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A Functional Inequality and a New Class of Probabilities in the N-Person Red-and-Black Game

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Abstract: In this paper, we explore a model of an N-player, non-cooperative stochastic game, drawing inspiration from the discrete formulation of the red-and-black gambling problem, as initially introduced by Dubins and Savage in 1965. We extend upon the work of Pontiggia from 2007, presenting a main theorem that broadens the conditions under which bold strategies by all players can achieve a Nash equilibrium. This is obtained through the introduction of a novel functional inequality, which serves as a key analytical tool in our study. This inequality enables us to circumvent the restrictive conditions of super-multiplicativity and super-additivity prevalent in the works of Pontiggia and others. We conclude this paper with a series of illustrative examples that demonstrate the efficacy of our approach, notably highlighting its ability to accommodate a broader spectrum of probability functions than previously recognized in the existing literature.

Keywords: red-and-black game; stochastic game; bold strategy; Nash equilibrium; functional inequality

MSC: 91A15; 39B62; 91A06; 91A10; 91A60

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Citation: Fechner, W.; Słomian, M. A Functional Inequality and a New Class of Probabilities in the N-Person Red-and-Black Game. *Symmetry* **2024**, *16*, 325. https://doi.org/10.3390/sym16030325

Academic Editors: Quanxin Zhu and Carlo Cattani

Received: 17 January 2024
Revised: 8 February 2024
Accepted: 5 March 2024
Published: 7 March 2024

1. Preliminaries

1.1. Introduction

Consider a simplified roulette game with two or more players. This game takes place on a board divided into red and black fields, with a special green "0" field where betting is prohibited. Each player starts with an equal number of identical tokens. Players can individually wager one token on either the red or black field, adhering to a "first come, first served" rule that allows only one token per field. After players have placed their bets, the wheel is spun. If the wheel lands on a red or black field matching a player's bet, that player collects all the tokens on the board. However, if the wheel lands on the green "0" field, all bets are forfeited and the tokens remain on the board for the next round. This simplified model captures the essence of risk-taking and potential rewards inherent in many real-world scenarios while providing a controlled and easily understandable framework for further analysis and experimentation.

The role of the board and wheel in this simplified model is to represent a controlled and random draw mechanism. Each player bets on a specific outcome (red or black), and the wheel's landing position determines the winner based on predetermined rules. This draw incorporates an element of chance, where the probability of winning depends on the following:

1. Number of tokens wagered: Players who bet on the winning color with more tokens stand to gain more as they collect all tokens on the board. However, this also translates to a higher risk of losing everything if they choose the wrong color.
2. Bets of other players: While not directly influencing the winning color, the distribution of bets can indirectly impact individual players' perceptions and strategies. If

other players bet more, then they increase their chances of victory in a given round, therefore decreasing ours.

By replacing the physical roulette wheel and board with a computer simulation, we remove the limitations of a physical setup. This allows for the following:

- Increased control and flexibility: We can precisely define the number of fields, initial token distribution, and winning conditions, enabling tailored simulations for specific research purposes.
- Efficiency in conducting a large number of trials: A computer simulation can run thousands or even millions of games in a short time, generating statistically significant data to analyze winning probabilities, player strategies, and game dynamics.
- Exploration of new game mechanics: Freed from the constraints of physical limitations, the computer simulation opens the door to experimenting with novel gameplay possibilities that might not be feasible with a physical setup. This forms the foundation for our "title game", where these new mechanics are implemented to create a unique and potentially more complex gaming experience.

1.2. Red-and-Black Game

The red-and-black game's rich history dates back to the groundbreaking 1965 work of Lester E. Dubins and Leonard J. Savage in their book "How to gamble if you must" [1]. Their approach fundamentally shifted the focus from the morally charged question of whether to gamble, to the more pragmatic issue of how to play strategically when a player seeks a specific monetary target and finds lesser outcomes unacceptable.

Dubins and Savage addressed this question by providing guidance on optimizing betting strategies based on the player' available capital and his desired winning. A key innovation presented in their work was a model where a player's probability of winning was directly influenced by the size of his bet. This insight added an intriguing layer of complexity and strategic consideration to the classic red-and-black game.

Furthermore, their analysis motivated other researchers and led to the identification of a Nash equilibrium for the red-and-black game when played between two opponents (see Secchi [2], Chen and Hsiau [3,4]). For those unfamiliar with game theory, a Nash equilibrium is a state where neither player gains any advantage by changing their strategy while holding the opponent's strategy constant. In this context, we consider the following strategies:

- Timid play: This strategy is best employed when the opponent wagers their entire fortune. In this scenario, the player should bet the minimum amount possible (one token), as this maximizes their probability of winning.
- Bold play: This strategy is ideal when the opponent bets the minimum amount, and the player cannot immediately reach their goal by wagering less. The player should bet all his money for the best chance to win (with an exception if a smaller bet is enough to win in one round).

These strategic concepts of "timid" and "bold" play (precisely defined in Definition 1 of their article) offer a foundation for a deeper analysis of the dynamics of the red-and-black game. This work continues to be highly relevant for understanding risk-taking and decision-making in scenarios where outcomes hinge on calculated choices under uncertainty.

The groundbreaking work of Dubins and Savage ignited further exploration of the red-and-black game, most notably inspiring Ashok P. Maitra and William D. Sudderth's influential book [5]. They delved deeper into the game's complexities, posing new challenges such as determining the consequences of players disregarding their game history when making strategic decisions in the present moment. These two seminal works laid the foundation for extensive research on the red-and-black game and its variations, fostering continued interest in this deceptively simple model.

