symmetry

Article

Analytic Functions in a Complete Reinhardt Domain Having Bounded L-Index in Joint Variables

Andriy Bandura [1,*], Tetyana Salo [2] and Oleh Skaskiv [3]

[1] Department of Advanced Mathematics, Ivano-Frankivsk National Technical University of Oil and Gas, 76019 Ivano-Frankivsk, Ukraine
[2] Department of Mathematics, Lviv Polytechnic National University, 79000 Lviv, Ukraine; tetiana.m.salo@lpnu.ua
[3] Department of Mechanics and Mathematics, Ivan Franko National University of Lviv, 79000 Lviv, Ukraine; oleh.skaskiv@lnu.edu.ua
* Correspondence: andriy.bandura@nung.edu.ua

Abstract: The manuscript is an initiative to construct a full and exhaustive theory of analytical multivariate functions in any complete Reinhardt domain by introducing the concept of **L**-index in joint variables for these functions for a given continuous, non-negative, non-vanishing, vector-valued mapping **L** defined in an interior of the domain with some behavior restrictions. The complete Reinhardt domain is an example of a domain having a circular symmetry in each complex dimension. Our results are based on the results obtained for such classes of holomorphic functions: entire multivariate functions, as well as functions which are analytical in the unit ball, in the unit polydisc, and in the Cartesian product of the complex plane and the unit disc. For a full exhaustion of the domain, polydiscs with some radii and centers are used. Estimates of the maximum modulus for partial derivatives of the functions belonging to the class are presented. The maximum is evaluated at the skeleton of some polydiscs with any center and with some radii depending on the center and the function **L** and, at most, it equals a some constant multiplied by the partial derivative modulus at the center of the polydisc. Other obtained statements are similar to the described one.

Keywords: bounded **L**-index in joint variables; analytic function; partial derivative; maximum modulus; complete Reinhardt domain; unit polydisc; local behavior; circular symmetry; unit ball; multiple-circular domain

MSC: 32A15; 32A17

check for updates

Citation: Bandura, A.; Salo, T.; Skaskiv, O. Analytic Functions in a Complete Reinhardt Domain Having Bounded **L**-Index in Joint Variables. *Symmetry* **2024**, *16*, 351. https://doi.org/10.3390/sym16030351

Academic Editor: Ioan Rașae

Received: 12 February 2024
Revised: 11 March 2024
Accepted: 11 March 2024
Published: 14 March 2024

1. Introduction, Main Notations, and Definitions

The Reinhardt domain of holomorphy [1–3] has attracted the attention of many investigators in multidimensional complex analysis. This interest is generated by its geometric and analytical properties and its universality because it overlaps the balls, the polydiscs, and the Thullen domains [4] as the partial cases. Moreover, these cases are not biholomorphic equivalent, but mathematicians [5] continue to find conditions proving that a pseudo-convex Reinhardt domain is biholomorphic to the bounded balanced convex domains in $\mathbb{C}^n$.

Therefore, the study of the Reinhardt domain allows us to discover a deep interplay between them. An increasing number of papers on various types of Reinhardt domains [6], on the Schwarz lemma [7], on the rigidity theorem [8], on Bohr radii [9,10], on Bergman kernels [11], and on the bounds of all the coefficients of homogeneous expansions [12] for the domain show the importance of this topic. A few recent papers [13,14] initiated an intensive study of functions that are analytical in a complete Reinhardt domain by methods of the Wiman–Valiron theory. We will take most of our notations from those papers: $\mathbb{R}^n$ and $\mathbb{C}^n$ have the typical meanings in complex analysis, i.e, they are the real and complex n-dimensional vector spaces, respectively. The notation $\mathcal{A}^n(\mathbb{G})$, $(n \in \mathbb{N})$ means the class of an analytic multivariate function f that is defined in a complete Reinhardt domain $\mathbb{G} \subset \mathbb{C}^n$ and that admits such a representation by the following multiple power series:

$$F(z) = f(z_1, \dots, z_n) = \sum_{\|J\|=0}^{+\infty} a_J z^J, \tag{1}$$

with the convergence domain $\mathbb{G}$, where $z^J = z_1^{j_1} z_2^{j_2} \dots z_n^{j_n}$ for $z = (z_1, z_2, \dots, z_n) \in \mathbb{G}$, $J = (j_1, \dots, j_n) \in \mathbb{Z}_+^n$, $\|J\| = \sum_{s=1}^{n} j_s$, $\mathbb{N} = \{1, 2, 3, \dots\}$, and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$.

For the domain $\mathbb{G} \subset \mathbb{C}^n$, we denote $|G| := \{(r_1, r_2, \dots, r_n) \colon r_j = |z_j| \text{ for } j \in \{1, 2, \dots, n\}, z = (z_1, z_2, \dots, z_n) \in \mathbb{G}\}$.

Recent publications [3,10,13,14] show the growth of adaptations of one-dimensional complex analytic methods to such a domain as the Reinhardt domain in multidimensional complex analysis. Its increasing value is justified by its properties. In particular, the domain of convergence of any multiple power series (1) has the following properties: a logarithmical convexity, a completeness, and a circular symmetry (multiple-circularity). This means that the domain is the logarithmically convex and complete Reinhardt domain with the center at the origin. Moreover, any analytical function given in the complete Reinhardt domain with the center at the origin can be developed in the multiple complex power series (1) in the domain. Moreover, the Reinhardt domain has circular symmetry in each dimension (see below, condition b) because it is a multiple-circular domain. Therefore, in view of these facts, it is important to build a complete theory of analytical functions that have special properties in this domain: growth estimates [15], boundedness of the partial logarithmic derivative modulus, uniform distribution of zero points in some sense, and some regular behavior expressed in the estimates of the maximum modulus by the minimum modulus at a polydisc. On the other hand, among such classes of analytic functions, those with a finite index occupy an important place. Additionally, these functions have applications in the analytic theory of differential equations. There are known sufficient conditions that preserve the finiteness of the index for entire solutions and analytical solutions in the unit ball for a system of partial differential equations [16]. But the mentioned properties are known for analytic functions in the unit ball and in the unit polydisc, as well as for entire functions within a theory of functions with finite $\mathbf{L}$-index in joint variables. We should like to observe that this class of functions is very wide because, for every function (entire or analytical in the unit ball) whose zero points have uniformly bounded multiplicities, a mapping $\mathbf{L}$ can be constructed for which the primary holomorphic function has finite $\mathbf{L}$-index in joint variables.

A separate consideration of the properties of analytic functions in the ball and the polydisc is inspiring and important because these domains are not conformally equivalent. In view of these facts, it is important to construct a general theory for the Reinhardt domain because the unit ball and the unit polydisc are partial cases of the domain.

The n-dimensional complex domain $\mathbb{G}$ is called the complete Reinhardt domain if:
(a) for every point $z = (z_1, \dots, z_n)$ from this domain $\mathbb{G}$ and for each n-dimensional radius-vector $R = (r_1, \dots, r_n) \in [0, 1]^n$, the point-wise product $Rz = (r_1 z_1, \dots, r_n z_n)$ also belongs to the domain $\mathbb{G}$ (*it is a condition of completeness of the domain*);
(b) for every point $(z_1, \dots, z_n) \in \mathbb{G}$, its n-dimensional rotation also belongs to the domain, i.e., for all angles $(\theta_1, \dots, \theta_n) \in [0; 2\pi]^n)$, the following component-wise rotation $(z_1 e^{i\theta_1}, \dots, z_n e^{i\theta_n})$ falls into this domain $\mathbb{G}$ (*it is a multiple-circular domain or condition of multiple-circularity*).

The domain $\mathbb{G}$ becomes a logarithmically convex domain if the image of the set G, excluding all coordinate hyperplanes $z_j = 0$ under the mapping $(\log|z_1|, \log|z_2|, \dots, \log|z_n|)$, is a convex set in n-dimensional real space. There are known examples [17] of the complete Reinhardt domain D that are not logarithmically convex. The most frequently considered complete Reinhardt domains ($n \geq 2$) are the following:

$$\mathbb{D}^n(R) \coloneqq \{z \in \mathbb{C}^n\colon |z_1| < r_1, |z_2| < r_2, \dots, |z_n| < r_n\},$$
$$R = (r_1, r_2, \dots, r_n) \in (0, +\infty)^n \ \text{ (polydisc with radii } R \text{ and center at the origin)},$$
$$\mathbb{B}^n(r) := \{z \in \mathbb{C}^n\colon |z| := \sqrt{|z_1|^2 + \ldots + |z_n|^2} < r\} \quad \text{(ball with radius } r \text{ and center at the origin)},$$
$$\Pi^n(r) := \{z \in \mathbb{C}^n\colon |z_1| + \ldots + |z_n| < r\}, \quad r > 0.$$

The polydisc $\mathbb{D}^n(R)$ with radius R and center at the origin is contained in the Reinhardt domain $\mathbb{G}$ for every point $w = (w_1, \dots, w_n)$ taken from the domain $\mathbb{G}$, where the radii R are evaluated by the component-wise modulus of the point w, i.e., $R = (|w_1|, \dots, |w_n|)$.

