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Dual Quaternion Matrix Equation $AXB = C$ with Applications

Yan Chen [1], Qing-Wen Wang [1,2,*] and Lv-Ming Xie [1]

[1] Department of Mathematics, Newtouch Center for Mathematics, Shanghai University, Shanghai 200444, China; 198383092@shu.edu.cn (Y.C.); xlm1817@shu.edu.cn (L.-M.X.)
[2] Collaborative Innovation Center for the Marine Artificial Intelligence, Shanghai 200444, China
* Correspondence: wqw@shu.edu.cn

Abstract: Dual quaternions have wide applications in automatic differentiation, computer graphics, mechanics, and others. Due to its application in control theory, matrix equation $AXB = C$ has been extensively studied. However, there is currently limited information on matrix equation $AXB = C$ regarding the dual quaternion algebra. In this paper, we provide the necessary and sufficient conditions for the solvability of dual quaternion matrix equation $AXB = C$, and present the expression for the general solution when it is solvable. As an application, we derive the ϕ-Hermitian solutions for dual quaternion matrix equation $AXA^{\phi} = C$, where the ϕ-Hermitian extends the concepts of Hermiticity and η-Hermiticity. Lastly, we present a numerical example to verify the main research results of this paper.

Keywords: dual quaternion; Moore–Penrose inverse; general solution; ϕ-Hermitian solution

MSC: 15A03; 15A09; 15A24; 15B33; 15B57

1. Introduction

Let $\mathbb{R}$ denote the set of real numbers and $\mathbb{H}^{m\times n}$ stand for the space of all $m \times n$ matrices over quaternions

$$\mathbb{H} = \{u_0 + u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k} | \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1,\ u_0, u_1, u_2, u_3 \in \mathbb{R}\}.$$

The symbols $r(A)$, A^*, I, and $\mathbf{0}$ are defined by the rank of a given quaternion matrix A, the conjugate transpose of A, identity matrix, and zero matrix with appropriate sizes, respectively. The Moore–Penrose inverse of $A \in \mathbb{H}^{l\times k}$ is denoted as $A^{\dagger}$, which is defined as the solution of $AYA = A$, $YAY = Y$, $(AY)^* = AY$, and $(YA)^* = YA$. Moreover, let $L_A = I - A^{\dagger}A$ and $R_A = I - AA^{\dagger}$ represent two projectors along A.

Since Hamilton's discovery of quaternions in 1843, quaternions and quaternion matrices have found a large amount of practical applications in fields such as computer science, statistics, quantum physics, signal and color image processing, flight mechanics, aerospace technology, and so on (see, e.g., [1–4]). Furthermore, quaternion matrix equations also have significant applications in many fields, such as system and control theory.

Up to this point, matrix equations have witnessed a large number of papers proposing various methods for solving some matrix equations (see, e.g., [5–10]). The classical matrix equation

$$AXB = C, \tag{1}$$

has been studied by many authors. Ben-Israel and Greville [11] eatablished the necessary and sufficient conditions for the solvability of matrix Equation (1). In 2003, Liao and Bai [12] investigated the least-squares solution of matrix Equation (1) over symmetric positive semidefinite matrices. Huang et al. [13] provided the skew-symmetric solution

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and the optimal approximate solution of matrix Equation (1). Peng [14] derived the centrosymmetric solution of matrix Equation (1). Deng et al. [15] studied the general expressions regarding the Hermitian solutions of matrix Equation (1). Xie and Wang [16] considered the reducible solution to matrix Equation (1) when it is solvable. As a special case of matrix Equation (1), the Hermitian solution X to matrix equation

$$AXA^{*} = B \tag{2}$$

has attracted extensive attention (see, e.g., [17,18]). Baksalary [19] and Größ [20] studied the nonnegative definite and positive definite solutions to matrix Equation (2), respectively. For $\eta \in \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$, a quaternion matrix A is called η-Hermitian and η-skew-Hermitian if $A = A^{\eta^{*}}$ and $A = -A^{\eta^{*}}$, respectively, where $A^{\eta^{*}} = -\eta A^{*} \eta$ [21]. The utilization of η-Hermitian matrices in linear modeling is extensively recognized [22]. Kyrchei [23] derived the explicit determinantal representation formulas of η-Hermitian and η-skew-Hermitian solutions to the quaternion matrix equation

$$AXA^{\eta *} = B.$$

It is well-known that dual numbers and dual quaternions have wide applications in computer graphics, automatic differentiation, geometry, mechanics, rigid body motions, and robotics (see, e.g., [24–26]). For the related definitions of dual numbers and dual quaternions, please see Section 2.

So far, there has been little information on matrix Equation (1) regarding dual quaternion algebra. Motivated by the work mentioned above, in this paper, we aim to investigate the general solution of dual quaternion matrix Equation (1) by using Moore–Penrose inverses and ranks of matrices. Since the ϕ-Hermitian serves as an extended form of both Hermiticity and η-Hermiticity over the quaternions [27], we also provide the definition of ϕ-Hermiticity over the dual quaternions. As an application, we establish the ϕ-Hermitian solution of a special dual quaternion matrix equation

$$AXA^{\phi} = C, \; C^{\phi} = C. \tag{3}$$

Further details regarding ϕ-Hermitian matrices will be illustrated in Section 2.

This paper is organized as follows. In Section 2, we provide an overview of essential definitions and lemmas that will be applied in the subsequent sections. In Section 3, we establish some necessary and sufficient conditions for solvability regarding dual quaternion matrix Equation (1) and consider some special cases of dual quaternion matrix Equation (1). As an application, we investigate the ϕ-Hermitian solution of dual quaternion matrix Equation (3) in Section 4. In Section 5, we present a numerical example to illustrate the results of this paper. Finally, a brief conclusion is provided in Section 6.

2. Preliminaries

In this section, we review some definitions of dual numbers, dual quaternions, and related propositions. Moreover, we introduce the definitions of dual quaternion matrix and ϕ-Hermitian matrix, which are fundamental for obtaining the main results.