Despite the numerous modifications and analyses applied to the red-and-black game throughout its evolution, one consistent theme stands out: the bold-timid strategy framework has proven remarkably resilient. Researchers have successfully employed and adapted this core concept across multiple game iterations, demonstrating its applicability to a wide range of strategic gambling scenarios. This resilience underscores the enduring power of the original model and the ingenuity of its creators.

The exploration of the red-and-black game continued to flourish in the early 2000s. In 2005, L. Pontiggia introduced a two-player game model in [6] where both players aimed to seize all their opponent's money. This model explored different winning probability structures:

- Proportional to bets: The probability of winning directly corresponded to the size of each player's bet, building upon the core concept of the original red-and-black game.
- Weighted probability: The model introduced an additional parameter, the "weight" ω, allowing for more nuanced control over the influence of bets on winning probabilities.

This model, along with earlier work by P. Secchi [2], provided fertile ground for further research. In 2006, M. Chen and S. Hsiau in [3] took a significant step by introducing a general winning probability function f dependent on a single variable. Their analysis yielded two key results:

1. Optimal strategy with known opponent strategy: They identified the optimal strategy for a player when their opponent's strategy was known, providing valuable insights into decision-making based on the anticipated actions of others.
2. Best strategy for specific properties of f: They further explored the best strategy for a player under specific properties of the winning probability function f, revealing crucial factors influencing optimal decision-making.

Additionally, Chen and Hsiau identified a counterexample to an assumption made by Pontiggia in their earlier work (reference [6]) for the N-person model with N players greater than or equal to 2. This highlighted the importance of careful analysis and refinement when extending existing models to more complex scenarios.

Building upon the evolving understanding of the red-and-black game, in 2007, L. Pontiggia introduced a novel model [7] that incorporated a gambling house into the N-person scenario. This model introduced a crucial twist: in every round, the gambling house held a positive probability of winning all players' bets. This new element fundamentally changed the dynamics of the game and led to a surprising conclusion. In this specific model with the inclusion of the house, Pontiggia demonstrated that the optimal strategy for all participants surprisingly became "bold play"—constantly betting their entire capital. This unexpected outcome underscores the importance of considering all participants and game mechanics when formulating optimal strategies.

1.3. Our Contribution

This paper delves deeper into the previously discussed N-person red-and-black game with a non-constant winning sum, expanding upon the work presented in [7] by L. Pontiggia. We draw inspiration from [8], which generalizes the results of Chen and Hsiau [3]. By leveraging these insights, we aim to construct a more general model that surpasses the limitations of previously explored scenarios.

We will establish a more general model that surpasses the generality of existing models by incorporating a non-constant winning sum and building upon the advancements outlined in [8].

By constructing this more general model and providing concrete examples, we aim to extend the key findings of Pontiggia's work [7]. This extension will shed light on the behavior of optimal strategies in a broader range of N-person red-and-black game scenarios with non-constant winning sums.

Finally, to demonstrate the applicability of our model, we will present several examples of winning probability functions that satisfy the assumptions of our main theorem, showcasing the model's flexibility and adaptability.

Through these steps, we hope to contribute significantly to the ongoing exploration of red-and-black game dynamics and further enhance our understanding of optimal strategies within the context of N-person games with non-constant winning sums.

Our model features N players, each possessing an initial fortune equal to a positive integer. Players engage in simultaneous betting, where each wager is an integer portion of their current capital. The outcome is determined by a probability function that depends on all players' bets. This can result in two scenarios:

1. Player victory: With a certain probability, one player wins the entire pool of combined bets.
2. Casino takeover: Alternatively, the casino claims the entire pool with a specific probability.

In this context, we focus on bold strategies, where players bet their entire fortune at each round. Our main result delves into the stability of bold strategies as a Nash equilibrium, which essentially means that no player has the incentive to deviate from this strategy, given that all other players adhere to it.

However, under specific conditions outlined in our main theorem, we demonstrate that a profile consisting solely of bold strategies for all players forms a Nash equilibrium. This implies that assuming everyone else plays boldly, no individual player gains any advantage by deviating from a bold strategy themselves. This finding provides valuable insights into the dynamics and potential stability of bold play in N-person red-and-black games with non-constant winning sums.

Our approach breaks new ground by introducing a general functional inequality (inequality (12) below), which significantly expands upon the limitations of previous works that relied on more restrictive assumptions like super-additivity and super-multiplicativity. These assumptions limit the applicability of existing models.

Functional inequalities play a crucial role in game theory and related fields, offering a powerful tool for analyzing strategic interactions. While often overlooked in the past, they have gained significant traction in recent years, as evidenced by the following works:

- J.C. Candeal et al. [9]: This comprehensive survey, published in 1997, offers valuable insights into the application of functional equations in game theory research up to that point, highlighting the potential of this approach.
- M. Chudziak [10]: This recent work emphasizes the importance of inequalities in utility theory, contributing to the growing body of research in this area.
- J. Chudziak and M. Chudziak [11] and J. Chudziak [12,13]: These publications showcase further advancements in applying inequalities to analyze and understand some concepts of mathematical economy.

By incorporating a general functional inequality, our model transcends the limitations of past approaches, allowing us to explore a broader spectrum of scenarios within N-person red-and-black games with non-constant winning sums. This opens doors for deeper understanding and potentially more generalizable results regarding optimal strategies and player behavior in these complex game environments.

For the sake of organization and clarity, we have structured this paper as follows:

- Game Description: We begin by introducing the specific N-person red-and-black game with a non-constant winning sum that forms the foundation of our analysis. This section clearly defines the players, their actions, and the mechanics of the game.
- Main Result: Following the game description, we present our core finding: the theorem demonstrating the conditions under which bold strategies form a Nash equilibrium for all players. This section includes a clear statement of the theorem accompanied by a rigorous proof.
- Functional Inequality Exploration: We then delve into the key element of our approach—the general functional inequality. This section unpacks the inequality itself (inequality (12) below), explains its significance, and explores its basic properties within the context of our model.

