symmetry

MDPI

Article

Bi-Concave Functions Connected with the Combination of the Binomial Series and the Confluent Hypergeometric Function

Hari M. Srivastava [1,2,3,4,5,6], Sheza M. El-Deeb [7,8], Daniel Breaz [9], Luminita-Ioana Cotîrlă [10,*] and Grigore Stefan Sălăgean [11]

1 Department of Mathematics and Statistics, University of Victoria, Victoria, BC V8W 3R4, Canada; harimsri@math.uvic.ca
2 Department of Medical Research, China Medical University Hospital, China Medical University, Taichung 40402, Taiwan
3 Center for Converging Humanities, Kyung Hee University, 26 Kyungheedae-ro, Dongdaemun-gu, Seoul 02447, Republic of Korea
4 Department of Applied Mathematics, Chung Yuan Christian University, Chung-Li, Taoyuan City 320314, Taiwan
5 Department of Mathematics and Informatics, Azerbaijan University, 71 Jeyhun Hajibeyli Street, AZ1007 Baku, Azerbaijan
6 Section of Mathematics, International Telematic University Uninettuno, I-00186 Rome, Italy
7 Department of Mathematics, College of Science, Qassim University, Buraydah 51452, Saudi Arabia; s.eldeeb@qu.edu.sa
8 Department of Mathematics, Faculty of Science, Damietta University, New Damietta 34517, Egypt
9 Department of Mathematics, "1 Decembrie 1918" University of Alba Iulia, RO-510009 Alba Iulia, Romania; dbreaz@uab.ro
10 Department of Mathematics, Technical University of Cluj-Napoca, RO-400114 Cluj-Napoca, Romania
11 Department of Mathematics, Faculty of Mathematics and Computer Science, Babeş-Bolyai University, RO-400084 Cluj-Napoca, Romania; salagean@math.ubbcluj.ro
* Correspondence: luminita.cotirla@math.utcluj.ro

Abstract: In this article, we first define and then propose to systematically study some new subclasses of the class of analytic and bi-concave functions in the open unit disk. For this purpose, we make use of a combination of the binomial series and the confluent hypergeometric function. Among some other properties and results, we derive the estimates on the initial Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions in these analytic and bi-concave function classes, which are introduced in this paper. We also derive a number of corollaries and consequences of our main results in this paper.

Keywords: analytic functions; hadamard product (or convolution); univalent and bi-univalent functions; Taylor-Maclaurin coefficients; bi-concave functions; pocchammer symbol; confluent hypergeometric function; gauss hypergeometric function; generalized hypergeometric function; binomial series

MSC: Primary 30C45; 33C15; Secondary 30C20; 33C50

check for updates

Citation: Srivastava, H.M.; El-Deeb, S.M.; Breaz, D.; Cotîrlă, L.-I.; Sălăgean, G.S. Bi-Concave Functions Connected with the Combination of the Binomial Series and the Confluent Hypergeometric Function. *Symmetry* **2024**, *16*, 226. https://doi.org/10.3390/sym16020226

Academic Editors: Charles F. Dunkl and Calogero Vetro

Received: 12 December 2023
Revised: 21 January 2024
Accepted: 24 January 2024
Published: 13 February 2024

Copyright: © 2024 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (https://creativecommons.org/licenses/by/4.0/).

1. Introduction and Preliminaries

The Geometric Function Theory of Complex Analysis is the study of the relationship of analytic properties of a given function and the geometric properties of its image domain. This subject has a remarkably rich literature due mainly to its potential for applications in the mathematical and physical sciences, and also in other areas. In particular, the estimate problems for the Taylor-Maclaurin coefficients of analytic and univalent functions have been investigated extensively and widely. As an example, we cite the celebrated de Branges theorem, which asserts the truths of the Milin conjecture of 1971, the Robertson conjecture of 1936 and the Bieberbach conjecture of 1916 (see, for details, [1]).

Investigations of the estimate problems for the Taylor-Maclaurin coefficients of the analytic functions, which are bi-univalent (that is, both the function and its inverse are univalent), were initiated by Brannan et al. (see [2,3]) (see also the related work by Netanyahu [4]). More recently, after the publication of the pioneering work on the subject by Srivastava et al. [5], analogous coefficient problems have been studied for various interesting subclasses of the class of analytic and bi-univalent functions. These subclasses include the classes of bi-starlike functions, bi-convex functions, bi-close-to-convex functions, and so on (see, for example, [6–11]).

The coefficient estimate problems, which we consider in this article, involve some new subclasses of analytic and bi-concave functions. Some earlier studies of the coefficient estimate problems for other subclasses of analytic and bi-concave functions include (for example) the works in [12–16].

We begin now by supposing that $\mathcal{A}$ represents the class of functions of the form given by

$$\mathcal{F}(z) = z + \sum_{t=2}^{\infty} a_t \, z^t \qquad (z \in \Lambda), \tag{1}$$

which are analytic in the open unit disk

$$\Lambda = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

As usual, we denote by $\mathcal{S}$ the subclass of the analytic function class $\mathcal{A}$, which consists of functions that are also univalent in Λ.

We also let the function $\mathcal{H} \in \mathcal{A}$ be given by

$$\mathcal{H}(z) = z + \sum_{t=2}^{\infty} c_t \, z^t \qquad (z \in \Lambda). \tag{2}$$

The Hadamard product (or convolution) of $\mathcal{F}$ and $\mathcal{H}$ is given by

$$(\mathcal{F} * \mathcal{H})(z) := z + \sum_{t=2}^{\infty} a_t \, c_t \, z^t =: (\mathcal{H} * \mathcal{F})(z) \qquad (z \in \Lambda).$$

The Koebe one-quarter theorem (see, for details, [17]) states that the image of Λ under every univalent function $\mathcal{F} \in \mathcal{S}$ contains the disk of radius $\frac{1}{4}$. Therefore, every function $\mathcal{F} \in \mathcal{S}$ has an inverse $\mathcal{F}^{-1}$ that satisfies the following property:

$$\mathcal{F}\left(\mathcal{F}^{-1}(w)\right) = w \qquad \left(|w| < r_0(\mathcal{F});\ r_0(\mathcal{F}) \geqq \frac{1}{4}\right),$$

where

$$\begin{aligned} \mathcal{G}(w) = \mathcal{F}^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 \\ - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots (w \in \Lambda), \end{aligned} \tag{3}$$

the function $\mathcal{G}$ being the analytic extension of $\mathcal{F}^{-1}$ to Λ.

