${}_{\alpha}\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}(z)$. (**a**) Image of $\mathbb{D}_{\frac{1}{2}}$ under ${}_{\alpha}\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}(z)$ for $\alpha = 1, p = -1.2, b = 1, c = -1, \xi = 1; \lambda = 1; \mu = 1.5$; (**b**) image of $\mathbb{D}_{\frac{1}{2}}$ under ${}_{\alpha}\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}(z)$ for $\alpha = 1, p = -0.6, b = 2, c = 1, \xi = 1; \lambda = 1; \mu = 1$.

4. *k*-Starlikeness and *k*-Uniform Convexity of GTSF

In this section, the k-$\mathcal{ST}$ and k-$\mathcal{UCV}$ are discussed.

Theorem 5. *Assume that* $\alpha \in \mathbb{N}$, $\xi, \lambda, \mu > 0$, $|c| < 4$, *such that* $2p + \xi(b+2) > 0$. *If the following holds true:*

(i) $(2\xi)^{\alpha} e^{\frac{k}{2+k} + \gamma(\lambda+\alpha)} < (\mu+\lambda)^{\lambda}(2p + \xi(b+2) + 2\xi\alpha)^{\alpha}$;

(ii) $|c|(2+k)\Gamma(\mu)\Gamma(\frac{p}{\xi} + \frac{b+2}{2}) < (4 - |c|)\Gamma(\mu+\lambda)\Gamma(\frac{p}{\xi} + \frac{b+2}{2} + \alpha)$,

then ${}_{\alpha}\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi} \in k$-$\mathcal{ST}$.

Proof. According to Lemma 4, it is enough to show that, under the given hypothesis, the following inequality holds:

$$\sum_{n=2}^{\infty} (n + k(n-1)) \left| \frac{\Gamma(\mu)\Gamma\left(\frac{p}{\xi} + \frac{b+2}{2}\right)\left(\frac{-c}{4}\right)^{n-1}}{\Gamma(\mu - \lambda + \lambda n)\Gamma\left(\frac{p}{\xi} + \frac{b+2}{2} - \alpha + \alpha n\right)} \right| < 1. \tag{31}$$

Let
$$y_n = y_n(\alpha, p, b, c, \xi, \lambda, \mu) = \frac{n + k(n-1)}{\Gamma(\mu - \lambda + \lambda n)\Gamma\left(\frac{p}{\xi} + \frac{b+2}{2} - \alpha + \alpha n\right)}, \quad n \geq 2.$$

Now, we define the function $\mathscr{Y}_1(s)$ as

$$\mathscr{Y}_1(s) := \frac{s + k(s-1)}{\Gamma(\mu - \lambda + \lambda s)\Gamma\left(\frac{p}{\xi} + \frac{b+2}{2} - \alpha + \alpha s\right)}, \quad s \in [2, \infty). \tag{32}$$

Therefore,
$$\mathscr{Y}_1'(s) = \mathscr{Y}_1(s)\mathscr{Y}_2(s), \tag{33}$$
where
$$\mathscr{Y}_2(s) = \frac{k}{s + k(s-1)} - \lambda\psi(\mu - \lambda + \lambda s) - \alpha\psi\left(\frac{p}{\xi} + \frac{b+2}{2} - \alpha + \alpha s\right), \quad s \in [2, \infty).$$

Applying Lemma 1, we obtain

$$\begin{aligned}\mathscr{Y}_2(s) \leq \frac{k}{s + k(s-1)} - \lambda\log(\mu - \lambda + \lambda s) - \alpha\log\left(\frac{p}{\xi} + \frac{b+2}{2} - \alpha + \alpha s\right)& \\ + \gamma(\lambda + \alpha) = \mathscr{Y}_3(s).&\end{aligned} \tag{34}$$

Thus, we have

$$\mathscr{Y}_3'(s) = -\frac{k(k+1)}{((1+k)s - k)^2} - \frac{\lambda^2}{\mu - \lambda + \lambda s} - \frac{\alpha^2}{\frac{p}{\xi} + \frac{b+2}{2} - \alpha + \alpha s} < 0, \quad s \in [2, \infty).$$

Hence, the function $\mathscr{Y}_3(s)$ is decreasing on $[2, \infty)$ and also by hypothesis (i), $\mathscr{Y}_3(2) < 0$. So, $\mathscr{Y}_3(s) < 0$ for all $s \geq 2$. Now, with the aid of (33) and (34), the function $\mathscr{Y}_1(s)$ is decreasing. Consequently, the sequence $\{y_n\}_{n\geq 2}$ is decreasing. Therefore,

$$\begin{aligned}&\sum_{n=2}^{\infty}(n + k(n-1))\left|\frac{\Gamma(\mu)\Gamma\left(\frac{p}{\xi} + \frac{b+2}{2}\right)\left(\frac{-c}{4}\right)^{n-1}}{\Gamma(\mu - \lambda + \lambda n)\Gamma\left(\frac{p}{\xi} + \frac{b+2}{2} - \alpha + \alpha n\right)}\right| \\ &= \sum_{n=2}^{\infty}\Gamma(\mu)\Gamma\left(\frac{p}{\xi} + \frac{b+2}{2}\right)y_n(\alpha, p, b, c, \xi, \lambda, \mu)\left|\frac{c}{4}\right|^{n-1} \\ &\leq \Gamma(\mu)\Gamma\left(\frac{p}{\xi} + \frac{b+2}{2}\right)y_2(\alpha, p, b, c, \xi, \lambda, \mu)\sum_{n=2}^{\infty}\left|\frac{c}{4}\right|^{n-1} \\ &= \frac{\Gamma(\mu)\Gamma\left(\frac{p}{\xi} + \frac{b+2}{2}\right)(2+k)|c|}{(4 - |c|)\Gamma(\mu + \lambda)\Gamma\left(\frac{p}{\xi} + \frac{b+2}{2} + \alpha\right)}, \quad (\forall z \in \mathbb{D}).\end{aligned} \tag{35}$$