- Experimental Data Analysis: As an additional layer of insight, we conclude the paper by presenting and discussing experimental data obtained through a computer program. These data simulate the number of wins achieved by players employing different strategies under a specific winning probability function. This analysis adds a practical dimension to our theoretical findings, providing further validation and potential avenues for future exploration.
- Illustrative Examples: To solidify the applicability and generality of our approach, we provide concrete examples of winning probability functions that satisfy the assumptions of our main theorem. These examples showcase scenarios not covered by previous works, highlighting the broader scope of our model.

We believe this structure effectively guides the reader through our research, ensuring a comprehensive understanding of the game, the main result, its underlying mathematical framework, and its practical implications.

2. Rules of the Game

Let $N \geq 2$ be a fixed integer, which denotes the number of players in the game. Next, assume that $(x_1^0, \ldots, x_N^0)$ is a vector of positive integers, where x_j^0 denotes the initial fortune of j-th player. Put $M := \sum_{i=1}^N x_i^0$ (the total amount of money at the beginning of the game) and let G be a positive integer equal to a fixed goal that the players aim to reach. We assume that the goal is the same for all players; only one player can win and at least some players have a chance to win. Therefore, we impose the following double inequality:

$$G \leq M < 2G. \tag{1}$$

Denote $S := \{0, 1, \ldots, M\}$. We define the state space for the game as

$$P := \{(x_1, \ldots, x_N) : x_j \in S, j = 1, \ldots, N, \sum_{i=1}^N x_i \leq M\}.$$

The absorbing states of the game consist of all vectors, with one of the coordinates greater or equal to G (when one of the players wins), and also of all vectors for which $\sum_{i=1}^N x_i < G$ (when it is no longer possible to win by any of the players).

Now, we define an action set of Player j when the current fortunes of the players are equal to $(x_1, \ldots, x_j, \ldots, x_N)$:

$$A_j(x_1, \ldots, x_N) := \begin{cases} \{1, \ldots, x_j\}, & \text{if } x_j \in \{1, \ldots, G-1\}, \\ \{0\}, & \text{if } x_j \in \{0, G, G+1, \ldots\}, \end{cases}$$

and his payoff function:

$$W_j(x_1, \ldots, x_N) := \begin{cases} 1, & \text{if } x_j \geq G, \\ 0, & \text{if } x_j < G. \end{cases} \tag{2}$$

Note that the game is non-cooperative, which means that each player does not know the actions that were simultaneously taken by the others.

Now, assume that we are given a function $\Phi \colon \{1, 2, \ldots, N\} \times P \to [0, 1]$, which represents the probability of winning for the players. More precisely, if the bets of the players are equal to $(a_1, \ldots, a_N)$, then the number $\Phi(j; a_1, \ldots, a_N)$ is the probability of victory of Player j (whose bet is a_j). Note that function Φ is defined on the product of the set $\{1, \ldots, N\}$ and the state space. This however does not mean that the players bet all their fortunes. But every possible vector of bets always belongs to the set P.

A special case of function Φ, namely

$$\Phi(j; a_1, \ldots, a_N) = f\left(\frac{a_j}{a_1 + \cdots + a_N}\right), \tag{3}$$

where f is super-additive and super-multiplicative, was studied by Pontiggia in [7]. Recall that a map $f\colon [0,1] \to \mathbb{R}$ is termed super-additive if

$$f(x+y) \geq f(x) + f(y), \quad x, y \in [0,1],\, x+y \leq 1.$$

Next, f is super-multiplicative, whenever

$$f(x \cdot y) \geq f(x) \cdot f(y), \quad x, y \in [0,1].$$

Typically, the graph of function f that satisfies both above inequalities jointly with $f(0) = 0$ and $f(1) = 1$ lies below the diagonal on the interval $(0,1)$ (for example, this is the case if f is continuous and not equal to the identity map). Therefore, in such a case at each round of the game, the expected value of victory for a player is smaller than his bet. This can be interpreted as follows: the casino is charging a tax or imposing a hidden fee for the players. Therefore, our model allows the casino to tax the players more flexibly, not necessarily depending only upon the quotient of the bid of the j-th player and the sum of all the bids made during this stage of the game. Another interpretation is given in [7] (Remark 3.1), where the idea of introducing function f is to penalize the players by reducing their probability of winning. Therefore, our model allows us to punish the players more flexibly. However, as has been pointed out by one of the reviewers, this interpretation can be misunderstood, since the word "punishment" suggests that the players are discouraged from playing at all.

We impose another assumption upon Φ, which is in particular fulfilled by all mappings of the form (3). Namely, we will assume that

$$\text{If } \sum_{i=1}^{N} a_i = \sum_{i=1}^{N} b_i \text{ and } a_j = b_j, \text{ then } \Phi(j; a_1, \ldots, a_N) = \Phi(j; b_1, \ldots, b_N). \tag{4}$$

Therefore, the winning probability of a player depends only on his bet and the sum of all in-game bets. Note that it can also depend on j. Therefore, our model allows for potential asymmetry between the players.

