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Solutions with a Flat Horizon in D Dimensions within the Cubic Form of $f(Q)$ Gravity

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Abstract: Given the AdS/CFT relationship, the study of higher-dimensional AdS black holes is extremely important. Furthermore, since the restriction derived from $f(Q)$'s field equations prevents it from deriving spherically symmetric black hole solutions, the result is either $Q' = 0$ or $f_{QQ} = 0$. Utilizing the cylindrical coordinate system within the context the cubic form of $f(Q)$ theory while imposing the condition of a coincident gauge, we establish the existence of static solutions in D-dimensions. The power-law ansatz, which is the most practical based on observations, will be used in this study, where $f(Q) = Q + \frac{1}{2}\gamma Q^2 + \frac{1}{3}\gamma Q^3 - 2\Lambda$ and the condition $D \geq 4$ are met. These solutions belong to a new solution class, the properties of which are derived only from the non-metricity Q modification, since they do not have a general relativity limit. We examine the singularities present in the solutions by calculating the non-metricity and curvature invariant values. In conclusion, we compute thermodynamic parameters such as Gibbs free energy, Hawking temperature, and entropy. These thermodynamic calculations confirm that our model is stable.

Keywords: $f(Q)$ theory; cylindrical black holes; singularities and thermodynamics

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1. Introduction

One notable and worrisome observation from the last 20 years is the universe's acceleration caused by Dark Energy (DE). This cosmological event is confirmed by recent developments in observational cosmology: cosmic microwave background radiation [1], Type Ia Supernovae [2–4], the Lyman-forest power spectrum from the Sloan Digital Sky Survey [5], large-scale structure observations [6–8], and the investigation of high-energy DE models with weak lensing data [9]. Consequently, scientists search for appropriate modifications to the GR, such as gravitation $f(R)$ [10–17], $f(R, G)$, where the Gauss–Bonnet and Ricci scalar expressions are denoted by R and G, respectively [18], $f(T)$ gravity, with the torsion scalar T [19–25], $f(G)$ gravity [26–28], Brans–Dicke (BD) gravity [29,30], and so on. The concept of higher-order curvature, or more precisely $f(R)$ gravity, is the most successful adaptation of GR for explaining the existence of dark matter [31].

Recently, $f(Q)$ gravity, a well-motivated theory of gravity, was put forth by Jiménez et al. [32]. Lagrangian density, on which it is based, produces a general function of the non-metricity scalar Q. Non-metricity drives the gravitational interaction in space-time in this theory. The modified theory of $f(Q)$ gravity leads to intriguing cosmic phenomenology at the background level [33–54]. It should also be noted that unlike $f(R)$ gravity, where the field equations are of the fourth order [55], $f(Q)$-gravity has field equations of second-order, which is free from pathologies. Hence, the building of this $f(Q)$ theory provides a novel starting point for several modified gravity theories. For the time being, the study of $f(Q)$ gravity is the most debatable phenomenon.

Moreover, it has effectively been tested against diverse observational data related to background and perturbations, Type Ia Supernovae (SNIa), including the Cosmic Microwave Background (CMB), Redshift Space Distortion (RSD), growth data, Baryonic

Acoustic Oscillations (BAO), and similar datasets [56–62]. Ultimately, the constraints of Big Bang Nucleosynthesis (BBN) are simply transcended by $f(Q)$ gravity [63]. Different yet comparable theories of gravity are produced depending on T (torsion) or Q (non-metricity). These are known as the teleparallel equivalent of GR, or $(TEGR)$ [64,65] and symmetric teleparallel GR (STGR) [66–68]. Instead of curvature and torsion, non-metricity underpins the concept of gravity in STGR. Inspired by the interesting qualities of $f(Q)$ gravity, we will derive a $D-dimensions$ flat horizons black hole using the cubic form of $f(Q)$, i.e., $f(Q) = Q + 1/2\gamma Q^2 + 1/3\gamma_1 Q^3 - 2\Lambda$, where γ and γ_1 are two constants of dimensions l$ength^2$, l$ength^4$, and Λ. Λ in this study represents the cosmological constant.

