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Pseudo-Finsler Radially Symmetric Spaces

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Abstract: We introduce the concept of radially symmetric pseudo-Finsler spaces, which generalize the notion of symmetric Finsler spaces, and prove that this concept is equivalent to the preservation of flag curvature by parallel transport together with reversibility. As a consequence, reversible pseudo-Finsler manifolds with constant flag curvature are radially symmetric.

Keywords: Finsler symmetric spaces; parallel transport; flag curvature

1. Introduction

The theory of symmetric spaces was successfully developed by E. Cartan in the 1920s (see [1,2]). The concept is very simple: a complete Riemannian space is symmetric if for all $p \in M$, the map ϕ_p, such that $\phi_p(\gamma(t)) = \gamma(-t)$ for all geodesic γ and all $t \in \mathbb{R}$, is an isometry. Locally, this is equivalent to M having a parallel curvature tensor, and the term *locally symmetric* has been reserved for the spaces with this property. The notion of symmetric space makes sense in the Finslerian realm, but it has been proved that in the global case, there is little novelty, as globally symmetric Finsler spaces (with positive definite fundamental tensors throughout) are of Berwald type (their geodesics can be computed as the auto-parallel curves of an affine connection), and there is a Riemannian symmetric space with the same geodesics (see [3], Theorem 2.7). Regarding the local case, as observed by Egloff (see the comments in [4] and also [5]), a non-Riemannian Hilbert geometry provides a counterexample of a reversible Finsler metric with parallel curvature (in the sense that we will explain later), which is not locally symmetric; namely, it does not locally admit an isometry map ϕ_p as defined above. Moreover, it seems that this concept of local symmetry is very restrictive and, in many cases, assuming some conditions on the flag curvature implies that the metric is Riemannian or Berwald (see [4,6–8]). For the study of Berwald symmetric spaces, see [3,9,10].

The main goal of this paper is to introduce a weaker notion of Finsler symmetric spaces, the so-called radially symmetric spaces, namely, those spaces with the property that the map ϕ_p defined above is an isometry for the osculating metrics g_v tangent to the radial geodesics from p (see Definition 6). Then, we prove that these spaces can be characterized by being reversible with flag curvature preserved by the parallel transport. At this point, it is important to point out that to define the parallel transport of the flag curvature, it is necessary to consider the parallel transport introduced in [11]. In this way, the flagpole is transported as an observer (taking as a reference the same vector field), while the transverse edge of the flag is transported with respect to the parallel observer determined by the flagpole (see Section 2.3). Unlike the Riemannian case, this condition does not imply that some data in the tangent map, namely, the curvature tensor and the Finsler metric, completely determines the Finsler metric in a neighborhood. Indeed, the Cartan–Ambrose–Hicks type theorem in Theorem 1 only provides information about the radial directions (compare this theorem with the one given in ([12], Theorem 2.1) without

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a complete proof). However, assuming that the metric is Berwald, we again recover this property (see Corollary 4).

To develop all the computations, we use the notion of anisotropic connection introduced in [13] rather than the more standard approach followed in [14]. The reasons for this choice will be explained at the beginning of Section 2. We organize the paper as follows: in Section 2, we define the concept of the anisotropic tensor field and introduce the notion of anisotropic connection and how it acts on anisotropic tensor fields. In Section 2.2, we introduce the curvature tensor of an anisotropic connection. In Section 2.3, we define the two types of parallel transport along a given curve, the observer-parallel transport of an admissible vector v with respect to an anisotropic connection and the parallel transport of a vector field along the curve with respect to a chosen admissible vector v. In the final Section 3, we prove in Proposition 3 that the Jacobi curvature operator is parallel if and only if it takes a parallel vector field into another parallel vector field in both cases with respect to an observer-parallel vector field V, and this is also equivalent to the flag curvature being invariant under parallel transport. To conclude this section, we prove the Cartan–Ambrose–Hicks type result mentioned in the beginning, a characterization of being radially symmetric as being reversible with parallel Jacobi operator (Corollary 2), and finally we prove that radially symmetric is equivalent to locally symmetric for Berwald metrics (Corollaries 4 and 5).