The next condition guarantees that the total probability does not exceed one:

$$\sum_{i=1}^{N} \Phi(i; a_1, \ldots, a_N) \leq 1. \tag{5}$$

For technical reasons, we will need another natural condition that the probability of victory of a given player is equal to zero if his bet is equal to zero (which, according to the rules of the game, is possible only if he lost all his capital at an earlier stage of the game):

$$\Phi(j; a_1, \ldots, a_N) = 0 \quad \text{if} \quad a_j = 0. \tag{6}$$

Now, we are at the point of precisely defining the game's law of motion. Let us fix a positive integer, m, a current stage of the game. Let $X_{m,j}$ be a random variable that is equal to the fortune of the j-th player at time m. By $a_{m,j}$, we denote an amount that he bids at this stage of the game. By the casino rules, $1 \leq a_{m,j} \leq X_{m,j}$ and $X_{1,j} < G$ for $j = 1, 2, \ldots, N$. The law of motion is as follows:

$$X_{1,1} = x_1^0, \qquad X_{1,2} = x_2^0, \qquad \ldots \qquad X_{1,N} = x_N^0 \tag{7}$$

$$(X_{m+1,1}, \ldots, X_{m+1,N}) = \begin{cases} (X_{m,1} - a_{m,1}, \ldots, X_{m,N} - a_{m,N}), & \text{w.p. } 1 - \sum_{i=1}^{N} \Phi(i; a_1, \ldots, a_N), \\ (X_{m,1} - a_{m,1}, \ldots, X_{m,j} + \sum_{i \neq j} a_{m,i}, \ldots, X_{m,N} - a_{m,N}), & \text{w.p. } \Phi(j; a_1, \ldots, a_N) \end{cases} \tag{8}$$

(here, "w.p." is an abbreviation for "with probability").

Note that inequality (5) together with the rules (7) and (8) implies that

$$\sum_{i=1}^{N} X_{m+1,i} \leq \sum_{i=1}^{N} X_{m,i}. \tag{9}$$

Sometimes we will omit double subscripts when it is clear which stage of the game is considered.

3. The Main Result

Our main result builds upon and generalizes a theorem established by L. Pontiggia [7] (Theorem 3.1). In her work, the analysis relied on assumptions of super-additivity and super-multiplicativity, which limit the applicability of the model. We overcome this limitation by introducing a less restrictive functional inequality (presented in detail in the following section), which allows for a broader range of scenarios.

This novel inequality plays a critical role in our analysis. It governs the relationship between the winning probabilities of individual players and their combined bets. By leveraging this inequality, we can derive conditions under which bold strategies, where players wager their entire fortune in each round, form a Nash equilibrium. In simpler terms, this means that when all players adopt the bold strategy, no individual player gains an advantage by deviating and choosing a different strategy.

The concept of a Nash equilibrium is crucial in game theory, signifying a state where rational players have no incentive to change their strategies given the strategies employed by others. Our main result demonstrates that under specific conditions, bold play becomes a stable and strategically sound approach for all players in the N-person red-and-black game with a non-constant winning sum.

Definition 1. *A strategy of Player j is called* bold *if for every $m = 1, 2, \ldots$ one has $a_{m,j} = X_{m,j}$, whenever $0 < a_{m,j} < G$. A strategy of Player j is called* timid *if for every $m = 1, 2, \ldots$, such that $0 < X_{m,j} < G$, one has $a_{m,j} = 1$.*

Theorem 1. *We consider an N-person red-and-black game with $N \geq 2$ and with the law of motion described by formulas (7) and (8), with probability function $\Phi\colon \{1, 2, \ldots, N\} \times P \to [0, 1]$ satisfying conditions (4) and (5).*

Assume that for every fixed $j \in \{1, \ldots, N\}$ and every choice of $X_{m,1}, \ldots, X_{m,N} \in S$, such that $X_{m,1} + \cdots + X_{m,N} \geq G$, the functions $f_j, g_j\colon S \to [0, 1]$ are given by

$$f_j(x) := \Phi(j; X_{m,1}, \ldots, X_{m,j-1}, x, X_{m,j+1}, \ldots, X_{m,N}), \quad x \in S, \tag{10}$$

$$g_j(x) := \sum_{i \neq j} \Phi(i; X_{m,1}, \ldots, X_{m,i-1}, x, X_{m,i+1}, \ldots, X_{m,N}), \quad x \in S, \tag{11}$$

and satisfy inequality

$$f_j(x) - f_j(a) \geq g_j(a) f_j(x - a), \tag{12}$$

for all $a, x \in S$, such that $a \leq x$. Then, the Nash equilibrium for all players is to play boldly.

Proof. We will follow the idea of the proof of [7] (Theorem 3.1).

Fix $j \in \{1, \ldots, N\}$ and assume that all players play boldly. Denote

$$Q_j(X_1, \ldots, X_N) = \mathbb{P}[\text{Player } j \text{ reaches } G, \text{when the game starts at } (X_1, \ldots X_N)]. \tag{13}$$

The law of motion of the game at time m is as follows:

$$(X_{m+1,1}, \dots, X_{m+1,j}, \dots, X_{m+1,N}) = \begin{cases} (0, \dots, 0), & \text{w.p. } 1 - \sum_{i=1}^{N} \Phi(i; X_{m,1}, \dots, X_{m,N}), \\ (0, \dots, 0, \sum_{i=1}^{N} X_{m,i}, 0, \dots, 0), & \text{w.p. } \Phi(j; X_{m,1}, \dots, X_{m,N}). \end{cases} \tag{14}$$

One can see that when all players adopt bold strategies, then the game terminates after the first round, i.e., $m = 1$. Clearly, since all the players bet their entire fortunes, then either one of them wins (say Player j) and reaches his goal, or all players go bankrupt and the casino collects their bets. Moreover, since $X_{1,1} + \cdots + X_{1,N} \geq G$, then obviously

$$Q_j(0, \dots, 0, \sum_{i=1}^{N} X_{1,i}, 0, \dots, 0) = 1,$$

since Player j won after the first round. Thus, we see that the expected return to Player j equals the value of his probability function at vector $(X_{1,1}, \dots, X_{1,N})$:

$$\begin{aligned} Q_j(X_{1,1}, \dots, X_{1,N}) &= \Phi(j; X_{1,1}, \dots, X_{1,N}) \cdot Q_j(0, \dots, \sum_{i=1}^{N} X_{1,i}, \dots, 0) \\ &= \Phi(j; X_{1,1}, \dots, X_{1,N}). \end{aligned} \tag{15}$$

Observe that

$$Q_j(0, \dots, 0, X_{1,j} - a_{1,j}, 0, \dots, 0) = 0,$$

since $\sum_{i=1}^{N} X_{1,i} \geq G$ and $X_{1,j} - a_{1,j} < G$, so Player j will not be able to increase his fortune and, as a consequence, reach his goal, G.