The structure of this investigation is outlined as follows: Section 2 deals with the examination of the field equations and a brief summary of the non-metricity formalism. Subsequently, we present the equation of motion for gravity within the framework of $f(Q)$. The ansatz of the metric with a flat horizon in D-dimensions is utilized to the equations of motion of $f(Q)$ gravity in Section 3. Applying this approach leads to deriving a new solution in D-dimensions. The asymptotic behavior of the solution corresponds to Anti-de-Sitter (AdS) space. The relevant physical properties of these solutions are discussed in Section 4. We explore the black hole's thermodynamics in Section 5. Finally, Section 6 has closing comments.

2. The Theory of $f(Q)$

This section covers some of the generic characteristics of $f(Q)$-gravity. We will restrict the scope of this explanation to components (the reader can turn to Refs. [69–71] for a more rigorous derivation in terms of forms).

For a parallelizable and differentiable manifold, the affine connection can be written in the form:

$$\Gamma^{\sigma}_{\mu\nu} = \tilde{\Gamma}^{\sigma}_{\mu\nu} + K^{\sigma}_{\mu\nu} + L^{\sigma}_{\mu\nu}\,, \tag{1}$$

where the Levi–Civita connection is represented by $\tilde{\Gamma}^{\sigma}_{\mu\nu}$ which has the following definition:

$$\tilde{\Gamma}^{\sigma}_{\mu\nu} = \frac{1}{2}g^{\sigma\rho}\left(\partial_{\mu}g_{\rho\nu} + \partial_{\nu}g_{\rho\mu} - \partial_{\rho}g_{\mu\nu}\right). \tag{2}$$

Also, the contortion $K^{\sigma}{}_{\mu\nu}$ is defined as:

$$K^{\sigma}_{\mu\nu} = \frac{1}{2}T^{\sigma}_{\mu\nu} + T_{(\mu}{}^{\sigma}{}_{\nu)}\,, \tag{3}$$

with the torsion tensor $T^{\sigma}_{\mu\nu} = 2\Gamma^{\sigma}_{[\mu\nu]}$. Lastly, the deformation is $L^{\sigma}_{\mu\nu}$, which reads,

$$L^{\sigma}_{\mu\nu} = \frac{1}{2}Q^{\sigma}_{\mu\nu} - Q_{(\mu}{}^{\sigma}{}_{\nu)}\,, \tag{4}$$

where the non-metricity tensor, $Q^{\sigma}_{\mu\nu}$, is provided by

$$Q_{\sigma\mu\nu} = \nabla_{\sigma}g_{\mu\nu} = \partial_{\sigma}g_{\mu\nu} - \Gamma^{\rho}_{\sigma\mu}g_{\nu\rho} - \Gamma^{\rho}_{\sigma\nu}g_{\mu\rho}\,. \tag{5}$$

Consequently, the scalar of the non-metricity is

$$Q = g^{\mu\nu}(L^{\alpha}_{\beta\nu}L^{\beta}_{\mu\alpha} - L^{\beta}_{\alpha\beta}L)^{\alpha}_{\mu\nu} \equiv Q_{\sigma\mu\nu}P^{\sigma\mu\nu}\,, \tag{6}$$

where $P^{\sigma\mu\nu}$, the conjugate of non-metricity, is given by

$$P^{\sigma}_{\mu\nu} = \frac{1}{4}\left(-Q^{\sigma}_{\mu\nu} + 2Q_{(\mu}{}^{\sigma}{}_{\nu)} + Q^{\sigma}g_{\mu\nu} - \tilde{Q}^{\sigma}g_{\mu\nu} - \delta^{\sigma}_{(\mu}Q_{\nu)}\right), \tag{7}$$

where $Q_{\sigma} = Q_{\sigma}{}^{\mu}{}_{\mu}$, and $\tilde{Q}_{\sigma} = Q^{\mu}{}_{\sigma\mu}$.

In the absence of torsion and non-metricity, the connection takes on the same form as the metrically compatible Levi–Civita connection. Curvature and torsion are both zero in symmetric teleparallel gravity STG, and non-metricity is contingent on the interaction between the metric and the connection.

In Ref. [32], the authors introduced modified symmetric teleparallel gravity, with the action reading

$$I = -\frac{1}{2\kappa^2}\int_{\mathcal{M}} f(Q)\sqrt{-g}d^4x + \int_{\mathcal{M}} \mathcal{L}_m\sqrt{-g}d^4x\,, \tag{8}$$

where g is the determinant of the metric tensor $g_{\mu\nu}$, $\mathcal{M}$ is the space-time manifold, $\mathcal{L}_m$ is the Lagrangian density of matter contents, and $f(Q)$ is a generic function of the non-metricity scalar Q.