We also need the following standard notations from the theory of holomorphic functions with finite index in all variables (see, for example, [15,16,18–20]). In particular, $\mathbb{R}_+$ means the non-negative real semi-axis, $\mathbf{0}= (0, \dots, 0)$ is the n-dimensional zero vector, $\mathbf{1}$ is the n-dimensional vector whose every component equals 1, $\mathbf{1}_j$ is the n-dimensional unit vector whose j-th component equals 1, and all other components are zeros. For two n-dimensional real (or, particularly, integer) vectors $A = (a_1, \dots, a_n) \in \mathbb{R}^n$ and $B = (b_1, \dots, b_n) \in \mathbb{R}^n$, the following formal notations are used in the text: the component-wise product $AB = (a_1b_1, \cdots, a_nb_n)$, the component-wise quotient $A/B = (a_1/b_1, \dots, a_n/b_n)$, and the vector exponentiation $A^B = a_1^{b_1} a_2^{b_2} \cdot \ldots \cdot a_n^{b_n}$. We do not violate the existence of these expressions. Under the norm $\|A\|$ of the integer vector A, we understand the sum of all its components, and all vector inequalities are understood as coordinate inequalities. This concerns the inequalities $A < B$, $A \leq B$, and so on. For the non-negative integer vector $K = (k_1, \dots, k_n) \in \mathbb{Z}^n_+$, we define the vector factorial $K!$ as the product of all component factorials. The arithmetic operations as addition, scalar multiplication, and conjugation for points from the n-dimensional complex space are given component-wise. For $z \in \mathbb{C}^n$ and $w \in \mathbb{C}^n$, we define:
$$\langle z, w\rangle = z_1\overline{w}_1 + \cdots + z_n\overline{w}_n,$$
where w_k is the complex conjugate of w_k. The open polydisc with radii R and center z^0 is defined as the Cartesian product of open discs $|z_j - z_j^0| < r_j$ in all $j \in \{1, \dots, n\}$, and it is denoted by $\mathbb{D}^n(z^0, R)$, while the polydisc skeleton $\{z \in \mathbb{C}^n : \ |z_j - z_j^0| = r_j, \ j \in \{1, \dots, n\}\}$ is written by $\mathbb{T}^n(z^0, R)$. Sometimes, we use the notation $\mathbb{D}^n[z^0, R]$ for the closed polydisc $\{z \in \mathbb{C}^n : \ |z_j - z_j^0| \leq r_j, \ j = 1, \dots, n\}$, while $\mathbb{D}^n$ means the unit polydisc with center at origin, and $\mathbb{D}$ is a usual open unit disc. In addition, $\mathbb{B}^n(z^0, r)$ stands for the n-dimensional complex open ball with radius r and center z^0, and its topological boundary is a sphere $\mathbb{S}^n(z^0, r)$. Similarly, $\mathbb{B}^n[z^0, r]$ indicates the n-dimensional complex closed ball with radius r and center z^0, and, finally, $\mathbb{B}^n$ means the open unit ball with its center at the origin. Obviously, the equality $\mathbb{D} = \mathbb{D}^1 = \mathbb{B}^1$ is valid.

For n-dimensional non-negative integer vector $J = (j_1, j_2, \dots, j_n)$, we will denote the J-th order partial derivatives of an analytic in $\mathbb{G} \subseteq \mathbb{C}^n$ function H as follows:
$$H^{(J)}(z) = \frac{\partial^{\|J\|} H}{\partial z^J}(z) = \frac{\partial^{j_1 + j_2 + \cdots + j_n} H}{\partial z_1^{j_1} \partial z_2^{j_2} \ldots \partial z_n^{j_n}}(z_1, z_2, \dots, z_n).$$

By $\overline{\mathbb{G}}$, we denote the closure of the complete Reinhardt domain $\mathbb{G}$ and $\partial\mathbb{G} = \overline{\mathbb{G}} \setminus \mathbb{G}$. We suppose that an auxiliary mapping $\mathbf{L}(z) = (l_1(z), l_2(z), \dots, l_n(z))$ satisfies the following conditions:

(1) for any $j \in \{1, 2, \dots, n\}$, the j-th component $l_j : \mathbb{G} \to \mathbb{R}_+$ of the mapping $\mathbf{L}$ has a continuity in all points from $\mathbb{G}$;

(2) for any $j \in \{1, 2, \dots, n\}$, the value of the j-th component l_j at every point z from the Reinhardt domain $\mathbb{G}$ is greater than $\frac{\beta}{\inf_{\substack{\widehat{R}_j z \in \partial\mathbb{G}, \\ r > 1}} (r|z_j|) - |z_j|}$, i.e.,

$$l_j(z) > \frac{\beta}{\inf_{\substack{\widehat{R}_j z \in \partial\mathbb{G},\\ r>1}}(r|z_j|) - |z_j|} \tag{2}$$

for some real $\beta > 1$. Here, $\widehat{R}_j = (1, \ldots, 1, \underbrace{r}_{j\text{-th item}}, 1 \ldots, 1)$. At the same time, if the set $\{z\widehat{R}_j \colon r > 1\}$ is unbounded for a given $z \in \mathbb{G}$, then we will only require that the condition $l_j(z) > 0$ be fulfilled. We will assume $l_j(z) > 0$ in the case when $\inf_{\substack{\widehat{R}_j z \in \overline{\mathbb{G}} \setminus \mathbb{G},\\ r>1}} r|z_j| = +\infty$. Such a case is possible, for example, if $\mathbb{G} = \mathbb{C} \times \mathbb{D}$. Examples of analytic functions in $\mathbb{C} \times \mathbb{D}$ are the deformed exponential function [21,22] and the partial theta function [23–25].

For simplicity, we also write $\mathcal{B} = (0, \beta]$ and $\mathcal{B}^n = (0, \beta]^n$, where the constant β is defined by the mapping $\mathbf{L}$, and $\mathcal{B}^n$ is obtained as the Cartesian product of the left-open interval $\mathcal{B}$.

Remark 1. *Suppose that $\mathbb{G}$ is a given complete Reinhardt domain. If $S \in \mathcal{B}^n$ is a set of radii and $z^0 = (z_1^0, \ldots, z_n^0)$ is a point belonging to the domain $\mathbb{G}$, then the polydisc $\mathbb{D}^n\left[z^0, \frac{S}{\mathbf{L}(z^0)}\right]$ is a subset of the domain $\mathbb{G}$. Indeed, for each $j \in \{1, \ldots, n\}$, we have:*

$$|z_j| \le |z_j - z_j^0| + |z_j^0| \le \frac{r_j}{l_j(z^0)} + |z_j^0| < \frac{r_j}{\beta}\left(\inf_{\substack{\widehat{R}_j z^0 \in \partial\mathbb{G},\\ r>1}} (r|z_j^0|) - |z_j^0|\right) + |z_j^0| <$$
$$< \inf_{\substack{\widehat{R}_j z^0 \in \partial\mathbb{G},\\ r>1}} (r|z_j^0|).$$

In other words, $|z_j| < r|z_j^0|$ for $z^0 \in \mathbb{G}$ and some $r > 1$. But, $\mathbb{G}$ is a complete domain, so the point z also lies within the domain $\mathbb{G}$.

Below we suppose everywhere that $\mathbb{G} \subset \mathbb{C}^n$ is the complete Reinhardt domain, and we will not repeat this assumption in the following assertions and definitions.

A multivariate holomorphic function $H \in \mathcal{A}^n(\mathbb{G})$ is called a function with *bounded (finite)* $\mathbf{L}$*-index (in joint variables)* if, for some non-negative integer n_0, the following inequality holds for every order J of partial derivatives in the whole domain $\mathbb{G}$:

$$\frac{|H^{(J)}(z)|}{J!\mathbf{L}^J(z)} \le \max\left\{\frac{|H^{(K)}(z)|}{K!\mathbf{L}^K(z)} : K \in \mathbb{Z}_+^n, \ \|K\| \le n_0\right\}. \tag{3}$$

The least corresponding number n_0 is the $\mathbf{L}$-*index in joint variables* for the function H, and $N(H, \mathbf{L}, \mathbb{G}) = n_0$ stands for the index. If the Reinhardt domain $\mathbb{G}$ matches with n-dimensional complex space $\mathbb{C}^n$, and if the mapping $\mathbf{L}$ identically equals 1, then it is a definition of an entire multivariate function of a bounded index [19,20,26,27]. These authors did not use the refinement "in joint variables". In addition, if $n = 1$ and $\mathbf{L} = l$, then it becomes the definition of the entire function of a single complex variable with bounded l-index [28], and if, finally, $l = 1$, then we obtain the definition of the entire function having a bounded (finite) index [29].

To achieve substantial results, we assume that the mapping $\mathbf{L}$ does not vary locally as soon as possible. With the phrase "the mapping does not vary locally as soon as possible", we understand that every component of the vector-valued mapping $\mathbf{L}$ has such a property that the following supremum describing the local variation

$$\lambda_j(R) = \sup_{z,w \in \mathbb{G}} \left\{\frac{l_j(z)}{l_j(w)} \colon |z_k - w_k| \le \frac{r_k}{\min\{l_k(z), l_k(w)\}}, k \in \{1, \ldots, n\}\right\} \tag{4}$$

is finite at least for one set of n radii R belonging to the domain $\mathcal{B}^n$ constructed as the Cartesian product of the half-open interval $(0, \beta]$. The class of these mapping $\mathbf{L} : \mathbb{G} \to \mathbb{R}^n_+$ satisfying (2) and (4) is denoted by $Q(\mathbb{G})$. It is easy to see that a validity of inequality (4) for some R from the domain $\mathcal{B}^n$ yields the validity of the same inequality for all values R from the same domain.

Example 1. *Let us consider the inequality (2) for different cases of* $\mathbb{G}$.