2. Anisotropic Tensor Calculus and Pseudo-Finsler Geometry

The anisotropic tensor calculus is an attempt to make computations in Finsler geometry closer to the techniques used in modern Riemannian geometry. Its origin dates back to the use of the osculating metric by A. Nazim [15] as early as 1936 in his Ph.D. thesis. A few years later, O. Varga [16] further studied the osculating metric and its Levi–Civita connection, which turned out to admit a good interpretation when a geodesic vector field V was chosen. In 1980, H. Matthias in his Ph.D. thesis [17] took a step forward defining the so-called family of affine connections constructed from the Chern connection, which is a connection on the vertical bundle, and a choice of a vector field V without zeroes. The use of this family of affine connections was clarified by Z. Shen in 2001 in his book about sprays [18], where he made clear that the family of affine connections is very useful to compute the Jacobi operator whenever the vector field V is geodesic ([18], Proposition 8.4.3). Around the same time, Álvarez-Paiva and Durán [19], Rademacher [20,21] and Kováks and Tóth [22] gave certain popularity to the consideration of the Chern connection as a family of affine connections. The notion of anisotropic tensor calculus has been recently introduced in [13,23]. Observe that a related notion of an anisotropic connection had previously appeared in ([18], Definition 7.1.1), but the notion used here is formally slightly different (see Definition 3). One of the main achievements of the anisotropic tensor calculus is that one can completely determine the Chern curvature tensor by means of affine connections without assuming that the prescribed vector field is geodesic (see Proposition 1). On the other hand, the covariant derivation of anisotropic tensor fields has a very natural interpretation using parallel transport, as explained in [11]. To obtain this interpretation, we need to define two different types of parallel transport, one which is already known, and it can be interpreted as the parallel transport of the non-linear connection, but we call it *observer-parallel transport*, and the another one with respect to a vector v, which takes as as reference the observer-parallel vector field obtained from v (see Definition 4). These definitions have been crucial to establishing the equivalences in Proposition 3. Even if it is possible to make all these definitions using connections on the vertical bundle of TM, they are very natural in the context of anisotropic tensor calculus, and as we state above, all the computations become quite similar to the modern treatment of Riemannian geometry. As this calculus is not yet a common ground in the Finslerian community, it is explained in the following. The explicit relation with the calculus using a connection in the vertical bundle is developed in ([13], Section 4.4).

Given a smooth manifold M of dimension n, with TM its tangent bundle and TM^* its cotangent bundle, let us denote by $\pi : TM \to M$ and $\tilde{\pi} : TM^* \to M$ the natural projections. If A is an open subset of the tangent bundle TM, the restriction $\pi|_A : A \subset TM \to M$ can be used to obtain two vector bundles over A by lifting π and $\tilde{\pi}$, which we denote, respectively, by $\pi_A^* : \pi_A^*(M) \to A$ and $\tilde{\pi}_A^* : \tilde{\pi}_A^*(M) \to A$:

$$\begin{tikzcd} \pi_A^*(M) \arrow[d, "\pi_A^*"'] & TM \arrow[d, "\pi"] \\ A \subset TM \arrow[r, "\pi|_A"] & M \end{tikzcd} \qquad \begin{tikzcd} \tilde{\pi}_A^*(M) \arrow[d, "\tilde{\pi}_A^*"'] & TM^* \arrow[d, "\tilde{\pi}"] \\ A \subset TM \arrow[r, "\pi|_A"] & M \end{tikzcd}$$

In particular, for every $v \in A$, one has that $(\pi_A^*)^{-1}(v) = T_{\pi(v)}M$ and $(\tilde{\pi}_A^*)^{-1}(v) = T_{\pi(v)}M^*$. Then, a section of π_A^* (resp. $\tilde{\pi}_A^*$) is a smooth map $A \ni v \to \mathcal{X}(v) \in TM$ (resp. $A \ni v \to \theta(v) \in TM^*$) such that $\mathcal{X}(v) \in T_{\pi(v)}M$ (resp. $\theta(v) \in T_{\pi(v)}M^*$). We will denote by $\mathfrak{T}_0^1(M_A)$ the space of (smooth) sections of $\pi_A^*(M)$, while the subset of smooth sections of $\tilde{\pi}_A^*(M)$ will be denoted by $\mathfrak{T}_1^0(M_A)$.

Definition 1. *An A-anisotropic tensor field T of type (r,s), $r,s \in \mathbb{N} \cup \{0\}$, $r+s > 0$ is defined as an $\mathcal{F}(A)$-multilinear map*

$$T : \mathfrak{T}_1^0(M_A)^r \times \mathfrak{T}_0^1(M_A)^s \to \mathcal{F}(A),$$

where $\mathcal{F}(A)$ is the subset of smooth real functions on A, namely, $f : A \to \mathbb{R}$.

The space of A-anisotropic tensor fields of type (r,s) is denoted by $\mathfrak{T}_s^r(M_A)$, while by convention $\mathfrak{T}_0^0(M_A) \equiv \mathcal{F}(A)$. The $\mathcal{F}(A)$-multilinearity implies that for every $v \in A$, T determines a multilinear map

$$T_v : (T_{\pi(v)}M^*)^r \times (T_{\pi(v)}M)^s \to \mathbb{R}.$$

By the $\mathcal{F}(A)$-multilinearity, it is enough to define the tensor field as

$$T : \mathfrak{X}^*(M)^r \times \mathfrak{X}(M)^s \to \mathcal{F}(A), \tag{1}$$

which then will be extended by $\mathcal{F}(A)$-multilinearity using a local frame in $\mathfrak{X}(M)$ (resp. $\mathfrak{X}^*(M)$); see also ([13], Remark 2).

