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Modified Tseng Method for Solving Pseudomonotone Variational Inequality Problem in Banach Spaces

Rose Maluleka [1], Godwin Chidi Ugwunnadi [1,2,*], Maggie Aphane [1], Hammed A. Abass [1] and Abdul Rahim Khan [3]

1 Department of Mathematics and Applied Mathematics, Sefako Makgatho Health Sciences University, Medunsa, P.O. Box 94, Pretoria 0204, South Africa; rose.maluleka@smu.ac.za (R.M.); maggie.aphane@smu.ac.za (M.A.); hammedabass548@gmail.com or hammed.abass@smu.ac.za (H.A.A.)

2 Department of Mathematics, Faculty of Science and Engineering, University of Eswatini, Private Bag 4, Kwaluseni M201, Eswatini

3 Department of Mathematics and Statistics, Institute of Southern Punjab, Multan 60800, Pakistan; abdulrahimkhan@isp.edu.pk

* Correspondence: ugwunnadi4u@yahoo.com or gcugwunnadi@uniswa.sz

Abstract: This article examines the process for solving the fixed-point problem of Bregman strongly nonexpansive mapping as well as the variational inequality problem of the pseudomonotone operator. Within the context of p-uniformly convex real Banach spaces that are also uniformly smooth, we introduce a modified Halpern iterative technique combined with an inertial approach and Tseng methods for finding a common solution of the fixed-point problem of Bregman strongly nonexpansive mapping and the pseudomonotone variational inequality problem. Using our iterative approach, we develop a strong convergence result for approximating the solution of the aforementioned problems. We also discuss some consequences of our major finding. The results presented in this paper complement and build upon many relevant discoveries in the literature.

Keywords: Bregman strongly nonexpansive mapping; variational inequality problem; strongly nonexpansive mapping; Banach space; fixed point

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1. Introduction

Let $\mathcal{E}$ be a real Banach space with its dual $\mathcal{E}^*$. Let $\mathcal{C}$ be a nonempty closed convex subset of $\mathcal{E}$ and $\langle \cdot, \cdot \rangle$ denote the duality pairing between $\mathcal{E}$ and $\mathcal{E}^*$. The variational inequality problem (VIP) with respect to $\mathcal{A}$ is a problem of finding $u \in \mathcal{C}$ such that

$$\langle \mathcal{A}u, v-u \rangle \geq 0, \quad \forall v \in \mathcal{C}, \tag{1}$$

in which the operator $\mathcal{A}$ maps $\mathcal{C}$ to $\mathcal{E}^*$. We write $VI(\mathcal{C}, \mathcal{A})$ to represent the set of (1) solutions. Variational inequality theory, independently developed in the mechanics and potential theory by Stampacchia and Fichera in the early 1960s (see [1,2]), can be used broadly to treat a wide class of unrelated linear and nonlinear problems in elasticity, economics, transportation, optimization, control theory, and engineering sciences. The development of variational inequality theory can be understood as the simultaneous pursuit of two different fields of research. Basic facts on the qualitative behavior of solutions to important kinds of issues are disclosed in the first aspect. However, it also makes it possible for us to develop highly efficient and powerful numerical techniques to deal with boundary value problems, including unilateral, moving, free, and obstacle problems (see [3]). It is commonly known that $VI(\mathcal{C}, \mathcal{A})$ is equivalent to the fixed-point problem:

$$\text{find } u^* \in \mathcal{C} \text{ such that } u^* = P_{\mathcal{C}}(u^* - \tau \mathcal{A}u^*),$$

where a metric projection onto $\mathcal{C}$ is denoted by $P_{\mathcal{C}}$ and τ is any positive real number. Solving $VI(\mathcal{C},\mathcal{A})$ has been approached in a variety of ways recently (see [4–13]). The gradient projection method (GPM) is the most basic projection method. The general concept of expanding the GPM to solve the $F(u)$ minimization issue pertaining to $u \in \mathcal{C}$ is provided by

$$u_{n+1} = P_{\mathcal{C}}(u_n - \alpha_n \nabla F(u_n)),\ n \geq 0, \tag{2}$$

where the gradient function is $\nabla F(u_n)$ and the positive real sequence $\{\alpha_n\}$ satisfies a given condition. The GPM is a direct expansion of the procedure in (2). It involves replacing operator F with the gradient function in order to produce a sequence $\{u_n\}$ in the way that follows:

$$u_{n+1} = P_{\mathcal{C}}(u_n - \alpha_n F u_n),\ \forall\, n \geq 0.$$

Nevertheless, this method's convergence necessitates a somewhat strong assumption that the operators are either strongly monotone or inversely monotonous. To loosen this constraint, the extragradient method (EM) for a monotone and L-Lipschitz continuous mapping $\mathcal{A}$ was suggested by Korpelvich [14] and Antipin [15] in finite-dimensional Euclidean spaces.