Definition 1 ([28])**.** *Suppose that $x_0, x_1 \in \mathbb{R}$; we say x is a dual number if x has the form*

$$x = x_0 + x_1 \epsilon,$$

where ϵ is the infinitesimal unit, satisfying $\epsilon^2 = 0$.

We call x_0 the real part or the standard part of x, x_1 as the dual part or the infinitesimal part of x. The infinitesimal unit ϵ is commutative in multiplication with real numbers, complex numbers, and quaternions. The set of dual numbers is denoted by

$$\mathbb{D} = \{x = x_0 + x_1\epsilon | \epsilon^2 = 0, x_0, x_1 \in \mathbb{R}\}.$$

Assume that $x = x_0 + x_1\epsilon, y = y_0 + y_1\epsilon \in \mathbb{D}$; we have $x = y$ if $x_0 = y_0$ and $x_1 = y_1$; regarding addition and multiplication, there is

$$x + y = x_0 + y_0 + (x_1 + y_1)\epsilon,$$
$$xy = x_0y_0 + (x_0y_1 + x_1y_0)\epsilon.$$

Definition 2 ([28])**.** *Let $z_0, z_1 \in \mathbb{H}$. We say z is a dual quaternion if z has the form*

$$z = z_0 + z_1\epsilon,$$

where ϵ is the infinitesimal unit, satisfying $\epsilon^2 = 0$, z_0, z_1 as the real part and the dual part of z, respectively.

The collection of dual quaternions is denoted by

$$\mathbb{DQ} = \{z = z_0 + z_1\epsilon | \epsilon^2 = 0, z_0, z_1 \in \mathbb{H}\}.$$

Now, we introduce the definition of dual quaternion matrix. Let $X_0, X_1 \in \mathbb{H}^{m\times n}$. X is said to be a dual quaternion matrix if X has the form $X = X_0 + X_1\epsilon$; the set of dual quaternion matrices is denoted by

$$\mathbb{DQ}^{m\times n} = \{X = X_0 + X_1\epsilon | \epsilon^2 = 0, X_0, X_1 \in \mathbb{H}^{m\times n}\}.$$

The conjugate transpose of X is defined as $X^* = X_0^* + X_1^*\epsilon$. For $Y = Y_0 + Y_1\epsilon \in \mathbb{DQ}^{m\times n}$, by analogy, we have $X = Y$ if $X_0 = Y_0$ and $X_1 = Y_1$; furthermore,

$$X + Y = X_0 + Y_0 + (X_1 + Y_1)\epsilon,$$
$$XY = X_0Y_0 + (X_0Y_1 + X_1Y_0)\epsilon.$$

To facilitate our study on the ϕ-Hermitian, we first review the concept of nonstandard involution over quaternions, and then proceed to generalize it to dual quaternions.

Definition 3 ([27])**.** *A map $\phi\colon \mathbb{H} \to \mathbb{H}$ is called an antiendomorphism if $\phi(pq) = \phi(q)\phi(p)$ and $\phi(p+q) = \phi(p) + \phi(q)$ for all $p, q \in \mathbb{H}$. An antiendomorphism ϕ is called an involution if $\phi(\phi(p)) = p$ for every $p \in \mathbb{H}$.*

Definition 4 ([27])**.** *Under the basis $(1, i, j, k)$, an involution ϕ is called nonstandard if and only if ϕ can be expressed as a real matrix*

$$\phi = \begin{pmatrix} 1 & 0 \\ 0 & T \end{pmatrix},$$

where T is a 3×3 real orthoganal symmetric matrix with eigenvalues $1, 1, -1$.

Proposition 1 ([27])**.** *Let $z \in \mathbb{H}$. Then, every nonstandard involution ϕ of $\mathbb{H}$ has the form $\phi(z) = \beta^{-1}z^*\beta$ for some $\beta \in \mathbb{H}$ with $\beta^2 = -1$.*

Definition 5 ([27])**.** *For a nonstandard involution ϕ, $Z \in \mathbb{H}^{m\times n}$, we denote by Z^ϕ the matrix obtained by applying ϕ entrywise to the transposed matrix of Z.*

For example, if ϕ is such that $\phi(i) = i$, $\phi(j) = -j$, $\phi(k) = k$, then

$$\begin{pmatrix} 1+k & j \\ i & 3+k \\ j+k & i-j \end{pmatrix}^{\phi} = \begin{pmatrix} 1+k & i & -j+k \\ -j & 3+k & i+j \end{pmatrix}.$$

Definition 6 ([27])**.** *For a nonstandard involution ϕ, a quaternion matrix Z is said to be ϕ-Hermitian if $Z = Z^{\phi}$. In fact, $Z^{\phi} = \beta^{-1}Z^{*}\beta$ for some $\beta \in \mathbb{H}$ with $\beta^{2} = -1$.*

Proposition 2. *Let $Z \in \mathbb{H}^{m\times n}$, $\beta \in \mathbb{H}$, and $\beta^2 = -1$. Then,*

(1) $(Z^{\phi})^{\dagger} = (Z^{\dagger})^{\phi}$,
(2) $(L_Z)^{\phi} = R_{Z^{\phi}}$,
(3) $(R_Z)^{\phi} = L_{Z^{\phi}}$.