From the given condition (ii), Inequality (31) is satisfied and, hence, the theorem is proved. □

Theorem 6. *Assume that $\alpha \in \mathbb{N}$, $\xi, \lambda, \mu > 0$, $|c| < 4$, such that $2p + \xi(b+2) > 0$. If the following holds true:*

(i) $3(2\xi)^{\alpha}e^{1+\gamma(\lambda+\alpha)} < (\mu + \lambda)^{\lambda}(2p + \xi(b+2) + 2\xi\alpha)^{\alpha}$;

(ii) $2|c|(2+k)\Gamma(\mu)\Gamma(\frac{p}{\xi} + \frac{b+2}{2}) < (4 - |c|)\Gamma(\mu + \lambda)\Gamma(\frac{p}{\xi} + \frac{b+2}{2} + \alpha)$,

then ${}_{\alpha}\mathcal{G}_{p,b,c,\xi}^{\lambda,\mu} \in k\text{-}\mathcal{UCV}$.

Proof. In view of Lemma 5, we show that

$$\sum_{n=2}^{\infty} n(n-1))\left|\frac{\Gamma(\mu)\Gamma\left(\frac{p}{\xi}+\frac{b+2}{2}\right)\left(\frac{-c}{4}\right)^{n-1}}{\Gamma(\mu-\lambda+\lambda n)\Gamma\left(\frac{p}{\xi}+\frac{b+2}{2}-\alpha+\alpha n\right)}\right| < \frac{1}{k+2}. \tag{36}$$

Let

$$u_n = u_n(\alpha, p, b, c, \xi, \lambda, \mu) = \frac{\Gamma(n+1)}{\Gamma(\mu-\lambda+\lambda n)\Gamma\left(\frac{p}{\xi}+\frac{b+2}{2}-\alpha+\alpha n\right)\Gamma(n-1)}, \quad n \geq 2.$$

Now, we define the function $\mathscr{U}_1(s)$ as

$$\mathscr{U}_1(s) := \frac{\Gamma(s+1)}{\Gamma(\mu-\lambda+\lambda s)\Gamma\left(\frac{p}{\xi}+\frac{b+2}{2}-\alpha+\alpha s\right)\Gamma(s-1)}, \quad s \in [2,\infty). \tag{37}$$

Therefore,

$$\mathscr{U}_1'(s) = \mathscr{U}_1(s)\mathscr{U}_2(s), \tag{38}$$

where

$$\mathscr{U}_2(s) = \psi(s+1) - \psi(s-1) - \lambda\psi(\mu-\lambda+\lambda s) - \alpha\psi\left(\frac{p}{\xi}+\frac{b+2}{2}-\alpha+\alpha s\right), \quad s \in [2,\infty).$$

By using Lemma 1, we have

$$\mathscr{U}_2(s) \leq \mathscr{U}_3(s), \tag{39}$$

where

$$\begin{aligned}\mathscr{U}_3(s) = {} & \log(s+1) - \log(s-1) - \lambda\log(\mu-\lambda+\lambda s) \\ & - \alpha\log\left(\frac{p}{\xi}+\frac{b+2}{2}-\alpha+\alpha s\right) + \gamma(1+\lambda+\alpha).\end{aligned}$$

Thus, we have

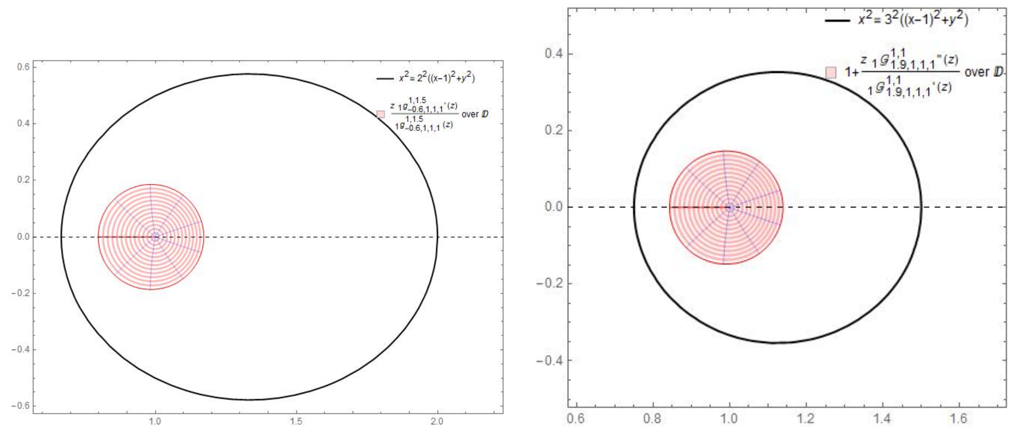
$$\mathscr{U}_3'(s) = -\frac{2}{s^2-1} - \frac{\lambda^2}{\mu-\lambda+\lambda s} - \frac{\alpha^2}{\frac{p}{\xi}+\frac{b+2}{2}-\alpha+\alpha s} < 0, \quad s \in [2,\infty).$$