If $\mathbb{G} = \mathbb{D}^n$ *(unit polydisc), then* $\inf_{\substack{\widehat{R}_j z \in \partial\mathbb{D}^n, \\ r>1}} (r|z_j|) = 1$ *because* $\widehat{R}_j z \in \partial\mathbb{D}^n$ *is equivalent to* $|rz_j| = 1$. *Thus, we obtain such a condition for the polydisc* $l_j(z) > \frac{\beta}{1-|z_j|}$. *It completely matches with a condition on the function* $\mathbf{L}$ *in paper [30]. This paper is an introductory paper on the term of bounded* $\mathbf{L}$*-index in joint variables for the function class whose domain of holomorphy is the unit polydisc.*

If $\mathbb{G} = \mathbb{B}^n$ *(i.e., the complete multiple-circular domain is the ball with unit radii) and* $n \geq 2$, *then* $\inf_{\substack{\widehat{R}_j z \in \overline{\mathbb{B}^n} \setminus \mathbb{B}^n, \\ r>1}} (r|z_j|) = \sqrt{1 - \sum_{s=1, s\neq j}^n |z_s|^2}$ *because* $\widehat{R}_j z \in \partial\mathbb{B}^n$ *is equivalent to* $|rz_j| = \sqrt{1 - \sum_{s=1, s\neq j}^n |z_s|^2}$. *Thus, we obtain such a condition* $l_j(z) > \frac{\beta}{\sqrt{1-\sum_{s=1,s\neq j}^n |z_s|^2} - |z_j|}$ *for the ball. We prove that*

$$\left(1 - \sum_{s=1, s\neq j}^{n} |z_s|^2\right)^{1/2} - |z_j| \geq \frac{1 - \sqrt{\sum_{s=1}^n |z_s|^2}}{\sqrt{n}}$$

for $|z| < 1$. *Denoting* $|z_j| = r_1$ *and* $\sum_{s=1, s\neq j}^n |z_s|^2 = r_2^2$, *we rewrite the last inequality as* $\sqrt{1 - r_2^2} - r_1 \geq \frac{1 - \sqrt{r_1^2 + r_2^2}}{\sqrt{n}}$. *Since* $n \geq 2$, *it is sufficient to prove* $\sqrt{1 - r_2^2} - r_1 \geq \frac{1 - \sqrt{r_1^2 + r_2^2}}{\sqrt{2}}$ *for* $r_1^2 + r_2^2 \leq 1$, $r_1 \geq 0$, $r_2 \geq 0$, *or* $\sqrt{1 - r_2^2} - r_1 - \frac{1 - \sqrt{r_1^2 + r_2^2}}{\sqrt{2}} \geq 0$. *Introducing the function* $h(r_1, r_2) = \sqrt{1 - r_2^2} - r_1 - \frac{1 - \sqrt{r_1^2 + r_2^2}}{\sqrt{2}}$ *and using optimization methods, it can be proved that*

$$\min_{\substack{r_1^2 + r_2^2 \leq 1 \\ r_1 \geq 0, r_2 \geq 0}} h(r_1, r_2) = 0.$$

In other words, we have

$$l_j(z) > \frac{\beta}{\sqrt{1 - \sum_{s=1, s\neq i}^n |z_s|^2} - |z_j|} \geq \frac{\beta\sqrt{n}}{(1 - |z|)}.$$

The right-hand side is the function used for conditions by the function $\mathbf{L}$ *in paper [15], where holomorphic functions having the unit ball as the domain of holomorphy within the theory of bounded* $\mathbf{L}$*-index in joint variables were investigated. In other words, condition (2) is no harder in the case of the unit ball than the standard condition* $l_j(z) > \frac{\beta\sqrt{n}}{1-|z|}$, *which appeared in paper [15]. But the condition (2) is universal for all complete Reinhardt domains. Besides the unit polydisc above, it completely matches with a condition by the mapping of* $\mathbf{L}$ *defined in the Cartesian product of a complex plane and the unit disc, i.e.,* $\mathbb{C} \times \mathbb{D}$. *Some results are known on the finiteness of* $\mathbf{L}$*-index in joint variables for analytic functions whose domain of holomorphy is the specified Cartesian product.*

If $n = 1$, *then* $\mathbb{G}^1 = \mathbb{B}^1 = \mathbb{D}^1$, *and the case is considered above for the polydisc.*

If $\mathbb{G} = \Pi^n(1)$, *then* $\inf_{\substack{\widehat{R}_j z \in \partial\Pi^n(1), \\ r>1}} (r|z_j|) = 1 - \sum_{s=1, s\neq i}^n |z_s|$. *Hence,* $l_j(z) > \frac{\beta}{1 - \sum_{s=1}^n |z_s|}$.

2. Behavior on Polydiscs of Mixed Derivatives of Holomorphic Functions

The analogues of the following theorem are fundamental to the theory of functions with finite index for various classes of holomorphic functions. For entire multivariate complex-valued functions, this was deduced by F. Nuray and R. Patterson in [18]. This is a starting point to establish more usable criteria providing finiteness of the index for various function classes. They characterize the maximum modulus of mixed derivatives on a polydisc or logarithmic derivative modulus in each variable separately, outside some exceptional sets (see [15,31,32]). The orders of partial derivatives are uniformly bounded by some positive integer depending only on the radii of the polydiscs, the analytical function, and the auxiliary function. In fact, the amount depends only on behavior characteristic of the vector-valued function $\mathbf{L}$ and the value of $\mathbf{L}$-index in joint variables. In this section, we also considered the replacement of the maximum modulus of partial derivatives by some fraction that matches with the Taylor–Maclaurin coefficient for multiple power series if the mapping $\mathbf{L}$ identically equals a vector consisting only of units.

Theorem 1. *Let $\mathbb{G}$ be a complete multiple-circular domain; the mapping $\mathbf{L}$ belongs to the class $Q(\mathbb{G})$. The joint $\mathbf{L}$-index for a function H belonging to the class $\mathcal{A}^n(\mathbb{G})$ is finite if and only if, for any vector-radius R taken from the Cartesian product $\mathcal{B}^n$, it is possible to find such a positive integer n_0 and a positive real d_0 that, for every point z^0 within the domain $\mathbb{G}$, there exists a mixed derivative order K^0 (n-dimensional positive integer vector), whose height $\|K^0\|$ does not exceed the integer n_0, and*

$$\max\left\{\frac{|H^{(k_1,\ldots,k_n)}(z)|}{k_1!\cdots k_n!(\mathbf{L}(z))^{(k_1,\ldots,k_n)}}: k_1+\cdots+k_n\le n_0,\ z\in\mathbb{D}^n\left[z^0,\frac{R}{\mathbf{L}(z^0)}\right]\right\}\le d_0\frac{|H^{(K^0)}(z^0)|}{K^0!\mathbf{L}^{K^0}(z^0)}. \quad (5)$$

Proof. We will start with the proof of necessity. Let H be a holomorphic function in the complete Reinhardt domain $\mathbb{G}$. Assume that $N=N(H,\mathbf{L},\mathbb{G})<\infty$, i.e., the analytic functions' class with finite $\mathbf{L}$-index in joint variables, at least, contains the function H. For every radii R chosen from the Cartesian product $\mathcal{B}^n$, we set

$$q=q(R)=q(r_1,r_2,\ldots,r_n)=1+\left[(2N+2)(r_1+r_2+\ldots r_n)\prod_{j=1}^n(\lambda_j(R))^{2N+1}\right].$$

Here, the square brackets $[b]$ mean the integer part of the real number b, i.e., it is the floor function. For every natural number p chosen from the finite set $\{0,1,2,\ldots,q\}$ and for every point $z^0=(z_1^0,\ldots,z_n^0)$ lying in the complete multiple-circular domain $\mathbb{G}$, we tag

$$S_p(z^0,R)=S_p((z_1^0,\ldots,z_n^0),(r_1,\ldots,r_n))=$$
$$=\max\left\{\frac{|H^{(k_1,\ldots,k_n)}(z_1,\ldots,z_n)|}{k_1!\cdots k_n!\mathbf{L}^{(k_1,\ldots,k_n)}(z_1,\ldots,z_n)}: k_1+\cdots+k_n\le N, z\in\mathbb{D}^n\left[z^0,\frac{p\cdot(r_1,\ldots,r_n)}{q\mathbf{L}(z_1^0,\ldots,z_n^0)}\right]\right\},$$
$$S_p^*(z^0,R)=\max\left\{\frac{|H^{(k_1,\ldots,k_n)}(z)|}{k_1!\ldots k_n!\mathbf{L}^{(k_1,\ldots,k_n)}(z^0)}: k_1+\cdots+k_n\le N, z\in\mathbb{D}^n\left[z^0,\frac{pR}{q\mathbf{L}(z^0)}\right]\right\}.$$

In these designations, the $S_p^*(z^0,R)$ differs on $S_p(z^0,R)$, the replacement value of mapping $\mathbf{L}$ at arbitrary point z within the polydisc by the value of mapping $\mathbf{L}$ at the center of the polydisc. We deduce estimate $S_p(z^0,R)$ by $S_p^*(z^0,R)$. By definitions of $S_p(z^0,R)$ and $S_p^*(z^0,R)$, we get

$$S_p(z^0,R)=\max\left\{\frac{|H^{(K)}(z)|}{K!\mathbf{L}^K(z)}\frac{\mathbf{L}^K(z^0)}{\mathbf{L}^K(z^0)}: \|K\|\le N, z\in\mathbb{D}^n\left[z^0,\frac{pR}{q\mathbf{L}(z^0)}\right]\right\}\le$$
$$\le S_p^*(z^0,R)\max\left\{\prod_{j=1}^n\frac{l_j^N(z^0)}{l_j^N(z)}: z\in\mathbb{D}^n\left[z^0,\frac{pR}{q\mathbf{L}(z^0)}\right]\right\}.$$

But, since $\mathbb{D}^n\left[z^0, \frac{pR}{q\mathbf{L}(z^0)}\right] \subset \mathbb{D}^n\left[z^0, \frac{R}{\mathbf{L}(z^0)}\right]$, by using definition (4), we obtain

$$S_p(z^0,R) \le S_p^*(z^0,R)\prod_{j=1}^{n}\left(\max\left\{\frac{l_j(z^0)}{l_j(z)} : z\in\mathbb{D}^n\left[z^0, \frac{(r_1,r_2,\dots,r_n)}{\mathbf{L}(z^0)}\right]\right\}\right)^N \le$$
$$\le S_p^*(z^0,R)\prod_{j=1}^{n}(\lambda_j(R))^N. \tag{6}$$

Similarly, in view of (4), we establish the estimate $S_p^*(z^0,R)$ by $S_p(z^0,R)$:

$$S_p^*(z^0,R)=\max\left\{\frac{|H^{(K)}(z)|}{K!\mathbf{L}^K(z)}\frac{\mathbf{L}^K(z)}{\mathbf{L}^K(z^0)} : \|K\|\le N, z\in\mathbb{D}^n\left[z^0, \frac{pR}{q\mathbf{L}(z^0)}\right]\right\} \le$$
$$\le\max\left\{\frac{|H^{(K)}(z)|}{K!\mathbf{L}^K(z)}\left(\Lambda\left(\frac{p}{q}R\right)\right)^K : \|K\|\le N, z\in\mathbb{D}^n\left[z^0, \frac{pR}{q\mathbf{L}(z^0)}\right]\right\} \le$$
$$\le S_p(z^0,R)\prod_{j=1}^{n}(\lambda_j(R))^N. \tag{7}$$