$$\begin{cases} u_0 \in \mathcal{C},\ \tau > 0, \\ v_n = P_{\mathcal{C}}(u_n - \tau \mathcal{A} u_n), \\ u_{n+1} = P_{\mathcal{C}}(u_n - \tau \mathcal{A} v_n),\ \forall\, n \geq 1, \end{cases} \tag{3}$$

where $\tau \in (0, \frac{1}{L})$. The sequence u_n produced by the EM (3) converges to an element of $VI(\mathcal{C},\mathcal{A})$ if $VI(\mathcal{C},\mathcal{A})$ is not empty. Note that each iteration in the EM requires the computation of two projections into the feasible set $\mathcal{C}$. Should the set $\mathcal{C}$ not be simple, implementing the EM becomes exceedingly complex and costly. Additionally, we stress that the stepsize defined by the process is excessively small and lowers the technique's convergence rate. Moreover, the method (3) requires a prior estimate of the Lipschitz constant, which is frequently difficult to estimate. To the best of our knowledge, these shortcomings can be addressed in certain ways. The first is the subgradient extragradient method (SEGM) [9], which was proposed by Censor et al. In this approach, a projection onto a certain constructible half-space is used in place of the second projection onto $\mathcal{C}$. Their approach takes this form:

$$\begin{cases} v_n = P_{\mathcal{C}}(u_n - \tau \mathcal{A} u_n), \\ \mathcal{T}_n = \{w \in H : \langle u_n - \tau \mathcal{A} u_n - v_n, w - v_n \rangle \leq 0\}, \\ u_{n+1} = P_{\mathcal{T}_n}(u_n - \tau \mathcal{A} v_n),\ \forall\, n \geq 0, \end{cases} \tag{4}$$

where $\tau \in (0, \frac{1}{L})$.

The second approach is Tseng's method from [16]. Their approach takes this form:

$$\begin{cases} v_n = P_{\mathcal{C}}(u_n - \tau \mathcal{A} u_n), \\ u_{n+1} = v_n - \lambda(\mathcal{A} v_n - \mathcal{A} u_n),\ \forall\, n \geq 0, \end{cases} \tag{5}$$

where $\tau \in (0, \frac{1}{L})$.

It is important to mention that the SEGM and TM algorithms explained earlier merely require the computation of a single projection onto $\mathcal{C}$ in each iteration, which has the potential to enhance the performance of these algorithms. Numerous researchers have made enhancements to the SEGM and TM through various approaches (refer to [4,7,9,16–18] and the related references). We want to emphasize that both methods (SEGM and TM) have been extensively studied by authors in the context of real Hilbert and Banach spaces. One of the most effective strategies to accelerate the rate of convergence for iterative algorithms is to incorporate the inertial term into the iterative scheme. This term, denoted by $\theta_n(u_n - u_{n-1})$, serves as a remarkable tool for enhancing the performance of the algorithm and is known for its favorable convergence properties. Consequently, there is a growing interest among researchers working in this field (see [4,19–23]). The concept of the inertial extrapolation method was initially introduced by Polyak [24] and was inspired by an implicit discretization of a second-order-in-time dissipative dynamical system, commonly referred to as the "heavy ball with friction".

$$\sqsubseteq''(t) + \gamma\sqsubseteq'(t) + \bigtriangledown g(\sqsubseteq(t)) = 0, \tag{6}$$

when $g : \mathbb{R}^n \to \mathbb{R}$ and $\gamma > 0$ are differentiable. The discretization of the system (6) allows for the determination of the following term using u_{n+1}:

$$\frac{u_{n+1} - 2u_n + u_{n-1}}{j^2} + \gamma\frac{u_n - u_{n-1}}{j} + \bigtriangledown g(u_n) = 0,\ n \geq 1, \tag{7}$$

where the step-size is denoted by j. The following iterative algorithm is produced by Equation (7):

$$u_{n+1} = u_n + \beta(u_n - u_{n-1}) - \alpha \bigtriangledown g(u_n),\ n \geq 1, \tag{8}$$

where the inertial approach $\beta(u_n - u_{n-1})$ and $\beta = 1 - \gamma_j, \alpha = j^2$ are used to accelerate the convergence of the sequence produced by (8). Using the proximal point algorithm (PPA), also known as the inertial PPA, Alvarez and Attouch used the inertial extrapolation method to set a general maximal monotone operator.