One can also consider an $\mathcal{F}(A)$-multilinear map

$$T : \mathfrak{T}_0^1(M_A)^s \to \mathfrak{T}_0^1(M_A), \tag{2}$$

which determines the A-anisotropic tensor field of type $(1,s)$ $\bar{T} : \mathfrak{T}_1^0(M_A) \times \mathfrak{T}_0^1(M_A)^s \to \mathcal{F}(A)$ defined by

$$\bar{T}(\theta, X_1, \ldots, X_s) = \theta(T(X_1, \ldots, X_s)). \tag{3}$$

As in classical tensor calculus, T will be considered as a tensor field itself, using the formula above only when necessary.

We will say that a vector field V defined on an open subset $\Omega \subset M$ is A-admissible (with $A \subset TM$) if $V_p \in A$ for every $p \in \Omega$. In such a case, we can define a (classical) tensor field $T_V \in \mathfrak{T}_s^r(\Omega)$ as a map

$$T_V : \mathfrak{X}^*(\Omega)^r \times \mathfrak{X}(\Omega)^s \to \mathcal{F}(\Omega),$$

such that

$$T_V(\theta^1, \ldots, \theta^r, X_1, \ldots, X_s)(p) = T_{V_p}(\theta^1, \ldots, \theta^r, X_1, \ldots, X_s),$$

where $\mathcal{F}(\Omega) = \{f : \Omega \to \mathbb{R} : f \in C^\infty\}$ and $\mathfrak{X}(\Omega)$ and $\mathfrak{X}^*(\Omega)$ denote, respectively, the space of vector fields and one-forms on Ω.

As a result of the dependence on directions of A-anisotropic tensor fields, one can define derivatives on the vertical bundle.

Definition 2. *Given an anisotropic tensor field $T \in \mathfrak{T}^r_s(M_A)$, its* vertical derivative *is defined as the tensor field $\dot{\partial} T \in \mathfrak{T}^r_{s+1}(M_A)$ given by*

$$(\dot{\partial} T)_v(\theta^1, \ldots, \theta^r, X_1, \ldots, X_s, Z) = \frac{\partial}{\partial t} T_{v+tZ_{\pi(v)}}(\theta^1, \ldots, \theta^r, X_1, \ldots, X_s)|_{t=0}$$

for any $v \in A$ and $(\theta^1, \theta^2, \ldots, \theta^r, X_1, \ldots, X_s, Z) \in \mathfrak{X}^*(M)^r \times \mathfrak{X}(M)^{s+1}$, and an analogous definition is made for anisotropic tensor fields of the type (2).

2.1. Anisotropic Connections

Let us introduce a central concept to make operations with anisotropic tensor fields.

Definition 3. *An* anisotropic (linear) connection *is a map*

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{T}^1_0(M_A), \qquad (X, Y) \mapsto \nabla_X Y,$$

such that

(i) $\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z$, *for any* $X, Y, Z \in \mathfrak{X}(M)$,
(ii) $\nabla_X(fY) = (X(f)Y) \circ \pi_A + (f \circ \pi_A)\nabla_X Y$ *for any* $f \in \mathcal{F}(M)$, $X, Y \in \mathfrak{X}(M)$,
(iii) $\nabla_{fX+hY} Z = (f \circ \pi_A)\nabla_X Z + (h \circ \pi_A)\nabla_Y Z$, *for any* $f, h \in \mathcal{F}(M)$, $X, Y, Z \in \mathfrak{X}(M)$,

where $\pi_A = \pi|_A$.

We will use the notation $\nabla^v_X Y := (\nabla_X Y)_v$. Furthermore, the *torsion* of ∇ is defined as the anisotropic tensor

$$\mathcal{T}_v(X, Y) = \nabla^v_X Y - \nabla^v_Y X - [X, Y], \quad \text{for any } X, Y \in \mathfrak{X}(M). \tag{4}$$

An anisotropic connection is said to be *torsion-free* if $\mathcal{T} = 0$. Given a system of coordinates (Ω, φ) of M, we define the Christoffel symbols of ∇ as the functions $\Gamma^k_{\ ij} : T\Omega \cap A \to \mathbb{R}$ determined by $\nabla^v_{\partial_i} \partial_j = \Gamma^k_{\ ij}(v)\partial_k$. Observe that ∇ is torsion-free if and only if $\Gamma^k_{\ ij}$ is symmetric in i and j.

An anisotropic connection ∇ induces an anisotropic tensor derivation ∇_X for every vector field $X \in \mathfrak{X}(M)$ (see [13], Section 2.2, for the general definition) in the space of anisotropic tensor fields $\mathfrak{T}^r_s(M_A)$ such that for any function $h \in \mathcal{F}(A)$, $\nabla_X h \in \mathcal{F}(A)$ is determined by

$$\nabla_X h(v) = X(h(V))(\pi(v)) - (\dot{\partial} h)_v(\nabla^v_X V), \tag{5}$$

where V is any A-admissible vector field extending v, namely, $V_{\pi(v)} = v$. Observe that the expression in (5) does not depend on the choice of V (see [13], Lemma 9). Moreover, if θ is a one-form, then $\nabla_X \theta \in \mathfrak{T}^0_1(M_A)$ is determined by

$$(\nabla_X \theta)_v(Y) = X(\theta(Y))(\pi(v)) - \theta_{\pi(v)}(\nabla^v_X Y), \quad \text{for any } Y \in \mathfrak{X}(M). \tag{6}$$