Proof. Regarding the proof of (1), it is obvious that $\beta^{-1} = -\beta, \beta^{*} = -\beta$; by the definition of Moore–Penrose inverse, we obtain

$$\begin{aligned} Z^{\phi}(Z^{\dagger})^{\phi}Z^{\phi} &= (-\beta Z^{*}\beta)[-\beta(Z^{\dagger})^{*}\beta](-\beta Z^{*}B) \\ &= -\beta Z^{*}(Z^{\dagger})^{*}Z^{*}\beta = -\beta(ZZ^{\dagger}Z)^{*}\beta \\ &= -\beta Z^{*}\beta = Z^{\phi}, \\ (Z^{\dagger})^{\phi}Z^{\phi}(Z^{\dagger})^{\phi} &= [-\beta(Z^{\dagger})^{*}\beta](-\beta Z^{*}\beta)[-\beta(Z^{\dagger})^{*}\beta] \\ &= -\beta(Z^{\dagger})^{*}Z^{*}(Z^{\dagger})^{*}\beta = -\beta(Z^{\dagger}ZZ^{\dagger})^{*}\beta \\ &= -\beta(Z^{\dagger})^{*}\beta = (Z^{\dagger})^{\phi}, \\ [Z^{\phi}(Z^{\dagger})^{\phi}]^{*} &= [(-\beta Z^{*}\beta)(-\beta(Z^{\dagger})^{*}\beta)]^{*} \\ &= [-\beta Z^{*}(Z^{\dagger})^{*}\beta]^{*} = (-\beta Z^{\dagger}Z\beta)^{*} \\ &= -\beta(Z^{\dagger}Z)^{*}\beta = (-\beta Z^{*}\beta)[-\beta(Z^{\dagger})^{*}\beta] \\ &= Z^{\phi}(Z^{\dagger})^{\phi}, \\ [(Z^{\dagger})^{\phi}Z^{\phi}]^{*} &= [-\beta(Z^{\dagger})^{*}\beta(-\beta Z^{*}\beta)]^{*} \\ &= [-\beta(Z^{\dagger})^{*}Z^{*}\beta]^{*} = (-\beta ZZ^{\dagger}\beta)^{*} \\ &= -\beta(ZZ^{\dagger})^{*}\beta = [-\beta(Z^{\dagger})^{*}\beta](-\beta Z^{*}\beta) \\ &= (Z^{\dagger})^{\phi}Z^{\phi}. \end{aligned} \tag{4}$$

Based on this, we can deduce that $(Z^{\dagger})^{\phi}$ is the Moore–Penrose inverse of Z^{ϕ}.
For (2), we have

$$(L_Z)^{\phi} = (I - Z^{\dagger}Z)^{\phi} = I - Z^{\phi}(Z^{\dagger})^{\phi} = I - Z^{\phi}(Z^{\phi})^{\dagger} = R_{Z^{\phi}}.$$

In a similar vein to (2), we can offer a demonstration for (3); therefore, we omit it here. □

By analogy, we propose the definition of ϕ-Hermiticity with respect to dual quaternion matrix, where ϕ is a nonstandard involution.

Definition 7. *For $X = X_0 + X_1\epsilon \in \mathbb{DQ}^{m\times n}$, X is called ϕ-Hermitian matrix if $X = X^{\phi}$, where*

$$X^{\phi} :\ = \beta^{-1}X^{*}\beta = \beta^{-1}X_0^{*}\beta + \beta^{-1}X_1^{*}\beta\epsilon = X_0^{\phi} + X_1^{\phi}\epsilon,$$

with $\beta \in \mathbb{H}$ and $\beta^2 = -1$.

For example, if ϕ is such that $\phi(i) = i,\ \phi(j) = -j,\ \phi(k) = k$,

$$X = \begin{pmatrix} 1 & 2+i \\ 2+i & 3+k \end{pmatrix}^{\phi} + \begin{pmatrix} 3+i & k \\ k & 2i+3k \end{pmatrix}^{\phi} \epsilon$$
$$= \begin{pmatrix} 1 & 2+i \\ 2+i & 3+k \end{pmatrix} + \begin{pmatrix} 3+i & k \\ k & 2i+3k \end{pmatrix} \epsilon,$$

then X is a ϕ-Hermitian matrix.

Proposition 3 ([27])**.** *Let $X, Y \in \mathbb{H}^{m\times n}, \alpha, \beta \in \mathbb{H}$. Then,*

(1) $(\alpha X + \beta Y)^{\phi} = X^{\phi}\phi(\alpha) + Y^{\phi}\phi(\beta)$,
(2) $(X\alpha + Y\beta)^{\phi} = \phi(\alpha)X^{\phi} + \phi(\beta)Y^{\phi}$,
(3) $(XY)^{\phi} = Y_{\phi}X_{\phi}$,
(4) $(X^{\phi})^{\phi} = X$.

Having outlined the properties of ϕ-Hermitian matrix over quaternions, we now present the corresponding properties of ϕ-Hermitian matrix over the dual quaternion algebra.

Proposition 4. *Let $X, Y \in \mathbb{DQ}^{n\times n}$. Then,*

(1) $(X + Y)^{\phi} = X^{\phi} + Y^{\phi}$,
(2) $(XY)^{\phi} = Y^{\phi}X^{\phi}$,
(3) $(X^{\phi})^{\phi} = X$.

Proof. By the algebraic properties of ϕ, we have

$$\begin{aligned}(X+Y)^{\phi} &= [(X_0+Y_0) + (X_1+Y_1)\epsilon]^{\phi} \\ &= (X_0+Y_0)^{\phi} + (X_1+Y_1)^{\phi}\epsilon \\ &= X_0^{\phi} + Y_0^{\phi} + X_1^{\phi}\epsilon + Y_1^{\phi}\epsilon \\ &= X_0^{\phi} + X_1^{\phi}\epsilon + Y_0^{\phi} + Y_1^{\phi}\epsilon \\ &= X^{\phi} + Y^{\phi}.\end{aligned}$$

In relation to (2), we obtain

$$\begin{aligned}(XY)^{\phi} &= [X_0Y_0 + (X_0Y_1 + X_1Y_0)\epsilon]^{\phi} \\ &= (X_0Y_0)^{\phi} + (X_0Y_1 + X_1Y_0)^{\phi}\epsilon \\ &= Y_0^{\phi}X_0^{\phi} + Y_1^{\phi}X_0^{\phi}\epsilon + Y_0^{\phi}X_1^{\phi}\epsilon \\ &= (Y_0^{\phi} + Y_1^{\phi}\epsilon)(X_0^{\phi} + X_1^{\phi}\epsilon) \\ &= Y^{\phi}X^{\phi}.\end{aligned}$$

In terms of (3), we have

$$\begin{aligned}(X^{\phi})^{\phi} &= [(X_0 + X_1\epsilon)^{\phi}]^{\phi} \\ &= (X_0^{\phi} + X_1^{\phi}\epsilon)^{\phi} \\ &= X_0 + X_1\epsilon \\ &= X.\end{aligned}$$

□

Now, we provide a few lemmas, which are basic tools for obtaining the key outcomes.

