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Quantum Chromodynamics of the Nucleon in Terms of Complex Probabilistic Processes

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Abstract: Despite the obvious progress made by the Feynman, Ravndal, and Kislinger relativistic model in describing the internal motion of a system with confinement of quarks in a nucleon, it turned out to be insufficiently realistic for a number of reasons. In particular, the model does not take into account some cornerstone properties of QCD, namely, gluon exchange between quarks, the influence of the resulting quark sea on valence quarks, and the self-interaction of colored gluons. It is these phenomena that spontaneously break the chiral symmetry of the quark system and form the bulk of the nucleon. To eliminate the above shortcomings of the model, the problem of self-organization of a three-quark dynamical system immersed in a colored quark–antiquark sea is considered within the framework of complex probabilistic processes that satisfy the stochastic differential equation of the Langevin–Kline–Gordon–Fock type. Taking into account the hidden symmetry of the internal motion of a dynamical system, a mathematically closed nonperturbative approach was developed, which makes it possible to construct the mathematical expectation of the wave function and other parameters of the nucleon in the form of multiple integral representations. It is shown that additional subspaces arising in a representation characterized by a noncommutative geometry with topological features participate in the formation of an effective interaction between valence quarks against the background of harmonic interaction between them.

Keywords: flavor physics; quantum chromodynamics; multiquark states; $4D$ relativistic quantum oscillator; three-quark system; colored quark–antiquark sea; gluon fields distribution; noncommutative geometry; mathematical expectation of nucleon wave function

Citation: Gevorkyan, A.S.; Bogdanov, A.V. Quantum Chromodynamics of the Nucleon in Terms of Complex Probabilistic Processes. *Symmetry* **2024**, *16*, 256. https://doi.org/10.3390/sym16030256

Academic Editor: Andrea Lavagno

Received: 23 January 2024
Revised: 8 February 2024
Accepted: 15 February 2024
Published: 20 February 2024

1. Introduction

The observable universe, ordinary matter, and star formations in particular, is composed primarily of strongly interacting particles of protons and neutrons called nucleons. In this regard, one of the most important tasks of modern nuclear physics is the study of the structure of the nucleon and its excited states from the point of view of effective degrees of freedom and, at a more fundamental level, the emergence of these from QCD [1]. After postulating the quark structure of strongly interacting particles by Gell-Mann, Zweig, and Fritzsch [2–4], based on the ideas of symmetry and invariance in the system of particles and fields, Feynman, Ravndahl, and Kislinger [5] proposed a three-quark representation of nucleons within the framework of a four-dimensional relativistic quantum oscillator model. Note that the main idea underlying the three-quark nucleon model is that quarks, interacting through the potential of a four-dimensional harmonic oscillator, cannot move away from each other and become free. In a series of papers [6–8], the authors discussed a quantum harmonic oscillator formalism to study the important features of hadronic structures in relativistic quark models. An important achievement of these models is that they allow a covariant-probabilistic interpretation of the wave functions under consideration [9].

Despite the obvious successes of the relativistic model of a four-dimensional oscillator in describing the structure and internal motion of a nucleon with the effect of quark confinement, it still remains insufficiently realistic. The fact is that the nucleon model under consideration does not take into account the continuous processes of colored gluon exchange between quarks. Moreover, the model does not take into account the spontaneous breaking of chiral symmetry, which is responsible for the generation of nucleon mass from more elementary light quarks. The latter is obviously a strong simplification of the problem. The difficulties of the model become even more obvious when we have to consider nucleons in nuclei or dense and superdense stellar formations. Recall that in this case, the main characteristics of the processes of gluon exchange between quarks, as well as the properties of the quark sea, change, which in turn directly affects the structure and other parameters of nucleons.

To explain the missing nucleon mass, Nambu and Jona-Lasinio proposed a simple model Hamiltonian in which the nucleon mass arises as the self-energy of some primordial fermion field, analogous to the energy gap in superconductivity theory [10]. In particular, a consequence of symmetry is that pseudoscalar bound states with zero mass of the nucleon–antinucleon pair arise, which can be considered as a pion.

In the last two decades, string models of hadrons have been intensively developed [11,12]; however, despite the very promising ideas underlying string approaches, these studies are still far from complete and are often unsuitable for describing various phenomena arising in experiments on hadron physics.

To solve a number of the above problems, this paper considers a relativistic three-quark dynamical system immersed in colored gluon fields, which, in turn, generate a quark–antiquark sea. At the same time, we describe the interactions between quarks using a four-dimensional harmonic oscillator, which ensures confinement of quarks, their asymptotic freedom at short distances, and chiral symmetry. We formulate the mathematical problem in terms of a complex probabilistic process that satisfies a *stochastic differential equation* (SDE) of the Langevin–Klein –Gordon–Fock type. Note that this formulation of the problem allows us to take into account both elastic and inelastic processes of gluon exchange between quarks as well as the self-action and type of gluon–antigluon interactions. It is shown that for the case when fluctuations of the colored quark–antiquark sea are characterized by complex processes of the Markov–Gauss type, it is possible to construct a mathematically closed nonperturbative theory of the nucleon with additional six-dimensional compact subspaces. In particular, when calculating the mathematical expectation of various parameters of the nucleon, we perform averaging over additional subspaces, which breaks the chiral symmetry [13] but at the same time maintains the color of the nucleon. In our opinion, it is very important that the averaging procedure over additional quantized subspaces against the background of the harmonic interaction between valence quarks forms a new, effective interaction between them. In this work, for all parameters of the nucleon, mathematical expressions are obtained in the form of double integral representations, where the integrand is a solution to a system of two coupled second-order *partial differential equations* (PDEs).

In conclusion, we note that considering the nucleon as a complex, self-consistent system of "three quarks + a sea of quarks–antiquarks", as a problem of self-organization within the framework of the developed concept, allows us to go beyond the framework of perturbation theory, which is very important for obtaining new nontrivial results in the field of *quantum chromodynamics* (QCD), which is essentially a nonperturbative theory.

This manuscript is organized as follows:
Section 2 briefly outlines the well-known formulation of the problem of the internal motion of a nucleon as a three-quark dynamical system within the framework of the Klein–Gordon–Fock equation using a four-dimensional model of a relativistic oscillator for quark interactions [6]. An exact, relativistically invariant solution of the wave function of the internal motion of a nucleon is presented.

In Section 3, the problem of the internal motion of a nucleon immersed in a colored quark sea is mathematically formulated in the framework of a complex probabilistic process that satisfies the Langevin–Klein–Gordon–Fock-type SDE. It is shown that a complex probabilistic process in the model of a four-dimensional relativistic oscillator, after a convenient transformation of coordinates, is written in factorized form, as a product consisting of three independent functions.

In Section 4, a system of stochastic equations for gluon fields is derived, taking into account the synchronization of four-dimensional events in the Minkowski space. The conditions of complex stochastic processes for generators of colored gluon fields, in the form of Markov–Gaussian processes, are determined.

In Section 5, taking into account the SDE system, an equation for the distribution of gluon fields in the limit of statistical equilibrium is derived. It is shown that the solution for the field distribution is factorized into the product of three two-dimensional distributions, each of which describes the states of gluon fields of a certain color and anticolor. It is shown that the additional six-dimensional subspace generated by the SDEs system is factorized as a direct product of three two-dimensional subspaces.

In Section 6, the geometric and topological features of the emerging two-dimensional subspaces are analyzed. It is shown that each of the sub-spaces is generated by an algebraic equation of the fourth degree and is described by noncommutative geometry. It is also proven that the topological features of these manifolds are characterized by the Betti number $n \leq 4$.

In Section 7, using a two-dimensional distribution equation for gluon fields, a Fokker–Planck measure of the functional space is constructed.

In Section 8, the mathematical expectation of the total wave function of a three-quark dynamical system immersed in a colored quark–antiquark sea is defined in the form of a functional integral representation. Using the generalized Feynman–Kac theorem, functional integrals are calculated. As a result, a factorized representation in the form of three double integrals is obtained for the mathematical expectation of the wave function of nucleon internal motion.

In Section 9, using the mathematical expectation of the total wave function of a nucleon, its radius and mass are determined depending on the constants characterizing the fluctuation powers of colored gluon fields. Two independent equations are derived that make it possible to uniquely calculate two independent constants characterizing the gluon fields of a nucleon when the nucleon is in a free state.

Section 10 discusses in detail the possibilities of representation for a more correct description of the quantum state of the nucleon, taking into account the processes of gluon exchange between quarks. The issues of the state of a nucleon in the case of their immersion in dense and superdense stellar formations are discussed in light of changes in the spectrum and power of fluctuations of gluon fields.

Appendix A discusses the issues of color synchronization of valence quarks or the problem of preserving the white color of the nucleon, which, in turn, is closely related to the problem of the three-particle interaction of valence quarks.

2. Relativistic Three-Quark Dynamical System

As is known, the main assumption about the dynamics of the internal motion of a nucleon is that the motion of three quarks is described by the relativistic Klein–Gordon–Fock equation in the light-front formalism, where the interaction between quarks is carried out through a four-dimensional harmonic potential, ensuring the confinement of quarks in the nucleon [5]. Taking this work into account, the following Lorentz-invariant Equation was proposed to describe the state of a three-quark dynamical system [6]:

$$\Big\{\sum_{\zeta=x_a,x_b,x_c}\Box_\zeta-\frac{1}{27}\,\Omega_0^2\big[(x_a-x_b)^2+(x_a-x_c)^2+(x_b-x_c)^2\big]+m_0^2\Big\}\Psi^{(0)}(x_a,x_b,x_c)=0, \quad (1)$$

where x_a, x_b, and x_c are the four-dimensional space-time coordinates of quarks a, b, and c (see Figure 1); in addition, Ω_0 is the sum constant, m_0 denotes the sum of the rest masses of three quarks, and $\Box_\zeta$ denotes the d'Alembert operator acting on the ζ-*th* quark:

$$\Box_\zeta = \partial_t^2 - \nabla^2(x_\zeta, y_\zeta, z_\zeta) = \partial_t^2 - \partial_{x_\zeta}^2 - \partial_{y_\zeta}^2 - \partial_{z_\zeta}^2.$$

Recall that below, all calculations will be carried out in units; $\hbar = c = 1$.

By performing the following coordinate transformations,

$$u = \frac{1}{\sqrt{3}}(x_a + x_b + x_c), \quad v = \frac{1}{\sqrt{2}}(x_a - x_b), \quad w = \frac{1}{\sqrt{6}}(x_a + x_b - 2x_c), \tag{2}$$

Equation (2) can be reduced to diagonal form:

$$\Big\{ \sum_{\zeta=u,v,w} \Box_\xi + m_0^2 - \frac{1}{9}\Omega_0^2(v^2 + w^2) \Big\} \Psi(u, v, w) = 0, \tag{3}$$

where $\xi = (\xi_0, \xi_1, \xi_2, \xi_3)$, and accordingly, the d'Alembert operator in new coordinates has the form $\Box_\xi = \partial_{\xi_0}^2 - \sum_{k=1}^3 \partial_{\xi_k}^2$.