As we mentioned above, a function $\mathcal{F} \in \mathcal{A}$ is said to be bi-univalent in Λ if both $\mathcal{F}$ and its inverse $\mathcal{F}^{-1}$ are univalent in Λ. Let Σ denote the class of analytic and bi-univalent functions in Λ given by (1). Remarkably, in their pioneering work in the year 2010, Srivastava et al. [5] actually revived the study of analytic and bi-univalent functions in latter years. Moreover, the current literature is flooded by many sequels to their paper [5]. We recall from [5] that the following functions:

$$\mathcal{F}_1(z) = \frac{z}{1-z}, \quad \mathcal{F}_2(z) = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right) \quad \text{and} \quad \mathcal{F}_3(z) = -\log(1-z),$$

together with their corresponding inverse functions:

$$\mathcal{F}_1^{-1}(w) = \frac{w}{1+w}, \quad \mathcal{F}_2^{-1}(w) = \frac{e^{2w}-1}{e^{2w}+1} \quad \text{and} \quad \mathcal{F}_3^{-1}(w) = \frac{e^w-1}{e^w},$$

are elements of Σ (see [5]). For a brief history and interesting examples of functions in the class Σ, see the work of Brannan et al. [2]. Brannan and Taha [3], and also the more recent work of Srivastava et al. [5]) who introduced certain subclasses of the bi-univalent function class Σ similar to the familiar subclasses $\mathcal{S}^*(\gamma)$ and $\mathcal{K}(\gamma)$ of starlike and convex functions of order γ $(0 \leqq \gamma < 1)$ in Λ, respectively.

A function $\mathcal{F} : \Lambda \to \mathbb{C}$ is said to belong to the family $\mathcal{C}_0(\gamma)$ if $\mathcal{F}$ satisfies the following conditions:

(i) $\mathcal{F}$ is analytic in Λ with the standard normalization given by

$$\mathcal{F}(0) = \mathcal{F}'(0) - 1 = 0.$$

(ii) $\mathcal{F}$ maps Λ conformally onto a set whose complement with respect to $\mathbb{C}$ is convex.
(iii) The opening angle of $\mathcal{F}(\Lambda)$ at ∞ is less than or equal to $\pi\gamma$ $\big(\gamma \in (1,2]\big)$.

The class of concave univalent functions in Λ is usually represented by the notation $\mathcal{C}_0(\gamma)$. For a detailed description of concave functions, see [14,18]. The following inequality:

$$\Re\left(1 + \frac{z\,\mathcal{F}''(z)}{\mathcal{F}'(z)}\right) < 0 \qquad (z \in \Lambda)$$

was used by Bhowmik et al. [19] in order to prove that an analytic function $\mathcal{F}$ maps Λ onto an angled concave domain $\pi\gamma$ if and only if

$$\Re\big(P_{\mathcal{F}}(z)\big) > 0 \qquad (z \in \Lambda),$$

where

$$P_{\mathcal{F}}(z) = \frac{2}{\gamma-1}\left(\frac{(\gamma+1)(1+z)}{2(1-z)} - 1 - \frac{z\,\mathcal{F}''(z)}{\mathcal{F}'(z)}\right).$$

The existing literature contains a number of investigations on various subclasses of the class $\mathcal{C}_0(\gamma)$ of concave univalent functions in Λ (see [12,16]).

Next, we denote by $(\lambda)_t$ the Pochhammer symbol or the *shifted factorial*, since

$$(1)_t = t! \qquad (t \in \mathbb{N}_0).$$

It is defined for $\lambda, \nu \in \mathbb{C}$, and in terms of the familiar Gamma function, by

$$(\lambda)_t := \frac{\Gamma(\lambda+t)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu = 0;\ \lambda \in \mathbb{C} \setminus \{0\}) \\ \\ \lambda(\lambda+1)\cdots(\lambda+t-1) & (t \in \mathbb{N};\ \lambda \in \mathbb{C}). \end{cases}$$

Here, and in what follows, it is understood *conventionally* that $(0)_0 := 1$ and assumed *tacitly* that the Γ-quotient exists.

In terms of the above-defined Pochhammer symbol $(\lambda)_t$, the generalized hypergeometric function ${}_pF_q$ $(p, q \in \mathbb{N}_0)$, with p numerator parameters

$$\alpha_j \in \mathbb{C} \quad (j = 1, \cdots, p)$$

and q denominator parameters

$$\beta_j \in \mathbb{C} \setminus \mathbb{Z}_0^- \quad (j = 1, \cdots, q),$$

is given by

$$ {}_pF_q\left[\begin{array}{c} \alpha_1,\cdots,\alpha_p; \\ \\ \beta_1,\cdots,\beta_q; \end{array} z\right] := \sum_{t=0}^{\infty} \frac{(\alpha_1)_t \cdots (\alpha_p)_t}{(\beta_1)_t \cdots (\beta_q)_t} \frac{z^t}{t!} $$
$$ =: {}_pF_q(\alpha_1,\cdots,\alpha_p;\beta_1,\cdots,\beta_q;z), \tag{4} $$

in which the infinite series

(i) converges absolutely for $|z| < \infty$ if $p \leqq q$,
(ii) converges absolutely for $|z| < 1$ if $p = q+1$, and
(iii) diverges for all z $(z \neq 0)$ if $p > q+1$.

For $p - 1 = q = 1$, the Equation (4) defines the Gauss hypergeometric function ${}_2F_1$, given by

$$ {}_2F_1(\alpha_1,\alpha_2;\beta_1;z) := \sum_{t=0}^{\infty} \frac{(\alpha_1)_t(\alpha_2)_t}{(\beta_1)_t} \frac{z^t}{t!} \qquad (z \in \Lambda), $$

so that, by the principle of confluence, we have the confulent hypergeometric function ${}_1F_1$ defined by (see, for details, [20,21])

$$ F(\alpha;\beta;z) := {}_1F_1(\alpha_1;\beta_1;z) = \lim_{\alpha_1\to\infty}\left\{{}_2F_1\left(\alpha_1,\alpha_2;\beta_1;\frac{z}{\alpha_1}\right)\right\} $$
$$ = \sum_{t=0}^{\infty} \frac{(\alpha)_t}{(\beta)_t} \frac{z^t}{t!} \qquad (z,\alpha \in \mathbb{C};\ \beta \in \mathbb{C}\setminus\{0,-1,-2,\cdots\}). $$