Finally, for an arbitrary anisotropic tensor field $T \in \mathfrak{T}^r_s(M_A)$, we define the derivation

$$\begin{aligned}(\nabla_X T)(\theta^1, \ldots, \theta^r, X_1, \ldots, X_s) =& \nabla_X(T(\theta^1, \ldots, \theta^r, X_1, \ldots, X_s)) \\ &- \sum_{i=1}^{r} T(\theta^1, \ldots, \nabla_X \theta^i, \ldots, \theta^r, X_1, \ldots, X_s) \\ &- \sum_{j=1}^{s} T(\theta^1, \ldots, \theta^r, X_1, \ldots, \nabla_X X_j, \ldots, X_s),\end{aligned} \tag{7}$$

for any $(\theta^1, \theta^2, \ldots, \theta^r, X_1, \ldots, X_s) \in \mathfrak{X}^*(M)^r \times \mathfrak{X}(M)^s$ (see [13], Theorem 11, and recall that ∇_X is an anisotropic derivation as in [13], Definition 8). Observe that the same formula (7) with $s = 0$ also holds for tensor fields of the type (2).

Finally, we can also define *the vertical derivative of* ∇ as the anisotropic tensor field given by:

$$P_v(X, Y, Z) = \frac{\partial}{\partial t}\left(\nabla_X^{v+tZ_{\pi(v)}} Y\right)|_{t=0},$$

where $v \in A$ and X, Y, Z are arbitrary smooth vector fields on M. Moreover, in a natural system of coordinates of the tangent bundle $(T\Omega, \tilde{\varphi})$, associated with a coordinate system (Ω, φ) on M, one has

$$P_v(u, w, z) = u^i w^j z^k \frac{\partial \Gamma^l{}_{ij}}{\partial y^k}(v)\partial_l \tag{8}$$

for every $v \in A$, and $u, w, z \in T_{\pi(v)}M$ and u^i, w^i and z^i being the coordinates of u, w, z. As usual, we denote the coordinates of a point $v \in T\Omega$ as

$$\tilde{\varphi} = (x, y) = (x^1, x^2, \ldots, x^n, y^1, y^2, \ldots, y^n), \tag{9}$$

and we use the *Einstein summation convention* when possible, omitting the coordinate functions φ and $\tilde{\varphi}$ to avoid clutter in equations. Thus, P is symmetric in the first two arguments if ∇ is torsion-free.

2.2. Curvature Tensor

It is possible to associate a curvature tensor $R_v : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to T_{\pi(v)}M$ with every anisotropic (linear) connection ∇ as follows

$$R_v(X, Y)Z = \nabla_X^v(\nabla_Y Z) - \nabla_Y^v(\nabla_X Z) - \nabla_{[X,Y]}^v Z, \tag{10}$$

for any $v \in A$ and $X, Y, Z \in \mathfrak{X}(M)$. Here, one has to take into account that $\nabla_Y Z, \nabla_X Z \in \mathfrak{T}_0^1(M_A)$, namely, they are anisotropic vector fields. One can check that R is an $\mathcal{F}(A)$-multilinear map, and then an anisotropic tensor as in (2), which is anti-symmetric in X and Y.

Recall that given an A-admissible vector field V in $\Omega \subset M$, the anisotropic connection ∇ provides an affine connection ∇^V on Ω defined as $(\nabla_X^V Y)_p = \nabla_X^v Y$ for any $X, Y \in \mathfrak{X}(M)$, being $v = V_p$. This affine connection determines its curvature tensor as

$$R^V(X, Y)Z = \nabla_X^V \nabla_Y^V Z - \nabla_Y^V \nabla_X^V Z - \nabla_{[X,Y]}^V Z,$$

where X, Y, Z are arbitrary smooth vector fields on Ω. The tensor R^V depends on the choice of V, but it can be used to get an expression of the curvature tensor of ∇.

Proposition 1. *Let ∇ be an anisotropic (linear) connection and $\Omega \subset M$, an open subset. Then, for any $v \in A$,*

$$R_v(X, Y)Z = (R^V(X, Y)Z - P_V(Y, Z, \nabla_X^V V) + P_V(X, Z, \nabla_Y^V V))(\pi(v)), \tag{11}$$

where $V, X, Y, Z \in \mathfrak{X}(\Omega)$, V being an A-admissible extension of v.

Proof. See [23], Proposition 2.5. $\square$

2.3. Parallel Transport

Proposition 2. *Given a smooth curve $\alpha : [a,b] \to M$, an anisotropic connection ∇ on M with admissible domain $A \subset TM \setminus \mathbf{0}$ and an A-admissible vector field X along α, there exists a unique covariant derivative*

$$\frac{D^X}{dt} : \mathfrak{X}(\alpha) \to \mathfrak{X}(\alpha)$$

such that $\frac{D^X}{dt}Y = \nabla^X_{\dot{\alpha}(t)}Y$ whenever $Y(t) = Y_{\alpha(t)}$ for some vector field Y on M.