One applies independent variations with respect to both the metric and the connection to Equation (8) in order to obtain the field equations of the theory, so

$$\xi_{\mu\nu} \equiv \frac{2}{\sqrt{-g}}\nabla_\alpha\left(\sqrt{-g}f_Q P^\alpha{}_{\mu\nu}\right) + \frac{1}{2}g_{\mu\nu}f + f_Q\left(P_{\mu\alpha\beta}Q_\nu{}^{\alpha\beta} - 2P_{\alpha\beta\mu}Q_\nu{}^{\alpha\beta}\right) = \kappa^2\mathcal{T}_{\mu\nu}\,, \tag{9}$$

$$\nabla_\mu\nabla_\nu\left(\sqrt{-g}f_Q P^\alpha{}_{\mu\nu}\right) = 0\,, \tag{10}$$

where $\mathcal{T}_{\mu\nu}$, as is traditional, represents the energy–momentum tensor of matter

$$\mathcal{T}_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{\mu\nu}}\,. \tag{11}$$

The above expression has two parts: $f_Q = \frac{df(Q)}{dQ}$ and $f \equiv f(Q)$. We observe that there is no hyper-momentum because the Lagrangian density of matter is calculated without consideration of the connection. Furthermore, it is well known that by presenting $f(Q) = Q$, the Lagrangian density $\mathcal{L} = -\frac{Q}{2\kappa^2} + \mathcal{L}_m$ may be produced, yielding the results of GR (in the STEGR framework).

3. Static Anti-de-Sitter Black Hole Solution

We investigate the cylindrical D-dimensional spacetime using the field equations of $f(Q)$ gravity, given by Equation (9). The line element that emerges from this analysis is shown in cylindrical coordinates $(t, r, \zeta_1, \zeta_2, \cdots, \zeta_{D-2})$, as elaborated in [72]:

$$ds^2 = \mu(r)dt^2 - \frac{dr^2}{\nu(r)} - r^2\sum_{i=1}^{D-2} d\zeta_i^2\,. \tag{12}$$

In this context, $\mu(r)$ and $\nu(r)$ denote two variables that depend on the radial coordinate. Additionally, the form of non-metricity, Q, given by Equation (12), yields the following form in D-dimension:

$$Q = -\frac{(D-2)\nu[(D-3)\mu + \mu']}{r^2\mu}\,. \tag{13}$$

By applying Equation (12) to the equations of motion (9), we derive the following non-zero components when the energy–momentum is vanishing, i.e., $\mathcal{T}_{\mu\nu} = 0$:

$$\xi_t{}^t \equiv \frac{1}{2r^2\mu^2}\Big[2(D-2)^2\nu^2r^2 f_{QQ}[\mu'^2-\mu\mu''] + (D-2)r\mu\nu\mu'[2(D-2)f_{QQ}\{\nu - r\mu'\} + r^2 f_Q] + \mu\Big\{(D-2)r\mu\mu'\Big[r^2 f_Q$$
$$-2(D-3)(D-2)f_{QQ}\nu\Big] + \mu(r^4 f(Q) + 2(D-2)(D-3)r^2\nu f_Q + 4(D-3)(N-2)^2\nu^2 f_{QQ})\Big\}\Big] = 0\,,$$
$$\xi_r{}^r \equiv \frac{f(Q)r^2\mu + 2(D-2)rf_Q\nu\mu' + 2(D-2)(D-3)f_Q\mu\nu}{2r^2\mu} = 0\,,$$
$$\xi_{\zeta_1}{}^{\zeta_1} = \xi_{\zeta_2}{}^{\zeta_2} = \cdots\cdots = \xi_{\zeta_{D-2}}{}^{\zeta_{D-2}} \equiv \frac{1}{4r^4\mu^3}\Big\{2r^2\nu\mu\mu''[\mu(r^2 f_Q - 2(D-2)(D-3)f_{QQ}\nu) - (D-2)r\nu\mu' f_{QQ}]$$
$$+2(D-2)f_{QQ}r^3\nu^2\mu'^3 - r^2\mu\nu\mu'^2[r^2 f_Q + 2f_{QQ}(D-2)(r\mu' - (2D-5)\nu)] + r\mu^2\mu'\Big[r\mu'(r^2 f_Q - 6(D-2)(D-3)f_{QQ}\nu)$$
$$+2(2D-5)r^2 f_Q\nu + 8(D-2)(D-3)f_{QQ}\nu^2\Big] + 2\mu^2\Big[(D-3)r\mu\mu'(r^2 f_Q - 2(D-2)(D-3)\nu f_{QQ}) + \mu\Big(r^4 f(Q)$$
$$+2(D-3)^2r^2\nu f_Q + 4(D-2)(D-3)^2 f_{QQ}\nu^2\Big)\Big]\Big\} = 0\,. \tag{14}$$