We chose the mixed derivative order $K^{(p)}$ and the point $z^{(p)}$ within the closed polydisc, with center at point z^0 and vector-radius $\frac{p}{q}\cdot\frac{R}{\mathbf{L}(z^0)}$, by the following conditions:

(1) the height of the n-dimensional positive integer vector $\|K^{(p)}\| = k_1^{(p)} + \ldots + k_n^{(p)}$ is not greater than N;

(2) as a two-parametric maximum above the total points within a polydisc and above a finite set of partial derivatives whose heights are bounded from above by N, the quantity $S_p^*(z^0,R)$ is attained at $z = z^{(p)}$ and $K = K^{(p)}$, i.e., the order $K^{(p)} \in \mathbb{Z}_+^n$ and the point $z^{(p)} \in \mathbb{D}^n\left[z^0, \frac{pR}{q\mathbf{L}(z^0)}\right]$ satisfy the equality

$$S_p^*(z^0,R) = \frac{|H^{(K^{(p)})}(z^{(p)})|}{K^{(p)}!\mathbf{L}^{K^{(p)}}(z^0)}. \tag{8}$$

We will apply the multidimensional maximum modulus principle. By this principle, the point $z^{(p)}$ must lie on the polydisc skeleton $\mathbb{T}^n(z^0, \frac{pR}{q\mathbf{L}(z^0)})$. This means that the point $z^{(p)}$ differs from the center z^0 of the polydisc. We construct an intermediate point $\widetilde{z}^{(p)}$ on a line between two specified points: the center z^0 of the polydisc and maximum point $z^{(p)}$ on the polydisc skeleton. The j-th coordinate of the point $\widetilde{z}^{(p)}$ is evaluated by the rule: $\widetilde{z}_j^{(p)} = z_j^0 + \frac{p-1}{p}(z_j^{(p)} - z_j^0)$. For further transformations, it needs to estimate consecutive distances between the points z^0, $\widetilde{z}^{(p)}$ and $z^{(p)}$. Making elementary calculations in each coordinate, we establish for every $j \in \{1,2,3,\dots,n\}$:

$$|\widetilde{z}_j^{(p)} - z_j^0| = \frac{p-1}{p}|z_j^{(p)} - z_j^0| = \frac{p-1}{p}\frac{pr_j}{ql_j(z^0)}, \tag{9}$$

$$|\widetilde{z}_j^{(p)} - z_j^{(p)}| = |z_j^0 + \frac{p-1}{p}(z_j^{(p)} - z_j^0) - z_j^{(p)}| = \frac{1}{p}|z_j^0 - z_j^{(p)}| =$$
$$= \frac{1}{p}\frac{pr_j}{ql_j(z^0)} = \frac{r_j}{ql_j(z^0)}. \tag{10}$$

The coordinate-wise estimate (9) of distance shows that the intermediate point $\widetilde{z}^{(p)}$ gets into the closed polydisc $\mathbb{D}^n\left[z^0, \frac{(p-1)R}{q(R)\mathbf{L}(z^0)}\right]$, i.e., it lies within the polydisc with the same center, and the vector-radius decreases by $\frac{R}{q(R)\mathbf{L}(z^0)}$. Therefore, $S_{p-1}^*(z^0,R)$ as the maximum of the expression $\frac{|H^{(K^{(p)})}(z)|}{K^{(p)}!\mathbf{L}^{K^{(p)}}(z^0)}$ above all points within the polydisc is greater than the value $\frac{|H^{(K^{(p)})}(\widetilde{z}^{(p)})|}{K^{(p)}!\mathbf{L}^{K^{(p)}}(z^0)}$, i.e., it exceeds the value at the specified point $\widetilde{z}^{(p)}$ lying on the skeleton of the same polydisc:

$$S^*_{p-1}(z^0,(r_1,r_2,\dots,r_n)) \geq \frac{|H^{(k_1^{(p)},\dots,k_n^{(p)})}(\widetilde{z}_1^{(p)},\dots,\widetilde{z}_n^{(p)})|}{k_1^{(p)}!\cdots k_n^{(p)}!l_1^{k_1^{(p)}}(z^0)\cdots l_n^{k_n^{(p)}}(z^0)}.$$

The last inequality and definition of $\widetilde{z}^{(p)}$ in equality (8) together imply such an upper estimate of difference between consecutive quantities $S^*_p(z^0,R)$ and $S^*_{p-1}(z^0,R)$:

$$\begin{gathered}0 \leq S^*_p(z^0,(r_1,r_2,\dots,r_n)) - S^*_{p-1}(z^0,(r_1,r_2,\dots,r_n)) \leq \\ \leq \frac{|H^{(k_1^{(p)},\dots,k_n^{(p)})}(z_1^{(p)},\dots,z_n^{(p)})| - |H^{(k_1^{(p)},\dots,k_n^{(p)})}(\widetilde{z}_1^{(p)},\dots,\widetilde{z}_n^{(p)})|}{k_1^{(p)}!\cdots k_n^{(p)}!l_1^{k_1^{(p)}}(z^0)\cdots l_n^{k_n^{(p)}}(z^0)}.\end{gathered}$$

Connecting the points $z^{(p)}$ and $\widetilde{z}^{(p)}$ by a parametric line $\widetilde{z}^{(p)}+t(z^{(p)}-\widetilde{z}^{(p)})$ for $t\in[0,1]$ and replacing the difference $|H^{(k_1^{(p)},\dots,k_n^{(p)})}(z_1^{(p)},\dots,z_n^{(p)})| - |H^{(k_1^{(p)},\dots,k_n^{(p)})}(\widetilde{z}_1^{(p)},\dots,\widetilde{z}_n^{(p)})|$ with an integral along the line, we deduce:

$$\begin{gathered}0 \leq S^*_p(z^0,(r_1,r_2,\dots,r_n)) - S^*_{p-1}(z^0,(r_1,r_2,\dots,r_n)) \leq \\ \leq \frac{1}{K^{(p)}!\mathbf{L}^{K^{(p)}}(z^0)}\int_0^1 \frac{d}{dt}|H^{(K^{(p)})}(tz^{(p)}+(1-t)\widetilde{z}^{(p)})|dt.\end{gathered}$$

Further, we evaluate the derivative in the parameter t. For such a goal, we use the derivative from the modulus of a function that is less than the modulus of the derivative of the function, i.e., $|\phi(t)|' \leq |\phi'(t)|$ for every t, excluding zeros, of the function ϕ. These transformations generate the following estimate:

$$\begin{gathered}0 \leq S^*_p(z^0,(r_1,r_2,\dots,r_n)) - S^*_{p-1}(z^0,(r_1,r_2,\dots,r_n)) \leq \\ \leq \frac{1}{K^{(p)}!\mathbf{L}^{K^{(p)}}(z^0)}\int_0^1 \sum_{j=1}^n |z_j^{(p)}-\widetilde{z}_j^{(p)}|\left|H^{(K^{(p)}+\mathbf{1}_j)}(tz^{(p)}+(1-t)\widetilde{z}^{(p)})\right|dt.\end{gathered}$$

Hence, by the mean value theorem, we can replace the sum under the integral in the real parametric variable t by a value at some point t^* belonging to the interval $[0,1]$:

$$\begin{gathered}0 \leq S^*_p(z^0,(r_1,r_2,\dots,r_n)) - S^*_{p-1}(z^0,(r_1,r_2,\dots,r_n)) \leq \\ \leq \frac{1}{K^{(p)}!\mathbf{L}^{K^{(p)}}(z^0)}\sum_{j=1}^n |z_j^{(p)}-\widetilde{z}_j^{(p)}|\left|H^{(K^{(p)}+\mathbf{1}_j)}(t^*z^{(p)}+(1-t^*)\widetilde{z}^{(p)})\right|,\end{gathered} \tag{11}$$

where the point $tz^{(p)}+(1-t)\widetilde{z}^{(p)}$ is contained in the polydisc $\mathbb{D}^n(z^0,\frac{pR}{q\mathbf{L}(z^0)})$. For every point z within the domain $\mathbb{D}^n(z^0,\frac{pR}{q\mathbf{L}(z^0)})$, and for each partial derivative order $J=(j_1,j_2,\dots,j_n)$ whose height $\|J\|=j_1+j_2+\cdots+j_n$ is less than the joint $\mathbf{L}$-index increased by 1, by using (4), we deduce that:

$$\begin{aligned}
&\frac{|H^{(j_1,\dots,j_n)}(z_1,\dots,z_n)|}{j_1!\cdots j_n!\mathbf{L}^J(z_1,\dots,z_n)}\frac{\mathbf{L}^J(z_1,\dots,z_n)}{\mathbf{L}^J(z_1^0,\dots,z_n^0)}\le\\
&\le(\Lambda(R))^J\max\left\{\frac{|H^{(k_1,\dots,k_n)}(z_1,\dots,z_n)|}{k_1!\cdots k_n!\mathbf{L}^K(z_1,\dots,z_n)}:\|K\|=k_1+k_2+\cdots+k_n\le N\right\}\le\\
&\le\prod_{j=1}^n(\lambda_j(R))^{2N+1}\max\left\{\frac{|H^{(k_1,\dots,k_n)}(z_1,\dots,z_n)|}{k_1!\cdots k_n!\mathbf{L}^K(z^0)}:\|K\|\le N\right\}\le\\
&\le\prod_{j=1}^n(\lambda_j(R))^{2N+1}S_p^*(z^0,R).
\end{aligned}$$

Here, $\Lambda(R)=(\lambda_1(R),\dots,\lambda_n(R))$. Above, we estimate the fraction $\frac{\mathbf{L}^J(z_1,\dots,z_n)}{\mathbf{L}^J(z_1^0,\dots,z_n^0)}$ by the $(\Lambda(R))^J$. This is possible because the mapping $\mathbf{L}$ belongs to the class $Q(\mathbb{G})$.