$$\begin{cases} v_n = u_n + \theta_n(u_n - u_{n-1}), \\ u_{n+1} = (I + r_n\mathcal{B})^{-1}v_n, n > 1. \end{cases} \tag{9}$$

Their demonstration showed that if $\{r_n\}$ is non-decreasing and $\{\theta_n\} \subset [0,1)$, then

$$\sum_{n=1}^{\infty} \theta_n||u_n - u_{n-1}||^2 < \infty, \tag{10}$$

Afterward, a weak convergence of Algorithm (9) to a zero of $\mathcal{B}$ is achieved. For $\theta_n < \frac{1}{3}$, more specifically, condition (10) holds true. An initial factor is denoted by θ_n.

The inertial extrapolation approach in Banach space has been updated by a number of writers by leaving out the calculation of the difference between the norms of the two neighboring iterates, u_n and u_{n-1}. Because of the geometry of the space, the inertial term must be modified when approximating solutions of various optimization problems using the inertial extrapolation method in Banach space using either the viscosity or Halpern method (see [7,20,25] and the references therein). The hybrid and shrinking procedures used in the Banach space setting is the only scenario in which the inertial terms remain unchanged (see [4,22,23]). As far as we are aware, there is not a result for the inertial extrapolation method in Banach space without utilizing the Halpern method modification.

Question 1: Without computing the difference between the norms of the two adjacent iterates, u_n and u_{n-1}, can we introduce an inertial Halpern method combined with the Tseng procedure for approximating the outcome of VIP in the context of p-uniformly convex real Banach spaces that are also uniformly smooth?

We propose a modified Halpern inertial iterative method, inspired by the work of [15–18] and others, combined with a Tseng-type technique to find a common solution of the pseudomonotone variational inequality problem and the fixed-point problem of Bregman strongly nonexpansive mapping in the context of uniformly smooth, p-uniformly real Banach space. We provide a strong convergence result for approximating the solution of the aforementioned problems using our iterative method. We stress that our iterative approach does not require any prior knowledge about the operator standard because of its architecture. To demonstrate the effectiveness of our solution, we provide a few numerical examples. Numerous relevant results in the literature are extended and enhanced by the results reported in this work.

2. Preliminaries

We state some known and useful results which will be needed in the proof of our main theorem. In the sequel, we denote strong and weak convergence by "$\rightarrow$" and "$\rightharpoonup$", respectively.

Given a Banach space $\mathcal{E}$, let its dual be $\mathcal{E}^*$. It is argued that an operator $\mathcal{A} : \mathcal{E} \rightarrow \mathcal{E}^*$ is $p - L$-Lipschitz if for all $u, v \in \mathcal{E}$,

$$||\mathcal{A}u - \mathcal{A}v|| \leq L||u - v||^p,$$

where two constants are $L \geq 0$ and $p \in [1, +\infty)$. The operator $\mathcal{A}$ is called L-Lipschitz if $p = 1$.

Suppose there is a nonempty set $\mathcal{C} \subseteq \mathcal{E}$. Next, let $\mathcal{A} : \mathcal{C} \rightarrow \mathcal{E}^*$ be a mapping. Then, for every $u, v \in \mathcal{C}$, $\mathcal{A}$ is

(a) *monotone* on $\mathcal{C}$ if $\langle \mathcal{A}u - \mathcal{A}v, u - v \rangle \geq 0$;
(b) *pseudomonotone* on $\mathcal{C}$ if $\langle \mathcal{A}u, v - u \rangle \geq 0 \implies \langle \mathcal{A}v, v - u \rangle \geq 0$;
(c) *Lipschitz continuous* on $\mathcal{C}$ if there is a number $L > 0$ such that $||\mathcal{A}u - \mathcal{A}v|| \leq L||u - v||$;
(d) weakly sequentially continuous if $\mathcal{A}u_n \rightharpoonup \mathcal{A}u$ is implied for all $\{u_n\} \subset \mathcal{E}$ such that $u_n \rightharpoonup u$.