Proof. See [23], Proposition 2.7. □

Definition 4. *Given a regular curve $\alpha : [a,b] \to M$, the* observer-parallel transport *of an admissible vector $v \in A$ with respect to the anisotropic connection ∇ is the map*

$$\mathcal{P}_p : T_{\alpha(a)}M \to T_{\alpha(b)}M$$

such that $\mathcal{P}_p(v) = V(b)$, where V is the vector field along α such that $\frac{D^V V}{dt} = 0$ and $V(a) = v$. Moreover, the parallel transport with respect to v is given by

$$\mathcal{P}^v_p : T_{\alpha(a)}M \to T_{\alpha(b)}M$$

such that $\mathcal{P}^v_p(w) = Z(b)$, being $Z \in \mathfrak{X}(\alpha)$ determined by $Z(a) = w$ and $\frac{D^V X}{dt} = 0$, with V as above.

Moreover, for any tensor $T \in \mathfrak{T}^1_s(M_A)$, interpreted as

$$T : \mathfrak{T}^1_0(M_A)^s \to \mathfrak{T}^1_0(M_A),$$

we can obtain the covariant derivative $(\nabla_Y T)_v$ using a curve α such that $\dot{\alpha}(0) = Y_{\pi(v)}$, an observer-parallel vector field V, with $V(a) = v$ and parallel vector fields $X_1, \ldots, X_s$ with respect to v. It turns out that

$$(\nabla_Y T)_v(X_1, \ldots, X_s) = \nabla^v_Y(T_V(X_1, \ldots, X_s)), \tag{12}$$

(see [11], Section 7).

2.4. Pseudo-Finsler Manifolds and Flag Curvature

We say that a function $L : TM \to \mathbb{R}$ is a pseudo-Finsler metric if

1. It is smooth on $TM \setminus \mathbf{0}$;
2. It is positive homogeneous of degree 2;
3. For every $v \in TM \setminus \mathbf{0}$, the fundamental tensor defined as

$$g_v(u,w) = \frac{1}{2}\frac{\partial^2}{\partial t \partial s} L(v + tu + sw)|_{t=s=0},$$

is a non-degenerate bilinear form.

Given a pseudo-Finsler manifold, there is a unique anisotropic connection ∇ such that it is torsion-free and $\nabla g = 0$, (see [13], Section 4.1). This connection can be identified with the Chern connection, which can be naturally interpreted as an anisotropic connection. Using the Chern connection and its associated curvature tensor introduced in (10), one can then introduce one of the main geometrical invariants of a pseudo-Finsler metric, the *flag curvature*, $K(v,w)$, which depends on a vector v, which plays the role of the flagpole, and a g_v-nondegenerate plane that contains v and w. It is defined as

$$K(v,w) = \frac{g_v(R_v(v,w)w,v)}{L(v)g_v(w,w) - g_v(v,w)^2}.$$

Definition 5. *The flag curvature is said to be invariant under parallel translation along a curve* $\alpha : [a,b] \to M$ *if*

$$K(v,w) = K(\mathcal{P}_p(v), \mathcal{P}_p^v(w)),$$

for all $v \in T_pM \setminus \{0\}$ *and* $w \in T_pM$*. When it is invariant along any regular curve, we will simply refer to it as invariant under parallel translation.*

3. Radially Symmetric Finsler Manifolds

Definition 6. *Let* (M,L) *be a pseudo-Finsler manifold with* $p \in M$ *a given point and* Ω *a neighborhood of* p *with the following property: for any geodesic* γ *with* $\gamma(0) = p$*, we have that if* $\gamma|_{[0,t]} \subset \Omega$*, then* γ *is defined at least in* $[-t,t]$*. Then,*

1. *We define* $I_p : \Omega \to M$ *as* $I_p(q) = \gamma(-t)$*, where* γ *is the only unit geodesic such that* $\gamma(0) = p$ *and* $\gamma(t) = q$*.*
2. *We say that* (M,L) *is* locally radially symmetric *if for all* $p \in M$ *and* Ω *as above, we have that*
$$d(I_p)_q : (T_qM, g_{\dot\gamma(t)}) \to (T_{\gamma(-t)}M, g_{-\dot\gamma(-t)})$$
is an isometry (as above γ *is a unit geodesic,* $\gamma(0) = p$ *and* $\gamma(t) = q$*) for all* $q \in \Omega$*.*

Recall that the Jacobi curvature operator R^J is defined as $(R^J)_v : T_{\pi(v)}M \to T_{\pi(v)}M$ with $R^J_v(w) = R_v(v,w)v$.