Now, we successively apply the last inequality to (11), substituting the distance estimate (10) and the expression from the definition of $q(R)$. It yields:

$$\begin{aligned}
&0\le S_p^*(z^0,(r_1,r_2,\dots,r_n))-S_{p-1}^*(z^0,(r_1,r_2,\dots,r_n))\le\\
&\le S_p^*(z^0,(r_1,r_2,\dots,r_n))\prod_{j=1}^n(\lambda_j(R))^{2N+1}\sum_{j=1}^n(k_j^{(p)}+1)l_j(z^0)|z_j^{(p)}-\tilde{z}_j^{(p)}|=\\
&=\prod_{j=1}^n(\lambda_j(r_1,r_2,\dots,r_n))^{2N+1}\frac{S_p^*(z^0,(r_1,r_2,\dots,r_n))}{q(r_1,r_2,\dots,r_n)}\sum_{j=1}^n(k_j^{(p)}+1)r_j\le\\
&\le\prod_{j=1}^n(\lambda_j(r_1,r_2,\dots,r_n))^{2N+1}\frac{S_p^*(z^0,R)}{q(R)}(N+1)\|R\|\le\frac{1}{2}S_p^*(z^0,(r_1,r_2,\dots,r_n)).
\end{aligned}$$

Combining the start and the end of our considerations above, we conclude that:

$$S_p^*(z^0,(r_1,r_2,\dots,r_n))-S_{p-1}^*(z^0,(r_1,r_2,\dots,r_n))\le\frac{1}{2}S_p^*(z^0,(r_1,r_2,\dots,r_n)).$$

After simplification and reducing similar summands, this inequality transforms into the following: $S_p^*(z^0,(r_1,r_2,\dots,r_n))\le 2S_{p-1}^*(z^0,(r_1,r_2,\dots,r_n))$. Applying successively the estimate $S_p(z^0,(r_1,r_2,\dots,r_n))$ by $S_p^*(z^0,(r_1,r_2,\dots,r_n))$, the last inequality and the converse estimate $S_p^*(z^0,(r_1,r_2,\dots,r_n))$ by $S_p(z^0,(r_1,r_2,\dots,r_n))$ (we take them from (6)–(7)), one has:

$$\begin{aligned}
S_p(z^0,(r_1,r_2,\dots,r_n))&\le 2\prod_{j=1}^n(\lambda_j(r_1,r_2,\dots,r_n))^N S_{p-1}^*(z^0,(r_1,r_2,\dots,r_n))\le\\
&\le 2\prod_{j=1}^n(\lambda_j(R))^{2N}S_{p-1}(z^0,(r_1,r_2,\dots,r_n)). \qquad (12)
\end{aligned}$$

Therefore, successively applying the inequality (12) firstly for $p=q$, then for $p=q-1$, and finally for $p=1$, we obtain:

$$\max\left\{\frac{|H^{(k_1,\ldots,k_n)}(z)|}{k_1!\cdots k_n!\mathbf{L}^K(z)} : k_1+\cdots+k_n \le N, z \in \mathbb{D}^n\left[z^0, \frac{p}{q}\cdot\frac{R}{\mathbf{L}(z^0)}\right]\right\} = S_q(z^0,(r_1,r_2,\ldots,r_n)) \le$$
$$\le 2\prod_{j=1}^{n}(\lambda_j(r_1,r_2,\ldots,r_n))^{2N}S_{q-1}(z^0,(r_1,r_2,\ldots,r_n)) \le \ldots \le$$
$$\le (2\prod_{j=1}^{n}(\lambda_j(r_1,r_2,\ldots,r_n))^{2N})^q S_0(z^0,R) =$$
$$= (2\prod_{j=1}^{n}(\lambda_j(r_1,r_2,\ldots,r_n))^{2N})^q \max\left\{\frac{|H^{(k_1,\ldots,k_n)}(z^0)|}{k_1!\ldots k_n!\mathbf{L}^K(z^0)} : k_1+\cdots+k_n \le N\right\}. \tag{13}$$

If we compare estimate (13) and inequality (5) among themselves, then we see that this form of necessity condition is valid with such a parameter

$$d_0 = 2^{q(R)}(\lambda_1(R)\cdot\lambda_2(R)\cdots\lambda_n(R))^{2Nq(R)}$$

and with some partial derivative order K^0 (non-negative integer n-dimensional vector), for which its height is less than N, i.e., it does not exceed the joint **L**-index. We chose the parameter K^0 as the vector, at which the maximum of the fraction

$$\frac{|H^{(k_1,\ldots,k_n)}(z^0)|}{k_1!\ldots k_n! l_1^{k_1}(z^0)\cdots l_n^{k_n}(z^0)}$$

is attained above a finite set of all possible partial derivative orders $(k_1,\ldots,k_n)$ whose height $k_1+\cdot+k_n$ is less than the joint **L**-index. It completely finishes the proof of necessity for condition (5).

Let us move on to the sufficiency proof for the same restriction, i.e., the maximum modulus estimate (5) in the polydiscs for the expressions containing the partial derivatives, the vector factorial, and the vector K-th degree of the mapping **L**. Assume that, for every real vector-radius $R = (r_1,r_2,\ldots,r_n) \in \mathcal{B}^n$, we can find an upper estimate of index n_0 belonging to the set of positive integer numbers and positive real multiplier d_0 greater than 1 such that, for any point $z_0 = (z_1^0, z_2^0, \ldots, z_n^0)$ from the domain $\mathbb{G}$, it is possible to fit the partial derivative order $K^0 \in \mathbb{Z}_+^n$ with its height less than the upper estimate of index n_0 and for which the sufficiency condition (5) holds.

For further goals, we need an integral Cauchy's formula for analytic multivariate function written by an integral on the polydisc skeleton as follows. For any point $z^0 \in \mathbb{G}$, for every partial derivative order $K \in \mathbb{Z}_+^n$, and for every partial derivative order $S \in \mathbb{Z}_+^n$, the following equality must be satisfied:

$$\frac{H^{(k_1+s_1,\ldots,k_n+s_n)}(z^0)}{s_1!\cdots s_n!} = (2\pi i)^{-n}\int_{\mathbb{T}^n\left(z^0,\frac{R}{\mathbf{L}(z^0)}\right)} \frac{H^{(k_1,\ldots,k_n)}(z)}{(z_1-z_1^0)^{s_1+1}\ldots(z_n-z_n^0)^{s_n+1}}dz.$$

Therefore, we apply (5) to the right-hand side of the integral Cauchy's formula:

$$\frac{|H^{(k_1+s_1,\dots,k_n+s_n)}(z^0)|}{s_1!\dots s_n!} \le (2\pi)^{-n}\int_{\mathbb{T}^n\left(z^0,\frac{R}{\mathbf{L}(z^0)}\right)} \frac{|H^{(k_1+\dots+k_n)}(z)|}{|z_1-z_1^0|^{s_1+1}\dots|z_n-z_n^0|^{s_n+1}}|dz| \le$$

$$\le \int_{\mathbb{T}^n\left(z^0,\frac{R}{\mathbf{L}(z^0)}\right)} |H^{(K)}(z)|\frac{\mathbf{L}^{S+\mathbf{1}}(z^0)}{(2\pi)^n R^{S+\mathbf{1}}}|dz| \le$$

$$\le \int_{\mathbb{T}^n\left(z^0,\frac{R}{\mathbf{L}(z^0)}\right)} |H^{(K^0)}(z^0)|\frac{K!d_0\prod_{j=1}^n \lambda_j^{n_0}(R)\mathbf{L}^{S+K+\mathbf{1}}(z^0)}{(2\pi)^n k_1^0!\cdots k_n^0! r_1{}^{s_1+1}\cdots r_n{}^{s_n+1}(\mathbf{L}(z_1^0,\dots,z_n^0))^{K^0}}|dz| =$$

$$= |H^{(k_1^0,\dots,k_n^0)}(z_1^0,\dots,z_n^0)|\frac{K!d_0\prod_{j=1}^n \lambda_j^{n_0}(R)\mathbf{L}^{S+K}(z_1^0,\dots,z_n^0)}{k_1^0!\cdots k_n^0! r_1{}^{s_1}\cdots r_n{}^{s_n}(\mathbf{L}(z_1^0,\dots,z_n^0))^{K^0}}.$$

After multiplication of the last expression by the fraction $\frac{S!}{(K+S)!\mathbf{L}(z^0)}$, it creates an estimate that implies:

$$\frac{|H^{(k_1+s_1,\dots,k_n+s_n)}(z^0)|}{(K+S)!\mathbf{L}^{S+K}(z^0)} \le \frac{\prod_{j=1}^n \lambda_j^{n_0}(R)p_0K!S!}{(K+S)!R^S}\frac{|H^{(K^0)}(z^0)|}{K^0!\mathbf{L}^{K^0}(z^0)}. \tag{14}$$

Obviously, $s! \le \frac{(k+s)!}{k!} = (k+1)(k+2)\cdots(k+s)$ for all natural numbers k, s, so

$$\frac{K!S!}{(K+S)!} = \frac{s_1!}{(k_1+1)(k_1+2)\cdot\ldots\cdot(k_1+s_1)}\cdots\frac{s_n!}{(k_n+1)(k_n+2)\cdots(k_n+s_n)} \le 1.$$

The j-th component r_j of radius-vector R is chosen from the half-open interval $(1,\beta]$, where β is given by condition (2) on the mapping $\mathbf{L}$. Combining this choice in each dimension up to n, we construct the radius-vector $R=(r_1,\dots,r_n)$, which belongs to the Cartesian product $\mathcal{B}^n$ of the specified half-closed interval. Since $R^S = r_1^{s_1}\cdots r_n^{s_n}$ increases for the chosen radius-vector R as $s_1+s_2+\dots+s_n \to \infty$, the multiplier $\frac{d_0\prod_{j=1}^n \lambda_j^{n_0}(R)}{R^S}$ in (14) is tending to zero as $\|S\| \to +\infty$. This implicitly confirms an existence of s_0 such that, for all $S\in\mathbb{Z}_+^n$, $\|S\|\ge s_0$, the next right-hand side multiplier

$$p_0\frac{k_1!\cdots k_n!s_1!\cdots s_n!\prod_{j=1}^n \lambda_j^{n_0}(r_1,\dots,r_n)}{r_1^{s_1}\cdots r_n^{s_n}(k_1+s_1)!\cdots(k_n+s_n)!}$$

concerning $\frac{|H^{(K^0)}(z^0)|}{K^0!\mathbf{L}^{K^0}(z^0)}$ at the end of (14) also belongs to the segment $[0,1]$. After replacement of the multiplier by inequality (14), it yields

$$\frac{|H^{(k_1+s_1,\dots,k_n+s_n)}(z^0)|}{l_1^{k_1+s_1}(z^0)\cdots l_n^{k_n+s_n}(z^0)(k_1+s_1)!\dots(k_n+s_n)!} \le \frac{|H^{(k_1^0,\dots,k_n^0)}(z^0)|}{k_1^0!\cdots k_n^0!\mathbf{L}^{K^0}(z^0)}.$$