Given a real Banach space $\mathcal{E}$ and a function $g : \mathcal{E} \rightarrow \mathbb{R}$, the function g is defined as follows:

(i) *Gâteaux differentiable* at $u \in \mathcal{E}$, denoted by $g'(u)$ or $\nabla g(u)$, if there exists an element v of $\mathcal{E}$ such that
$$\lim_{t \rightarrow 0} \frac{g(u + tv) - g(u)}{t} = \langle v, g'(u) \rangle, \quad v \in \mathcal{E},$$
where g is *Gâteaux differentiable* on $\mathcal{E}$ if g is *Gâteaux differentiable* at each $u \in \mathcal{E}$;
(ii) *weakly lower semicontinuous* at $u \in \mathcal{E}$ if $u_r \rightharpoonup u$ implies $g(u) \leq \liminf_{r \rightarrow \infty} g(u_r)$. g is *weakly lower semicontinuous* on $\mathcal{E}$ if g is *weakly lower semicontinuous* at each $u \in \mathcal{E}$.

Denote the unit sphere of $\mathcal{E}$ as $K(\mathcal{E}) := \{u \in \mathcal{E} : \|u\| = 1\}$. The function $\delta_{\mathcal{E}} : (0, 2] \rightarrow [0, 1]$ indicates the modulus of convexity described by

$$\delta_{\mathcal{E}}(\epsilon) = \inf \left\{ 1 - \frac{\|u + v\|}{2} : u, v \in K(\mathcal{E}),\ \|u - v\| \geq \epsilon \right\}.$$

If, for every $\epsilon \in (0, 2]$, $\delta_{\mathcal{E}}(\epsilon) > 0$, then $\mathcal{E}$ is considered uniformly convex. When $p > 1$, $\mathcal{E}$ is said to have a modulus of convexity of power type p, meaning that it is p-uniformly convex. If $c_p > 0$, for any $\epsilon \in (0, 2]$, $\delta_{\mathcal{E}}(\epsilon) \geq c_p \epsilon^p$. Keep in mind that any spaces that are p-uniformly convex are uniformly convex. For $\mathcal{E}$, the function $\rho_{\mathcal{E}} : \mathbb{R}^+ := [0, \infty) \rightarrow \mathbb{R}^+$ is the modulus of smoothness. It is defined by

$$\rho_{\mathcal{E}}(\tau) = \sup \left\{ \frac{\|u + \tau v\| + \|u - \tau v\|}{2} - 1 : u, v \in K(\mathcal{E}) \right\}.$$

Uniform smoothness of the space $\mathcal{E}$ is defined as $\frac{\rho_{\mathcal{E}}(\tau)}{\tau} \rightarrow 0$ as $\tau \rightarrow 0$. Assume $q > 1$. If, for every $\tau > 0$, there exists $\kappa_q > 0$ such that $\rho_{\mathcal{E}}(\tau) \leq \kappa_q \tau^q$, then a Banach space $\mathcal{E}$ is q-uniformly smooth. According to [26], $\mathcal{E}$ is p-uniformly convex if and only if $\mathcal{E}^*$ is q-uniformly smooth, where p and q satisfy $\frac{1}{p} + \frac{1}{q} = 1$.

Considering a real number $p > 1$, the generalized duality mapping $J_p^{\mathcal{E}} : \mathcal{E} \rightarrow 2^{\mathcal{E}^*}$ can be defined as follows:

$$J_p^{\mathcal{E}}(u) = \{\overline{u} \in \mathcal{E}^* : \langle u, \overline{u} \rangle = \|u\|^p, \|\overline{u}\| = \|u\|^{p-1}\},$$

where $\langle ., . \rangle$ represents the duality pairing between $\mathcal{E}$ and $\mathcal{E}^*$ elements. Specifically, the normalized duality mapping is denoted by J_2^E if $p = 2$. Assuming that $\mathcal{E}$ is uniformly smooth and p-uniformly convex, $\mathcal{E}^*$ is both uniformly smooth and q-uniformly convex. Here, $J_p^{\mathcal{E}}$ is a one-to-one, single-valued generalized duality mapping that satisfies the generalized duality mapping of $\mathcal{E}^*$ is $J_q^{\mathcal{E}^*}$, and $J_p^E = (J_q^{\mathcal{E}^*})^{-1}$. Moreover, the duality mapping J_p^E is norm-to-norm uniformly continuous on bounded subsets of E if $\mathcal{E}$ is uniformly smooth (see [27] for more information).

