symmetry

MDPI

Article

Pure Decoherence of the Jaynes–Cummings Model: Initial Entanglement with the Environment, Spin Oscillations and Detection of Non-Orthogonal States

Jerzy Dajka [1,2,3]

[1] Institute of Physics, University of Silesia in Katowice, 40-007 Katowice, Poland; jerzy.dajka@us.edu.pl
[2] The Professor Tadeusz Widła Interdisciplinary Research Centre for Forensic Science and Legislation, University of Silesia in Katowice, 40-007 Katowice, Poland
[3] Silesian Center for Education and Interdisciplinary Research, University of Silesia in Katowice, 41-500 Chorzow, Poland

Abstract: A model based on pure decoherence for the Jaynes–Cummings spin–boson system, coupled through its integral of motion to an infinite bosonic bath, is proposed and examined. The properties of the spin oscillation process suggest an initial entanglement between the environment and the spin–boson degrees of freedom. The study demonstrates that the potential applicability of the Jaynes–Cummings model in detecting non-orthogonal bosonic states is preserved in the presence of pure decoherence.

Keywords: Jaynes–Cummings model; pure decoherence; quantum state detection

check for updates

Citation: Dajka, J. Pure Decoherence of the Jaynes–Cummings Model: Initial Entanglement with the Environment, Spin Oscillations and Detection of Non-Orthogonal States. *Symmetry* **2024**, *16*, 250. https://doi.org/10.3390/sym16020250

Academic Editor: Luis L. Sánchez-Soto

Received: 21 January 2024
Revised: 14 February 2024
Accepted: 16 February 2024
Published: 18 February 2024

1. Introduction

The Jaynes–Cummings model [1] (JCM) serves as a natural starting point for describing matter–field interactions, being elementary yet far from trivial [2]. Since its inception, the JCM has garnered significant and sustained attention [3], and while its primary focus lies in quantum optical applications [4], its general form and structure render it highly effective across various scientific domains, spanning from quantum information [5,6] to quantum control [7] and the physics of open systems [8]. The JCM depicts a two-level quantum system (a spin or a qubit) interacting with a single bosonic mode of radiation. Despite its solvability [2,4,9], the model remains realistic enough to stimulate research in diverse fields, encompassing applied group theory [9–11], investigations related to quantum integrability [12,13], and extending to quantum information processing [14–16]. This text further explores a specific application of the Jaynes–Cummings model for detecting non-orthogonal quantum states, as proposed in Ref. [17]. The effectiveness of theoretical models, including Jaynes–Cummings modeling, in describing practical problems cannot be assumed unless their predictions remain robust against the natural and common imperfections inherent in real systems. One significant challenge is the omnipresent decoherence, which arises due to the interaction of a system with its environment, rendering it effectively "open" with respect to energy and information transfer. Numerous powerful models for describing open quantum systems exist [8,18–20], employing various problem-specific methods such as master Equations [18] or path integrals [20], to name a few. Striking a balance between mathematical rigor and physical soundness is, however, a non-trivial task [19,21]. The typical objective is to deduce, from the unitary evolution of the composite system (S) + environment (B), represented by $\rho_{SB}(t)$, a "reduced" evolution of the open subsystem $\rho_S(t) = \mathrm{Tr}B(\rho SB(t))$. This reduced evolution is determined by a non-unitary operator $\rho(t_j) = \Gamma(t_j, t_i)\rho(t_i)$, where $\rho(t_i)$ is the reduced density matrix of the evolving system [19]. It is crucial that the operator Γ satisfies minimal requirements; it must be completely positive [19,21,22] and adhere to the semi-group property, obeying a composition law: $\Gamma(t_2, t_0) = \Gamma(t_2, t_1)\Gamma(t_1, t_0)$. Notably, unless $\Gamma(t_j, t_i) = \Gamma(t_j - t_i)$, the time evolution is considered to be non-Markovian. Initial entanglement [23] of $\rho_{SB}(t=0)$ poses a well-known obstacle to the general construction of reduced dynamics meeting these natural requirements [19], explaining the continuous and considerable attention devoted to investigating this problem [24–31]. It is worth noting that quantum optical models are also eventually adapted to describe decoherence [8,9,32], and the Jaynes–Cummings model is no exception [33–37]. In addition to numerous approximate treatments of quantum systems affected by decoherence, a few exact models stand out due to their distinctive role in revealing the deep underlying symmetry of systems with significant physical consequences. One such model is pure decoherence [8,38,39] (dephasing), which describes a quantum system coupled to its environment via its integral of motion. The pure decoherence model enables the exploration of exact [38,39] reduced dynamics (traced with respect to the environment degrees of freedom) of an open system, going beyond Markovian or weak coupling approximations [19]. While pure decoherence or dephasing models may appear simple and somewhat artificial, they have proven to credibly describe realistic systems [40–43]. These models are studied extensively in the broader context of quantum information processing [44–47]. Notably, they provide a means to investigate the role played by initial system–environment correlation [25,26,48,49].

In this study, we present a modification of the Jaynes–Cummings model by introducing a pure decoherence framework. We assume that the decoherence-inducing infinite bosonic field (a bath) is coupled to the spin–boson Jaynes–Cummings (JC) system through its integral of motion, representing a conserved quantity—the total number of spin–boson excitations. This integral of motion, well-established for its central role in solving the JCM [9], is directly linked to the notion of integrability in quantum models [12,13]. The resulting pure decoherence model, formulated and analyzed using the coherent states technique, allows for exact predictions of the time evolution of the composite system, even for entangled initial states. Specifically, we explore spin oscillations and demonstrate that their collapses and revivals serve as a sensitive indicator of the initial entanglement between JC spin–boson degrees of freedom and the environment. Consequently, these properties can act as a hallmark, revealing the presence of such entanglement in the system's initial preparation. Furthermore, our investigation reveals that the properties of the spin oscillations can distinguish between bipartite entanglement (involving JC spin or JC boson and the environment) and genuinely tripartite entanglement. Additionally, we establish that the pure decoherence described by our proposed model does not hinder the potential application of the JCM [17] in detecting non-orthogonal states of bosonic modes, a common scenario in quantum communication.