Hence, the function $\mathscr{U}_3(s)$ is decreasing on $[2,\infty)$ and also by hypothesis (i), $\mathscr{U}_3(2) < 0$. So, $\mathscr{U}_3(s) < 0$ for all $s \geq 2$. Now, with the aid of (38) and (39), the function $\mathscr{U}_1(s)$ is decreasing. Consequently, the sequence $\{u_n\}_{n\geq 2}$ is decreasing. Therefore,

$$\begin{aligned}&\sum_{n=2}^{\infty} n(n-1)\left|\frac{\Gamma(\mu)\Gamma\left(\frac{p}{\xi}+\frac{b+2}{2}\right)\left(\frac{-c}{4}\right)^{n-1}}{\Gamma(\mu-\lambda+\lambda n)\Gamma\left(\frac{p}{\xi}+\frac{b+2}{2}-\alpha+\alpha n\right)}\right| \\ &= \sum_{n=2}^{\infty}\Gamma(\mu)\Gamma\left(\frac{p}{\xi}+\frac{b+2}{2}\right)u_n(\alpha,p,b,c,\xi,\lambda,\mu)\left|\frac{c}{4}\right|^{n-1} \\ &\leq \Gamma(\mu)\Gamma\left(\frac{p}{\xi}+\frac{b+2}{2}\right)u_2(\alpha,p,b,c,\xi,\lambda,\mu)\sum_{n=2}^{\infty}\left|\frac{c}{4}\right|^{n-1} \\ &= \frac{2|c|\Gamma(\mu)\Gamma\left(\frac{p}{\xi}+\frac{b+2}{2}\right)}{(4-|c|)\Gamma(\mu+\lambda)\Gamma\left(\frac{p}{\xi}+\frac{b+2}{2}+\alpha\right)}.\end{aligned} \tag{40}$$

From the given condition (*ii*), Inequality (36) is satisfied and, hence, the theorem is proved. □

Remark 2. *Figure 4a and 4b demonstrate that when the parameters adhere to the obtained results, the function* ${}_{\alpha}\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}(z)$ *belongs to the class k-*$\mathcal{ST}$ *and k-*$\mathcal{UCV}$*, respectively.*



(**a**) (**b**)

Figure 4. (**a**) Image of $\mathbb{D}$ under $\frac{z\, {}_{\alpha}\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}{}'(z)}{{}_{\alpha}\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}(z)}$ for $\alpha = 1, p = -0.6, b = 1, c = -1, \xi = 1; \lambda = 1; \mu = 1.5$. (**b**) Image of $\mathbb{D}$ under $1 + \frac{z\, {}_{\alpha}\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}{}''(z)}{{}_{\alpha}\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}{}'(z)}$ for $\alpha = 1, p = 1.9, b = 1, c = 1, \xi = 1; \lambda = 1; \mu = 1$.

5. Starlikeness and Convexity Associated with Exponential Function and Lemniscate of Bernoulli

Theorem 7. *Assume that* $\alpha \in \mathbb{N},\ \xi, \lambda, \mu > 0,\ |c| < 4$*, such that* $2p + \xi(b+2) > 0$*. If the following holds true:*

(i) $(2\xi)^{\alpha} e^{1+\gamma(\lambda+\alpha)} < (\mu+\lambda)^{\lambda}(2p + \xi(b+2) + 2\xi\alpha)^{\alpha}$;

(ii) $|c|(2e-1)\Gamma(\mu)\Gamma(\frac{p}{\xi} + \frac{b+2}{2}) < (e-1)(4-|c|)\Gamma(\mu+\lambda)\Gamma(\frac{p}{\xi} + \frac{b+2}{2} + \alpha)$,

then ${}_{\alpha}\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi} \in \mathcal{S}^*_e$ *in* $\mathbb{D}$*.*

Proof. To prove the result, it is sufficient to show that

$$\left| \frac{z\,{}_{\alpha}\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}{}'(z)}{{}_{\alpha}\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}(z)} - 1 \right| = \left| \frac{{}_{\alpha}\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}{}'(z) - \frac{{}_{\alpha}\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}(z)}{z}}{\frac{{}_{\alpha}\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}(z)}{z}} \right| < 1 - \frac{1}{e}. \tag{41}$$

Now, using the condition (*ii*) in (12), Inequality (41) follows, which concludes the proof. □

Theorem 8. *Assume that* $\alpha \in \mathbb{N},\ \xi, \lambda, \mu > 0,\ |c| < 4$*, such that* $2p + \xi(b+2) > 0$*. If the following holds true:*

(i) $(2\xi)^{\alpha} e^{\frac{3}{2}+\gamma(\lambda+\alpha)} < (\mu+\lambda)^{\lambda}(2p + \xi(b+2) + 2\xi\alpha)^{\alpha}$

(ii) $2|c|(2e-1)\Gamma(\mu)\Gamma(\frac{p}{\xi} + \frac{b+2}{2}) < (e-1)(4-|c|)\Gamma(\mu+\lambda)\Gamma(\frac{p}{\xi} + \frac{b+2}{2} + \alpha)$,

then ${}_{\alpha}\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi} \in \mathcal{C}^*_e$ *in* $\mathbb{D}$*.*

Proof. Condition (*ii*) implies

$$\frac{2\Gamma(\mu)\Gamma\left(\frac{p}{\xi}+\frac{b+2}{2}\right)|c|}{(4-|c|)\Gamma(\mu+\lambda)\Gamma\left(\frac{p}{\xi}+\frac{b+2}{2}+\alpha\right)-2|c|\Gamma(\mu)\Gamma\left(\frac{p}{\xi}+\frac{b+2}{2}\right)}<1-\frac{1}{e}. \tag{42}$$

Now combining, (20) and (42), we have

$$\left|\frac{z_\alpha\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}{}''(z)}{{}_\alpha\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}{}'(z)}\right|<1-\frac{1}{e}.$$

Thus, ${}_\alpha\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}\in\mathcal{C}^*_e$. □

Theorem 9. *Assume that* $\alpha\in\mathbb{N},\ \xi,\lambda,\mu>0,\ |c|<4$, *such that* $2p+\xi(b+2)>0$. *If the following holds true:*

(i) $(2\xi)^\alpha e^{1+\gamma(\lambda+\alpha)}<(\mu+\lambda)^\lambda(2p+\xi(b+2)+2\xi\alpha)^\alpha$;

(ii) $\dfrac{|c|\Gamma(\mu)\Gamma(\frac{p}{\xi}+\frac{b+2}{2})}{(4-|c|)\Gamma(\mu+\lambda)\Gamma(\frac{p}{\xi}+\frac{b+2}{2}+\alpha)}\left(2+\dfrac{|c|\Gamma(\mu)\Gamma(\frac{p}{\xi}+\frac{b+2}{2})}{(4-|c|)\Gamma(\mu+\lambda)\Gamma(\frac{p}{\xi}+\frac{b+2}{2}+\alpha)}\right)<\dfrac{1}{2}$,

then ${}_\alpha\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}\in\mathcal{S}^*_L$ *in* $\mathbb{D}$.