Lemma 1 ([29])**.** *Suppose that* A*,* B*, and* C *are provided for matrices with the adequate dimensions over* $\mathbb{H}$*; then, quaternion matrix Equation* (1) *is consistent if and only if*

$$R_A C = 0, \quad CL_B = 0.$$

In this case, the general solution can be expressed as

$$X = A^{\dagger} C B^{\dagger} + L_A U + V R_B,$$

where U, V *are any matrices over* $\mathbb{H}$ *with appropriate dimensions.*

Lemma 2 ([16])**.** *Let* A_1, A_2, B_1, B_2*, and* C_1 *have matrices with appropriate sizes. Set*

$$A = R_{A_1} C,\ B = B_1 L_{B_2},\ M = R_{A_1} A_2,\ C_1 = C L_{B_2}.$$

Then, the following descriptions are equivalent:

(1) *The quaternion matrix equation*

$$A_1 X_1 B_1 + A_1 X_2 B_2 + A_2 X_3 B_2 = C \tag{5}$$

is consistent.

(2) $R_M A = 0,\ R_{A_1} C L_{B_2} = 0,\ C_1 L_B = 0.$

(3)

$$\begin{aligned} r\begin{pmatrix} A_1 & A_2 & C \end{pmatrix} &= r\begin{pmatrix} A_1 & A_2 \end{pmatrix}, \\ r\begin{pmatrix} B_2 & 0 \\ C & A_1 \end{pmatrix} &= r(B_2) + r(A_1), \\ r\begin{pmatrix} C_1 \\ B_1 \\ B_2 \end{pmatrix} &= r\begin{pmatrix} B_1 \\ B_2 \end{pmatrix}. \end{aligned}$$

In this case, the general solution to (5) *can be expressed as follows:*

$$\begin{aligned} X_1 &= A_1^{\dagger} C_1 B^{\dagger} + L_{A_1} V_1 + V_2 R_B, \\ X_2 &= A_1^{\dagger}(C - A_1 X_1 B_1 - A_2 X_3 B_2) B_2^{\dagger} + T_1 R_{B_2} + L_{A_1} T_2, \\ X_3 &= M^{\dagger} A B_2^{\dagger} + L_M U_1 + U_2 R_{B_2}, \end{aligned}$$

where U_1, U_2, V_1, V_2, T_1*, and* T_2 *are arbitrary matrices over* $\mathbb{H}$ *with appropriate sizes.*

The following lemma, originally derived by Marsaglia and Styan [30], can be extended to $\mathbb{H}$.

Lemma 3. *Assume that* $A \in \mathbb{H}^{m\times n}$*,* $B \in \mathbb{H}^{m\times k}$*,* $C \in \mathbb{H}^{l\times n}$*,* $D \in \mathbb{H}^{j\times k}$*, and* $E \in \mathbb{H}^{l\times i}$*; then, we have the following rank equality:*

(1) $r(R_A B) = r\begin{pmatrix} A & B \end{pmatrix} - r(A).$

(2) $r(CL_B) = r\begin{pmatrix} B \\ C \end{pmatrix} - r(B).$

(3) $r(R_A B L_C) = r\begin{pmatrix} B & A \\ C & 0 \end{pmatrix} - r(A) - r(C).$

(4) $r\begin{pmatrix} R_A B \\ C \end{pmatrix} = r\begin{pmatrix} B & A \\ C & 0 \end{pmatrix} - r(A).$

(5) $r\begin{pmatrix} AL_B & C \end{pmatrix} = r\begin{pmatrix} A & C \\ B & 0 \end{pmatrix} - r(B).$

(6) $$r\begin{pmatrix} A & BL_D \\ R_E C & 0 \end{pmatrix} = r\begin{pmatrix} A & B & 0 \\ C & 0 & E \\ 0 & D & 0 \end{pmatrix} - r(D) - r(E).$$

3. The Solution of Matrix Equation (1)

In this section, we establish the necessary and sufficient conditions for the solvability of dual quaternion matrix Equation (1) and provide the expressions of its general solution. Additionally, we investigate some special cases of dual quaternion matrix Equation (1).

Theorem 1. *Let $A = A_0 + A_1\epsilon \in \mathbb{DQ}^{m\times n}$, $B = B_0 + B_1\epsilon \in \mathbb{DQ}^{k\times l}$, $C = C_0 + C_1\epsilon \in \mathbb{DQ}^{m\times l}$ be known. Put*

$$A_2 = A_1 L_{A_0},\ B_2 = R_{B_0} B_1,\ C_{11} = A_0 A_0^{\dagger} C_0 B_0^{\dagger} B_1, \tag{6}$$

$$C_{22} = A_1 A_0^{\dagger} C_0 B_0^{\dagger} B_0,\ C_2 = C_1 - C_{11} - C_{22}, \tag{7}$$

$$M = R_{A_0} A_2,\ N = R_{A_0} C_2,\ E = B_2 L_{B_0},\ F = C_2 L_{B_0}. \tag{8}$$

Then, the following statements are equivalent:

(1) Dual quaternion matrix Equation (1) is consistent.