The paper is structured as follows: First, we formulate the Jaynes–Cummings model with pure decoherence. The main results of the paper focus on the spin oscillation pattern in the presence of initial entanglement, categorized based on the entangled degrees of freedom: bipartite spin–environment, bipartite boson–environment, and genuine entanglement of JC degrees of freedom and the environment. We introduce an extension of an exact retrodiction method proposed in Ref. [17], utilizing the JCM to discriminate non-orthogonal states within a system affected by pure decoherence. Finally, we provide a summary and draw conclusions from our findings.

2. Materials and Methods: Pure Decoherence of the JCM

The Jaynes–Cummings model, describing the interaction of a single two-level system (spin) with a single-mode boson (electromagnetic field), is represented by the Hamiltonian (with $\hbar = 1$):

$$\hat{H}_{JC} \quad = \quad \omega\left(\hat{N}+\frac{1}{2}\right)+\frac{\Delta}{2}\hat{\sigma}_z+g\left(\hat{a}^\dagger\hat{\sigma}_-+\hat{a}\hat{\sigma}_+\right) \tag{1}$$

Here, $\hat{\sigma}_{z,+,-}$ and $\hat{a}, \hat{a}^\dagger$ correspond to the spin and boson (with $[\hat{a}, \hat{a}^\dagger] = 1$) degrees of freedom, respectively. The excitation number operator is given by

$$\hat{N} = \hat{a}^\dagger \hat{a} + \frac{\sigma_z}{2} \tag{2}$$

This operator is an integral of motion, as indicated by $[\hat{N}, \hat{H}_{JC}] = 0$, representing a conserved quantity. The dynamics of the system are influenced by the detuning parameter Δ and the spin–boson coupling strength g.

The spin–boson Jaynes–Cummings model given by Equation (1) necessitates an infinite-dimensional state space. This can be represented using bosonic number states $|n\rangle$, where $\hat{a}^\dagger \hat{a}|n\rangle = n|n\rangle$ with $n = 0, 1, \ldots$, and qubits $|k\rangle$, where $\hat{\sigma}_z|k\rangle = (2k-1)|k\rangle$ with $k = 0, 1$. The state space is then defined as $\mathcal{H} = \mathrm{span}|k\rangle \otimes |n\rangle$. However, due to the presence of the integral of motion given by Equation (2), the state space exhibits an isomorphic direct sum structure

$$\mathcal{H} = \mathbf{C}^1 \oplus \bigoplus_{N=1}^{\infty} \mathcal{H}_N \tag{3}$$

In this context, $\mathrm{span}|k=0\rangle \otimes |n=0\rangle \sim \mathbf{C}^1$, and each $\mathcal{H}N = \mathrm{span}|N,k\rangle, k = 0,1 \sim \mathbf{C}^2$ is two-dimensional. It is spanned by the so-called bare states [9] $|N,k\rangle := |k\rangle|n-k\rangle$, corresponding to eigenspaces of the integral of motion $\hat{N}|N,k\rangle = \left(N - \frac{1}{2}\right)|N,k\rangle$. Diagonalizing $\hat{H}JC$ within each subspace $\mathcal{H}N$ allows the construction of a basis of dressed states $|Nk\rangle$, where

$$\hat{H}_{JC}|N_k\rangle = \Lambda_{N,k}|N_k\rangle, \;\; k = 0, 1 \tag{4}$$

Here, $\Lambda_{N,k} = \omega N + (2k-1)\Omega_N$ with $\Omega_N^2 = g^2 N + \Delta^2/4$. A unitary equivalence between dressed and bare states can be expressed as follows:

$$|N_0\rangle = y_N|N,0\rangle + x_N|N,1\rangle \tag{5}$$

$$|N_1\rangle = x_N|N,0\rangle - y_N|N,1\rangle \tag{6}$$

with coefficients given by $x_N = g\sqrt{N}/\sqrt{\left(\Omega_N - \frac{\Delta}{2}\right)^2 + g^2 N}$ and $y_N = \left(\Omega_N - \frac{\Delta}{2}\right)/\sqrt{\left(\Omega_N - \frac{\Delta}{2}\right)^2 + g^2 N}$.

To describe pure decoherence [38,39,45] of the JCM, we assume that the spin–boson model (JC) given by Equation (1) is coupled to its environment (B) via the integral of motion $\hat{N}$ in Equation (2) with a coupling strength κ

$$\hat{H} = \hat{H}_{JC} + \hat{H}_B + \kappa \hat{N} \otimes \hat{A}_B \tag{7}$$

Among infinite environments, we consider the simplest one which is a one-dimensional bosonic field with a continuous energy spectrum given by an integrable and non–negative $h(q)$

$$\hat{H}_B = \int_0^\infty dq h(q) \hat{a}^\dagger(q) \hat{a}(q) \tag{8}$$

$$\hat{A}_B = \int_0^\infty dq g(q) \left(\hat{a}(q)^\dagger + \hat{a}(q)\right) \tag{9}$$

Here, the bosonic modes satisfy the commutation relation $[\hat{a}(q), \hat{a}^\dagger(q')] = i\delta(q - q')$. The field B is coupled to the JC system with a strength determined by a continuous and integrable function $g(q)$.

It is worth noting that while the Hamiltonian in Equation (7) exhibits high symmetry, it takes a standard Caldeira–Leggett form [8]. The proposed decoherence mechanism in

Equation (7) can be seen as a generalization of well-known pure dephasings. The coupling operator $\hat{N}$ given by Equation (2) represents the sum of the spin z-component [8,38,39,41,42] and the bosonic number operator [40,43]. These operators have a well-established physical meaning in the context of pure decoherence.