Recently, Porwal and Kumar [22] (see also [23,24]) introduced the confluent hypergeometric distribution (CHD) whose probability mass function is given by

$$ P(t) = \frac{(\alpha)_t}{(\beta)_t\ t!\ F(\alpha;\beta;m)}\ m^t \qquad (\alpha,\beta,m>0;\ t=0,1,2,\cdots). $$

On the other hand, El-Deeb [23] introduced the following series $\mathcal{I}(\alpha;\beta;m;z)$ whose coefficients are probabilities of the above-defined confluent hypergeometric distribution:

$$ \mathcal{I}(\alpha;\beta;m;z) = z + \sum_{t=2}^{\infty} \frac{(\alpha)_{t-1} m^{t-1}}{(\beta)_{t-1}\ (t-1)!\ F(\alpha;\beta;m)}\ z^t \qquad (\alpha,\beta,m>0). \tag{5} $$

El-Deeb [23] also defined the following linear operator $\mathcal{Q}^{\alpha;\beta;m}\mathcal{F} : \mathcal{A} \to \mathcal{A}$:

$$ \mathcal{Q}^{\alpha;\beta;m}\mathcal{F}(z) = \mathcal{I}(\alpha;\beta;m;z) * \mathcal{F}(z) $$
$$ = z + \sum_{t=2}^{\infty} \frac{(\alpha)_{t-1} m^{t-1}}{(\beta)_{t-1}\ (t-1)!\ F(\alpha;\beta;m)}\ a_t\ z^t \qquad (b,c,m>0). $$

Now, by making use of the binomial series:

$$ (1-z)^r = \sum_{j=0}^{r} \binom{r}{j} (-z)^j \qquad (r \in \mathbb{N}_0 := \mathbb{N}\cup\{0\}), $$

we introduce the following linear differential operator for $\mathcal{F} \in \mathcal{A}$ given by (1):

$$ \mathcal{D}_r^{\delta,0,\alpha,\beta,m}\mathcal{F}(z) = \mathcal{Q}^{\alpha;\beta;m}\mathcal{F}(z), \tag{6} $$

$$\begin{aligned}\mathcal{D}_r^{\delta,1,\alpha,\beta,m}\mathcal{F}(z) &= \mathcal{D}_r^{\delta,\alpha,\beta;m}\mathcal{F}(z) = (1-\delta)^r Q^{\alpha;\beta;m}\mathcal{F}(z)\\ &\quad + \left[1-(1-\delta)^r\right] z\Big(Q^{\alpha;\beta;m}\mathcal{F}\Big)'(z)\\ &= z + \sum_{t=2}^{\infty}[1+(t-1)c^r(\delta)]\left(\frac{(\alpha)_{t-1}m^{t-1}}{(\beta)_{t-1}\,(t-1)!\,F(\alpha;\beta;m)}\right) a_t\, z^t \end{aligned} \tag{7}$$

and

$$\begin{aligned}\mathcal{D}_r^{\delta,n,\alpha,\beta,m}\mathcal{F}(z) &= \mathcal{D}_r^{\delta,\alpha,\beta,m}\Big(\mathcal{D}_r^{\delta,n-1,b,c,m}\mathcal{F}(z)\Big)\\ &= (1-\delta)^r\mathcal{D}_r^{\delta,n-1,b,c,m}\mathcal{F}(z) + \left[1-(1-\delta)^r\right] z\Big(\mathcal{D}_r^{\delta,n-1,b,c,m}\mathcal{F}(z)\Big)'\\ &= z + \sum_{t=2}^{\infty}[1+(t-1)c^r(\delta)]^n\left(\frac{(\alpha)_{t-1}m^{t-1}}{(\beta)_{t-1}\,(t-1)!\,F(\alpha;\beta;m)}\right) a_t\, z^t\\ &= z + \sum_{t=2}^{\infty}\psi_t a_t\, z^t \end{aligned} \tag{8}$$

$$(\delta > 0;\ \alpha,\beta,m > 0;\ r,n \in \mathbb{N}_0),$$

where

$$\psi_t = [1+(t-1)c^r(\delta)]^n\left(\frac{(\alpha)_{t-1}m^{t-1}}{(\beta)_{t-1}\,(t-1)!\,F(\alpha;\beta;m)}\right) \tag{9}$$

and

$$c^r(\delta) = -\sum_{j=1}^{r}\binom{r}{j}(-\delta)^j \qquad (r \in \mathbb{N}). \tag{10}$$

We find from (8) that

$$c^r(\delta)\, z\left(\mathcal{D}_r^{\delta,n,\alpha,\beta,m}\mathcal{F}(z)\right)' = \mathcal{D}_r^{\delta,n+1,\alpha,\beta,m}\mathcal{F}(z) - [1-c^r(\delta)]\mathcal{D}_r^{\delta,n,\alpha,\beta,m}\mathcal{F}(z). \tag{11}$$

Remark 1. *Each of the following special cases is worthy of note:*

(i) *Putting $n = 0$, we obtain*

$$\mathcal{D}_r^{\delta,0,\alpha,\beta,m} = Q^{\alpha;\beta;m},$$

where $Q^{b;c;m}$ is given by

$$Q^{\alpha;\beta;m}\mathcal{F}(z) = z + \sum_{t=2}^{\infty}\left(\frac{(\alpha)_{t-1}m^{t-1}}{(\beta)_{t-1}\,(t-1)!\,F(\alpha;\beta;m)}\right) a_t\, z^t; \tag{12}$$

(ii) *Putting $r = 0$, we obtain*

$$\mathcal{D}_0^{\delta,n,\alpha,\beta,m} =: \mathcal{I}^{n,\alpha,\beta,m},$$

where $\mathcal{I}^{n,\alpha,\beta,m}$ is given by

$$\mathcal{I}^{n,\alpha,\beta,m}\mathcal{F}(z) = z + \sum_{t=2}^{\infty}(1+(t-1)]^n\left(\frac{(\alpha)_{t-1}m^{t-1}}{(\beta)_{t-1}\,(t-1)!\,F(\alpha;\beta;m)}\right) a_t\, z^t. \tag{13}$$

Definition 1. *We define the functions $h,\ p : \Lambda \to \mathbb{C}$, so that*

$$\min\left\{\Re\big(h(z)\big),\ \Re\big(p(z)\big)\right\} > 0$$

and

$$h(0) = p(0) = 1,$$

that is,

$$h(z) = 1 + \sum_{n=1}^{\infty} h_n z^n \qquad \text{and} \qquad p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \qquad (z \in \Lambda). \tag{14}$$

Under the above-mentioned conditions, the functions h and p are said to belong to the class $\mathcal{P}$.