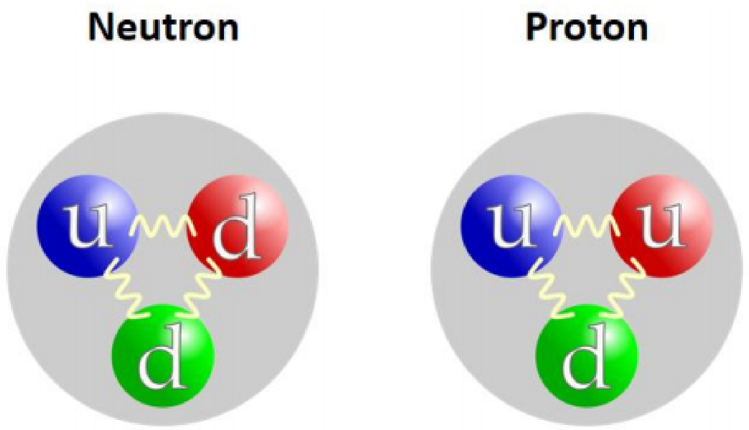

Figure 1. Figures show two nucleons (proton and neutron) in the form of valence quarks located in different quantum states, i.e., in different colors. Conventionally, we will assume that the a-quark is depicted in blue, the b-quark in red, and the c-quark in green.

Now, representing the solution of Equation (3) in factored form,

$$\Psi(u, v, w) = \Psi_1(u)\Psi_2(v)\Psi_3(w), \tag{4}$$

from (3), we obtain three new Equations:

$$\begin{gathered} \big[\Box_u + \lambda_{01}\big]\Psi_1(u) = 0, \\ \big[\Box_v + \lambda_{02} - (1/9)\Omega_0^2\, v^2\big]\Psi_2(v) = 0, \\ \big[\Box_w + \lambda_{03} - (1/9)\Omega_0^2 w^2\big]\Psi_3(w) = 0, \end{gathered} \tag{5}$$

where $m_0^2 = \lambda_{01} + \lambda_{02} + \lambda_{03}$.

The solution to the first Equation of system (5) can be represented as

$$\Psi_1(u) = \exp\{-iu \cdot P/\sqrt{3}\}, \tag{6}$$

where $P^\nu (\nu = 0, 1, 2, 3)$ is a four-vector.

After substituting this solution into the first Equation of system (5), it is easy to find that $P^2 = 3\lambda_{01}$. In addition, from (5), it follows that $\lambda_{01} = \lambda_{02} = (1/3)\Omega_0^2 = (1/3)m_0^2$.

The remaining two equations describe the quantum motions of two independent four-dimensional oscillators, described by the wave functions $\Psi_2(v)$ and $\Psi_2(w)$, respectively. In the rest frame, where $P^\nu = (m_0, \mathbf{0}) = (m_0, 0, 0, 0)$, the *ground-state* wave function normalized in relative coordinates v and w can be written as

$$\Psi(u, v, w; \mathbf{0}) = \left(\frac{\Omega_0}{3\pi}\right)^2 \exp\left\{-i\frac{m_0 u_0}{\sqrt{3}} - \frac{\Omega_0}{6}\left(v_0^2 + \mathbf{v}^2 + w_0^2 + \mathbf{w}^2\right)\right\}, \tag{7}$$

which may be written in covariant form:

$$\Psi(u, v, w; \mathbf{P}) = \left(\frac{\Omega_0}{3\pi}\right)^2 \exp\left\{-i\frac{m_0 u_0}{\sqrt{3}} - \frac{\Omega_0}{6m_0^2}\left(2\left[(P\cdot v)^2 + (P\cdot w)^2\right] - m_0^2\left[v^2 + w^2\right]\right)\right\}.$$

As for the wave function of the excited state, as shown in the work [6], it can be represented in the form

$$\Psi_{\mathbf{n}\,\mathbf{m}}(u, v, w) = N_{\mathbf{n}} N_{\mathbf{m}} \left[\prod_{k=1,2,3} H_{n_k}(v_k) H_{m_k}(w_k)\right] \Psi(u, v, w; \mathbf{0}), \quad N_{\mathbf{n}} = N_{n_1} N_{n_2} N_{n_3}, \tag{8}$$

where N_{n_k} is the normalization constant of the one-dimensional oscillator, $\mathbf{n} = (n_1, n_2, n_3)$ is the set of quantum numbers, and $H_n(x)$ denotes the Hermite polynomials. Recall that the wave function of a four-dimensional oscillator has the form

$$\Psi_2(\mathbf{n}; v) = N_{\mathbf{n}} \prod_{k=1,2,3} H_{n_k}(v_k) \exp\left\{-\frac{\Omega_0}{6} v_k^2\right\}. \tag{9}$$

Note that we obtain exactly the same function for another oscillator described by the function $\Psi_3(\mathbf{m}; w)$.

As can be seen from expressions (7) and (9), the wave function of the nucleon is localized along the coordinates v and w, while the three-quark dynamical system performs translational motion along the coordinate u.

3. Three-Quark Dynamical System Immersed in a Colored Quark–Antiquark Sea

Since nucleons consist of combinations of two types of light quarks **u** and **d** (see Figure 1), it is natural to expect that the interaction of these quarks should be carried out by gluons of different colors. In particular, when the colors of two quarks are known, the interaction between them will be carried out by gluons of these colors and corresponding antigluons. Note that by antigluon we mean the same gluon that has an anticolor. Having gone through all the combinations, it becomes obvious that a nucleon consisting of three quarks of different colors, without taking into account the spins of the quarks, is immersed in a six-color quark–antiquark sea (see Figure 2). Taking this into account, we must construct a consistent nonperturbative theory of self-organization of a quark system and its random multicolor quark–antiquark environment. Based on the experience of studying similar problems of nonrelativistic quantum mechanics [14], we can rewrite Equation (1) in the following form:

$$\left\{\sum_{\zeta = x_a, x_b, x_c} \Box_\zeta - \frac{1}{27}\Omega_0^2\left[(x_a - x_b)^2 + (x_a - x_c)^2 + (x_b - x_c)^2\right] + m^2(x_a, x_b, x_c)\right\}\Psi = 0, \tag{10}$$

where $m^2(q_a, q_b, q_c)$ is some space-time complex random function, the form of which will be indicated below. Recall that the valence quarks a, b, and c (see Figure 1) in a nucleon are in three different quantum states or colors, so the mathematical expectation of the mixed color of a nucleon should be white.

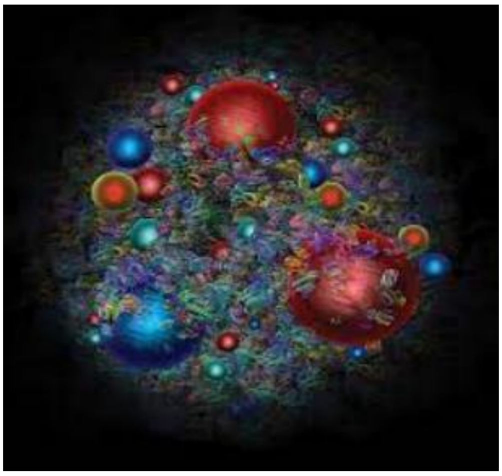

Figure 2. Three valence quarks are immersed in a colored quark–antiquark sea inside a single proton. As the numerical simulation of the problem (see, for example, [15,16]) as well as experimental studies show, with a decrease in space-time scales, the powers and frequency of gluon field fluctuations increase. As can be seen from the figure, for very short times, the color symmetry of valence quarks can be violated.

Using coordinate transformations (2), Equation (10) can be written as

$$\Big\{\sum_{\xi=u,v,w}\Box_\xi-\frac{1}{9}\Omega_0^2\big(v^2+w^2\big)+\breve{m}^2(u,v,w)\Big\}\breve{\Psi}(u,v,w)=0, \tag{11}$$

where $\breve{m}^2(u,v,w)=m^2(x_a,x_b,x_c)$.

To clarify the function $\breve{m}^2(u,v,w)$, we assume that it can be represented in the form

$$\breve{m}^2(u,v,w)=m_0^2+\{\lambda\},\qquad \{\lambda\}=\lambda_1(s_u)+\lambda_2(s_v)+\lambda_3(s_w), \tag{12}$$

where $\lambda_1(s_u)$, $\lambda_2(s_v)$, and $\lambda_3(s_w)$ are some complex random processes depending on proper space-time events:

$$s_\xi=\big(\xi_0^2-\sum_{\delta=x,y,z}\xi_\delta^2\big)^{1/2},\qquad \xi=u,v,w. \tag{13}$$

Recall that since we are considering a relativistic problem, it is natural to use a four-vector, which defines a chronologized sequence of space-time events in Minkowski space, as a parameter describing the evolution of a dynamical system.

Finally, based on the symmetry of Equations (11) and (12), the solution to the total wave function of the system can be represented as the following product:

$$\breve{\Psi}(u,v,w|\{\lambda\})=\breve{\Psi}_1\big(u,\lambda_1(s_u)\big)\cdot\breve{\Psi}_2\big(v,\lambda_2(s_v)\big)\cdot\breve{\Psi}_3\big(w,\lambda_3(s_w)\big). \tag{14}$$

Substituting (14) into (11), taking into account (12), we obtain three new equations:

$$\begin{aligned}
&\big[\Box_u+\lambda_{01}+\lambda_1(s_u)\big]\breve{\Psi}_1\big(u,\lambda_1(s_u)\big)=0,\\
&\big[\Box_v+\lambda_{02}-(1/9)\Omega_0^2v^2+\lambda_2(s_v)\big]\breve{\Psi}_2\big(v,\lambda_2(s_v)\big)=0,\\
&\big[\Box_w+\lambda_{03}-(1/9)\Omega_0^2w^2+\lambda_3(s_w)\big]\breve{\Psi}_3\big(w,\lambda_3(s_w)\big)=0.
\end{aligned} \tag{15}$$

It should be noted that Equation (15) due to the presence of random generators $\lambda_k(s_\xi)$, $(k=1,2,3)$ in them still require definition and reduction to canonical forms, that is to first-order stochastic differential equations of Langevin type.

Now our main task will be to construct the mathematical expectation of the total wave function of a nucleon on the light cone, taking into account gluon exchanges between quarks and the influence of the quark–antiquark sea:

$$\overline{\Psi}(x_a, x_b, x_c) = \overline{\Psi}(u, v, w) = \mathbb{E}[\check{\Psi}(u, v, w|\{\lambda\})], \tag{16}$$

where $\mathbb{E}[\cdots]$ denotes the mathematical expectation of a random variable; in addition, recall that the sets (x_a, x_b, x_c) and (u, v, w) denote four-vectors of Minkowski space-time.