To complete the proof we will show that Q_j is excessive. Then [5] (Theorem 3.3.10) implies that a bold strategy is optimal for Player j if all remaining players play boldly. Therefore, we need to prove that if at the first stage of the game, Player j, bets an amount $a_{1,j} < X_{1,j}$, i.e., less than his entire fortune, and then plays boldly for the rest of the game (if the game lasts until the second round), then the expected return for him is not greater than his expected return would be if he played boldly at the first stage as well. Denote the expected return for Player j, who adopts this strategy by $\sigma_j(X_{1,1}, \dots, X_{1,j}, \dots, X_{1,N})$. Staking an amount, $1 \leq a_{1,j} \leq X_{1,j}$, for him means that

$$\begin{aligned} &\sigma_j(X_{1,1}, \dots, X_{1,j}, \dots, X_{1,N}) \\ &\quad = \Phi(1; X_{1,1}, \dots, a_{1,j}, \dots, X_{1,N}) \cdot Q_j(\sum_{i \neq j} X_{1,i} + a_{1,j}, 0, \dots, X_{1,j} - a_{1,j}, \dots, 0) \\ &\quad + \ldots + \Phi(j; X_{1,1}, \dots, a_{1,j}, \dots, X_{1,N}) \cdot Q_j(0, \dots, \sum_{i=1}^{N} X_{1,i}, \dots, 0) + \ldots + \\ &\quad + \Phi(1; X_{1,1}, \dots, a_{1,j}, \dots, X_{1,N}) \cdot Q_j(0, \dots, X_{1,j} - a_{1,j}, \dots, 0, \sum_{i \neq j} X_{1,i} + a_{1,j}) \\ &\quad + \left[1 - \sum_{i \neq j} \Phi(i; X_{1,1}, \dots, a_{1,j}, \dots, X_{1,N}) - \Phi(j; X_{1,1}, \dots, a_{1,j}, \dots, X_{1,N})\right] \\ &\qquad \cdot Q_j(0, \dots, X_{1,j} - a_{1,j}, \dots, 0). \end{aligned}$$

In the above calculations, we counted all the possibilities of winning by each player, different from j (in this case, Player j continues to play with fortune $X_{1,j} - a_{1,j}$), the case when Player j wins (which is represented by the middle element), as well as the casino, rakes in the stake (the last term).

To proceed, we need to focus on the case where only two players remain in the game. By (4) and using (15), we have the following:

$$\begin{aligned} &Q_j(a_{1,j} + \sum_{i \neq j} X_{1,i}, 0, \ldots, X_{1,j} - a_{1,j}, \ldots, 0) \\ &\qquad = \ldots = Q_j(0, \ldots, X_{1,j} - a_{1,j}, \ldots, 0, \sum_{i \neq j} X_{1,i} + a_{1,j}) \\ &\qquad = \Phi(j; \sum_{i \neq j} X_{1,i} + a_{1,j}, 0, \ldots, 0, X_{1,j} - a_{1,j}, 0 \ldots, 0). \end{aligned}$$

Finally, joining the above estimates, we arrive at the following:

$$\begin{aligned} \sigma(X_{1,1}, \ldots, X_{1,j}, \ldots, X_{1,N}) &= \Phi(j; X_{1,1}, \ldots, a_{1,j}, \ldots, X_{1,N}) \cdot 1 \\ &+ \sum_{i \neq j} \Phi(i; X_{1,1}, \ldots, a_{1,j}, \ldots, X_{1,N}) \cdot \Phi(j; \sum_{i \neq j} X_{1,i} + a_{1,j}, 0, \ldots, 0, X_{1,j} - a_{1,j}, 0 \ldots, 0). \end{aligned}$$

To show that Q_j is excessive, we need to verify the inequality

$$Q_j(X_{1,1}, \ldots, X_{1,N}) \geq \sigma_j(X_{1,1}, \ldots, X_{1,N}).$$

Note that it is equivalent to the following:

$$\begin{aligned} \Phi(j; X_{1,1}, \ldots, X_{1,N}) &\geq \Phi(j; X_{1,1}, \ldots, a_{1,j}, \ldots, X_{1,N}) \\ &+ \sum_{i \neq j} \Phi(i; X_{1,1}, \ldots, a_{1,j}, \ldots, X_{1,N}) \cdot \Phi(j; \sum_{i \neq j} X_{1,i} + a_{1,j}, 0, \ldots, 0, X_{1,j} - a_{1,j}, 0 \ldots, 0). \end{aligned}$$

Now, introduce functions f_j, g_j as in the statement of the theorem, i.e.,

$$f_j(x_{1,j}) = \Phi(j; X_{1,1}, \ldots, X_{1,j-1}, x_{1,j}, X_{1,j+1}, \ldots, X_{1,N})$$

and

$$g_j(a_{1,j}) = \sum_{i \neq j} \Phi(i; X_{1,1}, \ldots, X_{1,j-1}, a_{1,j}, X_{1,j+1}, \ldots, X_{1,N}).$$