To describe the pure decoherence of the Jaynes–Cummings model (JCM), we introduce an environment (B) coupled to the JCM via the integral of motion $\hat{N}$ given by Equation (2). The coupled Hamiltonian is expressed as follows:

$$\hat{H} = \sum_{N=1}^{\infty} \sum_{k=0,1} \hat{H}_{N,k} |N_k\rangle\langle N_k| \tag{10}$$

$$\hat{H}_{N,k} = \Lambda_{N,k} \mathbf{1}_B + \hat{H}_B + \kappa\left(N - \frac{1}{2}\right)\hat{A}_B \tag{11}$$

We note that the term corresponding to $n = 0$ has been omitted as it does not lead to non-trivial evolution. The block-diagonal structure of Equation (10) is inherited by the corresponding time-evolution unitary operator

$$\hat{U}(t) = \exp(-iHt)) = \sum_{N=1}^{\infty} \sum_{k=0,1} \exp(-i\hat{H}_{N,k}t)|N_k\rangle\langle N_k| \tag{12}$$

which is of particular usefulness if it is applied jointly with the coherent states technique [39,45,50]. Coherent states $|F\rangle$ of a bosonic field are given by the displaced vacuum $|F\rangle = D(F)|V\rangle$, where the displacement operator [51,52] is defined as follows:

$$D(F) = \exp\left[\int_0^{\infty} dq(F(q)\hat{a}^{\dagger}(q) - h.c.)\right] \tag{13}$$

$$D(G)D(F) = \exp\left[i\int_0^{\infty} dqG(q)\bar{F}(q)\right]D(F+G) \tag{14}$$

Coherent states are overcomplete and non-orthogonal with a scalar product:

$$\langle G|F\rangle = \exp\left[\int_0^{\infty} dq[\bar{G}(q)F(q) - \frac{1}{2}|G(q)|^2 - \frac{1}{2}|F(q)|^2]\right] \tag{15}$$

This scalar product quantifies their overlap. The dynamic properties of coherent states [39,51] allow for the calculation of

$$\hat{U}(t)|N_k\rangle \otimes |F\rangle = |N_k\rangle \otimes |F^t_{N,k}\rangle, \quad k = 0,1 \tag{16}$$

$$|F^t_{N,k}\rangle = e^{-i\Lambda_{N,k}t}\hat{U}_N(t)D(F)|V\rangle \tag{17}$$

where, for the pure decoherence model, the unitary operator $\hat{U}_N$ can be explicitly evaluated and reads as follows:

$$\hat{U}_N(t)D(F)|V\rangle = e^{-i\Phi_N(t)}D\left(\frac{\kappa(N-\frac{1}{2})g(q)}{h(q)}(1-e^{-ih(q)t}) + \frac{F(q)}{h(q)}e^{-ih(q)t}\right)|V\rangle \tag{18}$$

$$\Phi_N(t) = \int_0^{\infty} dq\left[g^2(q)((h(q)t - \sin(h(q)t)) + 2g(q)F(q)\sin(h(q)t)\right] \tag{19}$$

3. Results

3.1. Entanglement-Assisted Spin Collapses and Revivals

Collapses and revivals of spin (atomic) oscillations, represented by $\langle\hat{\sigma}_z\rangle$, are among the most studied features of the Jaynes–Cummings model [2,4,9]. Here, we explore how initial entanglement in the pure-decoherence-assisted JCM affects the properties of spin inversion. It is noteworthy that in the absence of initial system–environment correlations, the pure decoherence given in Equation (7) does not modify the well-known spin inversion properties. To confirm this, let us consider a fully separable initial preparation of the JC spin–boson model and the environment in the vacuum state $|V\rangle$. The initial state of the system, considering a fully separable preparation, is given by

$$|\psi(0)\rangle \quad = \quad |0,z\rangle \otimes |V\rangle \tag{20}$$

Here, the spin is in its ground state $|0\rangle$, and the boson is in a coherent state [51] $|z\rangle = \sum_{N=0}^{\infty} z_N |N\rangle$, where $z_N = e^{-|z|^2/2} z^N / \sqrt{N!}$. Expressing the spin–boson initial state in terms of the dressed states given by Equation (5), it reads as follows:

$$|0,z\rangle \quad = \quad z_0|0,0\rangle + \frac{1}{\sqrt{2}} \sum_{N=1}^{\infty} z_N [|N_0\rangle + |N_1\rangle] \tag{21}$$

After applying Equation (12) and Equation (16), the state evolves in time as follows:

$$|\psi(t)\rangle \quad = \quad z_0|0,0\rangle \otimes |V\rangle + \frac{1}{\sqrt{2}} \sum_{N=1}^{\infty} z_N \sum_{k=0,1} \left[|N_k\rangle \otimes |V_{N,k}^t\rangle\right] \tag{22}$$

Despite the non-trivial effect of pure decoherence in Equation (22), the corresponding spin inversion in

$$\begin{aligned} \langle\hat{\sigma}_z\rangle(t) \quad &= \quad \langle\psi(t)|\hat{\sigma}_z|\psi(t)\rangle \\ &= \quad -|z_0|^2 - \sum_{N=1}^{\infty} |z_N|^2 \cos(2\Omega_N t) \end{aligned} \tag{23}$$

remains unmodified compared to the known decoherence-free formula [32]. This also holds true for a spin initially prepared in its excited state:

$$|1,z\rangle \quad = \quad \frac{1}{\sqrt{2}} \sum_{N=1}^{\infty} z_{N-1} [|N_0\rangle + |N_1\rangle] \tag{24}$$

An originally bipartite spin–boson model, extended by the presence of an environment causing its decoherence, becomes essentially tripartite. In doing so, three distinct classes of initial preparations are recognized, classified with respect to initial entanglement with an environment. The first two classes consist of bipartite entangled states with initial spin–environment or boson–environment entanglement but separable concerning a third part of the composite, which is either the boson or the spin, respectively. The third class comprises states with genuine tripartite initial entanglement, meaning that none of the parts of the composite can be separated (traced) without unavoidable information loss [53].