Proof. To prove the result, it is enough to establish the following inequality:

$$\left|\left(\frac{z_\alpha\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}{}'(z)}{{}_\alpha\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}(z)}\right)^2-1\right|=\frac{\left|z_\alpha\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}{}'(z)+\dfrac{{}_\alpha\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}(z)}{z}\right|\left|z_\alpha\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}{}'(z)-\dfrac{{}_\alpha\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}(z)}{z}\right|}{\left|\dfrac{{}_\alpha\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}(z)}{z}\right|^2}<1. \tag{43}$$

From a simple computation, we have

$$\left|{}_\alpha\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}{}'(z)+\frac{{}_\alpha\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}(z)}{z}\right|<2+\sum_{n=1}^{\infty}\Gamma(\mu)\Gamma\left(\frac{p}{\xi}+\frac{b+2}{2}\right)l_n(\alpha,p,b,c,\xi,\lambda,\mu)\left|\frac{c}{4}\right|^n, \tag{44}$$

where

$$l_n=l_n(\alpha,p,b,c,\xi,\lambda,\mu)=\frac{n+2}{\Gamma(\mu+\lambda n)\Gamma(\frac{p}{\xi}+\frac{b+2}{2}+\alpha n)}.$$

Now, consider the function

$$\mathscr{L}_1(s)=\frac{s+2}{\Gamma(\mu+\lambda s)\Gamma(\frac{p}{\xi}+\frac{b+2}{2}+\alpha s)},\ s\in[1,\infty).$$

Taking logarithmic differentiation,

$$\mathscr{L}_1'(s)=\mathscr{L}_1(s)\mathscr{L}_2(s), \tag{45}$$

where

$$\mathscr{L}_2(s)=\frac{1}{s+2}-\lambda\psi(\mu+\lambda s)-\alpha\psi\left(\frac{p}{\xi}+\frac{b+2}{2}+\alpha s\right).$$

By use of Lemma 1, we obtain

$$\begin{aligned}\mathscr{L}_2(s) &\leq \frac{1}{s+2} - \lambda\log(\mu+\lambda s) - \alpha\log\left(\frac{p}{\xi} + \frac{b+2}{2} + \alpha s\right) + \gamma(\lambda+\alpha) \\ &:= \mathscr{L}_3(s),\ s\in[1,\infty)\ \text{(say)}.\end{aligned} \tag{46}$$

Since

$$\mathscr{L}_3'(s) = -\frac{1}{(s+2)^2} - \frac{\lambda^2}{\mu+\lambda s} - \frac{\alpha^2}{\frac{p}{\xi}+\frac{b+2}{2}+\alpha s} < 0,$$

and $\mathscr{L}_3(1) < 0$, we, therefore, eventually obtain that $\mathscr{L}_1(s)$ is a decreasing function on $[1,\infty)$ and, hence, the sequence $\{l_n\}_{n\geq 1}$ is decreasing. Thus, from (44), the following holds:

$$\begin{aligned}\left|{}_\alpha\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}{}'(z) + \frac{{}_\alpha\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}(z)}{z}\right| &< 2 + \Gamma(\mu)\Gamma\left(\frac{p}{\xi}+\frac{b+2}{2}\right) l_1(\alpha,p,b,c,\xi,\lambda,\mu)\sum_{n=1}^{\infty}\left|\frac{c}{4}\right|^n \\ &= 2 + \Gamma(\mu)\Gamma\left(\frac{p}{\xi}+\frac{b+2}{2}\right)\frac{l_1(\alpha,p,b,c,\xi,\lambda,\mu)|c|}{4-|c|}.\end{aligned} \tag{47}$$

Combining (9), (11), and (47), we obtain

$$\begin{aligned}&\frac{\left|z{}_\alpha\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}{}'(z) + \frac{{}_\alpha\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}(z)}{z}\right|\left|z{}_\alpha\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}{}'(z) - \frac{{}_\alpha\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}(z)}{z}\right|}{\left|\frac{{}_\alpha\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}(z)}{z}\right|^2} \\ &< \frac{\left(2 + \frac{\Gamma(\mu)\Gamma\left(\frac{p}{\xi}+\frac{b+2}{2}\right)l_1|c|}{4-|c|}\right)\left(\frac{\Gamma(\mu)\Gamma\left(\frac{p}{\xi}+\frac{b+2}{2}\right)d_1|c|}{4-|c|}\right)}{\left(1 - \frac{r_1(\alpha,p,b,c,\xi,\lambda,\mu)|c|}{4-|c|}\right)^2} \\ &= \frac{\left(2 + \frac{\Gamma(\mu)\Gamma\left(\frac{p}{\xi}+\frac{b+2}{2}\right)3|c|}{\Gamma(\mu+\lambda)\Gamma\left(\frac{p}{\xi}+\frac{b+2}{2}+\alpha\right)(4-|c|)}\right)\left(\frac{\Gamma(\mu)\Gamma\left(\frac{p}{\xi}+\frac{b+2}{2}\right)|c|}{\Gamma(\mu+\lambda)\Gamma\left(\frac{p}{\xi}+\frac{b+2}{2}+\alpha\right)(4-|c|)}\right)}{\left(1 - \frac{\Gamma(\mu)\Gamma\left(\frac{p}{\xi}+\frac{b+2}{2}\right)|c|}{\Gamma(\mu+\lambda)\Gamma\left(\frac{p}{\xi}+\frac{b+2}{2}+\alpha\right)(4-|c|)}\right)^2}.\end{aligned} \tag{48}$$