The subclass $\mathcal{D}_r^{\delta,n,\alpha,\beta,m}\mathcal{C}_0(\gamma)$ of bi-concave functions in Λ is now introduced by Definition 2 below.

Definition 2. *A function* $\mathcal{F}$ *of the form* (1) *is said to be in the class* $\mathcal{D}_r^{\delta,n,\alpha,\beta,m}\mathcal{C}_0(\gamma)$ *of bi-concave functions* $\mathcal{F} \in \Sigma$ *if the following conditions are satisfied*:

$$\mathcal{F} \in \Sigma \quad \text{and} \quad \frac{2}{\gamma - 1}\left(\frac{(\gamma+1)(1+z)}{2(1-z)} - 1 - \frac{z\left(\mathcal{D}_r^{\delta,n,\alpha,\beta,m}\mathcal{F}(z)\right)''}{\left(\mathcal{D}_r^{\delta,n,\alpha,\beta,m}\mathcal{F}(z)\right)'}\right) \in h(\Lambda) \tag{15}$$

and

$$\mathcal{F} \in \Sigma \quad \text{and} \quad \frac{2}{\gamma - 1}\left(\frac{(\gamma+1)(1+w)}{2(1-w)} - 1 - \frac{w\left(\mathcal{D}_r^{\delta,n,\alpha,\beta,m}\mathcal{G}(w)\right)''}{\left(\mathcal{D}_r^{\delta,n,\alpha,\beta,m}\mathcal{G}(w)\right)'}\right) \in p(\Lambda), \tag{16}$$

where $\delta > 0$, $\alpha, \beta, m > 0$, $r \in \mathbb{N}$, $n \in \mathbb{N}_0$ *and* $\gamma \in (1, 2]$. *Moreover, the function* $\mathcal{G}$ *in the condition* (16) *is the analytic extension of* $\mathcal{F}^{-1}$ *to* Λ, *which is given by the Equation* (3), *and the functions h* $p \in \mathcal{P}$ *are defined and used as in Definition* 1.

Remark 2. *Our present investigation of the coefficient estimate problems makes use of Definition* 2 *involving the functions p and h of the class* $\mathcal{P}$ *for which the following important coefficient inequalities hold true for functions* $p,\ h \in \mathcal{P}$ *given by* (14):

$$|p_n| \leqq 2 \qquad \text{and} \qquad h_n| \leqq 2 \qquad (n \in \mathbb{N} := \{1, 2, 3, \cdots\}),$$

wherein the equality is valid for the extremal functions:

$$h(z) = \frac{1+z}{1-z} = 1 + 2z + 2z^2 + \cdots \quad \text{and} \quad p(z) = \frac{1-z}{1+z} = 1 - 2z + 2z^2 - \cdots \qquad (z \in \Lambda).$$

Several interesting examples of the functions p and h, which are motivated by the above extremal functions, and also such examples as those given below:

$$\left(\frac{1+z}{1-z}\right)^{\alpha} \qquad \text{and} \qquad \left(\frac{1-z}{1+z}\right)^{\alpha} \qquad (0 < \alpha \leqq 1)$$

and

$$\frac{1+(1-2\sigma)z}{1-z} \qquad \text{and} \qquad \frac{1-(1-2\sigma)z}{1+z} \qquad (0 < \sigma \leqq 1),$$

will be considered in the corollaries and consequences of our main results in Theorem 1 *below* (*see also Remark* 4).

It may be possible to suitably replace the functions $p,\ h \in \mathcal{P}$ *in Definition* 2 *by members of other function classes defined in* Λ. *However, the investigation of the analogous coefficient problems for the correspondingly-defined bi-concave function classes will presumably be much more involved than what we have presented in this article.*

Remark 3. *Each of the following special cases is worthy of note:*

(i) *Putting* $n = 0$, *we obtain*

$$\mathcal{D}_r^{\delta,0,\alpha,\beta,m}\mathcal{C}_0(\gamma) =: \mathcal{G}^{\alpha,\beta,m}\mathcal{C}_0(\gamma),$$

where $\mathcal{G}^{\alpha,\beta,m}\mathcal{C}_0(\gamma)$ *represents the functions* $\mathcal{F} \in \Sigma$ *that satisfy the conditions in* (15) *and* (16) *with* $\mathcal{D}_r^{\delta,n,\alpha,\beta,m}$ *replaced by* $Q^{\alpha;\beta;m}$, *which is given by* (12);
(ii) *Putting* $r = 0$, *we obtain*

$$\mathcal{D}_0^{\delta,n,\alpha,\beta,m}\mathcal{C}_0(\gamma) =: \mathcal{R}^{n,\alpha,\beta,m}\mathcal{C}_0(\gamma),$$

where $\mathcal{R}^{n,\alpha,\beta,m}\mathcal{C}_0(\gamma)$ *represents the functions* $\mathcal{F} \in \Sigma$ *that satisfy the conditions in* (15) *and* (16) *with* $\mathcal{D}_r^{\delta,n,\alpha,\beta,m}$ *replaced by* $\mathcal{I}^{n,\alpha,\beta,m}$, *which is given by* (13).

2. Coefficient Bounds for the Bi-Concave Function Class $\mathcal{D}_r^{\delta,n,\alpha,\beta,m}\mathcal{C}_0(\gamma)$

Throughout this section, we assume that $\delta > 0$, $\alpha, \beta, m > 0$, $t \in \mathbb{N}$ and $n \in \mathbb{N}_0$.