4. Equations of Motion of Gluon Fields under the Influence of Valence Quarks

Taking into account the previous section, the solution to each of the equations of the system (15) can be represented in the form:

$$\check{\Psi}_k(\xi, s_\xi|\lambda_k(s_\xi)) = \Psi_k(\xi) \exp\left(\int_0^{s_\xi} \Lambda_k(\xi, s')ds'\right), \qquad k = 1, 2, 3, \tag{17}$$

where $\Psi_k(\xi)$ is a regular function, a solution to one of the Equations in system (5), and $\Lambda_k(\xi, s_\xi)$ denotes a complex probabilistic process, which can be conveniently represented as a sum consisting of real and imaginary terms:

$$\Lambda_1 = \sum_{j=1,2} i^{j-1}\phi_j(s_u|u), \quad \Lambda_2 = \sum_{j=1,2} i^{j-1}\varphi_j(s_v|v), \quad \Lambda_3 = \sum_{j=1,2} i^{j-1}\theta_j(s_w|w). \tag{18}$$

Substituting a solution of the form (17) into the corresponding Equations of system (15), we obtain

$$\begin{cases} \dot{\Lambda}_1 + a_1\Lambda_1 + b_1\Lambda_1^2 + c_1\lambda_1(s_u) = 0, \\ \dot{\Lambda}_2 + a_2\Lambda_2 + b_2\Lambda_2^2 + c_2\lambda_2(s_v) = 0, \\ \dot{\Lambda}_3 + a_3\Lambda_3 + b_3\Lambda_3^2 + c_3\lambda_3(s_w) = 0, \end{cases} \tag{19}$$

where

$$\dot{\Lambda}_k = \partial\Lambda_k/\partial s_\xi, \qquad a_k = \frac{2\xi \cdot \nabla_M \ln \Psi_k - 2 - s_\xi}{\xi_0 - \sum_{k=1}^3 \xi_k}, \qquad b_k = c_k = \frac{s_\xi}{\xi_0 - \sum_{k=1}^3 \xi_k}. \tag{20}$$

Note that the operator $\nabla_M = (\partial_{\xi_0}, -\partial_{\xi_1}, -\partial_{\xi_2}, -\partial_{\xi_3})$ denotes the gradient in Minkowski space-time $\mathbb{R}^4$.

Now we define random functions $\lambda_k(s_\xi)$ that characterize the properties of colored and anticolored gluon fields, representing them as the sum of real and imaginary terms:

$$\lambda_k(s_\xi) = f_k^{(r)}(s_\xi) + i f_k^{(i)}(s_\xi), \qquad k = 1, 2, 3. \tag{21}$$

For definiteness, we assume that these functions satisfy Markov–Gaussian random processes or *white noise* correlation relations:

$$\mathbb{E}\big[f_j^{(v)}(s_\xi)\big] = 0, \qquad \mathbb{E}\big[f_j^{(v)}(s_\xi)f_j^{(v)}(s'_\xi)\big] = 2\varepsilon_j^{(v)}\,\delta(s_\xi - s'_\xi), \qquad j = 1, 2, \tag{22}$$

where $v = (i, r)$.

We assume that the random generators $f_j^{(r)}$ and $f_j^{(i)}$ characterize elastic and inelastic processes of exchange of gluons and antigluons of a given color between two specific quarks.

Finally, using expressions (17)–(21), we can obtain the following system of six nonlinear Langevin-type SDEs:

$$
\begin{cases}
\dot{\phi}_1 + a_1^{(r)}\phi_1 - a_1^{(i)}\phi_2 + b_1(\phi_1^2 - \phi_2^2) + c_1 f_1^{(r)}(s) = 0, \\
\dot{\phi}_2 + a_1^{(i)}\phi_1 + a_1^{(r)}\phi_2 + 2b_1\phi_1\phi_2 + c_1 f_1^{(i)}(s) = 0, \\
\dot{\varphi}_1 + a_2^{(r)}\varphi_1 - a_2^{(i)}\varphi_2 + b_2(\varphi_1^2 - \phi_2^2) + c_2 f_2^{(r)}(s) = 0, \\
\dot{\varphi}_2 + a_2^{(i)}\phi_1 + a_2^{(r)}\varphi_2 + 2b_2\varphi_1\varphi_2 + c_2 f_2^{(i)}(s) = 0, \\
\dot{\theta}_1 + a_3^{(r)}\theta_1 - a_3^{(i)}\theta_2 + b_3(\theta_1^2 - \theta_2^2) + c_3 f_3^{(r)}(s) = 0, \\
\dot{\theta}_2 + a_3^{(i)}\theta_1 + a_3^{(r)}\theta_2 + 2b_3\theta_1\theta_2 + c_3 f_3^{(i)}(s) = 0.
\end{cases} \tag{23}
$$

Recall that for further study of the system of Equation (23), we synchronized the evolutionary parameters, that is, we took the smallest parameter among them $s = \min\{s_u, s_v, s_w\}$.

Thus, the system of Equation (23) describes colored gluon fields under the influence of three valence quarks and also takes into account the self-actions of gluons. Now it is important to use the SDEs (23) to obtain a regular equation that describes the distribution of gluon fields in the limit of statistical equilibrium. Note that this is fundamentally important for a consistent analytical construction of the problem.

5. Distribution of Colored Gluon Fields in the Limit of Statistical Equilibrium

When gluons are exchanged between two valence quarks of different colors, since the overall white color of the nucleon must be conserved, this must affect the dynamics of the third valence quark to preserve the color of the nucleon. It follows that any processes of gluon exchange between two quarks make the interaction in the nucleon three-particle. The latter means that the probabilities of the distribution of gluon fields of all six colors should be interconnected and combined. Taking into account the above, it is necessary to represent the distribution of gluon fields in the following form:

$$
\mathcal{P}(\boldsymbol{\vartheta}, s|\boldsymbol{\vartheta}_0, s_0) = \Big\langle \prod_{j=1,2} \prod_{\vartheta=\phi,\varphi,\theta} \delta\big(\vartheta_j(s) - \vartheta_{0j}\big) \Big\rangle, \tag{24}
$$

where $\boldsymbol{\vartheta}(s) = \{\boldsymbol{\phi}(s), \boldsymbol{\varphi}(s), \boldsymbol{\theta}(s)\} \in \Xi_{\{\boldsymbol{\vartheta}(s)\}}$ and $\langle\cdots\rangle$ denotes the functional integration over the functional space $\Xi_{\{\boldsymbol{\vartheta}(s)\}}$, whose measure will be defined below; in addition, $\vartheta_{0j} = \vartheta_j(s_0)$ denotes the field value in $s_0 = 0$.

Using the SDE system (22), it is possible to strictly prove that the probability distribution of gluon fields satisfies the following Fokker–Planck-type PDE (see [17,18]):

$$
\frac{\partial \mathcal{P}}{\partial s} = \widehat{\mathcal{L}}(\boldsymbol{\vartheta}, s|u, v, w)\mathcal{P}, \tag{25}
$$

where the evolution operator $\widehat{\mathcal{L}}(\boldsymbol{\vartheta}, s|u, v, w)$ has the following form:

$$
\begin{aligned}
\widehat{\mathcal{L}} = \Bigg\{ \Bigg(\bar{\varepsilon}_1^{(r)} \frac{\partial^2}{\partial \phi_1^2} + \bar{\varepsilon}_1^{(i)} \frac{\partial^2}{\partial \phi_2^2} \Bigg) + \Bigg(\bar{\varepsilon}_2^{(r)} \frac{\partial^2}{\partial \varphi_1^2} + \bar{\varepsilon}_2^{(i)} \frac{\partial^2}{\partial \varphi_2^2} \Bigg) + \Bigg(\bar{\varepsilon}_3^{(r)} \frac{\partial^2}{\partial \theta_1^2} + \bar{\varepsilon}_3^{(i)} \frac{\partial^2}{\partial \theta_2^2} \Bigg) \Bigg\} \\
+ \sum_{j=1}^{2} \Bigg\{ \frac{\partial}{\partial \phi_j}\sigma_j(\boldsymbol{\phi}, s|u) + \frac{\partial}{\partial \varphi_j}\pi_j(\boldsymbol{\varphi}, s|v) + \frac{\partial}{\partial \theta_j}\omega_j(\boldsymbol{\theta}, s|w) \Bigg\}.
\end{aligned} \tag{26}
$$

where $\bar{\varepsilon}_k^{(v)} = c_k^2 \varepsilon_k^{(v)}$, $(k = 1, 2, 3)$.

Note that in Equation (26), the following notations are made: $\boldsymbol{\phi} = (\phi_1, \phi_2)$, $\boldsymbol{\varphi} = (\varphi_1, \varphi_2)$, and $\boldsymbol{\theta} = (\theta_1, \theta_2)$. In addition,

$$
\begin{aligned}
&\sigma_1(\boldsymbol{\phi}, s|u) = a_1^{(r)}\phi_1 - a_1^{(i)}\phi_2 + b_1[\phi_1^2 - \phi_2^2], \quad \sigma_2(\boldsymbol{\phi}, s|u) = a_1^{(i)}\phi_1 + a_1^{(r)}\phi_2 + 2b_1\phi_1\phi_2, \\
&\pi_1(\boldsymbol{\varphi}, s|v) = a_2^{(r)}\varphi_1 - a_2^{(i)}\varphi_2 + b_2[\varphi_1^2 - \varphi_2^2], \quad \pi_2(\boldsymbol{\varphi}, s|v) = a_2^{(i)}\varphi_1 + a_2^{(r)}\varphi_2 + 2b_2\varphi_1\varphi_2, \\
&\omega_1(\boldsymbol{\theta}, s|w) = a_3^{(r)}\theta_1 - a_3^{(i)}\theta_2 + b_3[\theta_1^2 - \theta_2^2], \quad \omega_2(\boldsymbol{\theta}, s|w) = a_3^{(i)}\theta_1 + a_3^{(r)}\theta_2 + 2b_3\theta_1\theta_2,
\end{aligned} \tag{27}
$$

where $a_k^{(r)} = Re\{a\}$ and $a_k^{(i)} = Im\{a\}$, $(k = 1,2,3)$.

It is important to note that Equations (25) and (26) take into account gluon–antigluon interactions. Recall that this is reflected in the mixed terms $\sigma_2(\boldsymbol{\phi}, s|u)$, $\pi_2(\boldsymbol{\varphi}, s|v)$, and $\omega_2(\boldsymbol{\theta}, s|w)$. It is important to note that since colored gluon fields generate a sea of colored quarks–antiquarks, the indicated distributions also describe a quark sea.

The symmetry of Equations (25) and (26) allows us to represent their solution in factorized form:

$$\mathcal{P}(\boldsymbol{\vartheta}, s|u, v, w) = \mathcal{P}_{\boldsymbol{\phi}}(\boldsymbol{\phi}, s|u) \cdot \mathcal{P}_{\boldsymbol{\varphi}}(\boldsymbol{\varphi}, s|v) \cdot \mathcal{P}_{\boldsymbol{\theta}}(\boldsymbol{\theta}, s|w), \tag{28}$$

where each of the probability distributions $\mathcal{P}_{\boldsymbol{\phi}}(\boldsymbol{\phi}, s|u)$, $\mathcal{P}_{\boldsymbol{\varphi}}(\boldsymbol{\varphi}, s|v)$, and $\mathcal{P}_{\boldsymbol{\theta}}(\boldsymbol{\theta}, s|w)$ is defined on the corresponding two-dimensional manifold. In other words, the additional six-dimensional subspace $\Xi^6_{\{\boldsymbol{\vartheta}\}}$ in the limit of the statistical equilibrium can be represented as the following decomposition:

$$\Xi^6_{\{\boldsymbol{\vartheta}\}} \cong \Xi^2_{\{\boldsymbol{\phi}\}} \bigotimes \Xi^2_{\{\boldsymbol{\varphi}\}} \bigotimes \Xi^2_{\{\boldsymbol{\theta}\}}. \tag{29}$$

If the nucleons are free, then we assume that the powers of gluon fluctuations are relatively small, i.e., $\bar{\varepsilon}_k^{(r)}, \bar{\varepsilon}_k^{(i)} \sim 1$ for all $k = 1,2,3$. In this case, we can safely assume that all additional two-dimensional subspaces are Euclidean; $\Xi^2_{\boldsymbol{\phi}} \cong \Xi^2_{\boldsymbol{\varphi}} \cong \Xi^2_{\boldsymbol{\theta}} \cong \mathbb{R}^2 = (-\infty, +\infty) \times (-\infty, +\infty)$.