3.1.1. Boson–Environment Initial Entanglement

Let us consider a decoherence-assisted Jaynes–Cummings model (JCM) with an initial state entangled in bosonic and environmental degrees of freedom only. We examine a family of entangled initial preparations given by

$$|\psi(0)\rangle \quad = \quad \frac{(1-\alpha)}{\mathcal{N}} |0,z\rangle \otimes |V\rangle + \frac{\alpha}{\mathcal{N}} |0,s\rangle \otimes |F\rangle \tag{25}$$

Here, α is a parameter in the range $[0,1]$, and the normalization factor is $\mathcal{N}^2 = (1-\alpha)^2 + \alpha^2 + 2\alpha(1-\alpha)\Re(\langle z|s\rangle\langle V|F\rangle)$. The states in Equation (25) are entangled with respect to boson–environment degrees of freedom. They represent entangled states of boson coherent states $|z\rangle$ and $|s\rangle$ with coherent states of the environment $|V\rangle$ (the vacuum) and $|F\rangle = D(F)|V\rangle$. The spin part of the composite system in Equation (25) is separated and given in its ground state $|0\rangle$.

It is worth noting that since coherent states are not orthogonal, the bipartite entanglement in Equation (25) is never maximal [23,53], unlike the celebrated Bell states [5]. In contrast to the decoherent JCM previously studied in the absence of initial correlations (Equation (20)), there is a non-trivial modification of spin inversion in $\langle\hat{\sigma}_z\rangle$ due to the initial entanglement in Equation (25). The expectation value of $\hat{\sigma}_z$ reads as follows:

$$\begin{aligned}\langle\hat{\sigma}_z\rangle(t) &= -\frac{(1-\alpha)^2}{\mathcal{N}^2}|z_0|^2 - \frac{\alpha^2}{\mathcal{N}^2}|s_0|^2 - 2\frac{\alpha(1-\alpha)}{\mathcal{N}^2}\Re\langle V|F\rangle \\ &- \frac{2}{\mathcal{N}^2}\sum_{N=1}^{\infty}\cos(2\Omega_N(t))\Big[(1-\alpha)^2|z_N|^2 + \alpha^2|s_N|^2 + 2\alpha(1-\alpha)\bar{z}_N s_N \Re\langle V|F\rangle\Big] \end{aligned} \quad (26)$$

A qualitative modification of the spin oscillations given in Equation (26) is directly related to the value of the parameter α as presented in Figure 1. The spin inversion, quantified by $\langle\hat{\sigma}_z\rangle(t)$ as a function of time t, is shown. It is evident that an increase in α leads to an increase in both the amplitude of oscillations and the time-averaged value of $\langle\hat{\sigma}_z\rangle$, i.e.,

$$\langle\langle\hat{\sigma}_z\rangle\rangle = \lim_{T\to\infty} T^{-1}\int_0^T \langle\hat{\sigma}_z\rangle(t)dt \quad (27)$$

which is a function of α with a maximal value for $\alpha = 1/2$. It is important to note that for $s \neq z$, there is a shift in the first oscillation revival, which appears earlier (later) for $|s| < |z|$ (for $|s| > |z|$, respectively). An impact of a choice of coherent states entering Equation (25) for the boson in the JCM is presented in Figure 2. However, the time average $\langle\langle\hat{\sigma}_z\rangle\rangle$ remains unchanged. Coherent states of the environment entering Equation (25) manifest themselves in Equation (26) solely via the real part of their scalar product, resulting in changes that are qualitatively the same as those induced by $\alpha \neq 0$.

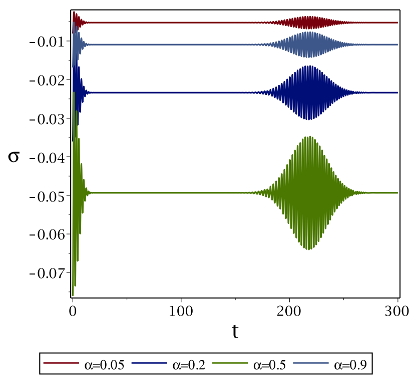

Figure 1. Expectation value $\sigma \equiv \langle\hat{\sigma}_z\rangle$ of spin and its oscillations as a function of time t given in Equation (26) for different values of α. The other parameters are $z = 6$, $s = 8$ and $\Re\langle V|F\rangle = 1/10$.

3.1.2. Spin–Environment Initial Entanglement

A second natural class of initially entangled states to be considered is given by the states as follows:

$$|\psi(0)\rangle = \frac{(1-\alpha)}{\mathcal{N}}|0,z\rangle \otimes |V\rangle + \frac{\alpha}{\mathcal{N}}|1,z\rangle \otimes |F\rangle \quad (28)$$

where $\mathcal{N}^2 = (1-\alpha)^2 + \alpha^2 + 2\alpha(1-\alpha)\Re(\langle V|F\rangle)$. In this case, the spin degree of freedom in Equation (28) is entangled with the environment. Note, however, that the bosonic part of the JC pair in a coherent state $|z\rangle$ remains separated.