Theorem 1. *If the function* $\mathcal{F}$ *given by* (1) *belongs to the class* $\mathcal{D}_r^{\delta,n,\alpha,\beta,m}\mathcal{C}_0(\gamma)$, *and if* $\gamma \in (1,2]$, *then*

$$|a_2| \leqq \min\left\{\sqrt{\frac{(\gamma+1)^2}{4\psi_2^2} + \frac{(\gamma-1)^2\left(|h'(0)|^2+|p'(0)|^2\right)}{32\psi_2^2} + \frac{(\gamma^2-1)(|h'(0)|+|p'(0)|)}{8\psi_2^2}}, \sqrt{\frac{(\gamma-1)(|h''(0)|+|p''(0)|)}{16(2\psi_2^2-3\psi_3)} + \frac{\gamma+1}{2(2\psi_2^2-3\psi_3)}}\right\} \tag{17}$$

and

$$|a_3| \leqq \min\left\{\frac{8(\gamma+1)^2+(\gamma-1)^2\left(|h'(0)|^2+|p'(0)|^2\right)}{32\psi_2^2} + \frac{(\gamma^2-1)(|h'(0)|+|p'(0)|)}{8\psi_2^2}, + \frac{(\gamma-1)(|h''(0)|+|p''(0)|)}{48\psi_3} \frac{(\gamma-1)(3\psi_3-\psi_2^2)|h''(0)|+(\gamma-1)\psi_2^2|p''(0)|}{24\psi_3(2\psi_2^2-3\psi_3)} + \frac{\gamma+1}{2(2\psi_2^2-3\psi_3)}\right\}, \tag{18}$$

where $\psi_t \ \ (t \in \{2,3\})$ *is given by* (9).

Proof. If $\mathcal{F} \in \mathcal{D}_r^{\delta,n,\alpha,\beta,m}\mathcal{C}_0(\gamma)$, then it follows from (15) and (16) that

$$\frac{2}{\gamma-1}\left(\frac{(\gamma+1)(1+z)}{2(1-z)} - 1 - \frac{z\left(\mathcal{D}_t^{\delta,n,b,c,m}\mathcal{F}(z)\right)''}{\left(\mathcal{D}_t^{\delta,n,b,c,m}\mathcal{F}(z)\right)'}\right) = h(z) \tag{19}$$

and

$$\frac{2}{\gamma-1}\left(\frac{(\gamma+1)(1+w)}{2(1-w)} - 1 - \frac{w\left(\mathcal{D}_t^{\delta,n,b,c,m}\mathcal{G}(w)\right)''}{\left(\mathcal{D}_t^{\delta,n,b,c,m}\mathcal{G}(w)\right)'}\right) = p(w), \tag{20}$$

where the functions h and p satisfy the conditions in Definition 1 and the function $\mathcal{G}$ in the assertion (20) is the analytic extension of $\mathcal{F}^{-1}$ to Λ, which is given by the Equation (3).

Furthermore, the functions $h(z)$ and $p(w)$ have the following Taylor-Maclaurin series expansions:

$$h(z) = 1 + h_1 z + h_2 z^2 + \cdots \tag{21}$$

and

$$p(w) = 1 + p_1 w + p_2 w^2 + \cdots, \tag{22}$$

respectively.

In view of the above Equations (21) and (22), we compare the coefficients of z and w in (19) and (20). We thus find that

$$\frac{2[(\gamma+1) - 2\psi_2 a_2]}{\gamma - 1} = h_1, \tag{23}$$

$$\frac{2\left[(\gamma+1) + 4\psi_2^2 a_2^2 - 6\psi_3 a_3\right]}{\gamma - 1} = h_2, \tag{24}$$

$$-\frac{2[(\gamma+1) - 2\psi_2 a_2]}{\gamma - 1} = p_1 \tag{25}$$

and

$$\frac{2\left[(\gamma+1) + 4\psi_2^2 a_2^2 - 6\psi_3\left(2a_2^2 - a_3\right)\right]}{\gamma - 1} = p_2. \tag{26}$$

Now, by using (23) and (25), we obtain

$$h_1 = -p_1. \tag{27}$$

From (23), we can write

$$a_2 = \frac{\gamma+1}{2\psi_2} - \frac{\gamma-1}{4\psi_2} h_1. \tag{28}$$

Upon squaring (23) and (25), if we add the resulting equations, we get

$$a_2^2 = \frac{(\gamma+1)^2}{4\psi_2^2} + \frac{(\gamma-1)^2\left(h_1^2 + p_1^2\right)}{32\psi_2^2} - \frac{\left(\gamma^2-1\right)}{8\psi_2^2}(h_1 - p_1) \tag{29}$$

Adding (24) and (26), we have

$$a_2^2 = \frac{(\gamma-1)(h_2 + p_2)}{8\left(2\psi_2^2 - 3\psi_3\right)} - \frac{\gamma+1}{2\left(2\psi_2^2 - 3\psi_3\right)}. \tag{30}$$

By taking the moduli in (29) and (30), we conclude that

$$|a_2| \leqq \sqrt{\frac{(\gamma+1)^2}{4\psi_2^2} + \frac{(\gamma-1)^2\left(|h'(0)|^2 + |p'(0)|^2\right)}{32\psi_2^2} + \frac{\left(\gamma^2-1\right)(|h'(0)| + |p'(0)|)}{8\psi_2^2}}$$

and

$$|a_2| \leqq \sqrt{\frac{(\gamma-1)(|h''(0)| + |p''(0)|)}{16\left(2\psi_2^2 - 3\psi_3\right)} + \frac{\gamma+1}{2\left(2\psi_2^2 - 3\psi_3\right)}},$$

which gives the bound for $|a_2|$ as we asserted in Theorem 1.

In order to find the bound for $|a_3|$, by subtracting (26) from (24), we get

$$a_3 = a_2^2 - \frac{(\gamma-1)(h_2 - p_2)}{24\psi_3}. \tag{31}$$

Also, upon substituting the value of a_2^2 from (29) and (30) into (31), we obtain

$$a_3 = \frac{(\gamma+1)^2}{4\psi_2^2} + \frac{(\gamma-1)^2(h_1^2+p_1^2)}{32\psi_2^2} - \frac{(\gamma^2-1)}{8\psi_2^2}(h_1-p_1) - \frac{(\gamma-1)(h_2-p_2)}{24\psi_3} \tag{32}$$

and

$$a_3 = \frac{(\gamma-1)(h_2+p_2)}{8(2\psi_2^2-3\psi_3)} - \frac{\gamma+1}{2(2\psi_2^2-3\psi_3)} - \frac{(\gamma-1)(h_2-p_2)}{24\psi_3}. \tag{33}$$

Finally, by taking the moduli in (32) and (33), we obtain

$$|a_3| \leqq \frac{8(\gamma+1)^2 + (\gamma-1)^2\left(|h'(0)|^2+|p'(0)|^2\right)}{32\psi_2^2} + \frac{(\gamma^2-1)(|h'(0)|+|p'(0)|)}{8\psi_2^2} + \frac{(\gamma-1)(|h''(0)|+|p''(0)|)}{48\psi_3} \tag{34}$$

and

$$|a_3| \leqq \frac{(\gamma-1)(3\psi_3-\psi_2^2)|h''(0)| + (\gamma-1)\psi_2^2|p''(0)|}{24\psi_3(2\psi_2^2-3\psi_3)} + \frac{\gamma+1}{2(2\psi_2^2-3\psi_3)}. \tag{35}$$

which completes the proof of Theorem 1. □

Putting $r = 0$ in Theorem 1, we obtain the following corollary.