Following that, we will find a complete solution to Equations (14) by using a specific expression for $f(Q)$, namely:

$$\mathrm{f}(Q) = Q + \frac{1}{2}\gamma Q^2 + \frac{1}{3}\gamma_1 Q^3 - 2\Lambda\,, \tag{15}$$

where γ and γ_1 are dimensional constants that have the unites of $length^2$, $length^4$, Λ is the cosmological constant. Given this specific $f(Q)$ configuration, the following results are obtained from Equations (13):

$$\xi_t{}^t \equiv \frac{1}{2r^2\mu^2}\Big\{2(D-2)^2\nu^2r^2[\gamma+2\gamma_1 Q][\mu'^2-\mu\mu''] + (D-2)r\mu\nu\mu'[2(D-2)[\gamma+2\gamma_1 Q]\{\nu - r\mu'\} + r^2(1+\gamma Q+\gamma_1 Q^2)]$$
$$+\mu\Big[(D-2)r\mu\mu'\Big[r^2[1+\gamma Q+\gamma_1 Q^2] - 2(D-3)(D-2)[\gamma+2\gamma_1 Q]\nu\Big] + \mu\Big\{r^4[Q+\frac{1}{2}\gamma Q^2 + \frac{1}{3}\gamma_1 Q^3 - 2\Lambda]$$
$$+2(D-2)(D-3)r^2\nu[1+\gamma Q+\gamma_1 Q^2] + 4(D-3)(D-2)^2\nu^2[\gamma+2\gamma_1 Q]\Big\}\Big]\Big\} = 0\,,$$
$$\xi_r{}^r \equiv \frac{[Q+\frac{1}{2}\gamma Q^2+\frac{1}{3}\gamma_1 Q^3 - 2\Lambda]r^2\mu + 2(D-2)r[1+\gamma Q+\gamma_1 Q^2]\nu\mu' + 2(D-2)(D-3)[1+\gamma Q+\gamma_1 Q^2]\mu\nu}{2r^2\mu} = 0\,,$$
$$\xi_{\zeta_1}{}^{\zeta_1} = \xi_{\zeta_2}{}^{\zeta_2} = \cdots\cdots = \xi_{\zeta_{D-2}}{}^{\zeta_{D-2}} \equiv \frac{1}{4r^4\mu^3}\Big\{2r^2\nu\mu\mu''\Big[\mu\Big(r^2[1+\gamma Q+\gamma_1 Q^2] - 2(D-2)(D-3)[\gamma+2\gamma_1 Q]\nu\Big)$$
$$-(D-2)r\nu\mu'(\gamma+\gamma_1 Q)\big] + 2(D-2)[\gamma+2\gamma_1 Q]r^3\nu^2\mu'^3 - r^2\mu\nu\mu'^2\Big[r^2[1+\gamma Q+\gamma_1 Q^2] + 2[\gamma+2\gamma_1 Q][r\mu' - (2D-5)\nu]$$
$$\times(D-2)\big] + r\mu^2\mu'\Big[r\mu'(r^2[1+\gamma Q+\gamma_1 Q^2] - 6(D-2)(D-3)[\gamma+\gamma_1 Q]\nu) + 2(2D-5)r^2[1+\gamma Q+\gamma_1 Q^2]\nu$$
$$+8(D-2)(D-3)[\gamma+\gamma_1 Q]\nu^2\Big] + 2\mu^2\Big[(D-3)r\mu\mu'(r^2[1+\gamma Q+\gamma_1 Q^2] - 2(D-2)(D-3)\nu[\gamma+2\gamma_1 Q])$$
$$+\mu\Big(r^4[Q+\frac{1}{2}\gamma Q^2+\frac{1}{3}\gamma_1 Q^3 - 2\Lambda] + 2(D-3)^2r^2\nu[1+\gamma Q+\gamma_1 Q^2] + 4(D-2)(D-3)^2[\gamma+2\gamma_1 Q]\nu^2\Big)\Big]\Big\} = 0\,. \tag{16}$$