An important feature of Equation (26), which describes the distribution of fields of colored gluons—or rather, the sea of quarks–antiquarks- in the limit of statistical equilibrium, is that it is quantized. Recall that this follows from the dependence of the function $a_k(\xi, s_\xi)$, $(k = 1,2,3)$ on the quantum state of quarks (see the definition of the function (19)), which is characterized by six quantum numbers.

6. Geometric and Topological Features of the Additional Two-Dimensional Subspaces

In the case when the fluctuation's powers for some reason take on large values, $\bar{\varepsilon}_k^{(r)}$, $\bar{\varepsilon}_k^{(i)} \gg 1$ for all $k = 1,2,3$, it is necessary to conduct a comprehensive analysis to identify the geometric and topological features of the two-dimensional manifolds: $\Xi^2_{\{\chi\}}$, $(\chi = \boldsymbol{\phi}, \boldsymbol{\varphi}, \boldsymbol{\theta})$.

Theorem 1. *A dynamical system described by any of the SDEs from (19) generates a functional space* $\Xi_{\{\chi(s)\}}$, $\{\chi(s) = \boldsymbol{\phi}(s), \boldsymbol{\varphi}(s), \boldsymbol{\theta}(s)\}$, *which in the limit of statistical equilibrium compactifies into a two-dimensional manifold* $\Xi^2_{\{\chi\}}$, $(\chi = \boldsymbol{\phi}, \boldsymbol{\varphi}, \boldsymbol{\theta})$, *characterized by non-commutative geometry.*

Proof. We will consider the distribution of two-component gluon fields or color–anticolor fields. In particular, the fields probability density equation $\boldsymbol{\phi} = (\phi_1, \phi_2)$ can be represented as the following (see Appendix A):

$$\frac{\partial \mathcal{P}_{\boldsymbol{\phi}}}{\partial s} = \widehat{\mathcal{L}}^{(0)}_{\boldsymbol{\phi}} \mathcal{P}_{\boldsymbol{\phi}}, \qquad \widehat{\mathcal{L}}^{(0)}_{\boldsymbol{\phi}} = \left\{ \left(\bar{\varepsilon}_1^{(r)} \frac{\partial^2}{\partial \phi_1^2} + \bar{\varepsilon}_1^{(i)} \frac{\partial^2}{\partial \phi_2^2} \right) + \sum_{j=1}^{2} \frac{\partial}{\partial \phi_j} \sigma_j(\boldsymbol{\phi}, s|u) \right\}. \tag{30}$$

Let us write the same Equation (30) in tensor form for further analysis [18,19]):

$$\frac{\partial \mathcal{P}_{\boldsymbol{\phi}}}{\partial s} = \left\{ \nabla^2 + k_{\boldsymbol{\phi}}(\phi_1, \phi_2, s) \right\} \mathcal{P}_{\boldsymbol{\phi}}, \qquad \nabla^2 = \frac{1}{\sqrt{|g|}} \sum_{i,j=1}^{2} \frac{\partial}{\partial \phi^i} \left(\sqrt{|g|} g^{ij} \frac{\partial}{\partial \phi^j} \right), \tag{31}$$

where $\phi_1 = \phi^1$ and $\phi_2 = \phi^2$; in addition, $k_{\boldsymbol{\phi}} = 2a_1^{(r)} + 4b_1\phi_1$.

To find the elements of the metric tensor, we write the two-dimensional Laplace–Beltrami operator ∇^2 in explicit form:

$$\begin{aligned}\nabla^2 = g^{11}\frac{\partial^2}{\partial\phi_1^2} + \frac{1}{\sqrt{|g|}}\left[\frac{\partial}{\partial\phi_1}\left(\sqrt{|g|}g^{11}\right) + \frac{\partial}{\partial\phi_2}\left(\sqrt{|g|}g^{21}\right)\right]\frac{\partial}{\partial\phi_1} + g^{12}\frac{\partial^2}{\partial\phi_1\partial\phi_2} \\ + g^{22}\frac{\partial^2}{\partial\phi_2^2} + \frac{1}{\sqrt{|g|}}\left[\frac{\partial}{\partial\phi_2}\left(\sqrt{|g|}g^{22}\right) + \frac{\partial}{\partial\phi_1}\left(\sqrt{|g|}g^{12}\right)\right]\frac{\partial}{\partial\phi_2} + g^{21}\frac{\partial^2}{\partial\phi_2\partial\phi_1}.\end{aligned} \tag{32}$$

Comparing the operator written in the forms (26) and (32), and requiring the equality of the corresponding terms in the Equations, we find

$$g^{11} = \bar{\varepsilon}^{(r)}, \quad g^{22} = \bar{\varepsilon}^{(i)}, \quad g^{12} = -g^{21}, \quad g = g^{11}g^{22} - g^{12}g^{21} = \bar{\varepsilon}^{(r)}\bar{\varepsilon}^{(i)} + \left(g^{12}\right)^2. \tag{33}$$

As can be seen from expression (33), the metric tensor of the additional sub-manifold $\Xi^2_{\{\boldsymbol{\phi}\}}$ is antisymmetric, which implies that the corresponding geometry is noncommutative. A similar comparison of Equations (26) and (32) allows us to obtain the following first-order differential equations for the non-diagonal element of the metric tensor $g^{12} = y$:

$$\begin{cases} \bar{\varepsilon}^{(r)}\eta\partial_{\phi_1}y - (1+y\eta)\partial_{\phi_2}y = \sigma_1(\boldsymbol{\phi}, s|u), \\ \bar{\varepsilon}^{(i)}\eta\partial_{\phi_2}y + (1+y\eta)\partial_{\phi_1}y = \sigma_2(\boldsymbol{\phi}, s|u), \qquad \partial_x = \partial/\partial x, \end{cases} \tag{34}$$

where

$$\eta(y,s|u) = \frac{y}{\varepsilon^{(r)}\varepsilon^{(i)} + y^2}.$$

Now our main task will be to use (34) to obtain an algebraic equation that allows us to determine the element of the metric tensor y.

Using Equation (34), we can find the following two expressions for the mixed second derivatives of the antisymmetric element of the metric tensor:

$$\begin{aligned} y_{12} = \frac{\partial^2 y}{\partial\phi_1\partial\phi_2} &= \frac{\bar{\epsilon}^{(i)}(\sigma_{1;2}\eta + \sigma_1\eta_2) + \sigma_{2;2}(1+y\eta) + \sigma_2(y_2\eta + y\eta_{;2})}{a\eta^2 + (1+y\eta)^2} \\ &\quad -2\frac{\bar{\epsilon}^{(i)}\sigma_1\eta + \sigma_2(1+y\eta)}{[a\eta^2 + (1+y\eta)^2]^2}\left[a\eta\eta_{;2} + (1+y\eta)(y_2\eta + y\eta_{;2})\right], \\ y_{21} = \frac{\partial^2 y}{\partial\phi_2\partial\phi_1} &= \frac{\bar{\epsilon}^{(r)}(\sigma_{2;1}\eta + \sigma_2\eta_{;1}) - \sigma_{1;1}(1+y\eta) - \sigma_1(y_1\eta + y\eta_{;1})}{a\eta^2 + (1+y\eta)^2} \\ &\quad -2\frac{\bar{\epsilon}^{(r)}\sigma_2\eta - \sigma_1(1+y\eta)}{[a\eta^2 + (1+y\eta)^2]^2}\left[a\eta\eta_{;1} + (1+y\eta)(y_1\eta + y\eta_{;1})\right], \end{aligned} \tag{35}$$

where $\eta_{;j} = \partial\eta/\partial\phi_j$ and $\sigma_{i;j} = \partial\sigma_i/\partial\phi_j$, $(i,j = 1,2)$.

It is important to note that the antisymmetry of the nondiagonal elements of the metric tensor arises at the stage of choosing a coordinate system and, accordingly, the orientation of the sub-manifold under consideration $\Xi^2_{\{\boldsymbol{\phi}\}}$.

Regarding the question of the symmetry of mixed second derivatives on any oriented manifolds, then, based on the basic requirement of mathematical analysis, the following identity must be satisfied at any point in four-dimensional Minkowski space (Schwartz's theorem, see [20]):

$$y_{12} = \frac{\partial^2 y}{\partial\phi_1\partial\phi_2} = y_{21} = \frac{\partial^2 y}{\partial\phi_2\partial\phi_1}, \tag{36}$$

which is a necessary condition for a twice continuously differentiable function. In the context of partial differential equations, it is called the Schwarz integrability condition.

Using Equations (32), (34), and (36), we can finally obtain the following fourth-degree algebraic equation for the asymmetric element of the metric tensor $g^{12} = -g^{21} = y$:

$$\sum_{n=0}^{4} A_n(\phi_1, \phi_2|u, s)y^n = 0, \tag{37}$$

where the coefficients of the algebraic equation $A_n(\phi_1, \phi_2|u, s)$ are defined by the following expressions:

$$\begin{aligned}
&A_0 = \varepsilon^{(r)}\varepsilon^{(i)}\{4\varepsilon^{(r)}\varepsilon^{(i)}b_1\phi_1 - 2\varepsilon^{(r)}[2a_1^{(i)}b_1\phi_1 + a_1^{(r)}(a_1^{(i)} + 2b_1\phi_2)]\phi_1\phi_2 + 2a_1^{(i)}b_1\varepsilon^{(i)}\phi_2(\phi_1^2 - \phi_2^2)\} \\
&-[a_1^{(i)}]^2(\varepsilon^{(r)}\phi_1^2 + \varepsilon^{(i)}\phi_2^2) - [a_1^{(r)}]^2(\varepsilon^{(i)}\phi_1^2 + \varepsilon^{(r)}\phi_2^2) + 2\varepsilon^{(i)}a^{(r)}\left[\varepsilon^{(r)} + a^{(i)}\phi_1\phi_2 - b_1\phi_1(\phi_1^2 - \phi_2^2)\right] \\
&-b_1^2[4\varepsilon^{(r)}\phi_1^2\phi_2^2 + \varepsilon^{(i)}(\phi_1^2 - \phi_2^2)^2]\}, \\
&A_1 = -\epsilon^{(r)}\epsilon^{(i)}\left(\epsilon^{(r)} + \epsilon^{(i)}\right)\left(a^{(i)} + 2b_1\phi_2\right), \\
&A_2 = 2\{4\varepsilon^{(r)} \times a_1^{(i)}b_1\phi_1^2\phi_2 + [a_1^{(i)}]^2(\varepsilon^{(r)}\phi_1^2 + \varepsilon^{(i)}\phi_2^2) + [a_1^{(r)}]^2(\varepsilon^{(i)}\phi_1^2 + \varepsilon^{(r)}\phi_2^2) + \\
&b_1^2\left[4\varepsilon^{(r)}\phi_1^2\phi_2^2 + \varepsilon^{(i)}(\phi_1^2 - \phi_2^2)^2\right] + 2b_1\varepsilon^{(i)}\left[6\varepsilon^{(r)}\phi_1 - a_1^{(i)}\phi_2(\phi_1^2 - \phi_2^2)\right] \\
&+ 2a^{(r)}\left[\varepsilon^{(r)}\phi_1\phi_2(a_1^{(i)} + 2b_1\phi_2) + \varepsilon^{(i)}(3\varepsilon^{(r)} - a^{(i)}\phi_1\phi_2 + b_1\phi_1(\phi_1^2 - \phi_2^2)\right]\}, \\
&A_3 = -4\left(\varepsilon^{(r)} + \varepsilon^{(i)}\right)\left(a^{(i)} + 2b_1\phi_2\right), \\
&A_4 = 16\left(a_1^{(r)} + 2b_1\phi_1\right).
\end{aligned}$$