Corollary 1. *If the function $\mathcal{F}$ given by (1) belongs to the class $\mathcal{R}^{n,\alpha,\beta,m}\mathcal{C}_0(\gamma)$, and if $\gamma \in (1,2]$, then*

$$|a_2| \leqq \min\left\{\left(\frac{[\beta(\gamma+1)F(\alpha;\beta;m)]^2}{2^{2(n+1)}\alpha^2m^2} + \frac{[\beta(\gamma-1)F(\alpha;\beta;m)]^2\left(|h'(0)|^2+|p'(0)|^2\right)}{2^{2(n+1)+3}\alpha^2m^2} + \frac{(\gamma^2-1)[\beta F(\alpha;\beta;m)]^2(|h'(0)|+|p'(0)|)}{2^{2(n+1)+1}\alpha^2m^2}\right)^{\frac{1}{2}}, \sqrt{\frac{8(\gamma+1)+(\gamma-1)(|h''(0)|+|p''(0)|)}{16\left[2^{n+1}\left(\frac{\alpha m}{\beta F(\alpha;\beta;m)}\right)^2 - 3^{n+1}\left(\frac{(\alpha)_2m^2}{2(\beta)_2\,F(\alpha;\beta;m)}\right)\right]}}\right\} \tag{36}$$

and

$$|a_3| \leqq \min\left\{\frac{[\beta(\gamma-1)F(\alpha;\beta;m)]^2}{2^{2(n+1)+3}\alpha^2m^2}\left[8(\gamma+1)^2 + (\gamma-1)^2\left(|h'(0)|^2+|p'(0)|^2\right) + 4\left(\gamma^2-1\right)(|h'(0)|+|p'(0)|)\right] + \frac{(\gamma-1)(\beta)_2\,F(\alpha;\beta;m)(|h''(0)|+|p''(0)|)}{(8)(3^{n+1})(\alpha)_2m^2}, \frac{(\gamma-1)\left[3^{n+1}\left(\frac{(\alpha)_2m^2}{2(\beta)_2\,F(\alpha;\beta;m)}\right) - 2^n\left(\frac{\alpha m}{\beta F(\alpha;\beta;m)}\right)^2\right]|h''(0)| + 2^n(\gamma-1)\left(\frac{\alpha m}{\beta F(\alpha;\beta;m)}\right)^2|p''(0)|}{8(3^{n+1})\left(\frac{(\alpha)_2m^2}{2(\beta)_2\,F(\alpha;\beta;m)}\right)\left[2^{n+1}\left(\frac{\alpha m}{\beta F(\alpha;\beta;m)}\right)^2 - 3^{n+1}\left(\frac{(\alpha)_2m^2}{2(\beta)_2\,F(\alpha;\beta;m)}\right)\right]} + \frac{\gamma+1}{2\left[2^{n+1}\left(\frac{\alpha m}{\beta F(\alpha;\beta;m)}\right)^2 - 3^{n+1}\left(\frac{(\alpha)_2m^2}{2(\beta)_2\,F(\alpha;\beta;m)}\right)\right]}\right\}. \tag{37}$$

Putting $n = 0$ in Theorem 1, we obtain the following corollary.

Corollary 2. *If the function $\mathcal{F}$ given by (1) belongs to the class $\mathcal{G}^{\alpha,\beta,m}\mathcal{C}_0(\gamma)$, and if $\gamma \in (1,2]$, then*

$$|a_2| \leqq \min\left\{\left(\frac{[c(\gamma+1)F(\alpha;\beta;m)]^2}{4b^2m^2} + \frac{[c(\gamma-1)F(\alpha;\beta;m)]^2\left(|h'(0)|^2+|p'(0)|^2\right)}{32b^2m^2} + \frac{(\gamma^2-1)[cF(\alpha;\beta;m)]^2(|h'(0)|+|p'(0)|)}{8b^2m^2}\right)^{\frac{1}{2}}, \sqrt{\frac{8(\gamma+1)+(\gamma-1)\left(|h''(0)|+|p''(0)|\right)}{16\left[2\left(\frac{bm}{cF(\alpha;\beta;m)}\right)^2-3\left(\frac{(b)_2m^2}{2(c)_2\,F(\alpha;\beta;m)}\right)\right]}}\right\} \tag{38}$$

and

$$|a_3| \leqq \min\left\{\frac{[\beta(\gamma-1)F(\alpha;\beta;m)]^2\left[8(\gamma+1)^2+(\gamma-1)^2\left(|h'(0)|^2+|p'(0)|^2\right)\right]}{32\alpha^2m^2} + \frac{[\beta(\gamma-1)F(\alpha;\beta;m)]^2\left[4(\gamma^2-1)(|h'(0)|+|p'(0)|)\right]}{32\alpha^2m^2} + \frac{(\gamma-1)(\beta)_2\,F(\alpha;\beta;m)(|h''(0)|+|p''(0)|)}{24(\alpha)_2m^2}, \frac{(\gamma-1)\left(3\left[\frac{(\alpha)_2m^2}{2(\beta)_2\,F(\alpha;\beta;m)}-\left(\frac{\alpha m}{\beta F(\alpha;\beta;m)}\right)^2\right]\right)|h''(0)|+(\gamma-1)\left(\frac{\alpha m}{\beta F(\alpha;\beta;m)}\right)^2|p''(0)|}{24\left(\frac{(\alpha)_2m^2}{2(\beta)_2\,F(\alpha;\beta;m)}\right)\left[2\left(\frac{\alpha m}{\beta F(\alpha;\beta;m)}\right)^2-3\left(\frac{(\alpha)_2m^2}{2(\beta)_2\,F(\alpha;\beta;m)}\right)\right]} + \frac{\gamma+1}{2\left[2\left(\frac{\alpha m}{\beta F(\alpha;\beta;m)}\right)^2-3\left(\frac{(\alpha)_2m^2}{2(\beta)_2\,F(\alpha;\beta;m)}\right)\right]}\right\}. \tag{39}$$