The Frenchel conjugate of g, denoted by $g^* : \mathcal{E}^* \to (-\infty, +\infty]$, is defined as follows if $g : \mathcal{E} \to (-\infty, +\infty]$ is a proper, lower semicontinuous, and convex function:

$$g^*(u^*) = \sup\{\langle u^*, u \rangle - g(u) : u \in \mathcal{E},\ u^* \in \mathcal{E}^*\}.$$

To represent the domain of g, wewrite $domg = \{u \in \mathcal{E} : g(x) < +\infty\}$. Since $v \in \mathcal{E}$ and $u \in int(domg)$, we may define and express the right-hand derivative of g at u in the direction of v as follows:

$$g^0(u, v) = \lim_{t \to 0^+} \frac{g(u + tv) - g(u)}{t}.$$

Definition 1 ([28])**.** *Given a convex function $g : \mathcal{E} \to (-\infty, +\infty]$, let g be Gâteaux differentiable. $\Delta_g : \mathcal{E} \times \mathcal{E} \to [0, +\infty)$ is a function defined by*

$$\Delta_g(u, v) := g(v) - g(u) - \langle \nabla g(u), v - u \rangle, \tag{11}$$

known as the Bregman distance with respect to g, where $\langle \nabla g(u), v \rangle = g^0(u, v)$.

It is commonly known that because Δ_g does not satisfy the symmetric and triangular inequality properties, and the Bregman distance Δ_g does not satisfy the properties of a metric. Furthermore, it is commonly known that, for $p > 1$, the sub-differential of the functional $g_p(.) = \frac{1}{p}||.||^p$ is the duality mapping $J_p^{\mathcal{E}}$ (see [29]). It is possible to demonstrate that the three-point identity, or the following equality, is satisfied by using (11):

$$\Delta_p(u, v) + \Delta_p(v, w) - \Delta_p(u, w) = \langle J_{\mathcal{E}}^p(w) - J_{\mathcal{E}}^p(v), u - v \rangle, \ \text{for all} \ u, v, w \in \mathcal{E}. \tag{12}$$

$$\Delta_p(u, v) = -\Delta_p(v, u) + \langle v - u, J_{\mathcal{E}}^p v - J_{\mathcal{E}}^p u \rangle, \ \forall\, u, v \in \mathcal{E}. \tag{13}$$

Moreover, if $g(u) = \frac{1}{p}||u||^p$, where $\frac{1}{p} + \frac{1}{q} = 1$, we obtain

$$\begin{aligned} \Delta_g(u, v) = \Delta_p(u, v) &= \frac{1}{p}\|y\|^p - \frac{1}{p}\|u\|^p - \langle v - u, J_{\mathcal{E}}^p(u) \rangle \\ &= \frac{1}{p}\|v\|^p + \frac{1}{q}\|u\|^p - \langle v, J_{\mathcal{E}}^p(u) \rangle. \end{aligned} \tag{14}$$

Suppose there is a nonlinear mapping $\mathcal{T} : \mathcal{C} \to \mathcal{C}$. Then, we have the following: Please check that intended meaning has been retained.

(i) An asymptotic fixed point of $\mathcal{T}$ is defined as $p \in \mathcal{C}$ if $\mathcal{C}$ contains a sequence $\{u_n\}$ that converges weakly to p, with the result that $\lim_{n \to \infty} \|\mathcal{T} u_n - u_n\| = 0$. By $\hat{F}(\mathcal{T})$, we represent the set of $\mathcal{T}$ asymptotic fixed points;

(ii) It is stated that $\mathcal{T}$ is Bregman relatively nonexpansive if

$$\hat{F}(\mathcal{T}) = F(\mathcal{T}) \neq \emptyset \text{ and } \Delta_p(x, \mathcal{T}u) \leq \Delta_p(x, u),\ \forall\, u \in \mathcal{C},\ x \in F(\mathcal{T});$$

(iii) Bregman relatively nonexpansive $\mathcal{T}$ is stated to exist if for all $u, v \in \mathcal{C}$,

$$\langle J^p_{\mathcal{E}}(\mathcal{T}u) - J^p_{\mathcal{E}}(\mathcal{T}v), \mathcal{T}u - \mathcal{T}v\rangle \leq \langle J^p_{\mathcal{E}}(u) - J^p_{\mathcal{E}}(v), \mathcal{T}u - \mathcal{T}v\rangle;$$