The condition (*ii*) and (48) leads to Inequality (43), which concludes the proof. □

Theorem 10. *Assume that* $\alpha\in\mathbb{N}$, $\xi,\lambda,\mu>0$, $|c|<4$, *such that* $2p+\xi(b+2)>0$. *If the following holds true:*

(i) $(2\xi)^\alpha e^{\frac{3}{2}+\gamma(\lambda+\alpha)} < (\mu+\lambda)^\lambda(2p+\xi(b+2)+2\xi\alpha)^\alpha$;

(ii)

$$\begin{aligned}&\frac{4|c|\Gamma(\mu)\Gamma(\frac{p}{\xi}+\frac{b+2}{2})}{(4-|c|)\Gamma(\mu+\lambda)\Gamma(\frac{p}{\xi}+\frac{b+2}{2}+\alpha) - 2|c|\Gamma(\mu)\Gamma(\frac{p}{\xi}+\frac{b+2}{2})} \\ &\times\left(1 + \frac{|c|\Gamma(\mu)\Gamma(\frac{p}{\xi}+\frac{b+2}{2})}{(4-|c|)\Gamma(\mu+\lambda)\Gamma(\frac{p}{\xi}+\frac{b+2}{2}+\alpha) - 2|c|\Gamma(\mu)\Gamma(\frac{p}{\xi}+\frac{b+2}{2})}\right) < 1,\end{aligned} \tag{49}$$

then ${}_{\alpha}\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi} \in \mathcal{C}^{*}_{L}$ *in* $\mathbb{D}$.

Proof. From (20), we have

$$\begin{aligned}
&\left|\frac{z_{\alpha}\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}{}''(z)}{{}_{\alpha}\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}{}'(z)}\right|\left|2+\frac{z_{\alpha}\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}{}''(z)}{{}_{\alpha}\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}{}'(z)}\right| \\
&< \frac{2\Gamma(\mu)\Gamma\left(\frac{p}{\xi}+\frac{b+2}{2}\right)|c|}{(4-|c|)\Gamma(\mu+\lambda)\Gamma\left(\frac{p}{\xi}+\frac{b+2}{2}+\alpha\right)-2|c|\Gamma(\mu)\Gamma\left(\frac{p}{\xi}+\frac{b+2}{2}\right)} \\
&\times\left(2+\frac{2\Gamma(\mu)\Gamma\left(\frac{p}{\xi}+\frac{b+2}{2}\right)|c|}{(4-|c|)\Gamma(\mu+\lambda)\Gamma\left(\frac{p}{\xi}+\frac{b+2}{2}+\alpha\right)-2|c|\Gamma(\mu)\Gamma\left(\frac{p}{\xi}+\frac{b+2}{2}\right)}\right),\ (\forall z\in\mathbb{D}).
\end{aligned} \tag{50}$$

Using condition (*ii*) in (50), we have the following inequality:

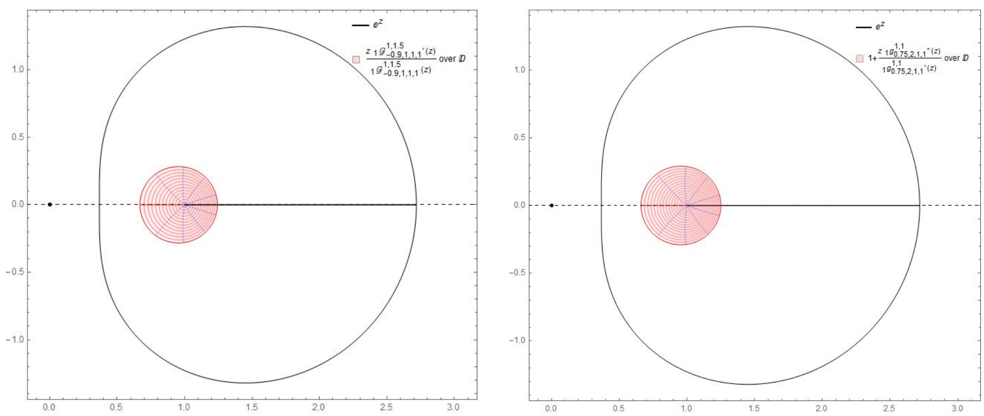
$$\left|\left(1+\frac{z_{\alpha}\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}{}''(z)}{{}_{\alpha}\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}{}'(z)}\right)^{2}-1\right|<1, (\forall z\in\mathbb{D}),$$

which completes the proof. □

Remark 3. *Interpretation of Figures 5 and 6 are given below:*

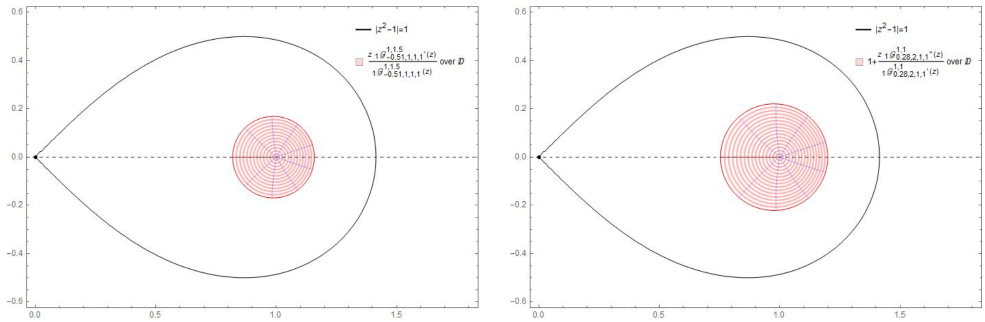
(i) *Figure 5a,b illustrate that* ${}_{\alpha}\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}(z)$ *satisfies the starlikeness and convexity properties associated with the exponential function when the values of the parameters are according to the obtained result.*