Remark 4. *The functions h and p, which we introduced in Definition 1 and used in Theorem 1, as well as in Corollarys 1 and 2 above, play a significant role in our investigation. For example, by appropriately specializing the functions h and p in Theorem 1, we are led to Corollarys 3 and 4 below.*

Corollary 3. *If the functions h and p are given by*

$$h(z) = \left(\frac{1+z}{1-z}\right)^{\alpha} = 1 + 2\alpha z + 2\alpha^2 z^2 + \cdots \qquad (0 < \alpha \leqq 1)$$

and

$$p(z) = \left(\frac{1-z}{1+z}\right)^{\alpha} = 1 - 2\alpha z + 2\alpha^2 z^2 - \cdots \qquad (0 < \alpha \leqq 1),$$

and if the function $\mathcal{F}$ given by (1) belongs to the class $\mathcal{D}_r^{\delta,n,\alpha,\beta,m}\mathcal{C}_0(\gamma)$ and $\gamma \in (1,2]$, then

$$|a_2| \leqq \min\left\{\sqrt{\frac{(\gamma+1)^2+(\gamma-1)^2\alpha^2+2(\gamma^2-1)\alpha}{4\psi_2^2}}, \sqrt{\frac{(\gamma+1)+(\gamma-1)\alpha^2}{2(2\psi_2^2-3\psi_3)}}\right\} \tag{40}$$

and

$$|a_3| \leqq \min\left\{ \frac{(\gamma+1)^2+(\gamma-1)^2\alpha^2+2(\gamma^2-1)\alpha}{4\psi_2^2}+\frac{(\gamma-1)\alpha^2}{6\psi_3}, \right. \\ \left. \frac{\left|(\gamma-1)(3\psi_3-\psi_2^2)\right|\alpha^2+(\gamma-1)\psi_2^2\alpha^2}{6\psi_3(2\psi_2^2-3\psi_3)}+\frac{\gamma+1}{2(2\psi_2^2-3\psi_3)} \right\}, \tag{41}$$

where ψ_t $(t \in \{2,3\})$ *is given by* (9).

Corollary 4. *If the functions h and p are given by*

$$h(z)=\frac{1+(1-2\sigma)z}{1-z}=1+2(1-\sigma)z+2(1-\sigma)z^2+\cdots \qquad (0 \leqq \sigma<1)$$

and

$$p(z)=\frac{1-(1-2\sigma)z}{1+z}=1-2(1-\sigma)z+2(1-\sigma)z^2+\cdots \qquad (0 \leqq \sigma<1),$$

and if the function $\mathcal{F}$ *given by* (1) *belongs to the class* $\mathcal{D}_r^{\delta,n,\alpha,\beta,m}\mathcal{C}_0(\gamma)$ *and* $\gamma \in (1,2]$, *then*

$$|a_2| \leqq \min\left\{ \sqrt{\frac{(\gamma+1)^2+(\gamma-1)^2(1-\sigma)^2+2(\gamma^2-1)(1-\sigma)}{4\psi_2^2}}, \right. \\ \left. \sqrt{\frac{(\gamma-1)(1-\sigma)-(\gamma+1)}{2(2\psi_2^2-3\psi_3)}} \right\} \tag{42}$$

and

$$|a_3| \leqq \min\left\{ \frac{8(\gamma+1)^2+(\gamma-1)^2\left(|h'(0)|^2+|p'(0)|^2\right)}{32\psi_2^2}+\frac{(\gamma^2-1)(|h'(0)|+|p'(0)|)}{8\psi_2^2} \right. \\ +\frac{(\gamma-1)(|h''(0)|+|p''(0)|)}{48\psi_3}, \frac{(\gamma-1)(3\psi_3-\psi_2^2)|h''(0)|+(\gamma-1)\psi_2^2|p''(0)|}{24\psi_3(2\psi_2^2-3\psi_3)} \\ \left. +\frac{\gamma+1}{2(2\psi_2^2-3\psi_3)} \right\}, \tag{43}$$

where ψ_t $(t \in \{2,3\})$ *is given by* (9).

3. Concluding Remarks and Observations

In our present investigation, we have introduced and studied the properties of some new subclasses of the class of analytic and bi-concave functions in the open unit disk Λ by using the combination of the binomial series and the confluent hypergeometric function. Among some other properties and results, we have derived the estimates on the initial Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions belonging to the bi-concave function classes, which are introduced in this paper. As some of the interesting consequences of our main results in Theorem 1, we have deduced a set of four corollaries. Each of these results is potentially useful in motivating further research on various other subclasses of the class of normalized analytic (or meromorphic) and univalent (or multivalent) bi-concave functions.

In this article, we have considered the coefficient estimate problems for the *initial* Taylor-Maclaurin coefficients for functions in some specified subclasses of the class of analytic and bi-concave functions in Λ. It would be of interest to investigate the applications of the Faber polynomial expansion method (see [25,26]) in order to tackle the coefficient estimate problems for the *general* Taylor-Maclaurin coefficients for these and other subclasses

of the class of analytic and bi-concave functions (see, for details, [27–29]; see also [30] and the references to the earlier literature on the subject, which are cited in each of these works).