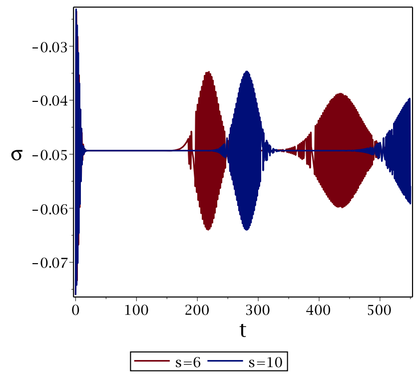

Figure 2. Expectation value $\sigma \equiv \langle\hat{\sigma}_z\rangle$ of spin and its oscillations as a function of time t given in Equation (26) for different values of s. The other parameters are $\alpha = 1/2$, $z = 8$ and $\Re\langle V|F\rangle = 1/10$

The spin oscillations can be inferred from an expectation value as follows:

$$\begin{aligned}\langle\hat{\sigma}z\rangle(t) &= -\frac{(1-\alpha)^2}{\mathcal{N}^2}|z_0|^2 \\ &- \frac{2}{\mathcal{N}^2}\sum_{N=1}^{\infty}\cos(2\Omega_N(t))[(1-\alpha)^2|z_N|^2+\alpha^2|z_{N-1}|^2 \\ &+ 2\alpha(1-\alpha)\bar{z}Nz(N-1)\Re\langle V|F\rangle]\end{aligned} \tag{29}$$

which is calculated for the initial state Equation (28). An influence of α on the properties of the spin inversion is presented in Figures 3 and 4. Time instants when the revivals of spin oscillations occur remain unchanged for different values of α. There is also a significant impact of α on the amplitude of spin oscillations, which is maximal for maximal entanglement, i.e., for $\alpha = 1/2$. However, there is a substantial difference indicated in a time-average $\langle\langle\hat{\sigma}_z\rangle\rangle$, which, contrary to the previously studied boson–environment initial entanglement Equation (25), does not depend on α for the initial preparation given in Equation (28).

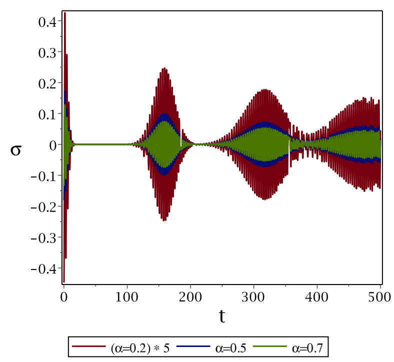

Figure 3. Expectation value $\sigma \equiv \langle\hat{\sigma}_z\rangle$ of spin and its oscillations as a function of time t given in Equation (29) for different values of α. The other parameters are $z = 6$ and $\Re\langle V|F\rangle = 1/10$. For the sake of clear presentation, the graph corresponding to $\alpha = 0.2$ is multiplied by a factor of 5.

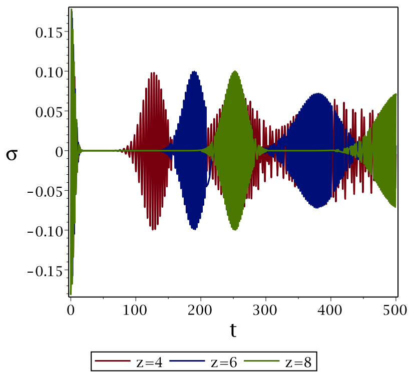

Figure 4. Expectation value $\sigma \equiv \langle \hat{\sigma}_z \rangle$ of spin and its oscillations as a function of time t given in Equation (29) for different values of z. The other parameters are $\alpha = 1/2$ $\Re\langle V|F\rangle = 1/10$.

3.1.3. 'Genuine' Initial Entanglement

For the sake of completeness, let us consider a fully three-partite entanglement of the spin–boson–environment system. This is a third type of multipartite entanglement which is 'genuine' since none of the possible bipartitions can lead to a separable state.

We consider an initial state

$$|\psi(0)\rangle = \frac{(1-\alpha)}{\sqrt{2}\mathcal{N}}|0,z\rangle \otimes |V\rangle + \frac{\alpha}{\sqrt{2}\mathcal{N}}|1,s\rangle \otimes |F\rangle \tag{30}$$

where $\mathcal{N}^2 = (1-\alpha)^2 + \alpha^2 + 2\alpha(1-\alpha)\Re(\langle z|s\rangle\langle V|F\rangle)$. Let us note, however, that despite the seemingly complicated form of a related expectation value of the spin,

$$\begin{aligned} \langle \hat{\sigma} z\rangle(t) &= -\frac{(1-\alpha)^2}{\mathcal{N}^2}|z_0|^2 - \frac{\alpha^2}{\mathcal{N}^2}|s_0|^2 - 2\frac{\alpha(1-\alpha)}{\mathcal{N}^2}\Re\langle V|F\rangle \\ &- \frac{2}{\mathcal{N}^2}\sum_{N=1}^{\infty}\cos(2\Omega_N(t))[(1-\alpha)^2|z_N|^2 + \alpha^2|s_{N-1}|^2 \\ &+ 2\alpha(1-\alpha)\bar{z}Ns(N-1)\Re\langle V|F\rangle], \end{aligned} \tag{31}$$

this expression captures the time-dependent behavior of the spin in the presence of genuine tripartite entanglement.

The spin oscillation properties, as described by the expectation value, appear to be qualitatively similar for the previously discussed spin–environment case and the current spin–boson–environment case. In other words, based solely on the spin oscillation properties, it is not possible to distinguish whether the initial entangled state is of the form given in Equation (28), i.e., bipartite, or Equation (30), i.e., genuinely tripartite.

3.2. *(Pure) Decoherence-Assisted Retrodiction*

The retrodiction method proposed in Ref. [17] utilizing the Jaynes–Cummings model involves using a spin variable as a probe to measure and postselect, aiming to achieve orthogonality of the reduced signal (bosonic) states. In the context of quantum communication [54–56], particularly with non-orthogonal pairs of coherent states $\mathcal{A} = \{|z\rangle, |-z\rangle\}$ forming an alphabet for encoding and transmitting messages, the Jaynes–Cummings model provides an effective and exact retrodiction method. As the states in $\mathcal{A}$ are not orthogonal $\langle z|-z\rangle \neq 0$, they cannot be with certainty distinguished. Receiving a signal encoded with $\mathcal{A}$, one is faced with a retrodictive decision of which of the two letters has been sent [57,58]. Here, the Jaynes–Cummings model turns out to provide an effective and exact retrodiction method [17]. The idea behind this is to utilize a spin variable of the JCM as a probe then measure it and postselect providing, after a time of interaction, orthogonality of the reduced signal (bosonic) states. Here, we explore an effect of pure decoherence Equation (7) on the potential ability of the JCM-based exact retrodiction proposed in Ref. [17]. We assume that initially the signal (boson) and the probe (spin) are separated from their environment (being in its vacuum state) and their state reads as follows:

$$|\psi(0)\rangle \quad = \quad |1,z\rangle \otimes |V\rangle \tag{32}$$

Due to interactions, applying Equation (12) and Equation (16), the signal+probe+environment system evolves in time

$$|\psi(t)\rangle \quad = \quad \frac{1}{\sqrt{2}} \sum_{N=1}^{\infty} z_{N-1} \sum_{k=0,1} \left[|N_k\rangle \otimes |V^t_{N,k}\rangle \right] \tag{33}$$