For Equation (16), a general D-dimensional case solution is:

$$\mu(r) = \nu(r) = \frac{1}{3}\left(\frac{1}{20}\frac{\sqrt[3]{600\,\gamma_1{}^2\Lambda - 90\,\gamma_1\,\gamma + 27\,\gamma^3 + 10\,\sqrt{80\,\gamma_1 - 27\,\gamma^2 - 1080\,\gamma_1\,\gamma\,\Lambda + 3600\,\gamma_1{}^2\Lambda^2 + 324\,\Lambda\,\gamma^3}\gamma_1}}{\gamma_1}\right.$$
$$\left.+\frac{r^2}{20}\frac{9\,\gamma^2 - 20\,\gamma_1}{\gamma_1\sqrt[3]{600\,\gamma_1{}^2\Lambda - 90\,\gamma_1\,\gamma + 27\,\gamma^3 + 10\,\sqrt{80\,\gamma_1 - 27\,\gamma^2 - 1080\,\gamma_1\,\gamma\,\Lambda + 3600\,\gamma_1{}^2\Lambda^2 + 324\,\Lambda\,\gamma^3}\gamma_1}} + \frac{3}{20}\frac{\gamma}{\gamma_1}\right) + \frac{c_1}{r}\,. \tag{17}$$

The general behavior of the above solution shows that μ and ν behave generally as Anti-de-Sitter(AdS) or de-Sitter (dS) spacetime. Here, c_1 stands for a dimensional integration constant. In an effort to streamline the computations, we shall assume that

$$\Lambda = \frac{1}{12\gamma}, \qquad \gamma_1 = \frac{2\gamma^2}{5}. \tag{18}$$

This presumption results in a special solution that has the following form:

$$\mu(r) = \nu(r) = \frac{r^2}{(D-1)(D-2)\gamma} + \frac{c_1}{r^{D-3}}. \tag{19}$$

It is evident from Equation (19) that in the case of cubic form, the higher order of $f(Q)$ serves as a cosmological constant.

4. The Fundamental Features of the Black Hole Solutions Given by Equation (19)

Let us now investigate certain relevant facets of the solution discussed in the previous section. The formulation of the solution's line element (19) is as follows:

$$ds^2 = \left[r^2\Lambda_1 - \frac{2M}{r^{D-3}}\right]dt^2 - \frac{dr^2}{r^2\Lambda_1 - \frac{2M}{r^{D-3}}} - r^2\sum_{i=1}^{D-2} d\zeta_i^2, \tag{20}$$

with Λ_1 being the cosmological constant related to the theory of $f(Q)$, and is defined as $\Lambda_1 = \frac{1}{(D-1)(D-2)\gamma}$ and $c_1 = -2M$. Equation (19) clearly signifies that the line element of the solution approaches AdS geometry. There is no counterpart for the linear form as $\Lambda_1 \to \infty$.

Singularity:

Physical singularities in this framework are identified by assessing all possible invariants within the domain of $f(Q)$ theory. The ansatz $\mu(r)$ might exhibit roots, represented as r_h. Consequently, one must investigate the invariant behavior around these roots. Following the evaluation from the different invariants, we obtain

$$\begin{aligned}
&Q = -\frac{1}{\gamma}, \\
&R^{\mu\nu\lambda\rho}R_{\mu\nu\lambda\rho} = \frac{2D}{(D-1)(D-2)^2\gamma^2} + \frac{(D-1)(D-2)^2(D-3)M^2}{r^{2(D-1)}}, \\
&R^{\mu\nu}R_{\mu\nu} = \frac{D}{(D-2)^2\gamma^2}, \qquad \mathbb{R} = \frac{D}{(D-2)\gamma}, \\
&Q^{\mu\nu\rho}Q_{\mu\nu\rho} \approx -\frac{(D-3)^2}{(D-1)\alpha} + \frac{10(D-3)M}{r^{D-1}} + \mathcal{O}\left(r^{D-1}\right), \\
&Q_\mu Q^\mu = \frac{4(D-2)}{(D-1)\gamma} + \frac{4(D-2)^2}{r^{(D-1)}},
\end{aligned} \tag{21}$$