The inequality in question takes the form

$$f_j(x_{1,j}) - f_j(a_{1,j}) \geq g_j(a_{1,j}) \cdot f_j(x_{1,j} - a_{1,j}).$$

Note also that $f(0) = 0$ by (6). Therefore, we reduced the problem to the inequality (12), and now the proof is completed. □

Remark 1. *A special case of the above theorem is when $N = 2$ and with no possibility of winning by the casino. In this situation, condition (5) simplifies, which together with inequality (12) applied to functions f_j and g_j for both players after an easy calculation, leads to the same functional inequality that appeared in [8] in a bit of a different situation, where the bold strategy was shown to be the best response to the timid strategy ([8] (Theorem 1 and inequality (7) therein)).*

4. Functional Inequality

We will begin this section with a fundamental observation about solutions to inequality (12). For simplicity, we will drop subscript j in f_j and g_j.

Proposition 1. *Assume that we are given two functions $f, g\colon S \to [0,1]$ and f is positive on S. Then f, g satisfy inequality (12) for all $a, x \in S$, such that $a \leq x$ if and only if*

$$g(y) \leq \min\left\{ \frac{f(x) - f(y)}{f(x - y)} : x \in \{y + 1, \ldots, M\} \right\}, \quad y \in S. \tag{16}$$

Proof. Straightforward. □

From the above proposition, we have an immediate corollary.

Corollary 1. *Assume that $f\colon S \to (0,1]$ is a non-decreasing function and $g\colon S \to [0,1]$ is defined by the following formula:*

$$g(y) = \min\left\{\frac{f(x) - f(y)}{f(x-y)} : x \in \{y+1,\ldots,M\}\right\}, \quad y \in S. \tag{17}$$

Then, the pair (f,g) satisfies inequality (12) for all $a, x \in S$, such that $a \le x$.

One can ask about solutions of a corresponding functional equation when one replaces the inequality sign in (12) by equality, i.e.,

$$f(x) - f(a) = g(a)f(x-a), \tag{18}$$

for all $a, x \in S$, such that $a \le x$. However, it is not difficult to find all solutions of (18). Let us note the following observation:

Proposition 2. *Assume we are given two functions, $f, g\colon S \to \mathbb{R}$. Then, f, g satisfy functional Equation (18) for all $a, x \in S$, such that $a \le x$ if and only if*

(i) $g = 0$ *and* f *is constant on* S, *or*
(ii) $f = 0$ *and* g *is arbitrary on* S, *or*
(iii) $f(x) = f(1)x$ *for all* $x \in S$ *and* $g = 1$ *on* S, *or*
(iv) $f(x) = \alpha(g(0)^x - 1)$ *and* $g(x) = g(1)^x$ *for all* $x \in S$ *with some constant* $\alpha \in \mathbb{R}$.

Proof. The "if" implication is straightforward; thus, we will justify the "only if" implication. Apply (18) with $x = a$ to deduce that $g(a)f(0) = 0$ for all $a \in S$. Thus, either $g = 0$ on S or $f(0) = 0$. We will now discuss the second case. Put $a = 0$ in (18) to obtain either $f = 0$ on S or $g(0) = 1$. So, assume that $g(0) = 1$. Next, observe that if for some $b \in S$, $b > 0$, we have $f = 0$ on $\{1,2,\ldots,b\}$, then by (18), we obtain $f(b+1) - f(1) = g(1)f(b) = 0$, so $f(b+1) = 0$. Therefore, we can assume that $f(1) \ne 0$. Without loss of generality, assume that $f(1) = 1$, by multiplying Equation (18) by a constant and replacing f by $f(1)^{-1}f$, if necessary. Denote $c := g(1)$ and use (18) to obtain the following:

$$f(x+1) - 1 = cf(x), \quad x \in S,\, x+1 \in S.$$

A straightforward induction leads to

$$f(y) = c^{y-1} + \cdots + 1, \quad y \in S,\, y > 0.$$

Thus, if $c = 1$, then $f(y) = y$ and $f(y) = \frac{c^y - 1}{c-1}$ if $c \ne 1$, corresponding to cases (iii) and (iv), respectively. □

Remark 2. *The first three cases of Proposition 2 correspond to rather uninteresting possibilities in the game. The first (when $g = 0$, and f is constant on S) occurs when the casino never wins and each player wins with the probability of $1/N$, regardless of the bets. The second (when $f = 0$) occurs when the casino collects all the bets with probability one. The third option spoken of in Proposition 2 is not applicable, since if $f(1) \ne 0$, then condition (5)—which guarantees that the sum of probabilities of winning does not exceed one—is not satisfied. The fourth case is—on the contrary—of definite interest since it corresponds to the power functions that were considered as win probability functions in [3]; see also [7] (Example 3.2).*

5. Numerical Simulations

We conducted a numerical experiment that illustrates our main theorem. Assume that there are four players, each with an initial capital equal to 500. They play red-and-black games in a casino and the target for each one is to achieve a capital of at least 1500. If the players bet amounts of $a_1, a_2, a_3, and a_4$, then in our experiment, the probability of victory for Player i is equal to

$$\left(\frac{a_i}{a_1 + a_2 + a_3 + a_4}\right)^{1.3}.$$

Therefore, the game is sub-fair for all the players, and, as a consequence, the casino has a positive probability of collecting all the bets. We considered four sets of strategies:

Strategy 1. Players 1, 2, and 3 adopt the bold strategy, and Player 4 bets randomly.
Strategy 2. Players 1, 2, and 3 bet randomly, and Player 4 adopts the bold strategy.
Strategy 3. All players bet randomly.
Strategy 4. All players adopt the bold strategy.