An amplitude that postselecting the probe+environment state $|i\rangle\langle i| \otimes |V\rangle\langle V|$ (with either $i = 0$ or $i = 1$), the signal is in the $|-z\rangle$ state and reads as follows:

$$K(i) \quad = \quad \frac{\langle -z|i \otimes V\rangle\langle i \otimes V|\psi\rangle}{|\langle i \otimes V|\psi\rangle|} = \frac{\exp(-|z|^2)}{P(i)} \Xi(i) \tag{34}$$

with

$$P(i) \quad = \quad \exp(-|z|^2) \sum_{N=0}^{\infty} f_i\left(gt\sqrt{N+1}\right) \frac{|z|^{2N}}{N!} \tag{35}$$

and

$$\Xi(i,t) \quad = \quad \sum_{N=0}^{\infty} (-1)^N f_i\left(gt\sqrt{N+1}\right) \xi(N) \frac{|z|^{2N}}{N!} \tag{36}$$

where $f_1(\cdot) = \cos(\cdot)$, $f_0(\cdot) = \sin(\cdot)$ and $\xi(N) = |\langle V|\hat{U}_N(t)|V\rangle|$. An exact retrodiction [17] is possible provided that $\Xi(i,t) = 0$ for some $t > 0$. Let us note that an effect of decoherence enters the amplitude $K(i)$ Equation (34) via Equation (36) only, whereas $P(i)$ remains not altered in comparison with the decoherence-free archetype, cf. Ref. [17]. As the pure decoherence modifies nothings but the magnitudes of summands in Equation (36) and $|\xi(N)| \leq 1$, one concludes that an effect of pure decoherence does not exclude vanishing of $K(i)$ and, in consequence, the exact retrodiction method of Ref. [17] is, in a presence of pure decoherence Equation (7), not obstructed. However, it changes if an arriving signal Equation (32) becomes modified by an unwanted initial entanglement with either a probe or its environment as an existence of roots $K(i), i = 0, 1$, for $t > 0$ cannot then in general be proved.

4. Discussion

We have developed a model to describe the impact of pure decoherence on the Jaynes–Cummings system. Our model assumes that the environment inducing decoherence couples to the spin–boson degrees of freedom of the Jaynes–Cummings model (JCM) through the number of excitation operators, a known integral of motion of the JCM. Despite its simplicity, the proposed model allows for a non-approximate and explicit solution for its time evolution, leading to exact predictions of observable effects. The model retains its solvability and exactness even in the presence of initial entanglement between the decohering environment and the JCM degrees of freedom. We examined three classes of entangled states characterized by the involved degrees of freedom: spin–environment entanglement, boson–environment entanglement, and genuine entanglement, involving all the JCM degrees of freedom in the presence of pure decoherence. Our analysis revealed that the process of spin inversion, typically studied in the context of collapses and revivals, is not only sensitive to the presence of initial entanglement but also enables discrimination between different types of entanglement in the initial state. Notably, we found qualitative differences in the properties of decoherence-assisted spin oscillations for bipartite spin–environment initial entanglement compared to a system initially in a state with entanglement between the JC boson and the environment. As a potential application of our pure decoherence model in the JCM, we explored its role in addressing a central problem in quantum communication: the detection of non-orthogonal states transmitted as letters of an alphabet encoding a message. Drawing on a method proposed in Ref. [17], we demonstrated that the Jaynes–Cummings interaction between the signal-carrying boson and a spin acting as a probe, along with postselection of the probe state, allows for the distinguishability of signal states. Importantly, this potential ability is, in principle, maintained for the decoherence model proposed in this paper, at least in the absence of entanglement with the decohering environment. The other possible applications of the results of the paper are related to the Hutner–Barnett model for polaritons [59] to describe spontaneous emission of a two-level atom in a dissipative environment. An application of that kind of modeling in the context of spontaneous emission is given, e.g., in Ref. [60].

While we acknowledge that pure decoherence is an approximation of real open systems influenced by dissipation, the idealistic nature of our model does not diminish its significance. The exact solvability and well-defined microscopic origin of our considered model can mitigate its limitations. We hope that our results can serve as a foundational approximation of the properties of general open spin–boson models, which are otherwise challenging to obtain. Moreover, our work may act as a potential benchmark for other, more general methods crucial for quantum information processing. The sensitivity of spin inversion to both the presence of initial entanglement and the type of entanglement in the initial state underscores the intriguing potential of our proposed model. Additionally, the application of our model to quantum communication showcases the versatility of the Jaynes–Cummings model, offering insights into maintaining distinguishability of signal states even in the presence of pure decoherence.

Overall, our work contributes to the understanding of the interplay between decoherence, entanglement, and quantum dynamics in the Jaynes–Cummings model, offering insights that can guide future research in this field.

Funding: This research received no external funding.

Data Availability Statement: Data are contained within the article.

Conflicts of Interest: The author declares no conflicts of interest.