(2)

$$R_{A_0} C_0 = 0,\ C_0 L_{B_0} = 0, \tag{9}$$

$$R_M N = 0,\ R_{A_0} C_2 L_{B_0} = 0,\ F L_E = 0. \tag{10}$$

(3)

$$r\begin{pmatrix} A_0 & C_0 \end{pmatrix} = r(A_0), r\begin{pmatrix} B_0 \\ C_0 \end{pmatrix} = r(B_0), \tag{11}$$

$$r\begin{pmatrix} A_1 & A_0 & C_1 \\ A_0 & 0 & C_0 \end{pmatrix} = r\begin{pmatrix} A_1 & A_0 \\ A_0 & 0 \end{pmatrix}, \tag{12}$$

$$r\begin{pmatrix} C_1 & A_0 \\ B_0 & 0 \end{pmatrix} = r(A_0) + r(B_0), \tag{13}$$

$$r\begin{pmatrix} B_1 & B_0 \\ B_0 & 0 \\ C_1 & C_0 \end{pmatrix} = r\begin{pmatrix} B_1 & B_0 \\ B_0 & 0 \end{pmatrix}. \tag{14}$$

In this case, the general solution X of dual quaternion matrix Equation (1) can be expressed as $X = X_0 + X_1\epsilon$, where

$$\begin{aligned} X_0 &= A_0^{\dagger} C_0 B_0^{\dagger} + L_{A_0} U + V R_{B_0}, \\ X_1 &= A_0^{\dagger}(C_2 - A_0 V B_2 - A_2 U B_0) B_0^{\dagger} + W_1 R_{B_0} + L_{A_0} W_2, \\ U &= M^{\dagger} N B_0^{\dagger} + L_M Q_1 + Q_2 R_{B_0}, \\ V &= A_0^{\dagger} F E^{\dagger} + L_{A_0} Q_3 + Q_4 R_E, \end{aligned} \tag{15}$$

and $Q_i(i = \overline{1,4})$, $W_i(i = \overline{1,2})$ are arbitrary matrices over $\mathbb{H}$ with appropriate dimensions.

Proof. (1) $\Leftrightarrow$ (2): Suppose that dual quaternion matrix Equation (1) is solvable and its solution is $X \in \mathbb{DQ}^{m\times n}$, which can be expressed as

$$X = X_0 + X_1\epsilon, \tag{16}$$

substituting (16) into (1), by the definition of equality of dual quaternion matrices, we can obtain that dual quaternion matrix Equation (1) is equivalent to the system of quaternion matrix equations

$$\begin{cases} A_0 X_0 B_0 = C_0, \\ A_0 X_0 B_1 + A_0 X_1 B_0 + A_1 X_0 B_0 = C_1. \end{cases} \tag{17}$$

The structure of the proof goes as follows. We first prove that $(1) \Leftrightarrow (2)$ and illustrate the general solution of (17) with the form of (15), and then we prove $(2) \Leftrightarrow (3)$.

We divide the system (17) into the following:

$$A_0 X_0 B_0 = C_0, \tag{18}$$

and

$$A_0 X_0 B_1 + A_0 X_1 B_0 + A_1 X_0 B_0 = C_1. \tag{19}$$

By Lemma 1, we obtain that (18) is consistent if and only if

$$R_{A_0} C_0, \; C_0 L_{B_0} = 0.$$

In this case, the general solution of (18) can be written as

$$X_0 = A_0^{\dagger} C_0 B_0^{\dagger} + L_{A_0} U + V R_{B_0}, \tag{20}$$

where U, V are any matrices over $\mathbb{H}$ with appropriate sizes.

Substituting Equation (20) into Equation (19) provides

$$A_0 V B_2 + A_0 X_1 B_0 + A_2 U B_0 = C_2, \tag{21}$$

where A_2, B_2, and C_2 are defined by (6) and (7). Using Lemmas 2 to (21), we know that matrix Equation (21) is solvable if and only if

$$R_M N = 0, \; R_{A_0} C_2 L_{B_0} = 0, \; F L_E = 0,$$

where M, N, E, and F are provided by (8). In this case, the general solution of (21) can be expressed as

$$X_1 = A_0^{\dagger}(C_2 - A_0 V B_2 - A_2 U B_0) B_0^{\dagger} + W_1 R_{B_0} + L_{A_0} W_2, \tag{22}$$

$$U = M^{\dagger} N B_0^{\dagger} + L_M Q_1 + Q_2 R_{B_0}, \tag{23}$$

$$V = A_0^{\dagger} F E^{\dagger} + L_{A_0} Q_3 + Q_4 R_E, \tag{24}$$

where Q_1, Q_2, Q_3, Q_4, W_1, and W_2 are any matrices with the suitable dimensions over $\mathbb{H}$. To sum up, we have shown that matrix Equation (1) has a dual solution $X \in \mathbb{DQ}^{m \times n}$ if and only if (2) holds.

$(2) \Leftrightarrow (3)$: We divide it into two parts to prove its equivalence.

- Firstly, we prove that (9) holds if and only if (11) holds. According to Lemma 3, it is easy to verify that $(9) \Leftrightarrow (11)$.
- Now, we turn to prove that $(10) \Leftrightarrow (12) - (14)$. Let $X_0 = A_0^{\dagger} C_0 B_0^{\dagger}$. Then, it is easy to verify that X_0 is a particular solution to the matrix equation $A_0 X_0 B_0 = C_0$. By Lemma 3 and block elementary operations, we obtain