(ii) *The lemniscate starlike and convexity properties are satisfied by* ${}_{\alpha}\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}(z)$ *when the values of the parameters adhere to the obtained results, as depicted in Figure 6a,b.*



(**a**) (**b**)

Figure 5. Starlikeness and convexity associated with exponential function of ${}_{\alpha}\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}(z)$. (**a**) Image of $\mathbb{D}$ under $\frac{z\,{}_{\alpha}\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}{}'(z)}{{}_{\alpha}\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}(z)}$ for $\alpha=1, p=-0.9, b=1, c=1, \xi=1; \lambda=1; \mu=1.5$. (**b**) Image of $\mathbb{D}$ under $1+\frac{z\,{}_{\alpha}\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}{}''(z)}{{}_{\alpha}\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}{}'(z)}$ for $\alpha=1, p=0.75, b=2, c=1, \xi=1; \lambda=1; \mu=1$.



(**a**) (**b**)

Figure 6. Lemniscate starlikeness and convexity of ${}_{\alpha}\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}(z)$. (a) Image of $\mathbb{D}$ under $\frac{z\, {}_{\alpha}\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}{}'(z)}{{}_{\alpha}\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}(z)}$ for $\alpha = 1, p = -0.51, b = 1, c = 1, \xi = 1; \lambda = 1; \mu = 1.5$. (b) Image of $\mathbb{D}$ under $1 + \frac{z\, {}_{\alpha}\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}{}''(z)}{{}_{\alpha}\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}{}'(z)}$ for $\alpha = 1, p = 0.28, b = 2, c = 1, \xi = 1; \lambda = 1; \mu = 1$.

6. Pre-Starlikeness

Another important class of function $\mathcal{L}_{\rho}$ known as pre-starlike functions, introduced by Ruscheweyh [27], is defined in the following manner:

$$\mathcal{L}_{\rho} = \{f \in \mathcal{A} : g_{\rho} * f \in \mathcal{ST}(\rho)\},\ (0 \le \rho < 1),$$

where $g_{\rho}(z) = \frac{z}{(1-z)^{2-2\rho}}$, $z \in \mathbb{D}$ and $g_{\rho} * f$ denote the Hadamard product of these functions. The concept of pre-starlikeness is extended in [28] by generalizing the class $\mathcal{L}_{\rho}$ to $\mathcal{L}[\rho, \delta]$, which is given by

$$\mathcal{L}[\rho, \delta] = \{f \in \mathcal{A} : g_{\rho} * f \in \mathcal{ST}(\delta)\},\ (0 \le \rho, \delta < 1).$$

In the following theorem, we obtain conditions for GTSF belonging to the class $\mathcal{L}_{\rho}$.

Theorem 11. *Assume that* $\alpha \in \mathbb{N}$, $\xi, \lambda, \mu > 0$, $|c| < 4$, *such that* $2p + \xi(b+2) > 0$ *and* $0 \le \rho < \frac{1}{2}$, $0 \le \delta < 1$. *If the following holds true:*

(i) $\xi^{\alpha}(3-2\rho)e^{1+\gamma(\lambda+\alpha)} < 2(\mu+\lambda)^{\lambda}(2p + \xi(b+2) + 2\xi\alpha)^{\alpha}$;

(ii) $|c|(1-\rho)(3-\delta)\Gamma(\mu)\Gamma(\frac{p}{\xi} + \frac{b+2}{2}) < (4-|c|)(1-\delta)\Gamma(\mu+\lambda)\Gamma(\frac{p}{\xi} + \frac{b+2}{2} + \alpha)$,

then ${}_{\alpha}\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi} \in \mathcal{L}_{[\rho,\delta]}$.

Proof. To prove the theorem, we show that $g_{\rho} * {}_{\alpha}\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi} = h \in \mathcal{ST}(\delta)$ by establishing the following inequality:

$$\left|\frac{zh'(z)}{h(z)} - 1\right| = \frac{\left|h'(z) - \frac{h(z)}{z}\right|}{\left|\frac{h(z)}{z}\right|} < 1 - \delta,\ (\forall z \in \mathbb{D}). \tag{51}$$

A calculation yields

$$\begin{aligned}\left|h'(z) - \frac{h(z)}{z}\right| &= \left|\sum_{n=1}^{\infty} \frac{\Gamma(\mu)\Gamma(\frac{p}{\xi} + \frac{b+2}{2})\Gamma(n+2-2\rho)n\left(\frac{-c}{4}\right)^n z^n}{\Gamma(2-2\rho)\Gamma(\mu+\lambda n)\Gamma(\frac{p}{\xi} + \frac{b+2}{2} + \alpha n)\Gamma(n+1)}\right| \\ &< \frac{\Gamma(\mu)\Gamma(\frac{p}{\xi} + \frac{b+2}{2})}{\Gamma(2-2\rho)} \sum_{n=1}^{\infty} v_n(\alpha, p, b, c, \xi, \lambda, \mu)\left|\frac{c}{4}\right|^n,\ \ (\forall z \in \mathbb{D}),\end{aligned} \tag{52}$$

where

$$v_n = v_n(\alpha, p, b, c, \xi, \lambda, \mu) = \frac{n\Gamma(n+2-2\rho)}{\Gamma(\mu+\lambda n)\Gamma(\frac{p}{\xi}+\frac{b+2}{2}+\alpha n)\Gamma(n+1)},\ n \geq 1.$$