References

1. Jaynes, E.; Cummings, F. Comparison of quantum and semiclassical radiation theories with application to the beam maser. *Proc. IEEE* **1963**, *51*, 89–109. [CrossRef]
2. Shore, B.W.; Knight, P.L. The Jaynes-Cummings Model. *J. Mod. Opt.* **1993**, *40*, 1195–1238. [CrossRef]
3. Greentree, A.D.; Koch, J.; Larson, J. Fifty years of Jaynes–Cummings physics. *J. Phys. At. Mol. Opt. Phys.* **2013**, *46*, 220201. [CrossRef]
4. Gerry, C.; Knight, P. *Introductory Quantum Optics*; Cambridge University Press: Cambridge, UK, 2004. [CrossRef]
5. Nielsen, M.A.; Chuang, I.L. *Quantum Computation and Quantum Information*; Cambridge University Press: Cambridge, UK, 2010.
6. Krantz, P.; Kjaergaard, M.; Yan, F.; Orlando, T.P.; Gustavsson, S.; Oliver, W.D. A quantum engineer's guide to superconducting qubits. *Appl. Phys. Rev.* **2019**, *6*, 021318. [CrossRef]
7. Jacobs, K. *Quantum Measurement Theory and Its Applications*; Cambridge University Press: Cambridge, UK, 1999.
8. Breuer, H.P.; Petruccione, F. *The Theory Od Open Quantum Systems*; Oxford University Press: Cambridge, UK, 2003.
9. Klimov, A.B.; Chumakov, S.M. *A Group-Theoretical Approach to Quantum Optics*; Wiley-VCH: Weinheim, Germany, 2009.
10. Chaichian, M.; Ellinas, D.; Kulish, P. Quantum algebra as the dynamical symmetry of the deformed Jaynes-Cummings model. *Phys. Rev. Lett.* **1990**, *65*, 980–983. [CrossRef] [PubMed]
11. Ruiz, A.M.; Frank, A.; Urrutia, L.F. AnSU(2)⊗SU(2) Jaynes–Cummings model with a maximum energy level. *Phys. Scr.* **2014**, *89*, 045103. [CrossRef]
12. Skrypnyk, T. Integrability and superintegrability of the generalized n-level many-mode Jaynes–Cummings and Dicke models. *J. Math. Phys.* **2009**, *50*, 103523. [CrossRef]

13. Carinena, J.F.; De Lucas, J. Quantum Lie systems and integrability conditions. *Int. J. Geom. Methods Mod. Phys.* **2009**, *6*, 1235–1252. [CrossRef]
14. Fasihi, M.A.; Mojaveri, B. Entanglement protection in Jaynes–Cummings model. *Quantum Inf. Process.* **2019**, *18*, 75. [CrossRef]
15. Quesada, N.; Sanpera, A. Bound entanglement in the Jaynes–Cummings model. *J. Phys. At. Mol. Opt. Phys.* **2013**, *46*, 224002. [CrossRef]
16. Raja, S.H.; Mohammadi, H.; Akhtarshenas, S.J. Geometric discord of the Jaynes-Cummings model: Pure dephasing regime. *Eur. Phys. J. D* **2015**, *69*, 14. [CrossRef]
17. Sasaki, M.; Usuda, T.S.; Hirota, O.; Holevo, A.S. Applications of the Jaynes-Cummings model for the detection of nonorthogonal quantum states. *Phys. Rev. A* **1996**, *53*, 1273–1279. [CrossRef]
18. Rivas, A.; Huelga, S.F. *Open Quantum Systems. An introduction*; Springer: Berlin/Heidelberg, Germany, 2012. [CrossRef]
19. Alicki, R.; Lendi, K. *Quantum Dynamical Semigroups and Applications*; Springer: Berlin/Heidelberg, Germany, 2007.
20. Zaikin, A.D.; Golubev, D.S. *Dissipative Quantum Mechanics of Nanostructures*; CRC Press: Boca Raton, FL, USA, 2019. [CrossRef]
21. Davies, E.B. *Quantum Theory of Open Systems*; Academic Press: London, UK, 1976.
22. Benatti, F.; Floreanini, R. Open quantum dynamics: Complete positivity and entanglement. *Int. J. Mod. Phys. B* **2005**, *19*, 3063–3139. [CrossRef]
23. Horodecki, R.; Horodecki, P.; Horodecki, M.; Horodecki, K. Quantum entanglement. *Rev. Mod. Phys.* **2009**, *81*, 865–942. [CrossRef]
24. Grabert, H.; Schramm, P.; Ingold, G.L. Quantum Brownian motion: The functional integral approach. *Phys. Rep.* **1988**, *168*, 115–207. [CrossRef]
25. Dajka, J.; Łuczka, J. Distance growth of quantum states due to initial system-environment correlations. *Phys. Rev. A* **2010**, *82*, 012341. [CrossRef]
26. Ban, M. Quantum master equation for dephasing of a two-level system with an initial correlation. *Phys. Rev. A* **2009**, *80*, 064103. [CrossRef]
27. Paz-Silva, G.A.; Hall, M.J.W.; Wiseman, H.M. Dynamics of initially correlated open quantum systems: Theory and applications. *Phys. Rev. A* **2019**, *100*, 042120. [CrossRef]
28. Chen, C.C.; Goan, H.S. Effects of initial system-environment correlations on open-quantum-system dynamics and state preparation. *Phys. Rev. A* **2016**, *93*, 032113. [CrossRef]
29. Chaudhry, A.Z.; Gong, J. Role of initial system-environment correlations: A master equation approach. *Phys. Rev. A* **2013**, *88*, 052107. [CrossRef]
30. Kitajima, S.; Ban, M.; Shibata, F. Expansion formulas for quantum master equations including initial correlation. *J. Phys. Math. Theor.* **2017**, *50*, 125303. [CrossRef]
31. Alipour, S.; Rezakhani, A.T.; Babu, A.P.; Mølmer, K.; Möttönen, M.; Ala-Nissila, T. Correlation-Picture Approach to Open-Quantum-System Dynamics. *Phys. Rev. X* **2020**, *10*, 041024. [CrossRef]
32. Saavedra, C.; Klimov, A.B.; Chumakov, S.M.; Retamal, J.C. Dissipation in collective interactions. *Phys. Rev. A* **1998**, *58*, 4078–4086. [CrossRef]
33. Fujii, K.; Suzuki, T. An approximate solution of the jaynes—Cummings model with dissipation. *Int. J. Geom. Methods Mod. Phys.* **2011**, *8*, 1799–1814. [CrossRef]
34. Fujii, K.; Suzuki, T. An approximate solution of the jaynes—Cummings model with dissipation ii: Another approach. *Int. J. Geom. Methods Mod. Phys.* **2012**, *9*, 1250036. [CrossRef]
35. Gangopadhyay, G.; Lin, S.H. The effect of pure decoherence on the Jaynes-Cummings model. *Phys. Scr.* **1997**, *55*, 425–430. [CrossRef]
36. Law, C.K.; Chen, T.W.; Leung, P.T. Jaynes-Cummings model in leaky cavities: An exact pure-state approach. *Phys. Rev. A* **2000**, *61*, 023808. [CrossRef]
37. Kitajima, S.; Ban, M.; Shibata, F. A solvable dissipative Jaynes–Cummings model with initial correlation. *J. Phys. At. Mol. Opt. Phys.* **2013**, *46*, 224004. [CrossRef]
38. Łuczka, J. Spin in contact with thermostat: Exact reduced dynamics. *Phys. A Stat. Mech. Appl.* **1990**, *167*, 919–934. [CrossRef]
39. Alicki, R. Pure Decoherence in Quantum Systems. *Open Syst. Inf. Dyn.* **2004**, *11*, 53. [CrossRef]
40. Schuster, D.I.; Houck, A.A.; Schreier, J.A.; Wallraff, A.; Gambetta, J.M.; Blais, A.; Frunzio, L.; Majer, J.; Johnson, B.; Devoret, M.H.; et al. Resolving photon number states in a superconducting circuit. *Nature* **2007**, *445*, 515–518. [CrossRef] [PubMed]
41. Roszak, K.; Machnikowski, P. Complete disentanglement by partial pure dephasing. *Phys. Rev. A* **2006**, *73*, 022313. [CrossRef]
42. Reina, J.H.; Quiroga, L.; Johnson, N.F. Decoherence of quantum registers. *Phys. Rev. A* **2002**, *65*, 032326. [CrossRef]
43. Chen, H.B.; Chen, H.B.; Lo, P.Y.; Gneiting, C.; Bae, J.; Chen, Y.N.; Nori, F. Quantifying the nonclassicality of pure dephasing. *Nat. Commun.* **2019**, *10*, 3794. [CrossRef] [PubMed]
44. Usui, R.; Ban, M. Temporal nonlocality of a two-level system interacting with a dephasing environment. *Quantum Inf. Process.* **2020**, *19*, 159. [CrossRef]
45. Dajka, J.; Mierzejewski, M.; Łuczka, J. Fidelity of asymmetric dephasing channels. *Phys. Rev. A* **2009**, *79*, 012104. [CrossRef]
46. Dajka, J.; Łuczka, J. Origination and survival of qudit-qudit entanglement in open systems. *Phys. Rev. A* **2008**, *77*, 062303. [CrossRef]
47. Łobejko, M.; Mierzejewski, M.; Dajka, J. Interference of qubits in pure dephasing and almost pure dephasing environments. *J. Phys. Math. Theor.* **2015**, *48*, 275302. [CrossRef]