where $R^{\mu\nu\lambda\rho}R\mu\nu\lambda\rho$, $R^{\mu\nu}Q\mu\nu$, Q, $Q^{\mu\nu\lambda}Q_{\mu\nu\lambda}$, $Q^\mu Q_\mu$, $\tilde{Q}^\mu\tilde{Q}_\mu$, and Q represent all the conceivable invariants that can be formulated within this theory, which demonstrates the singularity of the invariants at $r = 0$, which is described as a singularity in curvature. The invariants we used in this study are defined as $R^{\mu\nu\lambda\rho}R_{\mu\nu\lambda\rho}$, $R^{\mu\nu}Q_{\mu\nu}$, Q, $Q^{\mu\nu\lambda}Q_{\mu\nu\lambda}$ $Q^\mu Q_\mu$, $\tilde{Q}^\mu\tilde{Q}_\mu$, and Q, which are known as the Kretschmann scalar, the square of the Ricci tensor, the Ricci scalar, the square of the non-metricity tensor, the vectors of the non-metricity square, and the non-metricity, respectively.

5. The Black Holes Thermodynamic Properties as Expressed by Equation (19)

Using the recently found solution given in Equation (20), we explore the thermodynamic properties by introducing the concept of the Hawking temperature [73,74] as:

$$T_h = \frac{\mu'(r_h)}{4\pi}\,. \tag{22}$$

The notation $'$ in this scenario signifies a derivative in relation to the event horizon, r_h, which represents the most significant positive root of $\nu(r_h) = 0$ while ensuring that $\nu'(r_h)$ is not equal to zero. The $f(Q)$ theory's Bekenstein–Hawking entropy is expressed as [75,76] (remember that the way entropy is framed in $f(Q)$ geocentric theory is not the same as it is presented in linear non-metricity theory; we shall be aware of the non-metricity hypothesis when $f(Q) = Q$)

$$S(r_h) = \frac{1}{4} A f_Q(r_h)\,. \tag{23}$$

The event horizon's surface area in this frame is denoted by A. As per the heat capacity indicator C_h, the black hole will be thermodynamically stable; if $C_h > 0$, it will be stable, and if $C_h < 0$, it will not be stable. In the next study, we determine if these black hole solutions are thermally stable by looking at how each of their distinct heat capacity behaves [77,78].

$$H_h = \frac{dE_1}{dT_h} = \frac{\partial M}{\partial r_h}\left(\frac{\partial T}{\partial r_h}\right)^{-1}. \tag{24}$$

In this frame, E_1 describes the quasilocal energy. Within the framework of four dimensions and in relation to the solution given in Equation (20), the horizons are derived as follows:

$$r_h = \sqrt[3]{12M\gamma}. \tag{25}$$

Furthermore, we can obtain the following mass equation from Equation (20):

$$M = \frac{{r_h}^3}{12\gamma}. \tag{26}$$

The black hole's total mass is influenced by the horizon, as demonstrated by Equation (26). Figure 1a shows the relationship between $\nu(r)$ and r, illustrating the potential horizons.

The entropy of solution (20) takes the following manner:

$$S_h = \frac{2\pi {r_h}^2}{5}. \tag{27}$$

The patterns of entropy are depicted in Figure 1b, revealing a consistent behavior of the entropy. Figure 1c shows that as the dimensional quantity γ increases the entropy decreases as r increases.

The following formula is used to obtain the Hawking temperature of Equation (20):

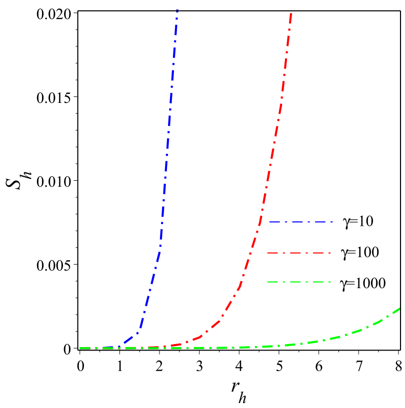
$$T_h = \frac{{r_h}^3 + 6M\gamma}{2\pi\, r_h(12M\gamma - {r_h}^3)}\,. \tag{28}$$

where T_h represents the Hawking temperature. Figure 1c shows the temperature, revealing that it is always positive. Figure 1c shows that as the dimensional quantity γ increases, the temperature also increases as r increases.