As can be seen, at each point of Minkowski space-time $u \in \mathbb{R}^4$, the coefficients of Equation (37) are functions of two coordinates (ϕ_1, ϕ_2), which are the coordinates of the tangent bundle of the two-dimensional sub-manifold $\Xi^2_{\{\boldsymbol{\phi}\}}$. Note that the submanifold $\Xi^2_{\{\boldsymbol{\phi}\}}$ generates the algebraic Equation (37) in the form of a two-dimensional quantized space (again due to the dependence on the function $a_k(u, s)$, see (19)). In particular, depending on the value of the fluctuation powers $\bar{\varepsilon}^{(r)}$ and $\bar{\varepsilon}^{(i)}$ in the continuum set of points defined by the coordinates (ϕ_1, ϕ_2), the solution to the algebraic equation can be complex. In this case, it is necessary to cut out such regions and leave only the regions in which the algebraic equation has real solutions. As a result of this procedure, the resulting submanifold will have topological features characterized by the Betti number $n \leq 4$ (see Figures 3 and 4), where 4 is the number of complex solutions of the algebraic Equation (37).

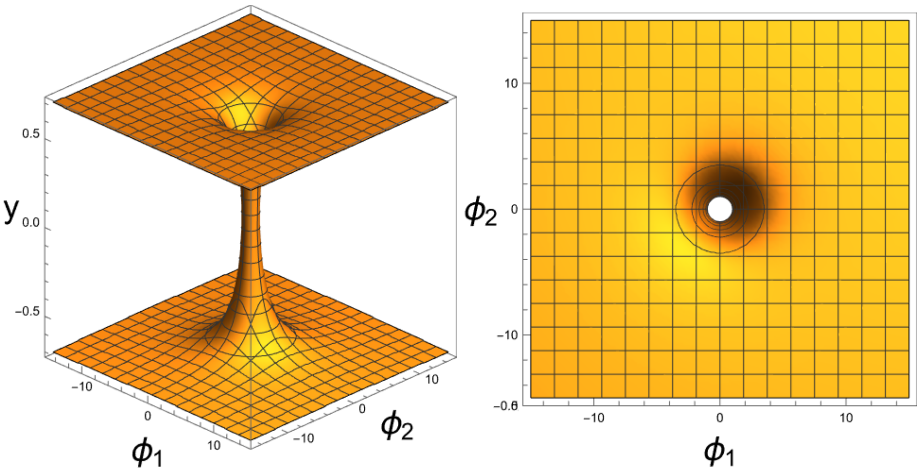

Figure 3. Left figure depicts a three-dimensional plot of the asymmetric element of the metric tensor $g^{12}(\phi_1, \phi_2)$ in the $s \to \infty$ limit, calculated in the nucleon ground state for the values of parameters $\varepsilon^{(r)} = \varepsilon^{(i)} = 1$ and $a^{(r)} = 0$, $a^{(i)} = b_1 = 1$. Right figure shows a two-dimensional projection of the sub-manifold $\Xi^2_{\{\boldsymbol{\phi}\}}$, where it is clearly seen that it has a singularity characterized by a Betti number of 1.

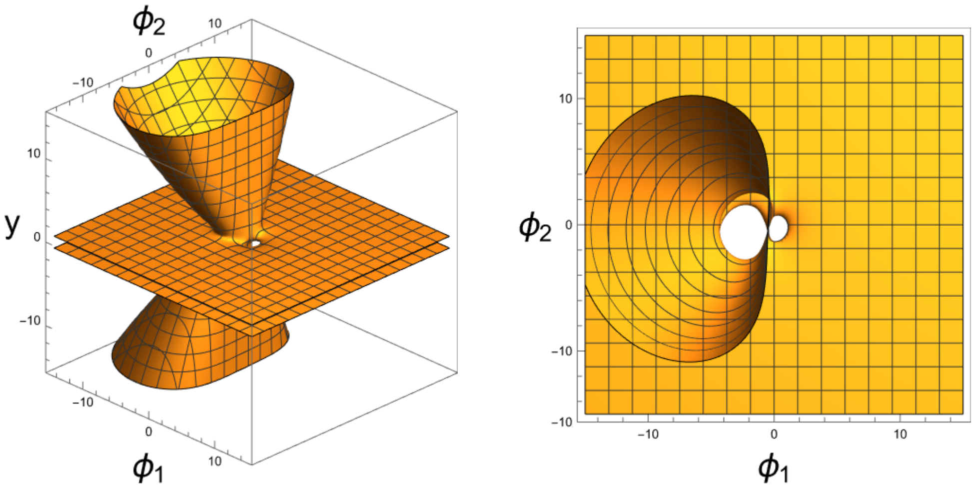

Figure 4. Left figure depicts a three-dimensional plot of the metric tensor element $g^{12}(\phi_1, \phi_2)$ in the limit $s \to \infty$, calculated in nucleon ground state for values of $\varepsilon^{(r)} = \varepsilon^{(i)} = 1.5$ and $a^{(r)} = 0,\ a^{(i)} = b_1 = 1$. Right figure shows a two-dimensional projection of the sub-manifold $\Xi^2_{\{\boldsymbol{\phi}\}}$, where it is clearly seen that the its has a singularity with a Betti number of 2.

Note that a similar proof can be carried out in the case of equations of distributions: $\mathcal{P}_{\boldsymbol{\varphi}}(\boldsymbol{\varphi}, s|v)$ and $\mathcal{P}_{\boldsymbol{\theta}}(\boldsymbol{\theta}, s|w)$. It is obvious that generating the additional two-dimensional submanifolds $\Xi^2_{\{\boldsymbol{\varphi}\}}$ and $\Xi^2_{\{\boldsymbol{\theta}\}}$ will have similar geometric and topological features as the sub-manifold $\Xi^2_{\{\boldsymbol{\phi}\}}$. $\square$

7. Construction of a Measure of the Functional Subspace

To calculate the mathematical expectation of different parameters of a dynamical system, we need to construct measures of three functional subspaces, $\Xi_{\{\boldsymbol{\phi}(s)\}}$, $\Xi_{\{\boldsymbol{\varphi}(s)\}}$, and $\Xi_{\{\boldsymbol{\theta}(s)\}}$. Since all these noted subspaces are similar in their geometric and topological properties, below, we will study only one of these subspaces and construct its measure.

Let the probability distribution in each point of the Minkowski space-time $u \in \mathbb{R}^4$ satisfy the following limiting condition:

$$\lim_{s \to s'} \mathcal{P}_{\boldsymbol{\phi}}(\boldsymbol{\phi}, s|\boldsymbol{\phi}', s') = \delta(\boldsymbol{\phi} - \boldsymbol{\phi}'), \qquad s = s' + \Delta s, \tag{38}$$

Taking into account (38) for small intervals of events, that is, for $\Delta s = s - s' \ll 1$, we can present the solution to Equations (25) and (26) in the following form (see also [18]):

$$P_{\boldsymbol{\phi}}(\boldsymbol{\phi}, s; u|\boldsymbol{\phi}', s') = \frac{1}{2\pi\sqrt{|\det \bar{\varepsilon}|}\,\Delta s} \times$$
$$\exp\left\{ -\frac{\left[\boldsymbol{\phi} - \boldsymbol{\phi}' - \boldsymbol{\sigma}(\boldsymbol{\phi}, s|u)\Delta s\right]^T \bar{\varepsilon}^{-1} \left[\boldsymbol{\phi} - \boldsymbol{\phi}' - \boldsymbol{\sigma}(\boldsymbol{\phi}, s|u)\Delta s\right]}{2\Delta s} \right\}, \tag{39}$$

where $\bar{\varepsilon}$ is the second-rank matrix with elements $\varepsilon_{11} = \bar{\varepsilon}^{(r)},\ \varepsilon_{22} = \bar{\varepsilon}^{(i)}$, and $\varepsilon_{12} = \varepsilon_{21} = 0$, while $[\cdots]^T$ denotes a vector transposition.

Additionally, in representation (39), a two-dimensional vector $\boldsymbol{\sigma}(\boldsymbol{\phi}, s|u)$ is defined as

$$\boldsymbol{\sigma}(\boldsymbol{\phi}, s|u) = \begin{cases} \sigma_1(\boldsymbol{\phi}, s|u) = a_1^{(r)}\phi_1 - a_1^{(i)}\phi_2 + b_1[\phi_1^2 - \phi_2^2], \\ \sigma_2(\boldsymbol{\phi}, s|u) = a_1^{(i)}\phi_1 + a_1^{(r)}\phi_2 + 2b_1\phi_1\phi_2, \end{cases} \tag{40}$$

where the functions $\sigma_1(\boldsymbol{\phi}, s|u)$ and $\sigma_2(\boldsymbol{\phi}, s|u)$ implicitly depend on the event interval "s" and parametrically on the points in Minkowski space $u \in \mathbb{R}^4$.

As can be seen from expression (39), the evolution of the system in the functional space $\Xi_{\{\boldsymbol{\phi}(s)\}}$ is characterized by a regular shift with a speed $\boldsymbol{\sigma}(\boldsymbol{\phi}, s|u)$ against the background of Gaussian fluctuations with the diffusion matrix ε_{ij}. Concerning the trajectory $\boldsymbol{\phi}(s)$ in the functional space $\Xi_{\{\boldsymbol{\phi}(s)\}}$, it is determined by the following Equations (see [21]):

$$\boldsymbol{\phi}(s) = \begin{cases} \phi_1(s+\Delta s) = \phi_1(s) + \sigma_1(\boldsymbol{\phi}, s|u)\Delta s + (\Delta s)^{1/2} f^{(r)}(s), \\ \phi_2(s+\Delta s) = \phi_2(s) + \sigma_2(\boldsymbol{\phi}, s|u)\Delta s + (\Delta s)^{1/2} f^{(i)}(s). \end{cases} \tag{41}$$

In order not to complicate the writing of formulas, we do not write the parametric dependence of the functions ϕ_1 and ϕ_2 on the variable "u".

As can be seen from (41), the trajectory $\boldsymbol{\phi}(s)$ is continuous everywhere, since $\boldsymbol{\phi}(s + \Delta s)\big|_{\Delta s \to 0} = \boldsymbol{\phi}(s)$, but nevertheless, it is nondifferentiable everywhere due to the presence of the term $(\Delta s)^{1/2}$. If the interval of events is represented as $\Delta s = s/N$, where $N \to \infty$, then expression (39) can be interpreted as the probability of transition from the vector field $\boldsymbol{\phi}_l(s)$ to the vector field $\boldsymbol{\phi}_{l+1}(s)$ during of the interval Δs within the Brownian motion model.