This means that, for every z^0 and $J\in\mathbb{Z}_+^n$,

$$\frac{|H^{(J)}(z^0)|}{J!}\mathbf{L}^{-J}(z^0) \le \max\left\{\frac{|H^{(K)}(z^0)|}{K!}\mathbf{L}^{-K}(z^0) : k_1+\cdots+k_n \le s_0+n_0\right\},$$

where the natural numbers s_0 and n_0 do not depend on z_0. Therefore, for the analytic in the complete multiple-circular domain function H, its joint $\mathbf{L}$-index must be finite, and it is bounded from above by the sum s_0+n_0. □

Theorem 2. *Let $\mathbb{G}$ be a complete Reinhardt domain; the mapping $\mathbf{L}$ belongs to the class $Q(\mathbb{G})$. In order that a holomorphic function $H\in\mathcal{A}^n(\mathbb{G})$ might be of finite $\mathbf{L}$-index in joint variables, it* *is necessary that, for every radius-vector $R \in \mathcal{B}^n$, there exists an upper estimate of joint index $n_0 \in \mathbb{Z}_+$, and there exists the uniform estimate $d \geq 1$ of the quotient of the maximum modulus of the partial derivative on a polydisc by the value of the modulus at the center of the polydisc. Additionally, for every point z^0 from the complete multiple-circular domain, one can find the partial derivative order K^0 (as an n-dimensional non-negative integer vector), whose height $\|K^0\| = k_1^0 + \cdots k_n^0$ is less than n_0 and*

$$\max\Big\{|H^{(K^0)}(z)| : z \in \mathbb{D}^n\Big[z^0, R/\mathbf{L}(z^0)\Big]\Big\} \leq d|H^{(K^0)}(z^0)|, \tag{15}$$

and it is sufficient that, for each radius-vector $R \in \mathcal{B}^n$, there exists $n_0 \in \mathbb{Z}_+$ and $d \geq 1$ such that, for all $z^0 \in \mathbb{G}$ and for every $j \in \{1, \ldots, n\}$, one can find the partial derivative in the j-th variable $K_j^0 = (0, \ldots, 0, \underbrace{k_j^0}_{j\text{-th place}}, 0, \ldots, 0)$, whose order k_j^0 is less than n_0 and whose maximum modulus of the K_j^0-th order partial derivative of the function H within the polydisc with the center z^0 and the radius $R/\mathbf{L}(z^0)$ is not greater than the value of the modulus of the derivative at the center of the polydisc, i.e.,

$$\max\Big\{|H^{(K_j^0)}(z)| : \ z \in \mathbb{D}^n\Big[z^0, \frac{R}{\mathbf{L}(z^0)}\Big]\Big\} \leq d|H^{(K_j^0)}(z^0)|. \tag{16}$$

Proof. Analyzing the proof of Theorem 1, we can discover that the inequality (5) is satisfied for some partial derivative order K^0. Rewriting (5) in the converse order and using a lower estimate concerning the behavior of the mapping $\mathbf{L}$, we establish that we have:

$$\begin{aligned}\frac{d_0}{K^0!}\frac{|H^{(K^0)}(z^0)|}{\mathbf{L}^{K^0}(z^0)} &\geq \max\Big\{\frac{|H^{(K^0)}(z)|}{K^0!\mathbf{L}^{K^0}(z)} : z \in \mathbb{D}^n\Big[z^0, R/\mathbf{L}(z^0)\Big]\Big\} = \\ &= \max\Big\{\frac{|H^{(K^0)}(z)|}{K^0!}\frac{\mathbf{L}^{K^0}(z^0)}{\mathbf{L}^{K^0}(z^0)\mathbf{L}^{K^0}(z)} : z \in \mathbb{D}^n\Big[z^0, R/\mathbf{L}(z^0)\Big]\Big\} \geq \\ &\geq \max\Big\{\frac{|H^{(K^0)}(z)|}{K^0!}\frac{\prod_{j=1}^n(\lambda_j(R))^{-n_0}}{\mathbf{L}^{K^0}(z^0)} : z \in \mathbb{D}^n\Big[z^0, R/\mathbf{L}(z^0)\Big]\Big\}.\end{aligned}$$

Multiplying this estimate by the product $\prod_{j=1}^n(\lambda_j(R))^{n_0}$, one has

$$\begin{aligned}&\frac{d_0\prod_{j=1}^n(\lambda_j(R))^{n_0}}{K^0!}\frac{|H^{(K^0)}(z^0)|}{\mathbf{L}^{K^0}(z^0)} \geq \\ &\geq \max\Big\{\frac{|H^{(K^0)}(z)|}{K^0!\mathbf{L}^{K^0}(z^0)} : z \in \mathbb{D}^n\Big[z^0, R/\mathbf{L}(z^0)\Big]\Big\}.\end{aligned} \tag{17}$$

Putting $d = d_0\prod_{j=1}^n(\lambda_j(R))^{n_0}$ above, we transform estimate (17) into inequality (15).

The sentence finishes the proof of necessity for condition (15).

To justify the sufficiency of (16), we suppose that, for each radius-vector $R \in \mathcal{B}^n$, one can find the upper estimate of index $n_0 \in \mathbb{Z}_+$ and $d > 1$ such that, for any point z_0 from the Reinhardt domain $\mathbb{G}$ and some $K_j^0 \in \mathbb{Z}_+^n$ with $k_j^0 \leq n_0$, inequality (16) holds.

As in the proof of the previous theorem, we again write the integral Cauchy's formula for an analytical function in the following form: for any point $z^0 \in \mathbb{G}$ and each partial derivative order $S \in \mathbb{Z}_+^n$:

$$\frac{H^{(K_j^0+S)}(z^0)}{S!} = \frac{1}{(2\pi i)^n}\int_{\mathbb{T}^n(z^0,R/\mathbf{L}(z^0))}\frac{H^{(K_j^0)}(z_1,\ldots,z_n)}{(z_1-z_1^0)^{s_1+1}\cdots(z_n-z_n^0)^{s_n+1}}dz.$$

We take the modulus from both parts and select the left-hand side of the sufficiency condition in each variable:

$$\frac{|H^{(K_j^0+S)}(z^0)|}{S!} \leq (2\pi)^{-n} \int_{\mathbb{T}^n(z^0,R/\mathbf{L}(z^0))} \frac{|H^{(K_j^0)}(z)|}{|z-z^0|^{S+\mathbf{1}}}|dz| \leq$$

$$\leq (2\pi)^{-n} \max\{|H^{(K_j^0)}(z)| : z \in \mathbb{D}^n\Big[z^0, R/\mathbf{L}(z^0)\Big]\} \frac{\mathbf{L}^{S+\mathbf{1}}(z^0)}{R^{S+\mathbf{1}}} \times$$
$$\times \int_{\mathbb{T}^n(z^0,R/\mathbf{L}(z^0))} |dz| = \max\{|H^{(K_j^0)}(z)| : z \in \mathbb{D}^n\Big[z^0, R/\mathbf{L}(z^0)\Big]\} \frac{\mathbf{L}^{S}(z^0)}{R^{S}}.$$

In the last expression, we substitute the maximum possible values of the radius-vector $R = (\beta, \ldots, \beta)$ and estimate the maximum modulus by the sufficiency condition (16) in each variable:

$$\frac{|H^{(K_j^0+S)}(z^0)|}{S!} \leq \frac{\mathbf{L}^S(z^0)}{\beta^{\|S\|}} \max\{|H^{(K_j^0)}(z)| : z \in \mathbb{D}^n\Big[z^0, R/\mathbf{L}(z^0)\Big]\} \leq$$
$$\leq \frac{d\mathbf{L}^S(z^0)}{\beta^{\|S\|}} |H^{(K_j^0)}(z^0)|. \tag{18}$$

The partial derivative order $S \in \mathbb{Z}_+^n$ will be chosen such that $s_1 + \ldots + s_n \geq s_0$ and such that s_0 is defined by the restriction $\frac{d}{\beta^{s_0}} \leq 1$. Therefore, (18) implies that, for all $j \in \{1, \ldots, n\}$ and $k_j^0 \leq n_0$:

$$\frac{|H^{(K_j^0+S)}(z^0)|}{\mathbf{L}^{K_j^0+S}(z^0)(K_j^0+S)!} \leq \frac{d}{\beta^{\|S\|}} \frac{S!K_j^0!}{(S+K_j^0)!} \frac{|H^{(K_j^0)}(z^0)|}{\mathbf{L}^{K_j^0}(z^0)K_j^0!} \leq \frac{|H^{(K_j^0)}(z^0)|}{\mathbf{L}^{K_j^0}(z^0)K_j^0!}.$$

Consequently, joint $\mathbf{L}$-index $N(H, \mathbf{L}, \mathbb{G})$ of the analytic function H in the Reinhardt domain is not greater than the sum of n_0 and s_0. □

Remark 2. *We write a few considerations concerning estimate* (15)*. It is a characterization property of finiteness of the l-index for the univariate complex-valued holomorphic function [28,31,32]. However, for some time, it was unknown whether this condition is sufficient so that the* $\mathbf{L}$*-index in joint variables for a holomorphic function is uniform bounded above all points from the holomorphy domain. At the present moment, there are examples of functions with finite* $\mathbf{L}$*-index in joint variables and unbounded l-index in each variable for any positive continuous function l. The presented conditions* (16) *in each variable are a certain multidimensional counterpart for the sufficient conditions.*