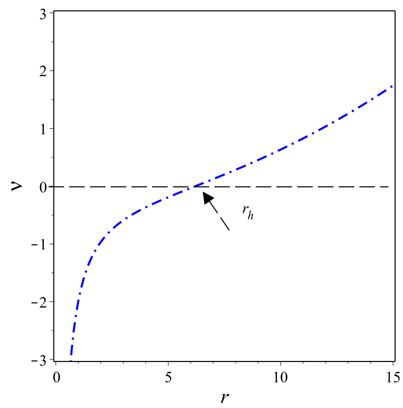
Equations (26) and (28) are substituted into (24) to yield

$$H_h = \frac{\pi {r_h}^4({r_h}^3 - 12M\gamma)^2}{2\gamma({r_h}^6 + 48{r_h}^3 M\gamma - 72M^2\gamma^2)}\,. \tag{29}$$

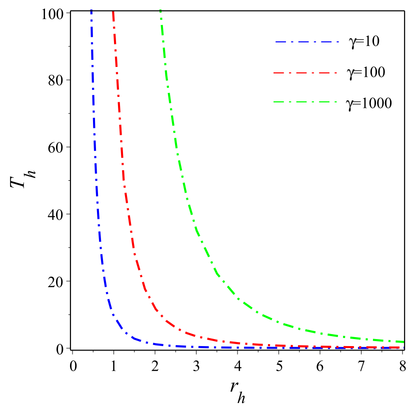
Figure 1d shows the patterns of the heat capacity for solution (20) for several values of the model parameters. The heat capacity is consistently positive as long as $r > r_h$, suggesting higher global stability, as seen in Figure 1d. Moreover, Figure 1c also shows that as the dimensional quantity γ increases, the heat capacity decreases until we obain a stable model.



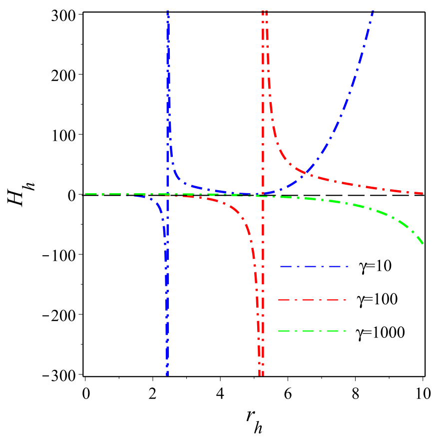
(**a**) The horizons' placements inside the metric potential g_{rr}



(**b**) The features of the entropy



(**c**) The temperature



(**d**) The Heat capacity

Figure 1. (**a**) The overall trend of g_{rr} is depicted in Figure 1; (**b**) highlights the entropy behavior; (**c**) depicts changes in the temperature; and (**d**) shows the heat capacity behavior. The model parameters are consistently set to $M = 1$.

6. Conclusions and Discussion

In this study, we delved into the intricate realm of the cubic form of the $f(Q)$ gravity theory, aiming to unravel its implications and characteristics. The cubic form, encapsulated by the function $f(Q)$ where Q represents the non-metricity, introduces a compelling dimension to our understanding of gravitational dynamics. Throughout our investigation, we scrutinized various aspects including the solutions to the field equations, the behavior of invariants, and the implications for spacetime geometry.

One of the pivotal findings of our study is the reality that the dimensional quantities related to the quadratic and cubic higher-order theories, i.e., γ and γ_1, are finally unified to show behavior of cosmological constants, as is clear from Equation (18). This sheds light on the nuanced interplay between the cubic form of $f(Q)$ gravity and the fundamental aspects of gravitational physics which indeed ensures that $f(Q)$ will generally not be different from the first-order approximation of $f(Q)$ in the case of static geometry. This is due to the fact that both the constants of the cubic form of $f(Q)$, γ and γ_1, can be rewritten in terms of the cosmological constant Λ, and generally the solution of the coinciding cubic case reproduces the Schwarzschild AdS/dS, which can be generated in the linear case, i.e., $f(Q) = Q + \Lambda$.

It is crucial to extend our special study to include the charge, i.e., study the charged field equations of $f(Q)$ in the cubic domain. This aspect warrants further investigation and refinement in future research endeavors.

In conclusion, our study on the cubic form of $f(Q)$ gravity theory represents a step forward in comprehending the complexities of alternative gravitational theories. The intriguing patterns and phenomena uncovered in this exploration pave the way for continued research and offer valuable contributions to the broader landscape of gravitational physics.

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