Let

$$\mathcal{V}_1(s) = \frac{n\Gamma(n+2-2\rho)}{\Gamma(\mu+\lambda n)\Gamma(\frac{p}{\xi}+\frac{b+2}{2}+\alpha n)\Gamma(n+1)},\ s \in [1,\infty).$$

Differentiating logarithmically,

$$\mathcal{V}_1'(s) = \mathcal{V}_1(s)\mathcal{V}_2(s), \tag{53}$$

where

$$\mathcal{V}_2(s) = \frac{1}{s} + \psi(s+2-2\rho) - \psi(s+1) - \lambda\psi(\mu+\lambda s) - \alpha\psi\left(\frac{p}{\xi}+\frac{b+2}{2}+\alpha s\right).$$

In view of Lemma 1, the inequality follows:

$$\begin{aligned}\mathcal{V}_2(s) \leq &\frac{1}{s} + \log\left(\frac{s+2-2\rho}{s+1}\right) - \lambda\log(\mu+\lambda s)\\ &- \alpha\log\left(\frac{p}{\xi}+\frac{b+2}{2}+\alpha s\right) + \gamma(1+\lambda+\alpha) := \mathcal{V}_3(s),\ s \in [1,\infty)\ \text{(say)}.\end{aligned} \tag{54}$$

Differentiating $\mathcal{V}_3(s)$, we obtain

$$\mathcal{V}_3'(s) = -\frac{1}{s^2} + \frac{2\rho-1}{(s+2-2\rho)(s+1)} - \frac{\lambda^2}{\mu+\lambda s} - \frac{\alpha^2}{\frac{p}{\xi}+\frac{b+2}{2}+\alpha s} < 0.$$

Thus, $\mathcal{V}_3(s)$ is decreasing on $s \in [1,\infty)$. Also, by the hypothesis *(i)*, $\mathcal{V}_3(1) < 0$. Hence, from (54) and (53), $\mathcal{V}_1(s)$ is a decreasing function on $s \in [1,\infty)$. Consequently, $\{v_n\}_{n\geq 1}$ is a decreasing sequence. Therefore, from (52),

$$\begin{aligned}\left|h'(z) - \frac{h(z)}{z}\right| &< \frac{\Gamma(\mu)\Gamma(\frac{p}{\xi}+\frac{b+2}{2})}{\Gamma(2-2\rho)}\sum_{n=1}^{\infty} v_1(\alpha,p,b,c,\xi,\lambda,\mu)\left|\frac{c}{4}\right|^n\\ &= \frac{\Gamma(\mu)\Gamma(\frac{p}{\xi}+\frac{b+2}{2})v_1(\alpha,p,b,c,\xi,\lambda,\mu)|c|}{\Gamma(2-2\rho)(4-|c|)},\ \ (\forall z \in \mathbb{D}).\end{aligned} \tag{55}$$

A simple computation leads to

$$\left|\frac{h(z)}{z}\right| > 1 - \frac{\Gamma(\mu)\Gamma(\frac{p}{\xi}+\frac{b+2}{2})}{\Gamma(2-2\rho)}\sum_{n=1}^{\infty} q_n(\alpha,p,b,c,\xi,\lambda,\mu)\left|\frac{c}{4}\right|^n, \tag{56}$$

where

$$q_n(\alpha,p,b,c,\xi,\lambda,\mu) = \frac{\Gamma(n+2-2\rho)}{\Gamma(\mu+\lambda n)\Gamma\left(\frac{p}{\xi}+\frac{b+2}{2}+\alpha n\right)\Gamma(n+1)},\ n \in \mathbb{N}.$$

By similar arguments, it can be shown that $\{q_n\}_{n\geq 1}$ is a decreasing sequence. Now, using (56), we obtain

$$\left|\frac{h(z)}{z}\right| > 1 - \frac{\Gamma(\mu)\Gamma(\frac{p}{\xi} + \frac{b+2}{2})}{\Gamma(2-2\rho)} q_1(\alpha, p, b, c, \xi, \lambda, \mu) \sum_{n=1}^{\infty} \left|\frac{c}{4}\right|^n$$
$$= 1 - \frac{\Gamma(\mu)\Gamma(\frac{p}{\xi} + \frac{b+2}{2}) q_1(\alpha, p, b, c, \xi, \lambda, \mu)|c|}{\Gamma(2-2\rho)(4-|c|)}. \quad (57)$$

Combining (55) and (57), we have

$$\frac{\left|h'(z) - \frac{h(z)}{z}\right|}{\left|\frac{h(z)}{z}\right|} < \frac{\frac{\Gamma(\mu)\Gamma(\frac{p}{\xi} + \frac{b+2}{2}) v_1(\alpha, p, b, c, \xi, \lambda, \mu)|c|}{\Gamma(2-2\rho)(4-|c|)}}{1 - \frac{\Gamma(\mu)\Gamma(\frac{p}{\xi} + \frac{b+2}{2}) q_1(\alpha, p, b, c, \xi, \lambda, \mu)|c|}{\Gamma(2-2\rho)(4-|c|)}} \quad (58)$$
$$= \frac{2|c|(1-\rho)\Gamma(\mu)\Gamma\left(\frac{p}{\xi} + \frac{b+2}{2}\right)}{(4-c)\Gamma(\mu+\lambda)\Gamma\left(\frac{p}{\xi} + \frac{b+2}{2} + \alpha\right) - |c|(1-\rho)\Gamma(\mu)\Gamma\left(\frac{p}{\xi} + \frac{b+2}{2}\right)}.$$

Applying the condition (ii) on (58), Inequality (51) holds, which proves the theorem. □

Remark 4. *In Figure 7, it can be observed that for suitable parameter values consistent with the obtained results,* $\left(g_\rho * z\, {}_\alpha\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}\right)(z)$ *maps the unit disk* $\mathbb{D}$ *onto a starlike domain. Consequently,* ${}_\alpha\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}(z)$ *satisfies the pre-starlikeness property.*