$$\begin{aligned}
R_M N = 0 &\Leftrightarrow r(R_M N) = 0 \Leftrightarrow r\begin{pmatrix} R_{A_0}A_2 & R_{A_0}C_2 \end{pmatrix} = r(R_{A_0}A_2),\\
&\Leftrightarrow r\begin{pmatrix} A_0 & A_2 & C_2 \end{pmatrix} = r\begin{pmatrix} A_0 & A_2 \end{pmatrix},\\
&\Leftrightarrow r\begin{pmatrix} A_0 & A_1L_{A_0} & C_2 \end{pmatrix} = r\begin{pmatrix} A_0 & A_1L_{A_0} \end{pmatrix},\\
&\Leftrightarrow r\begin{pmatrix} A_1 & A_0 & C_2 \\ A_0 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} A_1 & A_0 \\ A_0 & 0 \end{pmatrix},\\
&\Leftrightarrow r\begin{pmatrix} A_1 & A_0 & C_1 - A_0A_0^{\dagger}C_0B_0^{\dagger}B_1 - A_1A_0^{\dagger}C_0B_0^{\dagger}B_0 \\ A_0 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} A_1 & A_0 \\ A_0 & 0 \end{pmatrix},\\
&\Leftrightarrow r\begin{pmatrix} A_1 & A_0 & C_1 \\ A_0 & 0 & A_0A_0^{\dagger}C_0B_0^{\dagger}B_0 \end{pmatrix} = r\begin{pmatrix} A_1 & A_0 \\ A_0 & 0 \end{pmatrix},\\
&\Leftrightarrow r\begin{pmatrix} A_1 & A_0 & C_1 \\ A_0 & 0 & C_0 \end{pmatrix} = r\begin{pmatrix} A_1 & A_0 \\ A_0 & 0 \end{pmatrix},
\end{aligned}$$

$$\begin{aligned}
R_{A_0}C_2L_{B_0} = 0 &\Leftrightarrow r(R_{A_0}C_2L_{B_0}) = 0 \Leftrightarrow r\begin{pmatrix} C_2 & A_0 \\ B_0 & 0 \end{pmatrix} = r(A_0) + r(B_0),\\
&\Leftrightarrow r\begin{pmatrix} C_1 - C_{11} - C_{22} & A_0 \\ B_0 & 0 \end{pmatrix} = r(A_0) + r(B_0),\\
&\Leftrightarrow r\begin{pmatrix} C_1 - A_0A_0^{\dagger}C_0B_0^{\dagger}B_1 - A_1A_0^{\dagger}C_0B_0^{\dagger}B_0 & A_0 \\ B_0 & 0 \end{pmatrix} = r(A_0) + r(B_0),\\
&\Leftrightarrow r\begin{pmatrix} C_1 & A_0 \\ B_0 & 0 \end{pmatrix} = r(A_0) + r(B_0),\\
FL_E = 0 &\Leftrightarrow r(FL_E) = 0 \Leftrightarrow r\begin{pmatrix} E \\ F \end{pmatrix} = r(E) \Leftrightarrow r\begin{pmatrix} B_2L_{B_0} \\ C_2L_{B_0} \end{pmatrix} = r(B_2L_{B_0}),\\
&\Leftrightarrow r\begin{pmatrix} B_0 \\ B_2 \\ C_2 \end{pmatrix} = r\begin{pmatrix} B_0 \\ B_2 \end{pmatrix} \Leftrightarrow r\begin{pmatrix} B_0 \\ R_{B_0}B_1 \\ C_2 \end{pmatrix} = r\begin{pmatrix} B_0 \\ R_{B_0}B_1 \end{pmatrix},\\
&\Leftrightarrow r\begin{pmatrix} B_1 & B_0 \\ B_0 & 0 \\ C_2 & 0 \end{pmatrix} = r\begin{pmatrix} B_1 & B_0 \\ B_0 & 0 \end{pmatrix},\\
&\Leftrightarrow r\begin{pmatrix} B_1 & B_0 \\ B_0 & 0 \\ C_1 - A_0A_0^{\dagger}C_0B_0^{\dagger}B_1 - A_1A_0^{\dagger}C_0B_0^{\dagger}B_0 & 0 \end{pmatrix} = r\begin{pmatrix} B_1 & B_0 \\ B_0 & 0 \end{pmatrix},\\
&\Leftrightarrow r\begin{pmatrix} B_1 & B_0 \\ B_0 & 0 \\ C_1 & A_0A_0^{\dagger}C_0B_0^{\dagger}B_0 \end{pmatrix} = r\begin{pmatrix} B_1 & B_0 \\ B_0 & 0 \end{pmatrix},\\
&\Leftrightarrow r\begin{pmatrix} B_1 & B_0 \\ B_0 & 0 \\ C_1 & C_0 \end{pmatrix} = r\begin{pmatrix} B_1 & B_0 \\ B_0 & 0 \end{pmatrix}.
\end{aligned}$$

□

Now, we consider some special cases of dual quarernion matrix Equation (1).

Corollary 1 ([31]). *Assume that* $A = A_0 + A_1\epsilon \in \mathbb{DQ}^{m\times n}$, $C = C_0 + C_1\epsilon \in \mathbb{DQ}^{m\times l}$ *are given. Put*

$$A_2 = A_1L_{A_0},\ C_{22} = A_1A_0^{\dagger}C_0,\ C_2 = C_1 - C_{22},\ M = R_{A_0}A_2,\ N = R_{A_0}C_2. \tag{25}$$

Then, the following statements are equivalent:

(1) Dual quaternion matrix equation $AX = C$ *is consistent.*

(2)

$$R_{A_0}C_0 = 0,\ R_M N = 0. \tag{26}$$

(3)

$$r\begin{pmatrix} A_0 & C_0 \end{pmatrix} = r(A_0),\ r\begin{pmatrix} A_1 & A_0 & C_1 \\ A_0 & 0 & C_0 \end{pmatrix} = r\begin{pmatrix} A_1 & A_0 \\ A_0 & 0 \end{pmatrix}. \tag{27}$$

In this case, the general solution X of dual quaternion matrix equation $AX = C$ can be expressed as $X = X_0 + X_1\epsilon$, where

$$\begin{aligned} X_0 &= A_0^{\dagger}C_0 + L_{A_0}U, \\ X_1 &= A_0^{\dagger}(C_2 - A_2U) + L_{A_0}W_1, \\ U &= M^{\dagger}N + L_M W_2, \end{aligned} \tag{28}$$

and W_1, W_2 are arbitrary matrices over $\mathbb{H}$ with appropriate dimensions.