Finally, we can define the measure of the function space $\Xi_{\{\boldsymbol{\phi}(s)\}}$, which we will conventionally call the Fokker–Planck measure:

$$D\mu(\boldsymbol{\phi}) = d\mu(\boldsymbol{\phi}_0) \lim_{N\to\infty} \Bigg\{ \bigg(\frac{1}{2\pi} \frac{N/s}{\sqrt{\bar{\varepsilon}^{(r)}\bar{\varepsilon}^{(i)}}} \bigg)^N \prod_{l=0}^{N} d\phi_{1(l+1)} d\phi_{2(l+1)} \exp\Bigg[-\frac{N/s}{2\bar{\varepsilon}^{(r)}} \bigg(\phi_{1(l+1)} \\ -\phi_{1(l)} - \sigma_{1(l+1)} \frac{s_{l+1}}{N} \bigg)^2 - \frac{N/s}{2\bar{\varepsilon}^{(i)}} \bigg(\phi_{2(l+1)} - \phi_{2(l)} - \sigma_{2(l+1)} \frac{s_{l+1}}{N} \bigg)^2 \Bigg] \Bigg\}, \tag{42}$$

where $d\mu(\boldsymbol{\phi}_0) = \delta(\phi_1 - \phi_{1(0)})\delta(\phi_2 - \phi_{2(0)}) d\phi_1 d\phi_2$ denotes the measure of the initial distribution. In addition, in representation (42), the following notations are made:

$$\phi_{1(l)} = \phi_1(s_l), \quad \phi_{2(l)} = \phi_2(s_l), \quad \sigma_{1(l)} = \sigma_1(\phi_{1(l)}, \phi_{2(l)}, s_l), \quad \sigma_{2(l)} = \sigma_2(\phi_{1(l)}, \phi_{2(l)}, s_l).$$

Thus, we have constructed a measure (42) of the functional subspace $\Xi_{\{\boldsymbol{\phi}(s)\}}$, which is necessary for further analytical constructions of the theory. In a similar way, we can construct expressions for the measures of the subspaces $\Xi_{\{\boldsymbol{\varphi}(s)\}}$ and $\Xi_{\{\boldsymbol{\theta}(s)\}}$, respectively.

8. Mathematical Expectation of the Nucleon Wave Function

To calculate the mathematical expectation of the full wave function, we first need to average the complex probabilistic process (17) over the functional sub-space $\Xi_{\{\boldsymbol{\chi}(s)\}}$.

Definition 1. *The mathematical expectation of a complex probabilistic process, taking into account the influence of colored gluon fields, will be determined by the following integral representation:*

$$\overline{\Psi}_k(\xi) = \mathbb{E}[\check{\Psi}_k(\xi|\boldsymbol{\chi})] = \frac{\Psi_k(\xi)}{\alpha_k(s,\xi)} \int \cdots \int_{\Xi_{\{\boldsymbol{\chi}(s)\}}} D\mu(\boldsymbol{\chi}) \exp\bigg(\int_0^s \Lambda_k(s';\xi) ds' \bigg), \tag{43}$$

where $\alpha_k(s,\xi) = \int \cdots \int_{\Xi_{\{\boldsymbol{\chi}(s)\}}} D\mu(\boldsymbol{\chi}) = \int \int_{\Xi^2_{\{\chi\}}} \mathcal{P}_\chi(\boldsymbol{\chi}, s) d\chi_1 d\chi_2$ *is a normalizing constant.*

Recall that we consider that $\check{\Psi}_k(\xi|\lambda_1(s_\xi)) = \check{\Psi}_k(\xi|\boldsymbol{\chi})$ (see expression (16)).
Using the generalized Feynman–Katz theorem [18] and given the expression (17), we can integrate the functional integral in (43) and find the following two-dimensional integral representation for the mathematical expectation of the nucleon wave function:

$$\overline{\Psi}_1(u) = \mathbb{E}[\check{\Psi}_1(u|\boldsymbol{\phi})] = \frac{\Psi_1(u)}{\alpha_1(s,u)} \int \int_{\Xi^2_{\{\boldsymbol{\phi}\}}} Q_{\boldsymbol{\phi}}(\boldsymbol{\phi}, s|u) d\phi_1 d\phi_2, \tag{44}$$

where the integrand function $Q_{\boldsymbol{\phi}}(\boldsymbol{\phi}, s|u)$ is the solution of the following complex PDE:

$$\frac{\partial Q_{\boldsymbol{\phi}}}{\partial s} = \{\widehat{\mathcal{L}}^{(0)}_{\boldsymbol{\phi}}(\boldsymbol{\phi}, s|u) + \phi_1 + i\phi_2\}Q_{\boldsymbol{\phi}}. \tag{45}$$

Let us represent the solution to Equation (45) as the sum of the real and imaginary parts

$$Q_{\boldsymbol{\phi}}(\boldsymbol{\phi}, s|u) = Q^{(r)}_{\boldsymbol{\phi}}(\boldsymbol{\phi}, s|u) + iQ^{(i)}_{\boldsymbol{\phi}}(\boldsymbol{\phi}, s|u). \tag{46}$$

Substituting (46) into Equation (45), we obtain two real coupled PDEs:

$$\begin{cases} \partial_s Q^{(r)}_{\boldsymbol{\phi}} = \{\widehat{\mathcal{L}}^{(0)}_{\boldsymbol{\phi}} + \phi_1\}Q^{(r)}_{\boldsymbol{\phi}} - \phi_2 Q^{(i)}_{\boldsymbol{\phi}}, \\ \partial_s Q^{(i)}_{\boldsymbol{\phi}} = \{\widehat{\mathcal{L}}^{(0)}_{\boldsymbol{\phi}} + \phi_1\}Q^{(i)}_{\boldsymbol{\phi}} + \phi_2 Q^{(r)}_{\boldsymbol{\phi}}. \end{cases} \tag{47}$$

Similar to definition (43), we can construct mathematical expectations of complex probabilistic processes $\check{\Psi}_2(v|\boldsymbol{\varphi})$ and $\check{\Psi}_3(w|\boldsymbol{\theta})$.

In particular, in averaging the wave function $\check{\Psi}_2(v|\boldsymbol{\varphi})$, we find the following expression:

$$\overline{\Psi}_2(v) = \mathbb{E}[\check{\Psi}_2(v|\boldsymbol{\varphi})] = \frac{\Psi_2(v)}{\alpha_2(s,v)} \int\int_{\Xi^2_{\{\boldsymbol{\varphi}\}}} [Q^{(r)}_{\boldsymbol{\varphi}}(\boldsymbol{\varphi}, s|v) + iQ^{(i)}_{\boldsymbol{\varphi}}(\boldsymbol{\varphi}, s|v)]d\varphi_1 d\varphi_2, \tag{48}$$

where $\alpha_2(s,v) = \int \cdots \int_{\Xi_{\{\boldsymbol{\varphi}(s)\}}} D\mu(\boldsymbol{\varphi}) = \int\int_{\Xi^2_{\{\boldsymbol{\varphi}\}}} \mathcal{P}_{\boldsymbol{\varphi}}(\boldsymbol{\varphi}, s)d\varphi_1 d\varphi_2$ and, accordingly, $Q^{(r)}_{\boldsymbol{\varphi}}(\boldsymbol{\varphi}, s|v)$ and $Q^{(i)}_{\boldsymbol{\varphi}}(\boldsymbol{\varphi}, s|v)$ are solutions to the following PDEs' system:

$$\begin{cases} \partial_s Q^{(r)}_{\boldsymbol{\varphi}} = \{\widehat{\mathcal{L}}^{(0)}_{\boldsymbol{\varphi}} + \varphi_1\}Q^{(r)}_{\boldsymbol{\varphi}} - \varphi_2 Q^{(i)}_{\boldsymbol{\varphi}}, \\ \partial_s Q^{(i)}_{\boldsymbol{\varphi}} = \{\widehat{\mathcal{L}}^{(0)}_{\boldsymbol{\varphi}} + \varphi_1\}Q^{(i)}_{\boldsymbol{\varphi}} + \varphi_2 Q^{(r)}_{\boldsymbol{\varphi}}. \end{cases} \tag{49}$$

The procedure for functional averaging of the wave function $\check{\Psi}_3(w|\boldsymbol{\theta})$ leads to the following result:

$$\overline{\Psi}_3(w) = \mathbb{E}[\check{\Psi}_3(w|\boldsymbol{\theta})] = \frac{\Psi_3(w)}{\alpha_3(s,w)} \int\int_{\Xi^2_{\{\boldsymbol{\theta}\}}} [Q^{(r)}_{\boldsymbol{\theta}}(\boldsymbol{\theta}, s|w) + iQ^{(i)}_{\boldsymbol{\theta}}(\boldsymbol{\theta}, s|w)]d\theta_1 d\theta_2, \tag{50}$$

where $\alpha_3(s,w) = \int \cdots \int_{\Xi_{\{\boldsymbol{\theta}(s)\}}} D\mu(\boldsymbol{\theta}) = \int\int_{\Xi^2_{\{\boldsymbol{\theta}\}}} \mathcal{P}_{\boldsymbol{\theta}}(\boldsymbol{\theta}, s)d\theta_1 d\theta_2$; in addition, the functions $Q^{(r)}_{\boldsymbol{\theta}}(\boldsymbol{\theta}, s|w)$ and $Q^{(i)}_{\boldsymbol{\theta}}(\boldsymbol{\theta}, s|w)$ are solutions to the following PDEs' system:

$$\begin{cases} \partial_s Q^{(r)}_{\boldsymbol{\theta}} = \{\widehat{\mathcal{L}}^{(0)}_{\boldsymbol{\theta}} + \theta_1\}Q^{(r)}_{\boldsymbol{\theta}} - \theta_2 Q^{(i)}_{\boldsymbol{\theta}}, \\ \partial_s Q^{(i)}_{\boldsymbol{\theta}} = \{\widehat{\mathcal{L}}^{(0)}_{\boldsymbol{\theta}} + \theta_1\}Q^{(i)}_{\boldsymbol{\theta}} + \theta_2 Q^{(r)}_{\boldsymbol{\theta}}. \end{cases} \tag{51}$$

Now, regarding the functions $Q^{(v)}_{\chi}(\chi, s|\xi)$, where $(v = r, i)$, $\chi = (\boldsymbol{\phi}, \boldsymbol{\varphi}, \boldsymbol{\theta})$, and $\xi = (u, v, w)$, giving them the meaning of density probabilities, we can normalize them:

$$\bar{Q}^{(v)}_{\chi}(\chi, s|\xi) = \alpha^{-1}_{\chi}(\xi; s)Q^{(v)}_{\chi}(\chi, s|\xi), \tag{52}$$

where $\alpha_{\chi}(\xi; s) = \int\int_{\Xi^2_{\{\chi\}}} \sum_{v=r,i} Q^{(v)}_{\chi}(\chi, s|\xi)d\chi_1 d\chi_2$.