48. Gao, Y. The dynamical role of initial correlation in the exactly solvable dephasing model. *Eur. Phys. J. D* **2013**, *67*, 183. [CrossRef]
49. Dajka, J.; Łuczka, J.; Hänggi, P. Distance between quantum states in the presence of initial qubit-environment correlations: A comparative study. *Phys. Rev. A* **2011**, *84*, 032120. [CrossRef]
50. Dajka, J. Faint trace of a particle in a noisy Vaidman three-path interferometer. *Sci. Rep.* **2021**, *11*, 1123. [CrossRef]
51. Perelomov, A. *Generalized Coherent States and Their Applications*; Springer: Berlin/Heidelberg, Germany, 1986.
52. Bratteli, O.; Robinson, D.W. *Operator Algebras and Quantum Statistical Mechanics: Equilibrium States. Models in Quantum Statistical Mechanics*; Springer: Berlin/Heidelberg, Germany, 2003.
53. Gühne, O.; Tóth, G. Entanglement detection. *Phys. Rep.* **2009**, *474*, 1–75. [CrossRef]
54. Gazeau, J.P. Coherent states in Quantum Information: An example of experimental manipulations. *J. Phys. Conf. Ser.* **2010**, *213*, 012013. [CrossRef]
55. Olivares, S.; Paris, M.G.A. Binary optical communication in single-mode and entangled quantum noisy channels. *J. Opt. Quantum Semiclassical Opt.* **2003**, *6*, 69–80. [CrossRef]
56. Ban, M. Quantum dense coding of continuous variables in a noisy quantum channel. *J. Opt. Quantum Semiclassical Opt.* **2000**, *2*, 786–791. [CrossRef]
57. Ivanovic, I. How to differentiate between non-orthogonal states. *Phys. Lett. A* **1987**, *123*, 257–259. [CrossRef]
58. Peres, A. How to differentiate between non-orthogonal states. *Phys. Lett. A* **1988**, *128*, 19. [CrossRef]
59. Huttner, B.; Barnett, S.M. Quantization of the electromagnetic field in dielectrics. *Phys. Rev. A* **1992**, *46*, 4306–4322. [CrossRef]
60. Drezet, A. Description of spontaneous photon emission and local density of states in the presence of a lossy polaritonic inhomogeneous medium. *Phys. Rev. A* **2017**, *95*, 043844. [CrossRef]