(iv) When $\hat{F}(\mathcal{T}) \neq \emptyset$, then $\mathcal{T}$ is a Bregman strongly nonexpansive mapping (BSNE) if, for all $v \in \hat{F}(\mathcal{T})$,

$$\Delta_p(v, \mathcal{T}u) \leq \Delta_p(v, u),$$

and for every bounded sequence $\{u_n\}_{n\geq 1} \subset \mathcal{C}$,

$$\lim_{n\to\infty} (\Delta_p(v, u_n) - \Delta_p(v, \mathcal{T}u_n)) = 0$$

implies

$$\lim_{n\to\infty} \Delta_p(\mathcal{T}u_n, u_n) = 0.$$

Assume that $\mathcal{C}$ is a closed, nonempty, convex subset of $\mathcal{E}$. The projection metric $P_{\mathcal{C}} : \mathcal{E} \to \mathcal{C}$ is defined as

$$P_{\mathcal{C}}u := arg \min_{v\in\mathcal{E}} ||u - v||, \; u \in \mathcal{E},$$

the one and only minimizer of the norm distance, which has the following variational inequality:

$$\langle J^p_{\mathcal{E}}(u - P_{\mathcal{C}}u), w - P_{\mathcal{C}}u\rangle \leq 0, \; \forall\, w \in \mathcal{C}. \tag{15}$$

Additionally, the Bregman projection represented by $\Pi_{\mathcal{C}}$ from $\mathcal{E}$ onto $\mathcal{C}$ satisfies the following property:

$$\Delta_p(x, \Pi_{\mathcal{C}}(u)) = \inf_{v\in\mathcal{C}} \Delta_p(u, v), \;\text{ for all }\; u \in \mathcal{E}. \tag{16}$$

Assume that $u \in \mathcal{E}$ and $\mathcal{C}$ are nonempty, closed, convex subsets of a p-uniformly convex and uniformly smooth Banach space $\mathcal{E}$. Then, the following claims are true [26]:
$w = \Pi_{\mathcal{C}}u$ if and only if

$$\langle J^p_{\mathcal{E}}(u) - J^p_{\mathcal{E}}(w),\; v - w\rangle \leq 0, \; \forall\, y \in \mathcal{C}; \tag{17}$$

$$\Delta_p(\Pi_{\mathcal{C}}u, v) + \Delta_p(u, \Pi_{\mathcal{C}}u) \leq \Delta_p(u, v), \; \forall\, v \in \mathcal{C}. \tag{18}$$

We now present a few findings that support our main result.

Lemma 1 ([29])**.** *Consider a Banach space $\mathcal{E}$ with $u, v \in \mathcal{E}$. There exists $C_q > 0$ such that, if $\mathcal{E}$ is q-uniformly smooth,*

$$\|u - v\|^q \leq \|u\|^q - q\langle J^q_{\mathcal{E}}(u), v\rangle + C_q\|v\|^q.$$

Let u, v, and w be in $\mathcal{E}$. With $\frac{1}{p} + \frac{1}{q} = 1$, we therefore have

$$\Delta_p(u, v) = \Delta_p(u, z) + \Delta_p(z, v) + \langle u - z, J^p_{\mathcal{E}}(z) - J^p_{\mathcal{E}}(v)\rangle, \tag{19}$$

$$\Delta_p(u, v) = -\Delta_p(v, u) + \langle v - u, J^p_{\mathcal{E}}(v) - J^p_{\mathcal{E}}(u)\rangle, \tag{20}$$

and

$$\Delta_p(u,v) = \frac{\|u\|^p}{p} + \frac{\|v\|^q}{p} - \langle u, J_{\mathcal{E}}^p v\rangle. \quad (21)$$

Lemma 2 ([30]). *Consider a p-uniformly convex Banach space, $\mathcal{E}$. For any $u, v \in \mathcal{E}$, the relationship between the metric and the Bregman distance is as follows:*

$$\pi_p\|u-v\|^p \leq \Delta_p(u,v) \leq \langle u-v, J_{\mathcal{E}}^p(u) - J_{\mathcal{E}}^p(v)\rangle. \quad (22)$$

For any $q > 1$, if $\frac{1}{p} + \frac{1}{q} = 1$, we have Young's inequality, where $\pi_p > 0$ is a fixed number.

$$\begin{aligned}\langle J_E^p(u), v\rangle \leq ||J_E^p(u)||||v|| &\leq \frac{1}{q}||J_E^p(u)||^q + \frac{1}{p}||v||^p \\ &= \frac{1}{q}||u||^p + \frac{1}{p}||v||^p. \quad (23)\end{aligned}$$