Obviously, for the normalized probability distributions, the following condition will occur:

$$\int\int_{\Xi^2_{\{\chi\}}} \sum_{v=r,i} \bar{Q}^{(v)}_{\chi}(\chi, s|\xi)d\chi_1 d\chi_2 = 1.$$

Finally, taking into account the obtained results (16), (44), (48), and (50), we write down the mathematical expectation of the wave nucleon functions, taking into account the continuous gluon exchange between quarks and the influence of the sea of colored quarks:

$$\overline{\Psi}(u,v,w) = \mathbb{E}[\check{\Psi}(u,v,w|\{\lambda\})] = \overline{\Psi}_1(u)\overline{\Psi}_2(v)\overline{\Psi}_3(w). \tag{53}$$

By carrying out this type of normalization, we actually take into account two-quark interactions in a three-quark system, which is a simplification of the real problem. To take into account three-quark interactions that preserve the color of the nucleon, it is necessary to carry out more complex three-color synchronization in the dynamical system, which will be equivalent to taking into account three-quark interactions (see Appendix A for details).

In the end, we note that the mathematical algorithm for the numerical study of a complex PDE of type (45) has been studied in detail in the works of the authors [14,18].

9. Mathematical Expectation of Nucleon Radius and Mass

The average radius and average mass of a nucleon are formed as mathematical expectations of the corresponding quantities in the ground state of the nucleon.

Definition 2. *Average value of the square root of the squared radii of quark displacements, calculated in the ground state of the nucleon, will be called the nucleon radius:*

$$\mathsf{R}_{nuc}\big(\varepsilon_1^{(r)},\cdots,\varepsilon_3^{(i)}\big) = \lim_{s\to\infty}\sqrt{\int\cdots\int(\mathbf{v}^2+\mathbf{w}^2)\varrho(u,v,w)d^4vd^4w}, \tag{54}$$

where $\mathbf{v},\mathbf{w}\in\mathbb{R}^3$ *denote the spatial three-dimensional vectors,* $\varrho_0(u,v,w) = |\overline{\Psi}_0(u,v,w)|^2$ *is the probability density, and* $\overline{\Psi}_0(u,v,w)$ *denotes the mathematical expectation of the wave function of the nucleon in the ground state.*

The fact that there is no integration over the coordinates of the four-dimensional vector u is due to the fact that the system performs translational motion along these coordinates. Expression (54) can be written explicitly:

$$\mathsf{R}_{nuc}\big(\varepsilon_1^{(r)},\cdots,\varepsilon_3^{(i)}\big) = \lim_{s\to\infty}\sqrt{\int\cdots\int e^{-\frac{\Omega_0}{3}\left(v_0^2+\mathbf{v}^2+w_0^2+\mathbf{w}^2\right)}(\mathbf{v}^2+\mathbf{w}^2)\frac{\varrho_0(v,w,s)d^4vd^4w}{\alpha_1^2(u,s)\alpha_2^2(v,s)\alpha_3^2(w,s)}}, \tag{55}$$

where the function of the probability density $\varrho_0(v,w,s)$ is defined as follows:

$$\begin{aligned}\varrho_0(v,w,s) = \Big|\int\int_{\Xi^2_{\{\boldsymbol{\phi}\}}}\big[Q^{(r)}_{\boldsymbol{\phi}}(\boldsymbol{\phi},s|u)+iQ^{(i)}_{\boldsymbol{\phi}}(\boldsymbol{\phi},s|u)\big]d\phi_1d\phi_2\times\int\int_{\Xi^2_{\{\boldsymbol{\varphi}\}}}\big[Q^{(r)}_{\boldsymbol{\varphi}}(\boldsymbol{\varphi},s|v)\\ +iQ^{(i)}_{\boldsymbol{\varphi}}(\boldsymbol{\varphi},s|v)\big]d\varphi_1d\varphi_2\times\int\int_{\Xi^2_{\{\boldsymbol{\theta}\}}}\big[Q^{(r)}_{\boldsymbol{\theta}}(\boldsymbol{\theta},s|w)+iQ^{(i)}_{\boldsymbol{\theta}}(\boldsymbol{\theta},s|w)\big]d\theta_1d\theta_2\Big|^2.\end{aligned} \tag{56}$$

Finally, we can calculate an important parameter of the nucleon—its mass.

Definition 3. *Following expression will be called the mathematical expectation of the nucleon mass in the ground state*

$$m_{nuc}\big(\varepsilon_1^{(r)},\cdots,\varepsilon_3^{(i)}\big) = \lim_{s\to\infty}\sqrt{\int\cdots\int\Big|\overline{\Psi}^*(u,v,w)\frac{\partial^2}{\partial t^2}\overline{\Psi}(u,v,w)d^4vd^4w\Big|}. \tag{57}$$

Having calculated (57), we obtain the following expression for the nucleon mass:

$$m_{nuc}\big(\varepsilon_1^{(r)},\cdots,\varepsilon_3^{(i)}\big) = \\ m_0 \lim_{s\to\infty}\sqrt{\int\cdots\int e^{-\frac{\Omega_0}{3}\left(v_0^2+\mathbf{v}^2+w_0^2+\mathbf{w}^2\right)}\frac{\varrho_0(v,w,s)d^4v d^4w}{\alpha_1^2(u,s)\alpha_2^2(v,s)\alpha_3^2(w,s)}}. \quad (58)$$

After we have determined the radius and mass of the nucleon through the fluctuation powers of the gluon fields $(\varepsilon_1^{(r)},...,\varepsilon_3^{(i)})$, a natural question arises: can we find these constants, knowing only the masses of the nucleon and quarks, respectively? Based on the fact that two of the three quarks in a nucleon are the same and the fluctuation powers of the gluon and antigluon fields are equal, we can conclude that there are only two independent constants that characterize six-color gluon fields. The latter means that if the mass of a nucleon in a free state is known, then in addition to Equation (58), another equation is needed to completely determine these constants. The missing second equation in this case may be the total energy of the colored gluon fields, which is equivalent to the effective mass of the sea of quarks $m_{qs} = m_{nuc} - m_0 \simeq 0.99 \cdot m_{nuc}$. This energy is determined by the probability distributions on the corresponding two-dimensional sub-manifolds and can be represented as a sum of three terms (see expression (A9)):

$$\lim_{s\to\infty}\sum_{\chi=\boldsymbol{\phi},\boldsymbol{\varphi},\boldsymbol{\theta}}\ \sum_{v=i,r}\int\int_{\Xi^2_{\{\chi\}}}\bar{Q}^{(v)}_{\chi}(\boldsymbol{\phi},s|\xi)d\chi_1 d\chi_2 = m_{qs}/m_{nuc} \simeq 0.99. \quad (59)$$

Thus, we have obtained the two Equations (58) and (59), which allow us to uniquely determine two constants characterizing the powers of color gluon fluctuations in a free nucleon when the nucleon is in the *ground state*.

10. Conclusions

Although it has long been known that nucleons consist of valence quarks and a quark–antiquark sea, and although we know the rules of local gauge invariance with respect to the $SU(3)$ symmetry group, it can be argued that we are still far from a satisfactory understanding of the problem [22]. This is largely due to the problem of the strong nonlinearity of QCD, due to which perturbative methods are often inapplicable. In this regard, new, especially nonperturbative theoretical studies on this topic remain highly relevant and continue to serve as a source of ideas for new experiments in hadron physics.

The model of the nucleon as a three-quark relativistic system with the interaction potentials of a four-dimensional harmonic oscillator, proposed by Feynman et al. [5], at one time turned out to be very useful for studying the structure and properties of the internal motion of the nucleon. However, as further experimental studies have shown, the rest mass of a three-quark system is only a percentage of the total rest mass of the nucleon. The missing main mass of the nucleon turns out to be due to the QCD binding energy, which arises as a result of the breaking of QCD chiral symmetry. In other words, intense interactions of valence quarks with colored gluon fields generate a sea of virtual quarks and antiquarks, which is ultimately recorded by measuring instruments as the rest mass of a nucleon.

To overcome this difficulty, we generalized the relativistic model [5], considering the problem of the internal motion of a nucleon as a self-organization problem of a complex three-quark dynamical system in a colored sea of quarks–antiquarks. We formulated the mathematical problem within the framework of a complex probabilistic process, satisfying an equation of the Langevin–Kline–Gordon–Fock type (see Equations (10)–(15)). Using this equation, we obtained a system of SDEs that describes the motion of a six-color gluon field under the influence of a valence three-quark system (see system of Equation (23)). Note that the stochastic extension of the Klein–Gordon–Fock equation does not allow including all eight gluons: in the quark–gluon interaction scheme due to the neglect of the spin part of the motion. In this representation, the contributions of two colorless gluons are not taken into account. It is obvious that a complete description of the problem can only be achieved by stochastic extension of the Yang–Mills equations within the framework of the gauge symmetry group $SU(3)$, similar to what was performed in the work of the author [23] for the gauge symmetry group $SU(2) \otimes U(1)$. Recall that such work would be very important for quantum field theory also because it would make it possible to theoretically substantiate the existence of eight-color massless gluons, carriers of strong interactions.

One of the most interesting and important results of the developed approach is the appearance of additional quantized subspaces with Betti topological singularities. By averaging over these subspaces in order to calculate the mathematical expectation of the nucleon wave function, the chiral symmetry is spontaneously broken, but most importantly, effective interactions between valence quarks arise with respect to which the harmonic interactions between these particles are only a background.

Thus, the developed approach represents a significant rethinking of the concept of the relativistic quark model [5] and takes into account all the physical processes known to us that lead to the formation of a nucleon in its modern understanding. In a mathematical sense, this representation combines elements of complex probabilistic processes, relativistic wave mechanics, path integrals, and noncommutative geometry with Betti singularities, which makes it possible to overcome all known difficulties for a rigorous description of the quantum state of the nucleon.

In the future, we plan to numerically study the features of nucleons depending on these fluctuation powers. Recall that this will give us important information not only about nucleons but also about macroscopic objects in which these particles are immersed. We emphasize that this kind of information about nucleons and their environment can only be obtained through the development of a nonperturbative theory.

Finally, it is important to note that all results obtained as the mathematical expectation of various nucleon parameters are formed in times $\Delta t \geq \hbar/2\Delta E = 2,19 \cdot 10^{-24}\, sec$, where $\Delta E \approx m_{nuc} c_0^2$ is the energy of the quark–antiquark sea and c_0 is the speed of light in vacuum.

Author Contributions: Conceptualization and writing—original draft preparation, A.S.G.; Writing—review and editing, A.V.B. All authors have read and agreed to the published version of the manuscript.

Funding: The work was partially supported by grant No. 21T-1B059 of the Science Committee of Armenia.

Data Availability Statement: Data are contained within the article.

Acknowledgments: The authors also thank Movsesyan K.A. for the numerical study and visualization of calculations of two-dimensional submanifolds in Figures 3 and 4.

Conflicts of Interest: The authors declare no conflicts of interest.