Lemma 1. *Assume that the mappings* $\mathbf{L}_1$ *and* $\mathbf{L}_2$ *belong to the class* $Q(\mathbb{G})$ *and that, for every point* $z \in \mathbb{G}$*, the inequality* $\mathbf{1} \leq \mathbf{L}_1(z) \leq \mathbf{L}_2(z)$ *holds as a component-wise inequality. If the joint index for an analytic function* $H \in \mathcal{A}^n(\mathbb{G})$ *is bounded for the lesser function* L_1*,. i.e.,* $N(F, \mathbf{L}_1, \mathbb{G}) < +\infty$*, then the joint index for the analytic function* $H \in \mathcal{A}^n(\mathbb{G})$ *is also bounded for the greater function* L_2*, i.e.,* $N(F, \mathbf{L}_2, \mathbb{G}) < +\infty$*, and the joint index concerning the greater auxiliary function does not exceed the joint index concerning the lesser auxiliary function increased by n times, i.e.,* $N(H, \mathbf{L}_2, \mathbb{G}) \leq nN(H, \mathbf{L}_1, \mathbb{G})$*.*

Proof. For simplicity of notation, we set $N(H, \mathbf{L}_1, \mathbb{G}) = n_0$. Using inequality (3) from the definition of joint index, we write the appropriate expression for the function $\mathbf{L}_2$ and replace it with the function $\mathbf{L}_1$:

$$\frac{|H^{(j_1,\ldots,j_n)}(z)|}{j_1!\cdots j_n!\mathbf{L}_2^J(z)} = \frac{\mathbf{L}_1^J(z)}{\mathbf{L}_2^J(z)}\frac{|H^{(j_1,\ldots,j_n)}(z)|}{j_1!\cdots j_n!\mathbf{L}_1^J(z)} \le$$
$$\le \frac{\mathbf{L}_1^J(z)}{\mathbf{L}_2^J(z)}\max\left\{\frac{|H^{(k_1,\ldots,k_n)}(z)|}{k_1!\cdots k_n!\mathbf{L}_1^K(z)} :\ k_1+\cdots+k_n \le n_0\right\} \le$$
$$\le \frac{\mathbf{L}_1^J(z)}{\mathbf{L}_2^J(z)}\max\left\{\frac{\mathbf{L}_2^K(z)}{\mathbf{L}_1^K(z)}\frac{|H^{(k_1,\ldots,k_n)}(z)|}{k_1!\cdots k_n!\mathbf{L}_2^K(z)} :\ k_1+\cdots+k_n \le n_0\right\} \le$$
$$\le \max_{k_1+\cdots+k_n\le n_0}\left(\frac{\mathbf{L}_1(z)}{\mathbf{L}_2(z)}\right)^{J-K}\max\left\{\frac{|H^{(k_1,\ldots,k_n)}(z)|}{k_1!\cdots k_n!\mathbf{L}_2^K(z)} :\ k_1+\ldots+k_n \le n_0\right\}.$$

Since $1 \le \mathbf{L}_1(z) \le \mathbf{L}_2(z)$, we can bound the expression $\max_{k_1+\cdots+k_n\le n_0}\left(\frac{\mathbf{L}_1(z)}{\mathbf{L}_2(z)}\right)^{J-K}$ by one from above. Then, for every $j_1+\cdots+j_n \ge n\cdot n_0$:

$$\frac{|H^{(J)}(z)|}{J!\mathbf{L}_2^J(z)} \le \max\left\{\frac{|H^{(K)}(z)|}{K!\mathbf{L}_2^K(z)} :\ K\in\mathbb{Z}_+^n,\ \|K\|\le n_0\right\}.$$

The last inequality means finiteness of the joint $\mathbf{L}_2$-index for the holomorphic function H in the whole complete multiple-circular domain, and $N(H,\mathbf{L}_2,\mathbb{G}) \le nN(H,\mathbf{L}_1,\mathbb{G})$. □

Let us introduce the second auxiliary function $\widetilde{\mathbf{L}}(z) = (\widetilde{l}_1(z),\ldots,\widetilde{l}_n(z))$. The notation $\mathbf{L} \asymp \widetilde{\mathbf{L}}$ stands for the existence of two n-dimensional positive real vectors $\Theta_1 = (\theta_{1,j},\ldots,\theta_{1,n})$ and $\Theta_2 = (\theta_{2,j},\ldots,\theta_{2,n})$, for which $\theta_{1,j}\widetilde{l}_j(z) \le l_j(z) \le \theta_{2,j}\widetilde{l}_j(z)$ for every $j\in\{1,2,3,\ldots,n\}$ in the whole Reinhardt domain concerning the variable z.

Theorem 3. *Let* $\mathbf{L}\in Q(\mathbb{G})$, $\mathbf{L}\asymp\widetilde{\mathbf{L}}$, $\beta\Theta_1 > \mathbf{1}$. *A function H belonging to the class $\mathcal{A}^n(\mathbb{G})$ of analytic functions has bounded $\widetilde{\mathbf{L}}$-index in joint variables if and only if the function is of finite joint* $\mathbf{L}$*-index.*

Proof. Using the definition of the auxiliary class $Q(\mathbb{G})$, it can be checked that, if $\mathbf{L}\in Q(\mathbb{G})$ and $\mathbf{L}\asymp\widetilde{\mathbf{L}}$, then $\widetilde{\mathbf{L}}\in Q(\mathbb{G})$.

As above, for simplicity, we put $N(H,\widetilde{\mathbf{L}},\mathbb{G}) = \widetilde{n}_0$ as finite. Then, by Theorem 1 for every radius-vector $\widetilde{R} = (\widetilde{r}_1,\ldots,\widetilde{r}_n)$ taken from the Cartesian product $\mathcal{B}^n$, there must exist a real $\widetilde{d}\ge 1$ such that inequality (5) holds at all points $z^0\in\mathbb{G}$ chosen as centers of polydiscs of domain exhaustion and some K^0 dependent on z^0 with the height less than $\widetilde{n}_0$. In addition, we replace the auxiliary function $\mathbf{L}$ with $\widetilde{\mathbf{L}}$ and change the positive real radius-vector R by $\widetilde{R}$ in (5). Hence:

$$\frac{\widetilde{d}}{K^0!}\frac{|H^{(K^0)}(z^0)|}{\mathbf{L}^{K^0}(z^0)} = \frac{\widetilde{d}}{K^0!}\frac{\Theta_2^{K^0}|H^{(K^0)}(z^0)|}{\Theta_2^{K^0}\mathbf{L}^{K^0}(z^0)} \ge \frac{\widetilde{d}}{\Theta_2^{K^0}}\frac{|H^{(K^0)}(z^0)|}{\widetilde{\mathbf{L}}^{K^0}(z^0)K^0!} \ge$$
$$\ge \frac{1}{\Theta_2^{K^0}}\max\left\{\frac{|H^{(K)}(z)|}{K!\widetilde{\mathbf{L}}^K(z)} : k_1+\ldots+k_n\le\widetilde{n}_0, z\in\mathbb{D}^n\left[z^0,\widetilde{R}(\widetilde{\mathbf{L}}(z))^{-1}\right]\right\} \ge$$
$$\ge \frac{1}{\Theta_2^{K^0}}\max\left\{\frac{\Theta_1^K|H^{(K)}(z)|}{K!\mathbf{L}^K(z)} : k_1+\ldots+k_n\le\widetilde{n}_0, z\in\mathbb{D}^n\left[z^0,\Theta_1\widetilde{R}(\mathbf{L}(z))^{-1}\right]\right\} \ge$$
$$\ge \frac{\min\limits_{0\le\|K\|\le n_0}\{\Theta_1^K\}}{\Theta_2^{K^0}}\max\left\{\frac{|H^{(K)}(z)|}{K!\mathbf{L}^K(z)} : k_1+\ldots+k_n\le\widetilde{n}_0, z\in\mathbb{D}^n\left[z^0,\Theta_1\widetilde{R}(\widetilde{\mathbf{L}}(z))^{-1}\right]\right\}.$$

Again using Theorem 1 in the converse direction, we conclude that the function H is of finite $\mathbf{L}$-index in joint variables. □

Theorem 4. *Let an auxiliary function* $\mathbf{L}$ *belong to* $Q(\mathbb{G})$*. A function* H *from the class* $\mathcal{A}^n(\mathbb{G})$ *is of finite joint* $\mathbf{L}$*-index if and only if there exist a radius-vector* $R \in \mathcal{B}^n$*, an upper estimate of index* $n_0 \in \mathbb{Z}_+$*, a uniform estimate* $d_0 > 1$ *of local growth of the maximum modulus such that, for every point* $z^0 \in \mathbb{G}$ *and for some* $K^0 \in \mathbb{Z}^n_+$ *with* $\|K^0\| \le n_0$*, inequality* (5) *holds.*

Proof. The necessity of Theorem 1 is proved above for all possible values of radius-vector from the Cartesian product $\mathcal{B}^n$. In the present theorem, it is required for one radius-vector. Therefore, it follows from the specified theorem. We will organize the proof of the sufficiency by the schema of sufficiency proof from Theorem 1. The proof of Theorem 1 with $R = (\beta, \ldots, \beta)$ implies that $N(H, \mathbf{L}, \mathbb{G}) < +\infty$.