Lemma 3 ([31]). *Consider a real p-uniformly smooth and convex Banach space, $\mathcal{E}$. Let us define $V_p : \mathcal{E}^* \times \mathcal{E} \to [0, +\infty)$ as*

$$V_p(u^*, u) = \frac{1}{q}\|u^*\|^q - \langle u^*, u\rangle + \frac{1}{p}\|u\|^p,\ \forall\, u \in \mathcal{E}, u^* \in \mathcal{E}^*.$$

The following claims hold:

(i) *In the first variable, V_p is nonnegative and convex.*
(ii) $\Delta_p(J_q^{\mathcal{E}^*}(u^*), u) = V_p(u^*, u),\ \forall\, u \in E,\ u^* \in E^*$.
(iii) $V_p(u^*, u) + \langle v^*, J_q^{\mathcal{E}^*}(u^*) - u\rangle \leq V_p(u^* + v^*, u), \forall\, u \in \mathcal{E},\ u^*, v^* \in \mathcal{E}^*$.

Lemma 4 ([26]). *Let $\mathcal{E}$ be a real p-uniformly convex and uniformly smooth Banach space. Suppose that $\{u_n\}$ and $\{v_n\}$ are bounded sequences in $\mathcal{E}$. Then $\lim_{n\to\infty} \Delta_p(u_n, v_n) = 0$ implies $\lim_{n\to\infty} ||u_n - v_n|| = 0$.*

Lemma 5 ([32]). *Assume that $\mathcal{E}$ is a real reflexive Banach space and that $\mathcal{C}$ is a nonempty, closed, convex subset of $\mathcal{E}$. We also define $\mathcal{A}$ as a continuous pseudomonotone mapping from $\mathcal{C}$ into E^*. In such a case, $VI(\mathcal{C}, \mathcal{A})$ is convex and closed. Moreover, for any $u \in \mathcal{C}$, $u^* \in VI(\mathcal{C}, \mathcal{A})$ if and only if $\langle \mathcal{A}u, u - u^*\rangle \geq 0$.*

Lemma 6 ([33]). *Define $\{t_n\}$ as a nonnegative real number and $\{\alpha_n\}$ as a real number sequence in $(0,1)$ with the following condition: $\sum_{n=1}^{\infty} \alpha_n = \infty$, and $\{f_n\}$ as a real number sequence. Suppose that*

$$t_{n+1} \leq (1-\alpha_n)t_n + \alpha_n f_n, \forall n \geq 1.$$

If, for each subsequence $\{t_{n_k}\}$ of $\{t_n\}$ satisfying the condition, $\limsup_{k\to\infty} f_{n_k} \leq 0$

$$\liminf_{k\to\infty}(t_{n_k+1} - t_{n_k}) \geq 0,$$

then $\lim_{n\to\infty} t_n = 0$.

3. Main Result

Assumption 1. *(L1) A nonempty, closed, and convex subset of $\mathcal{E}$ is $\mathcal{C}$. $\mathcal{E}$ is a p-uniformly convex real Banach space that is also uniformly smooth. Afterward, the definition of $\mathcal{C}$ is as below:*

$$\mathcal{C} := \bigcap_{i=1}^{m} C^i,$$

where $C^i := \{v \in E : g_i(v) \leq 0\}$.

(L2) *On* $\mathcal{E}$, $\mathcal{F} : \mathcal{C} \to \mathcal{E}^*$ *is pseudomonotone and L-Lipschitz continuous.*

(L3) *Given any* $\{a_n\} \subset \mathcal{E}$, $a_n \rightharpoonup a^*$ *implies* $\mathcal{F}a_n \rightharpoonup \mathcal{F}a^*$. *This indicates that* $\mathcal{F}$ *is weakly sequentially continuous.*

(L4) *In* $\left(0, \frac{p\pi_p}{2^{p-1}}\right)$, $\{\mu_n\}$ *is a positive sequence.* π_p *is defined in* (22), *and* $\mu_n = \circ(\varrho_n)$, *where* ϱ_n *is a sequence in* $(0,1)$ *such that* $\lim_{n\to\infty} \varrho_n = 0$. *Both* $\sum_{n=1}^{\infty} \varrho_n = \infty$ *and* $\varrho_n + \eta_n + \sigma_n = 1$ *is the relationship between* $\{\eta_n\}$ *and* $\{\sigma_n\}$ *sequences in* $(0,1)$. $\varrho_n \in (a,b) \subset (0,1)$, *and* $\sigma_n \in (c,d) \subset (0,1)$ *since* $\sum_{n=1}^{\infty} \delta_n < +\infty$ *for all* $n \geq 1$, *and* $\{\delta_n\}$ *are nonnegative real numbers sequences.*

(L5) *We indicate* $T : \mathcal{E} \to \mathcal{E}$, *a Bregman strongly nonexpansive mapping, by* $\Psi := SOl(\mathcal{F}, \mathcal{C}) \cap F(T)$, *where* $\Omega \neq \emptyset$.