Corollary 2 ([31])**.** *Let $B = B_0 + B_1\epsilon \in \mathbb{DQ}^{k\times l}$, $C = C_0 + C_1\epsilon \in \mathbb{DQ}^{m\times l}$ be known. Denote*

$$B_2 = R_{B_0}B_1,\ C_{11} = C_0B_0^{\dagger}B_1, C_2 = C_1 - C_{11},\ E = B_2L_{B_0},\ F = C_2L_{B_0}. \tag{29}$$

Then, the following statements are equivalent:

(1) *Dual quaternion matrix equation $XB = C$ is consistent.*

(2)

$$C_0L_{B_0} = 0,\ FL_E = 0. \tag{30}$$

(3)

$$r\begin{pmatrix} B_0 \\ C_0 \end{pmatrix} = r(B_0),\ r\begin{pmatrix} B_1 & B_0 \\ B_0 & 0 \\ C_1 & C_0 \end{pmatrix} = r\begin{pmatrix} B_1 & B_0 \\ B_0 & 0 \end{pmatrix}. \tag{31}$$

In this case, the general solution can be expressed as $X = X_0 + X_1\epsilon$, where

$$\begin{aligned} X_0 &= C_0B_0^{\dagger} + VR_{B_0}, \\ X_1 &= (C_2 - VB_2)B_0^{\dagger} + W_1R_{B_0}, \\ V &= FE^{\dagger} + W_2R_E, \end{aligned} \tag{32}$$

and W_1, W_2 are arbitrary matrices over $\mathbb{H}$ with appropriate dimensions.

Remark 1. *Matrix equations $AX = B$ and $XC = D$ have significant applications in eigenvalue problems, image processing, and solving linear systems. However, the matrix equation $AXB = C$ is a general case that encompasses either matrix equation $AX = B$ or $XC = D$. Therefore, the applications regarding matrix equations $AX = B$ and $XC = D$ are applicable to matrix equation $AXB = C$.*

4. Applications

As an application of Theorem 1, we can investigate dual quaternion matrix Equation (3).

Theorem 2. *Suppose that $A = A_0 + A_1\epsilon \in \mathbb{DQ}^{m\times n}$, $C = C_0 + C_1\epsilon = C^{\phi} \in \mathbb{DQ}^{m\times m}$ are provided; denote*

$$\begin{aligned} &B_2 = R_{A_0^{\phi}}A_1^{\phi},\ C_{11} = A_0A_0^{\dagger}C_0(A_0^{\phi})^{\dagger}A_1^{\phi},\ C_{22} = A_1A_0^{\dagger}C_0(A_0^{\phi})^{\dagger}A_0^{\phi}, \\ &C_2 = C_1 - C_{11} - C_{22},\ M = R_{A_0}B_2^{\phi},\ N = R_{A_0}C_2. \end{aligned}$$

Then, the following statements are equivalent:

(1) *Dual quaternion matrix Equation (3) is consistent.*

(2) *The following equalities are satisfied:*

$$R_{A_0}C_0 = 0,\ R_M N = 0,\ R_{A_0}C_2 L_{A_0^\phi} = 0.$$

(3) *The following rank equalities hold:*

$$\begin{aligned}
&r\begin{pmatrix} A_0 & C_0 \end{pmatrix} = r(A_0),\\
&r\begin{pmatrix} A_1 & A_0 & C_1 \\ A_0 & 0 & C_0 \end{pmatrix} = r\begin{pmatrix} A_1 & A_0 \\ A_0 & 0 \end{pmatrix},\\
&r\begin{pmatrix} C_1 & A_0 \\ A_0^\phi & 0 \end{pmatrix} = r(A_0) + r\left(A_0^\phi\right) = 2r(A_0).
\end{aligned}$$

In this case, the general solution X of (3) can be expressed as $X = X_0 + X_1\epsilon$, *where*

$$X_0 = \frac{\widetilde{X_0} + \widetilde{X_0}^\phi}{2}, X_1 = \frac{\widetilde{X_1} + \widetilde{X_1}^\phi}{2}$$

and

$$\begin{aligned}
X_0 &= A_0^\dagger C_0 (A_0^\phi)^\dagger + L_{A_0}U + VR_{A_0^\phi},\\
X_1 &= A_0^\dagger (C_2 - A_0 V B_2 - B_2^\phi U A_0^\phi)(A_0^\phi)^\dagger + W_1 R_{A_0^\phi} + L_{A_0} W_2,\\
U &= M^\dagger N (A_0^\phi)^\dagger + L_M Q_1 + Q_2 R_{A_0^\phi},\\
V &= A_0^\dagger N^\phi (M^\phi)^\dagger + L_{A_0} Q_3 + Q_4 R_{M^\phi},
\end{aligned}$$

$Q_i(i = \overline{1,4})$, $W_i(i = \overline{1,2})$ *are any matrices with appropriate dimensions over* $\mathbb{H}$.

Proof. By using the definitions of equality of dual quaternion matrices and dual quaternion matrix multiplication, we can conclude that the consistency of dual quaternion matrix Equation (3) is contingent on the existence of the solutions to the system of quaternion matrix equations

$$\begin{cases} A_0 \widetilde{X_0} A_0^\phi = C_0, \\ A_0 \widetilde{X_0} A_1^\phi + A_0 \widetilde{X_1} A_0^\phi + A_1 \widetilde{X_0} A_0^\phi = C_1. \end{cases} \tag{33}$$

In fact, if matrix Equation (3) has a ϕ-Hermitian solution $X = X_0 + X_1\epsilon$, it is obvious that X_0 and X_1 must be solutions to (33). Conversely, if the system (33) has solutions $\widetilde{X_0}$ and $\widetilde{X_1}$, then matrix Equation (3) has solution $X = X_0 + X_1\epsilon$, where

$$X_0 = \frac{\widetilde{X_0} + \widetilde{X_0}^\phi}{2}, X_1 = \frac{\widetilde{X_1} + \widetilde{X_1}^\phi}{2}.$$

According to Theorem 1, we can present the necessary and sufficient conditions for the solvability of (33), along with the general expression for its solutions. □

5. Numerical Example

Now, we provide a numerical example to illustrate the main results of this paper.