Proposition 3. *Given a pseudo-Finsler manifold, the following conditions are equivalent:*

(i) $\nabla R^J = 0$.
(ii) *If* V *is an observer-parallel vector field along a regular curve* α *and* X *is parallel with respect to* V*, then* $R^J_V(X)$ *is also parallel with respect to* V *along* α*.*
(iii) *Flag curvature is invariant under parallel translation.*

Proof. $(i) \Leftrightarrow (ii)$. It follows from the characterization of the covariant derivative in terms of parallel vector fields. Indeed, given a regular curve α, a vector field Y such that $Y_{\alpha(t)} = \dot\alpha(t)$ and an observer-parallel vector field V along α with $V(a) = v$, then

$$(\nabla_Y R^J)_v(X) = \nabla^v_Y(R^J_V(X)) = \frac{D^V}{dt}(R^J_V(X))$$

taking into account (12).

$(ii) \Rightarrow (iii)$. Observe that if $\alpha : [a,b] \to M$ is a regular curve, V is an observer-parallel vector field with $V(a) = v$ and X is a parallel vector field with respect to v, then, by applying ([23], Equation (46)), and using that $g_V(X,X)$, $g_V(V,V)$ and $g_V(V,X)$ are constant as a consequence of the choice of V and X as observer-parallel and parallel vector fields, respectively, we deduce that

$$\begin{aligned}
\frac{d}{dt}K(V,X) &= \frac{d}{dt}\frac{g_V(R_V(V,X)X,V)}{g_V(X,X)g_V(V,V) - g_V(V,X)^2} \\
&= -\frac{\frac{d}{dt}g_V(R_V(V,X)V,X)}{g_V(X,X)g_V(V,V) - g_V(V,X)^2} \\
&= -\frac{\frac{d}{dt}g_V(R^J_V(X),X)}{g_V(X,X)g_V(V,V) - g_V(V,X)^2} \\
&= -\frac{g_V(\nabla^V_{\dot\alpha}(R^J_V(X)),X)}{g_V(X,X)g_V(V,V) - g_V(V,X)^2} = 0,
\end{aligned}$$

as required.

$(iii) \Rightarrow (i)$. We have to check that if V is an observer-parallel vector field and X is parallel with respect to V, then $g_V((\nabla_Y R^J)(X), W) = 0$ for all W parallel with respect to V, but this is equivalent to proving that

$$A_V(X, W) = g_V(R_V^J(X), W)$$

is constant along α (with α as in the proof of $(i) \Leftrightarrow (ii)$). As $g_V(X, X)g_V(V, V) - g_V(V, X)^2$ is constant because V is observer-parallel and X is parallel with respect to V, we have that $A_V(X, X)$ is constant because, by hypothesis in part (iii), the flag curvature

$$K(V, X) = -\frac{A_V(X, X)}{g_V(X, X)g_V(V, V) - g_V(V, X)^2}$$

is constant (see the centered formula in the above implication). In particular, $A_V(X + W, X + W)$ is also constant (as $X + W$ is parallel with respect to V), and therefore, we conclude that

$$A_V(X, W) = \frac{1}{2}(A_V(X + W, X + W) - A_V(X, X) - A_V(W, W))$$

is also constant, as required. □

Corollary 1. *A pseudo-Finsler manifold of constant flag curvature has a parallel Jacobi curvature operator.*

Observe that using Proposition 3 and ([24], Theorem 2), it follows that the property of having a flag curvature invariant by parallel transport is invariant under Zermelo deformations using a killing field. Recall that a Zermelo deformation of a Finsler metric F with a vector field W is obtained as the Finsler metric Z, which has as indicatrix the translation of the indicatrix of F_p with W_p at each point $p \in M$.

Definition 7. *Let (M, L) and $(\bar{M}, \bar{L})$ be pseudo-Finsler manifolds and $\ell : T_pM \to T_q\bar{M}$ an isometry. Let us denote with $\exp_p$ the exponential map of M at p and $\overline{\exp}_q$ the exponential map of $\bar{M}$ at q, and let $\mathcal{U}$ be a normal neighborhood of p small enough such that $\ell \circ \exp_p^{-1}(\mathcal{U})$ is contained in a domain of $\overline{\exp}_q$ where it is a diffeomorphism. Then, we define the polar map as*

$$\phi_\ell = \overline{\exp}_q \circ \ell \circ \exp_p^{-1} : \mathcal{U} \to \bar{M}.$$

The next theorem is a local Finslerian version of the Cartan–Ambrose–Hicks Theorem using the osculating metrics (see [1,2] for the original local version by Cartan and [25,26] for the global extensions).

Theorem 1. *If (M, L) and $(\bar{M}, \bar{L})$ are pseudo-Finsler manifolds that have parallel Jacobi curvature operators, and $\ell : T_pM \to T_q\bar{M}$ is a linear isometry that preserves the Jacobi curvature operator, then ϕ_ℓ is an isometry with respect to the osculating metrics g_V and $\bar{g}_{V^*}$, V being the tangent vector to the unit radial geodesics from p and $V^* = d\phi_\ell(V)$.*

Proof. Let $v \in T_pM$ and denote $v^* = \ell(v)$. Then, $\gamma_v(t) = \exp_p(tv)$ is the radial geodesic with $\gamma(0) = p$ and $\dot{\gamma}_v(0) = v$. By the definition of ϕ_ℓ, we have that $(\phi_\ell \circ \gamma_v)(t) = \tilde{\gamma}_{v^*}(t) = \overline{\exp}_q(tv^*)$ is the radial geodesic in $\bar{M}$ starting at q and with initial velocity tv^*. The tangent vector V to γ_v is an observer-parallel vector field, i.e., $\frac{D^V V}{dt} = 0$. Similarly, the tangent vector field to the geodesic $\tilde{\gamma}_{v^*}$ is an observer-parallel vector field, i.e., $\frac{\bar{D}^{V^*} V^*}{dt} = 0$.