Appendix A

As mentioned above, Equations (25) and (26) describe the distribution of gluon fields in the limit of statistical equilibrium, taking into account their interaction with the three-quark dynamical system, as well as gluon–antigluon interactions (see terms $\sigma_2(\boldsymbol{\phi}, s|u)$, $\pi_2(\boldsymbol{\varphi}, s|u)$ and $\omega_2(\boldsymbol{\theta}, s|u)$ in Equations (26) and (27)). This equation can be represented as

$$\frac{\partial \mathcal{P}}{\partial s} = \Big\{ \Big(\bar{\varepsilon}_1^{(r)} \frac{\partial^2}{\partial \phi_1^2} + \bar{\varepsilon}_1^{(i)} \frac{\partial^2}{\partial \phi_2^2} \Big) + \Big(\bar{\varepsilon}_2^{(r)} \frac{\partial^2}{\partial \varphi_1^2} + \bar{\varepsilon}_2^{(i)} \frac{\partial^2}{\partial \varphi_2^2} \Big) + \Big(\bar{\varepsilon}_3^{(r)} \frac{\partial^2}{\partial \theta_1^2} + \bar{\varepsilon}_3^{(i)} \frac{\partial^2}{\partial \theta_2^2} \Big) + \sum_{j=1}^{2} \Big[\frac{\partial}{\partial \phi_j} \sigma_j(\boldsymbol{\phi}, s|u) + \frac{\partial}{\partial \varphi_j} \pi_j(\boldsymbol{\varphi}, s|v) + \frac{\partial}{\partial \theta_j} \omega_j(\boldsymbol{\theta}, s|w) \Big] + \mathcal{K}(\phi_1, \phi_2; \varphi_1, \varphi_2; \theta_1, \theta_2) \Big\} \mathcal{P}, \quad \text{(A1)}$$

where $\mathcal{K}(\phi_1, \phi_2; \varphi_1, \varphi_2; \theta_1, \theta_2) = \mathcal{K}_{\boldsymbol{\phi}} + \mathcal{K}_{\boldsymbol{\varphi}} + \mathcal{K}_{\boldsymbol{\theta}} \equiv 0$ is the color-mixing member. In addition,

$$\mathcal{K}_{\boldsymbol{\phi}} = \varphi_1 + \varphi_2 - \theta_1 - \theta_2, \quad \mathcal{K}_{\boldsymbol{\theta}} = \phi_1 + \phi_2 - \varphi_1 - \varphi_2, \quad \mathcal{K}_{\boldsymbol{\varphi}} = \theta_1 + \theta_2 - \phi_1 - \phi_2.$$

Substituting the solution (28) into Equations (25) and (26), we find the following system of three loosely coupled PDEs:

$$\begin{cases} \partial_s \mathcal{P}_{\boldsymbol{\phi}} = \widehat{\mathcal{L}}_{\boldsymbol{\phi}}(\boldsymbol{\phi}, s|u)\mathcal{P}_{\boldsymbol{\phi}}, & \widehat{\mathcal{L}}_{\boldsymbol{\phi}} = \big\{\big(\bar{\varepsilon}_1^{(r)}\frac{\partial^2}{\partial \phi_1^2} + \bar{\varepsilon}_1^{(i)}\frac{\partial^2}{\partial \phi_2^2}\big) + \sum_{j=1}^{2}\frac{\partial}{\partial \phi_j}\sigma_j(\boldsymbol{\phi}, s|u) + \mathcal{K}_{\boldsymbol{\phi}}\big\}, \\ \partial_s \mathcal{P}_{\boldsymbol{\varphi}} = \widehat{\mathcal{L}}_{\boldsymbol{\varphi}}(\boldsymbol{\varphi}, s|v)\mathcal{P}_{\boldsymbol{\varphi}}, & \widehat{\mathcal{L}}_{\boldsymbol{\varphi}} = \big\{\big(\bar{\varepsilon}_2^{(r)}\frac{\partial^2}{\partial \varphi_1^2} + \bar{\varepsilon}_2^{(i)}\frac{\partial^2}{\partial \varphi_2^2}\big) + \sum_{j=1}^{2}\frac{\partial}{\partial \varphi_j}\pi_j(\boldsymbol{\varphi}, s|v) + \mathcal{K}_{\boldsymbol{\varphi}}\big\}, \\ \partial_s \mathcal{P}_{\boldsymbol{\theta}} = \widehat{\mathcal{L}}_{\boldsymbol{\theta}}(\boldsymbol{\theta}, s|w)\mathcal{P}_{\boldsymbol{\theta}}, & \widehat{\mathcal{L}}_{\boldsymbol{\theta}} = \big\{\big(\bar{\varepsilon}_3^{(r)}\frac{\partial^2}{\partial \theta_1^2} + \bar{\varepsilon}_3^{(i)}\frac{\partial^2}{\partial \theta_2^2}\big) + \sum_{j=1}^{2}\frac{\partial}{\partial \theta_j}\sigma_j(\boldsymbol{\theta}, s|w) + \mathcal{K}_{\boldsymbol{\theta}}\big\}. \end{cases} \tag{A2}$$

Now we can give the distributions $\mathcal{P}_{\boldsymbol{\phi}}(\boldsymbol{\phi}, s|u)$, $\mathcal{P}_{\boldsymbol{\varphi}}(\boldsymbol{\varphi}, s|v)$, and $\mathcal{P}_{\boldsymbol{\theta}}(\boldsymbol{\theta}, s|w)$ the meaning of probability densities and normalize them:

$$\begin{aligned} \bar{\mathcal{P}}_{\boldsymbol{\phi}}(\boldsymbol{\phi}, s|u, v, w) &= C^{-1}(u, v, w, s)\mathcal{P}_{\boldsymbol{\phi}}(\boldsymbol{\phi}, s|u), \\ \bar{\mathcal{P}}_{\boldsymbol{\varphi}}(\boldsymbol{\varphi}, s|u, v, w) &= C^{-1}(u, v, w, s)\mathcal{P}_{\boldsymbol{\varphi}}(\boldsymbol{\varphi}, s|v), \\ \bar{\mathcal{P}}_{\boldsymbol{\theta}}(\boldsymbol{\theta}, s|u, v, w) &= C^{-1}(u, v, w, s)\mathcal{P}_{\boldsymbol{\theta}}(\boldsymbol{\theta}, s|w), \end{aligned} \tag{A3}$$

where $C(u, v, w, s)$ is the normalization constant defined by the expression

$$C = \int\int_{\Xi^2_{\{\boldsymbol{\phi}\}}} \mathcal{P}_{\boldsymbol{\phi}}(\boldsymbol{\phi}, s|u)d\phi_1 d\phi_2 + \int\int_{\Xi^2_{\{\boldsymbol{\varphi}\}}} \mathcal{P}_{\boldsymbol{\varphi}}(\boldsymbol{\varphi}, s|v)d\varphi_1 d\varphi_2 + \int\int_{\Xi^2_{\{\boldsymbol{\theta}\}}} \mathcal{P}_{\boldsymbol{\theta}}(\boldsymbol{\theta}, s|w)d\theta_1 d\theta_2.$$

Obviously, in this case, the following equality holds:

$$\begin{aligned} \int\int_{\Xi^2_{\{\boldsymbol{\phi}\}}} \bar{\mathcal{P}}_{\boldsymbol{\phi}}(\boldsymbol{\phi}, s|u, v, w)d\phi_1 d\phi_2 + \int\int_{\Xi^2_{\{\boldsymbol{\varphi}\}}} \bar{\mathcal{P}}_{\boldsymbol{\varphi}}(\boldsymbol{\varphi}, s|u, v, w)d\varphi_1 d\varphi_2 \\ + \int\int_{\Xi^2_{\{\boldsymbol{\theta}\}}} \bar{\mathcal{P}}_{\boldsymbol{\theta}}(\boldsymbol{\theta}, s|u, v, w)d\theta_1 d\theta_2 = 1. \end{aligned} \tag{A4}$$

Now we can similarly construct a measure of the function space $\Xi_{\{\chi(s)\}}$ and perform functional integration of the full wave function (see Sections 7 and 8):

$$\begin{aligned} \overline{\Psi}(u, v, w) &= \mathbb{E}[\check{\Psi}(u, v, w|\{\lambda\})] = \frac{\Psi_1(u)\Psi_2(v)\Psi_3(w)}{C(u, v, w, s)} \times \\ &\prod_{\chi=\boldsymbol{\phi},\boldsymbol{\varphi},\boldsymbol{\theta}} \int\int_{\Xi^2_{\{\chi\}}} \big[Q^{(r)}_{\chi}(\chi, s|\xi) + iQ^{(i)}_{\chi}(\chi, s|\xi)\big]d\chi_1 d\chi_2, \qquad \xi = u, v, w, \end{aligned} \tag{A5}$$

where the solutions $Q^{(r)}_{\chi}$ and $Q^{(i)}_{\chi}$ satisfy the following system of PDEs:

$$\begin{cases} \partial_s Q^{(r)}_{\chi} = \{\widehat{\mathcal{L}}_{\chi} + \chi_1\}Q^{(r)}_{\chi} - \chi_2 Q^{(i)}_{\chi}, \\ \partial_s Q^{(i)}_{\chi} = \{\widehat{\mathcal{L}}_{\chi} + \chi_1\}Q^{(i)}_{\chi} + \chi_2 Q^{(r)}_{\chi}. \end{cases} \tag{A6}$$

As in the case of functions $\mathcal{P}_{\boldsymbol{\phi}}$, $\mathcal{P}_{\boldsymbol{\varphi}}$, and $\mathcal{P}_{\boldsymbol{\theta}}$, we can give these solutions the meaning of probabilities density and normalize them. In particular, we can write

$$\begin{aligned} \bar{Q}^{(r)}_{\chi}(\chi, s|u, v, w) &= C_*^{-1}(u, v, w, s)Q^{(r)}_{\chi}(\chi, s|\xi), \\ \bar{Q}^{(i)}_{\chi}(\chi, s|u, v, w) &= C_*^{-1}(u, v, w, s)Q^{(i)}_{\chi}(\chi, s|\xi), \end{aligned} \tag{A7}$$

where $C_*(u, v, w, s)$ is the normalization constant, which is defined as follows:

$$C_*(u, v, w, s) = \sum_{\chi=\boldsymbol{\phi},\boldsymbol{\varphi},\boldsymbol{\theta}} \sum_{v=i,r} \int\int_{\Xi^2_{\{\chi\}}} Q^{(v)}_{\chi}(\boldsymbol{\phi}, s|\xi)d\chi_1 d\chi_2. \tag{A8}$$

In addition, there will be conservation of total probability:

$$\sum_{\chi=\boldsymbol{\phi},\boldsymbol{\varphi},\boldsymbol{\theta}} \sum_{v=i,r} \int\int_{\Xi^2_{\{\chi\}}} \bar{Q}^{(v)}_{\chi}(\boldsymbol{\phi}, s|\xi)d\chi_1 d\chi_2 = 1. \tag{A9}$$

Thus, we have combined and normalized the probability distributions of all quark colors, which allows us to be confident that we have synchronized the quark colors so that the nucleon color will be invariant with respect to the internal motion of the nucleon. Obviously, such synchronization will mean taking into account three-quark interactions in the system.

Note that the choice of a model for connecting the probabilities $Q^{(r)}_{\chi}(\chi, s|u, v, w)$ and $Q^{(i)}_{\chi}(\chi, s|u, v, w)$ or the parameter $\mathcal{K}(\phi_1, \phi_2; \varphi_1, \varphi_2; \theta_1, \theta_2)$ is not limited by any physical principle and therefore can be further refined, taking into account the requirements of the considered problem. In the end, we note that if in the calculations we replace the operator $\widehat{\mathcal{L}}_{\chi}$ with $\widehat{\mathcal{L}}^{(0)}_{\chi} = \widehat{\mathcal{L}}_{\chi}|_{\mathcal{K}_{\chi}=0}$ (see Equation (A2)), then our consideration of the problem will have a two-particle approximation (see Section 8).

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