By "betting randomly", we mean that the player chooses his bet from the set of all allowed bets according to the discrete uniform probability.

The game has been played 100,000 times. The number of victories for each player and the casino is presented in Table 1.

Table 1. Victories for each player and the casino.

	Player 1	**Player 2**	**Player 3**	**Player 4**	**Casino**
Strategy 1.	19,687	20,065	19,691	7848	32,709
Strategy 2.	12,211	12,359	12,050	20,944	42,436
Strategy 3.	11,677	11,604	11,612	11,839	53,268
Strategy 4.	16,346	16,618	16,567	16,550	33,919

The experimental data obtained through the computer program paint a clear picture. Players who adopted the bold strategy, consistently wagering their entire fortune, achieved a significantly higher number of wins compared to those who employed a randomized betting approach. This finding aligns with our theoretical predictions, suggesting that bold play provides a substantial advantage in this specific game setting.

Furthermore, the data reveal a particularly noteworthy observation. The scenario where all players adhered to the bold strategy (Strategy 1) resulted in the lowest number of wins for the casino. This implies that coordinated bold play significantly minimizes the casino's advantage, further solidifying the strategic benefit of this approach for the players.

Therefore, based on both the theoretical analysis and the experimental data, we can confidently conclude that the bold strategy emerges as the optimal choice for players in the N-person red-and-black game with a non-constant winning sum, outperforming random betting and posing a significant challenge to the casino's advantage, especially when adopted by all players in a coordinated manner. Future research could explore potential counter-strategies for the casino or investigate the dynamics of mixed strategies (combining bold play with other approaches) in this game context.

6. Examples and Final Remarks

A calculation of [7] (p. 552, before Remarks 3.1) leads us to the following corollary:

Corollary 2. *Assume that $\varphi\colon [0,1] \to [0,1]$ is a super-multiplicative and super-additive function and $x_1, \ldots, x_N \in S$ are fixed. Then function $\Phi\colon \{1, 2, \ldots, N\} \times P \to [0,1]$ given by*

$$\Phi(j; a_1, \ldots a_N) := \varphi\left(\frac{a_j}{\sum_{j=1}^{N} a_j}\right)$$

satisfy assumptions of Theorem 1.

Two particular examples of function φ that fulfill assumptions of the above corollary are given in [7] (Examples 3.1 and 3.2), such as $\varphi(s) = ws$ with $w \leq 1$, and $f(s) = s^p$ with $p \geq 1$.

We will conclude the paper with a few simple examples of probability functions that satisfy (3) (except for the last one, which describes a situation where the assumptions of our model are not satisfied). Then, we will compute corresponding functions, f and g, and check whether inequality (12) is satisfied or not. This shows that our approach is more general than the one presented in the literature on red-and-black gambling.

Example 1. *Assume that function Φ is defined as follows:*

$$\Phi(j; a_1, \dots, a_N) := \frac{1}{M \cdot N}[a_1 + \cdots + a_N - a_j]$$

when $a_j > 0$ and $\Phi(j; a_1, \dots, a_N) = 0$ if $a_j = 0$. Factor $1/M \cdot N$ guarantees that Φ is a probability function and condition (5) holds. Moreover, (3) is satisfied by Φ. It is easy to check that f and g are constant functions. Thus, inequality (12) is satisfied if and only if $f = 0$ or $g = 0$. This corresponds to uninteresting cases of the game (which should be of no surprise, since in this example, the chance of winning for Player j does not depend on his bet).

Example 2. *Assume that function Φ is defined as follows:*

$$\Phi(j; a_1, \dots, a_N) := \frac{a_j}{M}$$

when $a_j > 0$ and $\Phi(j; a_1, \dots, a_N) = 0$ if $a_j = 0$. Again, (3) is satisfied by Φ, f and g are also easy to find, and inequality (12) is always satisfied. If (and only if), all the players choose the bold strategy, i.e., $a_j = X_j$; then the sum of all Φ's equals 1, which means that the casino has no chance to rack in their bets. Therefore, in this example, the bold strategy is dominant.

Further examples are easy and show that in the context of non-constant sum games with bet-dependent probabilities, it is no longer relevant whether the game is sub-fair or super-fair. Namely, it is possible to construct a super-fair game in which a player should adopt a bold strategy and a sub-fair game with a timid strategy being optimal.

Example 3. *Assume that for every $j \in \{1, \dots, N\}$ and every choice of bets $(a_1, \dots, a_N)$, we have*

$$\Phi(j; a_1, \dots, a_N) := \frac{1}{N}.$$

In this situation, a timid strategy is always dominant for each player.

One can modify this example by introducing a positive probability for the casino to collect all the bets, as follows:

$$\Phi(j; a_1, \dots, a_N) := \frac{1}{2N}.$$

Now the situation changes, and with the probability of $1/2$, all the bets are taken by the casino. Still, it is unwise for a single player to deviate from a dominant timid strategy and increase his bet. Consequently, with the timid strategies of all players, the probability that any of them will win decreases as the initial sum is large.

However, we note that if the players are allowed to cooperate, then they will adopt bold strategies to increase their expected payoffs.

Example 4. *Assume that the win probability function for the first player is equal to*

$$\Phi(1; a_1, \ldots, a_N) := \begin{cases} 0, & \text{if } a_1 < x_1^0, \\ 1, & \text{if } a_1 \geq x_1^0. \end{cases}$$

Thus, without any additional restrictions upon the remaining probability functions, we see that the bold strategy is a dominant strategy, guaranteeing her a sure victory. Here, equilibrium is not unique (provided the game lasts more than one turn) since any bet of Player 1 that is greater than or equal to x_1^0 *is a dominant strategy. Note that, in this example, assumptions of our model are violated. Therefore, our conditions are by no means necessary for the bold profile.*

7. Conclusions

In this paper, we significantly expanded the theoretical basis of the N-person red-and-black gambling model established in prior game theory research. Our key innovation—introducing a functional inequality —elegantly sidesteps the restrictions inherent in the super-multiplicativity and super-additivity assumptions that constrained earlier models.