Let us introduce the auxiliary new function $\mathbf{L}^*(z) = \frac{R_0\mathbf{L}(z)}{R}$, where $R^0 = (\beta, \ldots, \beta)$ and where the parameter β is chosen by the property of the function $\mathbf{L}$. We will try to justify validity of (5) for any radius-vector if it is true for H, $\mathbf{L}$, and some $R = (r_1, \ldots, r_n) \in \mathcal{B}^n$ with $R \neq R^0$; we obtain:

$$\begin{aligned}
&\max\left\{ \frac{|H^{(K)}(z)|}{K!(\mathbf{L}^*(z^0))^K} : \|K\| \le n_0, z \in \mathbb{D}^n\left[z^0, R_0/\mathbf{L}^*(z^0)\right] \right\} = \\
&= \max\left\{ \frac{|H^{(K)}(z)|}{K!(R_0\mathbf{L}(z)/R)^K} : \|K\| \le n_0, z \in \mathbb{D}^n\left[z^0, R_0/(R_0\mathbf{L}(z)/R)\right] \right\} \le \\
&\le \max\left\{ \frac{n^{\|K\|/2}|H^{(K)}(z)|}{K!\mathbf{L}^K(z)} : \|K\| \le n_0, z \in \mathbb{D}^n\left[z^0, R/\mathbf{L}(z^0)\right] \right\} \le \\
&\le \frac{d_0}{K^0!}\frac{n^{n_0/2}|H^{(K^0)}(z^0)|}{\mathbf{L}^{K^0}(z^0)} = \frac{n^{n_0/2}\beta^{\|K^0\|}d_0}{R^{K^0}K^0!}\frac{|H^{(K^0)}(z^0)|}{(R_0\mathbf{L}(z)/R)^{K^0}} < \\
&< n^{n_0/2}d_0 \max\left\{ \frac{\beta^{\|K^0\|}}{R^{K^0}} : \|K^0\| \le n_0 \right\} \frac{|H^{(K^0)}(z^0)|}{K^0!(\mathbf{L}^*(z))^{K^0}}.
\end{aligned}$$

Therefore, (5) is satisfied for the holomorphic function H, the auxiliary mapping $\mathbf{L}^*$, and radius-vector $R_0 = (\beta, \ldots, \beta)$. Further, we will apply Theorem 1 to the holomorphic function $H(z)$ and the following mapping: $\mathbf{L}^*(z) = R_0/R\mathbf{L}(z)$. This application leads us to the conclusion that H has finite joint $\mathbf{L}^*$-index. Then, we can refer to Theorem 3 to justify the boundedness of joint $\mathbf{L}$-index for the function H, which is analytic in the complete multiple-circular domain. □

3. Discussion

The obtained results are the basis for further investigations of analytic functions in a complete Reinhardt domain by usage of the notion of $\mathbf{L}$-index in joint variables. For entire multivariate complex-valued functions and analytical functions in the unit ball, there are known results describing estimates of partial logarithmic derivatives [15], value distribution [20,33,34], and applications in analytic theory of systems of partial differential equations [16,35,36]. The Reinhardt domain has the fine property of so-called circular symmetry in differential geometry (or multiple-circularity in multidimensional complex analysis). The property means that rotations by a circle in each variable do not move points outside the Reinhardt domain. Taking this into account, we hope that most of the known results for entire functions can be generalized for analytic functions in a complete Reinhardt domain in the framework of the theory of functions having a bounded index. Moreover, we do not know lower growth estimates for entire functions with finite $\mathbf{L}$-index in joint variables. Meanwhile, the symmetry and completeness of the Reinhardt domain require new, more difficult, and powerful methods for investigations. In view of this, these newly developed methods will allow us to solve the problems that are still unsolved for n-dimensional complex space, such as the complete regular growth of entire functions of several complex variables [37], lower growth estimates for this class of functions [15], index estimates for solutions of infinite order linear differential equations [29], etc.

Author Contributions: Conceptualization, A.B. and O.S.; methodology, A.B.; validation, O.S.; investigation, A.B. and T.S.; writing—original draft preparation, T.S.; writing—review and editing, A.B. and O.S.; supervision, O.S. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Data are contained within the article.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Fu, Q.; Deng, G. The weighted reproducing kernels of the Reinhardt domain. *Compl. Var. Ell. Equ.* **2023**. [CrossRef]
2. Cho, G. Comparing the Carathéodory Pseudo-Distance and the Kähler–Einstein Distance on Complete Reinhardt Domains. *Asian J. Math.* **2022**, *26*, 37–44. [CrossRef]
3. Beberok, T. Explicit Formulas of the Bergman Kernel for Some Reinhardt Domains. *J. Geom. Anal.* **2016**, *26*, 2883–2892. [CrossRef]
4. Wang, A.; Zhong, C. Lu Qi-Keng Problem and Topological Properties of Zero Variety on the Thullen Domain. *Front. Math.* **2023**, *18*, 817–836. [CrossRef]
5. Chatterjee, S.; Mondal, G.M. A characterization of bounded balanced convex domains in $\mathbb{C}^n$. *J. Math. Anal. Appl.* **2024**, *533*, 128008. [CrossRef]
6. Jarnicki, M.; Pflug, P. Properties of Bounded Holomorphic Functions: A Survey. *J. Geom. Anal.* **2024**, *34*, 18. [CrossRef]
7. Wang, J.; Zhang, Y. The boundary Schwarz lemma and the rigidity theorem on Reinhardt domains B_p^n of $\mathbb{C}^n$. *Acta Math. Sci.* **2024**, *44*, 839–850. [CrossRef]
8. Liu, T.; Tang, X.; Wang, J. Rigidity for convex mappings of Reinhardt domains and its applications. *Sci. China Math.* **2019**, *62*, 901–920. [CrossRef]
9. Das, N. A logarithmic lower bound for the second Bohr radius. *Canad. Math. Bull.* **2024**, *67*, 90–93. [CrossRef]
10. Das, N. The p-Bohr radius for vector-valued holomorphic and pluriharmonic functions. *Forum Math.* **2023**. [CrossRef]
11. Chakrabarti, D.; Cinzori, I.; Gaidhane, I.; Gregory, J.; Wright, M. Bergman kernels of monomial polyhedra. *J. Math. Anal. Appl.* **2024**, *531*, 127723. [CrossRef]
12. Sima, X.; Tu, Z.; Xiong, L. Some results of homogeneous expansions for a class of biholomorphic mappings defined on a Reinhardt domain in $\mathbb{C}^n$. *Demonstr. Math.* **2023**, *56*, 20220242. [CrossRef]
13. Kuryliak, A.; Skaskiv, O. Wiman's type inequality in multiple-circular domain. *Axioms* **2021**, *10*, 348. [CrossRef]
14. Kuryliak, A.O.; Skaskiv, O.B. Wiman-Type Inequality in Multiple-Circular Domains: Levy's Phenomenon and Exceptional Sets. *Ukr. Math. J.* **2022**, *74*, 743–756. [CrossRef]
15. Bandura, A.I.; Skaskiv, O.B. Analytic functions in the unit ball of bounded **L**-index: Asymptotic and local properties. *Mat. Stud.* **2017**, *48*, 37–73. [CrossRef]
16. Nuray, F.; Patterson, R.F. Vector-valued bivariate entire functions of bounded index satisfying a system of differential equations. *Mat. Stud.* **2018**, *49*, 67–74. [CrossRef]
17. Range, R.M. Domains of Holomorphy and Pseudoconvexity. In *Holomorphic Functions and Integral Representations in Several Complex Variables. Graduate Texts in Mathematics*; Springer: New York, NY, USA, 1986; Volume 108. [CrossRef]
18. Patterson, R.; Nuray, F. A characterization of holomorphic bivariate functions of bounded index. *Math. Slov.* **2017**, *67*, 731–736. [CrossRef]
19. Nuray, F.; Patterson, R.F. Entire bivariate functions of exponential type. *Bull. Math. Sci.* **2015**, *5*, 171–177. [CrossRef]
20. Nuray, F.; Patterson, R.F. Multivalence of bivariate functions of bounded index. *Le Mat.* **2015**, *70*, 225–233. [CrossRef]
21. Wang, L.; Zhang, C. Zeros of the deformed exponential function. *Adv. Math.* **2018**, *332*, 311–348. [CrossRef]
22. Zhang, C. An asymptotic formula for the zeros of the deformed exponential function. *J. Math. Anal. Appl.* **2016**, *441*, 565–573. [CrossRef]
23. Sokal, A. Roots of a Formal Power Series, with Applications to Graph Enumeration and q-Series. Available online: https://webspace.maths.qmul.ac.uk/p.j.cameron/csgnotes/sokal11.pdf (accessed on 11 February 2024).
24. Sokal, A. Some Wonderful Conjectures (but Almost No Theorems) at the Boundary between Analysis, Combinatorics and Probability. Available online: http://ipht.cea.fr/statcomb2009/misc/Sokal_20091109.pdf (accessed on 11 February 2024).
25. Sokal, A. The leading root of the partial theta function. *Adv. Math.* **2012**, *229*, 2603–2621. [CrossRef]
26. Krishna, G.J.; Shah, S.M. Functions of bounded indices in one and several complex variables. In *Mathematical Essays Dedicated to A.J. Macintyre*; Ohio University Press: Athens, OH, USA, 1970; pp. 223–235.
27. Salmassi, M. Functions of bounded indices in several variables. *Indian J. Math.* **1989**, *31*, 249–257.
28. Sheremeta, M.N.; Kuzyk, A.D. Logarithmic derivative and zeros of an entire function of bounded l-index. *Sib. Math. J.* **1992**, *33*, 304–312. [CrossRef]
29. Lepson, B. Differential equations of infinite order, hyperdirichlet series and entire functions of bounded index. *Proc. Sympos. Pure Math.* **1968**, *11*, 298–307.
30. Bandura, A.; Petrechko, N.; Skaskiv, O. Maximum modulus in a bidisc of analytic functions of bounded **L**-index and an analogue of Hayman's theorem. *Mat. Bohem.* **2018**, *143*, 339–354. [CrossRef]

31. Kushnir, V.O.; Sheremeta, M.M. Analytic functions of bounded l-index. *Mat. Stud.* **1999**, *12*, 59–66.
32. Sheremeta, M. *Analytic Functions of Bounded Index*; VNTL Publishers: Lviv, Ukraine, 1999.
33. Sheremeta, M.M. On the univalence of entire functions of bounded l-index. *Mat. Stud.* **2015**, *43*, 185–188. [CrossRef]
34. Bandura, A.I. Analytic functions in the unit ball of bounded value L-distribution in a direction. *Mat. Stud.* **2018**, *49*, 75–79. [CrossRef]
35. Sheremeta, M.M.; Trukhan, Y.S. Properties of analytic solutions of three similar differential equations of the second order. *Carp. Math. Publ.* **2021**, *13*, 413–425. [CrossRef]
36. Sheremeta, M.M.; Trukhan, Y.S. Properties of analytic solutions of a differential equation. *Mat. Stud.* **2019**, *52*, 138–143. [CrossRef]
37. Ronkin, L.I. *Functions of Completely Regular Growth*; Mathematics and Its Applications: Soviet Series; Springer: Dordrecht, The Netherlands, 1992; Volume 81.