Author Contributions: Conceptualization, H.M.S., S.M.E.-D. and L.-I.C.; data curation, H.M.S., S.M.E.-D., D.B. and G.S.S.; formal analysis, S.M.E.-D., L.-I.C. and D.B.; funding acquisition, S.M.E.-D. and L.-I.C.; investigation, H.M.S., S.M.E.-D. and L.-I.C.; methodology, H.M.S. and S.M.E.-D.; project administration, H.M.S., S.M.E.-D. and L.-I.C.; resources, S.M.E.-D.; software, S.M.E.-D.; supervision, H.M.S., D.B. and G.S.S.; validation, H.M.S., D.B. and G.S.S.; visualization, S.M.E.-D.; writing—original draft, S.M.E.-D.; writing—review and editing, H.M.S. and D.B. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Data are contained within the article.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. de Branges, L. A proof of the Bieberbach conjecture. *Acta Math.* **1985**, *154*, 137–152. [CrossRef]
2. Brannan, D.A.; Clunie, J.; Kirwan, W.E. Coefficient estimates for a class of star-like functions. *Can. J. Math.* **1970**, *22*, 476–485. [CrossRef]
3. Brannan, D.A.; Taha, T.S. On some classes of bi-univalent functions. In *Mathematical Analysis and Tts Applications (Kuwait; 18–21 February 1985)*; Mazhar, S.M., Hamoui, A., Faour, N.S., Eds.; KFAS Proceedings Series; Pergamon Press (Elsevier Science B.V.): Oxford, UK, 1988; Volume 3, pp. 53–60; see also *Studia Univ. Babeş-Bolyai Math.* **1986**, *31*, 70–77.
4. Netanyahu, E. The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in $|z| < 1$. *Arch. Ration. Mech. Anal.* **1969**, *32*, 100–112.
5. Srivastava, H.M.; Mishra, A.K.; Gochhayat, P. Certain subclasses of analytic and bi-univalent functions. *Appl. Math. Lett.* **2010**, *23*, 1188–1192. [CrossRef]
6. Barik, S.; Mishra, A.K. Estimates for initial coefficients of certain subclasses of bi-close-to-convex analytic functions. *Kragujev. J. Math.* **2023**, *47*, 387–401. [CrossRef]
7. Magesh, N.; Nirmala, J.; Yamini, J.; Swamy, S.R. Initial estimates for certain subclasses of bi-univalent functions with κ-Fibonacci numbers. *Afr. Mat.* **2023**, *34*, 35. [CrossRef]
8. Murugusundaramoorthy, G.; Cho, N.E.; Vijaya, K. A class of ϑ-bi-pseudo-starlike functions with respect to symmetric points associated with telephone numbers. *Afr. Mat.* **2024**, *35*, 17. [CrossRef]
9. Sabir, P.O. Some remarks for subclasses of bi-univalent functions defined by Ruscheweyh derivative operator. *Filomat* **2024**, *38*, 1255–1264.
10. Srivastava, H.M.; Sabir, P.O.; Atshan, W.G.; Mohammed, P.O.; Chorfi, N.; Vivas-Cortez, M. A family of holomorphic and m-fold symmetric bi-univalent functions endowed with coefficient estimate problems. *Mathematics* **2023**, *11*, 3970.
11. Vijaya, K.; Murugusundaramoorthy, G.; Güney, H.Ö. On λ-pseudo bi-starlike functions related with Fibonacci numbers. *Commun. Math.* **2024**, *32*, 117–127. [CrossRef]
12. Altinkaya, Ş. Bi-Concave functions defined by Al-Oboudi differential operator. *Iran. J. Math. Sci. Inform.* **2022**, *17*, 207–217. [CrossRef]
13. Altinkaya, Ş.; Yalçin, S. Some applications of generalized Srivastava-Attiya operator to the bi-concave functions. *Miskolc Math. Notes* **2020**, *21*, 51–60. [CrossRef]
14. Bayram, H.; Altinkaya, Ş. General properties of concave functions defined by the generalized Srivastava-Attiya operator. *J. Comput. Anal. Appl.* **2017**, *23*, 408–416.
15. Güney, H.Ö. Coefficient estimates for bi-concave functions of Sakaguchi type. *Acta Univ. Apulensis Math. Inform.* **2019**, *59*, 53–62.
16. Sakar, F.M.; Güney, H.Ö. Coefficient estimates for bi-concave Functions. *Commun. Fac. Sci. Univ. Ank. Ser. Math. Statist.*. **2019**, *68*, 53–60. [CrossRef]
17. Duren, P.L. *Univalent Functions, Grundlehren der Mathematischen Wissenschaften, Band 259*; Springer: New York, NY, USA; Berlin/Heidelberg, Germany; Tokyo, Japan, 1983.
18. Cruz, L.; Pommerenke, C. On Concave Univalent Functions. *Complex Var. Elliptic Equ.* **2007**, *52*, 153–159. [CrossRef]
19. Bhowmik, B.; Ponnusamy, S.; Wirths, K.-J. Characterization and the pre-Schwarzian norm estimate for concave univalent functions. *Monatsh. Math.* **2010**, *161*, 59–75. [CrossRef]
20. Rainville, E.D. *Special Functions*; The Macmillan Company: New York, NY, USA, 1960; Reprinted by Chelsea Publishing Company: New York, NY, USA, 1971.

21. Srivastava, H.M. A survey of some recent developments on higher transcendental functions of analytic number theory and applied mathematics. *Symmetry* **2021**, *13*, 2294. [CrossRef]
22. Porwal, S.; Kumar, S. Confluent hypergeometric distribution and its applications on certain classes of univalent functions. *Afr. Mat.* **2017**, *28*, 1–8. [CrossRef]
23. El-Deeb, S.M. On unified subclass of complex order connected with q-confluent hypergeometric distribution. *Surv. Math. Appl.* **2022**, *16*, 327–337.
24. Porwal, S. Confluent hypergeometric distribution and its applications on certain classes of univalent functions of conic regions. *Kyungpook Math. J.* **2018**, *58*, 495–505.
25. Airault, H.; Bouali, A. Differential calculus on the Faber polynomials. *Bull. Sci. Math.* **2006**, *130*, 179–222. [CrossRef]
26. Airault, H.; Ren, J. An algebra of differential operators and generating functions on the set of univalent functions. *Bull. Sci. Math.* **2002**, *126*, 343–367. [CrossRef]
27. Attiya, A.A.; Albalahi, A.M.; Hassan, T.S. Coefficient estimates for certain families of analytic functions associated with Faber polynomial. *J. Funct. Spaces* **2023**, *2023*, 4741056. [CrossRef]
28. Khan, S.; Altinkaya, Ş.; Xin, Q.; Tchier, F.; Khan, S.N.M.N. Faber Polynomial coefficient estimates for Janowski type bi-close-to-convex and bi-quasi-convex functions. *Symmetry* **2023**, *15*, 604. [CrossRef]
29. Srivastava, H.M.; Al-Shbeil, I.; Xin, Q.; Tchier, F.; Khan, S.; Malik, S.N. Faber polynomial coefficient estimates for bi-close-to-convex functions defined by the q-fractional derivative. *Axioms* **2023**, *12*, 585. [CrossRef]
30. Pommerenke, C. *Univalent Functions*; Vanderhoeck and Ruprecht: Göttingen, Germany, 1975.

Disclaimer/Publisher's Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.