This broader theoretical framework offers a more nuanced and flexible understanding of the game's dynamics. It allows us to consider a wider spectrum of winning probability functions, opening up new avenues for strategic analysis and exploration. We are confident that this contribution will ignite further research in this area, furthering the understanding of strategic decision-making in similar gambling scenarios.

While our work paves the way for a deeper understanding of the N-person red-and-black game with a non-constant winning sum, it also opens doors for exciting avenues in future research:

1. Exploring alternative functional inequalities: While our functional inequality proves effective in this specific context, further exploration of alternative inequalities could broaden the scope of applicable scenarios and potentially reveal new insights. Examining the properties of different inequalities and their impact on optimal strategies could be a fruitful direction.
2. Analyzing N-person games with incomplete information: In this work, players possessed complete information about the winning probability function. However, real-world scenarios often involve incomplete information. Investigating how the introduction of incomplete information (where players have limited knowledge about the winning probability function) affects optimal strategies and Nash equilibria would be a valuable extension.
3. Incorporating player dynamics and learning: Our model assumed players to be static entities with fixed strategies. However, introducing elements of player dynamics and learning capabilities could significantly increase the model's complexity and realism. Examining how players adapt their strategies over time based on past experiences and observations could offer valuable insights into long-term behavior and potential evolutionary dynamics within the game.
4. Exploring generalizations to other game settings: The core concepts explored in this paper, such as the use of functional inequalities to analyze strategic interactions, could hold potential for application in other game settings beyond the N-person red-and-black game. Investigating the applicability of our framework and the development of analogous inequalities for different games could lead to broader advancements in game theory research.
5. Integrating the model with real-world data: While the current work focused on theoretical analysis and experimental simulations, future research could explore the integration of real-world data from actual gambling environments. By analyzing betting patterns and outcomes in real-world scenarios, researchers could potentially validate or refine the theoretical findings and gain further insights into player behavior in practice.

These are just a few potential avenues for future research inspired by our work. By delving deeper into these directions, researchers can continue to enrich our understanding of strategic decision-making in complex game environments and contribute to the ongoing development of game theory.

Author Contributions: Conceptualization, W.F. and M.S.; methodology, W.F. and M.S.; validation, M.S.; formal analysis, W.F. and M.S.; investigation, W.F. and M.S.; writing—original draft preparation, W.F. and M.S.; writing—review and editing, W.F. and M.S.; supervision, W.F.; project administration, W.F. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: No new data were created or analyzed in this study. Data sharing does not apply to this article.

Acknowledgments: This article is based on the Master's Thesis of Maria Słomian, which was defended under the supervision of Włodzimierz Fechner in July 2021. This thesis was awarded the first prize in the Marian Rejewski Competition, organized at the Lodz University of Technology for the best Master's Thesis in applied mathematics for that year. The authors would like to express their most sincere gratitude to the anonymous reviewers for several valuable comments regarding the previous versions of this manuscript.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Dubins, L.E.; Savage, L.J. *How to Gamble If You Must. Inequalities for Stochastic Processes*; McGraw-Hill Book Co.: New York, NY, USA , 1965; p. xiv+249.
2. Secchi, P. Two-person red-and-black stochastic games. *J. Appl. Probab.* **1997**, *34*, 107–126. [CrossRef]
3. Chen, M.R.; Hsiau, S.R. Two-person red-and-black games with bet-dependent win probability functions. *J. Appl. Probab.* **2006**, *43*, 905–915. [CrossRef]
4. Chen, M.R.; Hsiau, S.R. Two new models for the two-person red-and-black game. *J. Appl. Probab.* **2010**, *47*, 97–108. [CrossRef]
5. Maitra, A.P.; Sudderth, W.D. *Discrete Gambling and Stochastic Games*; Applications of Mathematics (New York); Springer-Verlag: New York, NY, USA, 1996; Volume 32, p. xii+244. [CrossRef]
6. Pontiggia, L. Two-person red-and-black with bet-dependent win probabilities. *Adv. Appl. Probab.* **2005**, *37*, 75–89. [CrossRef]
7. Pontiggia, L. Nonconstant sum red-and-black games with bet-dependent win probability function. *J. Appl. Probab.* **2007**, *44*, 547–553. [CrossRef]
8. Fechner, W. New inequalities for probability functions in the two-person red-and-black game. *J. Math. Anal. Appl.* **2020**, *491*, 124339. [CrossRef]
9. Candeal, J.C.; De Miguel, J.R.; Induráin, E.; Oloriz, E.; Tala, J.E. Functional equations in utility and game theory. *Rev. Union Mat. Argent.* **1997**, *40*, 113–124.
10. Chudziak, M. On a disparity between willingness to pay and willingness to accept under the rank-dependent utility model. *Math. Inequal. Appl.* **2022**, *25*, 621–630. [CrossRef]
11. Chudziak, J.; Chudziak, M. On some applications of quasideviation means. *J. Differ. Equ. Appl.* **2019**, *25*, 1429–1437. [CrossRef]
12. Chudziak, J. On quasi-convexity of the zero utility principle. *J. Nonlinear Convex Anal.* **2018**, *19*, 749–754.
13. Chudziak, J. On applications of inequalities for quasideviation means in actuarial mathematics. *Math. Inequal. Appl.* **2018**, *21*, 601–610. [CrossRef]