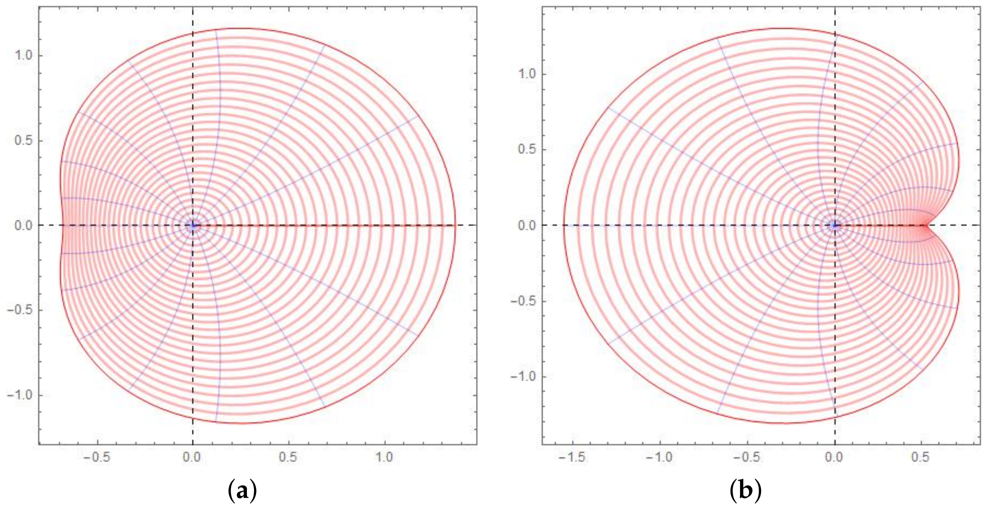


Figure 7. Pre-starlikeness of ${}_\alpha\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}(z)$. (**a**) Image of $\mathbb{D}$ under $\left(g_\rho * z\, {}_\alpha\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}\right)(z)$ for $\alpha = 1, p = -1.7, b = 3, c = -1, \xi = 1; \lambda = 1; \mu = 1$. (**b**) Image of $\mathbb{D}$ under $\left(g_\rho * z\, {}_\alpha\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}\right)(z)$ for $\alpha = 1, p = -1.01, b = 1, c = 1, \xi = 1; \lambda = 1; \mu = 1.5$.

7. Hardy Space of GTSF

Let $\mathcal{H}^\infty$ represent the space of all bounded functions in $\mathbb{D}$. We also assume that h is in the class of the analytic functions in domain $\mathbb{D}$ and set

$$\mathcal{M}_p(r,h) = \begin{cases} \left(\frac{1}{2\pi}\int_0^{2\pi}\left|h(re^{i\theta})\right|^p d\theta\right)^{\frac{1}{p}} & (0 < p < \infty) \\ \max\{|h(z)| : |z| \leq r\} & (p = \infty). \end{cases}$$

As per [29], the function h is considered to belong to the Hardy space, denoted as $\mathcal{H}^p(0 < p \leq \infty)$, if $\mathcal{M}_p(r,h)$ is bounded for all $r \in [0,1)$ and

$$\mathcal{H}^\infty \subset \mathcal{H}^q \subset \mathcal{H}^p \qquad (0 < p < q < \infty).$$

The study of the Hardy space of Mittag-Leffler functions is presented in [30]. In [31], the results related to the Hardy space for the Fox–Wright function are derived. Additionally, [17] establishes the conditions for generalized Struve functions belonging to the Hardy space. In the following, we demonstrate a direct consequence of convexity for GTSF connected to the Hardy space of analytic functions.

Theorem 12. *Under the conditions in Theorem 2, the following holds:*

$${}_\alpha\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi} \in \begin{cases} \mathcal{H}^{\frac{1}{1-2\delta}} \text{ if } 0 \leq \delta < \frac{1}{2}, \\ \mathcal{H}^\infty \text{ if } \frac{1}{2} \leq \delta < 1. \end{cases}$$

Also, ${}_\alpha\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}{}' \in \mathcal{H}^{\frac{1}{2(1-\delta)}}$.

Proof. Applying Lemma 6, for any $\delta \in [0, \frac{1}{2})$, there exists $\sigma > 0$, such that

$${}_\alpha\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi} \in \mathcal{H}^{\sigma + \frac{1}{1-2\delta}},$$

and if $\delta \geq \frac{1}{2}$, then

$${}_\alpha\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi} \in \mathcal{H}^\infty.$$

Also, there exists ρ, such that

$${}_\alpha\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}{}' \in \mathcal{H}^{\rho + \frac{1}{2(1-\delta)}}.$$

Now, since for any q, r with $0 < q < r < \infty$, it is implied that $\mathcal{H}^\infty \subset \mathcal{H}^r \subset \mathcal{H}^q$. Hence,

$${}_\alpha\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi} \in \mathcal{H}^{\frac{1}{(1-2\delta)}} \text{ for } 0 \leq \delta < \frac{1}{2}.$$

Also,

$${}_\alpha\mathcal{G}^{\lambda,\mu}_{p,b,c,\xi}{}' \in \mathcal{H}^{\frac{1}{2(1-\delta)}}.$$

Hence, the proof is completed. □

8. Concluding Remarks and Observations

In this article, we established various geometric properties for the normalized Galué type Struve function (GTSF), including the starlikeness of order δ, convexity of order δ, k-starlikeness, k-uniform convexity, lemniscate starlikeness and convexity, exponential starlikeness and convexity, and pre-starlikeness. Moreover, Theorem 12 illustrates a direct implication of the convexity of GTSF connected to the Hardy space of analytic functions. Several outcomes derived herein generalize the findings available in prior literature. The findings of this study were supported by interesting examples and graphical representations.

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