To show that ϕ_ℓ is an isometry with respect to the osculating metrics g_V and $\bar{g}_{V^*}$, it suffices to show that for any $x \in T_{\tilde{p}}M \setminus \{0\}$, $\tilde{p} \in \mathcal{U}$ with $\tilde{p} = \gamma_v(1)$ for $v \in T_pM$, we have $\bar{g}_{V^*}(d\phi_\ell(x), d\phi_\ell(x)) = g_V(x, x)$.

We know, from the description of Jacobi fields using the exponential map (see, for example, [27], Lemma 3.14 and Proposition 3.15) that $x = (d\ \exp_p)_v(y_v)$ for some $y_v \in T_v(T_pM)$, and then $x = Z(1)$, where $Z(t) = (d\ exp_p)_{tv}(ty_v)$ is the unique Jacobi field with $Z(0) = 0$ and $\frac{D^V Z}{dt}(0) = y \in T_pM$ (here $y_v \equiv y$ using $T_v(T_pM) \equiv T_pM$).

Now, we look at the manifold $\bar{M}$. Since $\phi_\ell(\tilde{p}) = \overline{\exp}_q \circ \ell \circ \exp_p^{-1}(\tilde{p})$; therefore, $d\phi_\ell \circ d\ \exp_p = d\ \overline{\exp}_q \circ d\ell$ and together with the fact that $d\ell(y_v) = \ell(y)_{v^*}$, since ℓ is linear, gives that $d\phi_\ell(x) = (d\ \overline{\exp}_q)(\ell(y)_{v^*}) = \tilde{Z}(1)$, where $\tilde{Z}(t) = (d\ \overline{\exp}_q)_{tv^*}(t\ell(y)_{v^*})$ is the unique Jacobi field with $\tilde{Z}(0) = 0$ and $\frac{\bar{D}^{V^*}\tilde{Z}}{dt}(0) = \ell(y) \in T_qM$.

Let $\{E_1(0), \dots, E_n(0)\}$ be an orthonormal basis of T_pM with regard to the metric g_v and let $\{E_1(t), \dots, E_n(t)\}$ be its parallel transport with respect to v along γ_v. This parallel frame will be orthonormal with regard to the metric g_V. Since ℓ is a linear isometry, $\{\ell(E_1(0)), \dots, \ell(E_n(0))\}$ is an orthonormal basis of T_qM with regard to the metric $\bar{g}_{v^*}$. Let $\{\tilde{E}_1(t), \dots, \tilde{E}_n(t)\}$ be its parallel transport with regard to the vector v^* along the geodesic $\tilde{\gamma}_{v^*}$. This parallel frame will remain parallel with regard to the metric $\bar{g}_{V^*}$, and we have that $\tilde{E}_i(0) = \ell(E_i(0))$, $i = 1, \dots, n$.

We write $v = a^i E_i(0)$, which gives $\ell(v) = a^i \tilde{E}_i(0)$. Similarly, if $y = b^i E_i(0)$, then $\ell(y) = b^i \tilde{E}_i(0)$. We also write the Jacobi fields Z along γ and $\tilde{Z}$ along $\tilde{\gamma}$ using these parallel frames as $Z = z^i E_i$ and $\tilde{Z} = \tilde{z}^i \tilde{E}_i$, respectively. They satisfy the Jacobi field equation $\frac{D^V}{dt}\left(\frac{D^V Z}{dt}\right) = R_V(V, Z)V$, which can be written as $\frac{D^V}{dt}\left(\frac{D^V Z}{dt}\right) = R^J_V(Z)$, where R^J is the Jacobi operator. So, the coordinate functions $z_i(t)$ satisfy the system of ODEs

$$(z^i)''(t) = R^j_i(V) z^i(t),$$

where $R^J_V(E_i) = R^j_i(V) E_j$. Since R^J is parallel and V is the observer-parallel translation of v, it follows that $R^j_i(V)$ is constant, since by Proposition 3, $R^J_V(E_i)$ is also parallel. A similar argument, using the Jacobi equation satisfied by $\tilde{Z}$ on the manifold $\bar{M}$, gives

$$(\tilde{z}^i)''(t) = \bar{R}^j_i(V^*) \tilde{z}^i(t)$$

where $\bar{R}^J_V{}^*(\tilde{E}_i) = \bar{R}^j_i(V^*)\tilde{E}_j$ and $\bar{R}^J$ is the Jacobi operator associated with $\tilde{\gamma}_{v^*}$. Since $\bar{R}^J$ is parallel and V^* is the observer-parallel translation of v^*, it follows that the components $\bar{R}^j_i(V^*)$ are also constant. However, $R^j_i(V) = \bar{R}^j_i(V^*)$ on the common domain I, since the isometry ℓ preserves the Jacobi curvature operator, i.e., $R^j_i(v) = \bar{R}^j_i(v^*)$ at $t = 0$.