Example 1. *Given the dual quaternion matrices:*

$$A = A_0 + A_1\epsilon = \begin{pmatrix} 2\boldsymbol{i}+\boldsymbol{k} & 3\boldsymbol{i}+\boldsymbol{j} \\ \boldsymbol{j} & 0 \\ 3\boldsymbol{j}-4\boldsymbol{k} & \boldsymbol{i}+\boldsymbol{k} \end{pmatrix} + \begin{pmatrix} 2-3\boldsymbol{i}+\boldsymbol{k} & \boldsymbol{i} \\ -3\boldsymbol{k} & \boldsymbol{i}-\boldsymbol{j} \\ 0 & 4\boldsymbol{i}+\boldsymbol{j} \end{pmatrix}\epsilon,$$

$$B = B_0 + B_1\epsilon = \begin{pmatrix} 1+\boldsymbol{i}+\boldsymbol{j} & -\boldsymbol{j} & -3\boldsymbol{i}+\boldsymbol{k} \\ 0 & \boldsymbol{k}+\boldsymbol{j} & 0 \end{pmatrix} + \begin{pmatrix} -\boldsymbol{i}-2\boldsymbol{k} & \boldsymbol{j}+3\boldsymbol{k} & 0 \\ \boldsymbol{i} & \boldsymbol{k} & \boldsymbol{i}+\boldsymbol{k} \end{pmatrix}\epsilon,$$

$$\begin{aligned} C &= C_0 + C_1\epsilon \\ &= \begin{pmatrix} -6-3\boldsymbol{j}+3\boldsymbol{k} & -9+3\boldsymbol{i}+4\boldsymbol{j}-6\boldsymbol{k} & 9+9\boldsymbol{i}-3\boldsymbol{j}-3\boldsymbol{k} \\ -1+2\boldsymbol{i}-\boldsymbol{j} & -\boldsymbol{i}-\boldsymbol{k} & 4+2\boldsymbol{j} \\ 4+10\boldsymbol{i}-2\boldsymbol{j}+6\boldsymbol{k} & -7-\boldsymbol{i}-\boldsymbol{j}-3\boldsymbol{k} & 14-16\boldsymbol{i}+2\boldsymbol{j}-8\boldsymbol{k} \end{pmatrix} \\ &+ \begin{pmatrix} -2\boldsymbol{i}+9\boldsymbol{j}+5\boldsymbol{k} & -16+28\boldsymbol{i}-16\boldsymbol{j}-18\boldsymbol{k} & 2+14\boldsymbol{i}-6\boldsymbol{j}-10\boldsymbol{k} \\ 2+5\boldsymbol{i}+\boldsymbol{j}+7\boldsymbol{k} & -3+\boldsymbol{i}-6\boldsymbol{j}+\boldsymbol{k} & 9-10\boldsymbol{i}-3\boldsymbol{j}-10\boldsymbol{k} \\ -11+24\boldsymbol{i}+17\boldsymbol{j}-2\boldsymbol{k} & -11-6\boldsymbol{i}-11\boldsymbol{j}+4\boldsymbol{k} & 42-6\boldsymbol{i}-12\boldsymbol{j}+10\boldsymbol{k} \end{pmatrix}\epsilon. \end{aligned}$$

Computing directly yields

$$\begin{aligned} & r\begin{pmatrix} A_0 & C_0 \end{pmatrix} = r(A_0) = 2,\ r\begin{pmatrix} B_0 \\ C_0 \end{pmatrix} = r(B_0) = 2, \\ & r\begin{pmatrix} A_1 & A_0 & C_1 \\ A_0 & 0 & C_0 \end{pmatrix} = r\begin{pmatrix} A_1 & A_0 \\ A_0 & 0 \end{pmatrix} = 4, \\ & r\begin{pmatrix} C_1 & A_0 \\ B_0 & 0 \end{pmatrix} = r(A_0) + r(B_0) = 4, \\ & r\begin{pmatrix} B_1 & B_0 \\ B_0 & 0 \\ C_1 & C_0 \end{pmatrix} = r\begin{pmatrix} B_1 & B_0 \\ B_0 & 0 \end{pmatrix} = 4. \end{aligned}$$

All rank equations are satisfied and a solution of dual quaternion matrix Equation (1) can be expressed as

$$X = X_0 + X_1\epsilon = \begin{pmatrix} \boldsymbol{i}+\boldsymbol{k} & 0 \\ 1 & \boldsymbol{j}-2\boldsymbol{k} \end{pmatrix} + \begin{pmatrix} \boldsymbol{j}+2\boldsymbol{k} & -1 \\ \boldsymbol{i} & 2+3\boldsymbol{i}-4\boldsymbol{k} \end{pmatrix}\epsilon.$$

6. Conclusions

Matrix equations $AX = B$ and $XC = D$ have specific applications in areas such as eigenvalue problems, image processing, and linear system solving. On the other hand, $AXB = C$ is a more general matrix equation that has broader use. In this paper, we have established the solvability conditions for dual quaternion matrix Equation (1) by using Moore–Penrose inverses and ranks of matrices; we have also derived the expressions of its general solution to (1) when the solvability conditions are met. As special cases, some dual quaternion matrix equations have also been discussed. Moreover, we have investigated the ϕ-Hermitian matrix over dual quaternion algebra and provided its related properties. As an application of the aforementioned research, we have considered a special case of (1) and provided the ϕ-Hermitian solutions to (3). Finally, we have presented an example to illustrate the main results. In the future, we will focus on researching more complex matrix and tensor equations over the dual quaternion algebra.

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