It follows that the functions z^i and $\tilde{z}^i$, $i = 1, \dots, n$ satisfy the same system of linear ODEs with the same initial conditions, so it follows by uniqueness of such solutions that $z^i(t) = \tilde{z}^i(t), i = 1 \dots n$. Therefore $g_{V*}(d\phi_\ell(x), d\phi_\ell(x)) = g_{V*}(\tilde{Z}(1), \tilde{Z}(1)) = \sum_{i=1}^n \varepsilon_i (\tilde{z}^i)^2 = \sum_{i=1}^n \varepsilon_i (z^i)^2 = g_V(Z(1), Z(1)) = g_V(x, x)$, where $\varepsilon_i = g_V(E_i, E_i) = \bar{g}_{V^*}(\tilde{E}_i, \tilde{E}_i)$ for $i = 1, \dots, n$. $\square$

Corollary 2. *A reversible pseudo-Finsler manifold (M, L) has parallel Jacobi operator if and only if it is locally radially symmetric.*

Proof. As (M, L) is reversible, at every point $p \in M$, the map $\ell : (T_pM, L_p) \to (T_pM, L_p)$, $v \to -v$ is an isometry. Moreover, the reversibility also implies that ℓ preserves the Jacobi operator, namely, $R^J_v = R^J_{-v}$ for all $v \in T_pM \setminus \{0\}$, and then we can apply Theorem 1 to conclude the implication to the right. For the converse, observe that the radial isometry preserves $(\nabla R^J)_v$, and therefore

$$-(\nabla_y R^J)_v(x) = d\phi_\ell((\nabla_y R^J)_v(x)) = (\nabla_{-y} R^J)_{-v}(-x) = (\nabla_y R^J)_{-v}(x),$$

for any $x, y \in T_pM$, since $d\phi_\ell = -\mathrm{Id}$. As $\nabla^v = \nabla^{-v}$, because F is reversible, we also have that $(\nabla R^J)_{-v} = (\nabla R^J)_v$, and then we conclude that $\nabla R^J = 0$. □

Corollary 3. *A reversible pseudo-Finsler metric with constant flag curvature is locally radially symmetric.*

Proof. A constant flag curvature manifold always satisfies part (iii) of Proposition 3, and then it has a parallel Jacobi operator. Therefore, the result follows from Corollary 2. □

In particular, Hilbert geometries are always radially symmetric, as they have constant flag curvature and are reversible. Recall that a Hilbert–Finsler metric is defined by symmetrizing a Berwald metric, which is obtained applying a Zermelo deformation with the position vector field to the unit ball of a Minkowski norm.

Corollary 4. *If (M, L) and $(\bar{M}, \bar{L})$ are Berwald manifolds that have parallel Jacobi curvature operator, and $\ell : T_pM \to T_q\bar{M}$ is an isometry that preserves the Jacobi curvature operators, then ϕ_ℓ is an isometry.*

Proof. Observe that as (M, L) is Berwald, and V is the tangent vector field to the unit geodesics from a certain $p \in M$, then the anisotropic connection $\nabla = \nabla^V$ is indeed an affine connection on M and coincides with the Chern connection of L (see [18], p. 100). Moreover, as V is geodesic, ∇^V is the Levi–Civita connection of g_V, and then $(\phi_\ell)_*(\nabla^V) = \bar{\nabla}^{V^*}$, where V^* is the tangent vector field to the unit geodesics from $\phi_\ell(p)$, since ϕ_ℓ is an isometry of g_V and $\bar{g}_{V^*}$, the osculating metrics of (M, L) and $(\bar{M}, \bar{L})$, respectively. Finally, using that the parallel transport of ∇ preserves the Berwald metric L and that ℓ also preserves L, we easily conclude that ϕ_ℓ is an isometry for L and $\bar{L}$ (see also [28], Theorem 5.2). □

Corollary 5. *If (M, L) is a reversible Berwald manifold that has a parallel Jacobi curvature operator, then it is locally symmetric.*

4. Conclusions

In this paper, we have introduced the class of locally radially symmetric pseudo-Finsler manifolds, a new family of pseudo-Finsler manifolds with some symmetric properties with respect to an osculating metric associated with the exponential map. The advantage of this family with respect to the classical symmetric Finsler manifolds is that these metrics can be characterized as those which are reversible and have parallel flag curvature (see Corollary 2), mimicking what occurs in the family of locally symmetric Riemannian manifolds. Moreover, they include reversible pseudo-Finsler manifolds with constant flag curvature (see Corollary 3). The key result to prove all these properties is Theorem 1, which is a Cartan–Ambrose–Hicks Theorem, and it does not hold in general when the isometry of the osculating metrics is replaced with an isometry of the pseudo-Finsler metrics.

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