In this section, we present Algorithm 1 for finding a common solution to the pseudomonotone variational inequality problem and the fixed-point problem of Bregman strongly nonexpansive mapping by combining the Tseng and Halpern-type methods with inertial extrapolation:

Algorithm 1 Inertial Tseng-type method for pseudomonotone VIP.

Initialization: Assume the following: $\theta \in (0, \pi_p)$, $0 < \varepsilon < 1$, and $u, a_0, a_1 \in \mathcal{E}, \lambda_0 > 0$. Create the family of half spaces for $i = 1, 2, \cdots, m$ using the current iteration u_n.

$$C_n^i := \{v \in \mathcal{E} : g_i(u_n) + \langle g_i^{'}(u_n), v - u_n \rangle \leq 0\},$$

and set

$$C_n := \bigcap_{i=1}^{m} C_n^i.$$

Iterative Steps: Calculate a_{n+1} as follows:

Step 1. Assuming that $n \geq 1$ and $\theta > 0$ for each iterate a_{n-1} and a_n, determine θ_n such that $0 \leq \theta_n \leq \bar{\theta}_n$.

$$\bar{\theta}_n = \begin{cases} \min\{\theta, \frac{\mu_n}{\|J_{\mathcal{E}}^p(a_n) - J_{\mathcal{E}}^p(a_{n-1})\|}\}, & if \ a_n \neq a_{n-1}, \\ \theta, & otherwise. \end{cases} \tag{24}$$

Step 2. Compute

$$\begin{cases} u_n = J_{\mathcal{E}^*}^q[J_{\mathcal{E}}^p(a_n) + \theta_n(J_{\mathcal{E}}^p(a_n) - J_{\mathcal{E}}^p(a_{n-1}))], \\ z_n = \Pi_{C_n}(J_{\mathcal{E}^*}^q[J_{\mathcal{E}}^p(u_n) - \lambda_n \mathcal{F}(u_n)]). \end{cases} \tag{25}$$

$$\lambda_{n+1} = \begin{cases} \min\{\lambda_n + \delta_n, \frac{\varepsilon\|z_n - u_n\|}{\|\mathcal{F}(z_n) - \mathcal{F}(u_n)\|}\}, & if \ \mathcal{F}(z_n) \neq \mathcal{F}(u_n), \\ \lambda_n + \delta_n, & otherwise. \end{cases} \tag{26}$$

When $z_n = u_n$ for a given $n \geq 1$, the problem VIP has been addressed. If not, proceed to step 3.

Step 3. Compute

$$\begin{cases} w_n = J_{\mathcal{E}^*}^q[J_{\mathcal{E}}^p(z_n) - \lambda_n(\mathcal{F}(z_n) - \mathcal{F}(u_n))], \\ a_{n+1} = J_{\mathcal{E}^*}^q(\varrho_n J_{\mathcal{E}}^p(u) + \eta_n J_{\mathcal{E}}^p(w_n) + \sigma_n J_{\mathcal{E}}^p(Tw_n)), \end{cases} \tag{27}$$

Stopping Criterion: For any $n \geq 1$, if $a_{n+1} = u_n = z_n$ and $w_n = Tw_n$, then end the process. Alternatively, assign $n := n + 1$ and go back to **Step 1**.

Remark 1. *Note that z_n is a VIP solution if* (1) *stops in a finite step of iterations. Therefore, we assume for the remainder of our demonstration that* (1) *generates an infinite sequence and continues without stopping in any finite number of iterations.*

Remark 2. *Suppose $\delta_n = 0$, then the stepsize in* (1) *is like the ones in [4,34,35]. Additionally, the stepsize used in* (1) *increases from iteration to iteration, reducing the reliance on the starting step size lambda$_0$. As δ_n is a summable sequence, $\lim_{n\to\infty} \delta_n = 0$. For big n, the stepsize λ_n may not be growing.*

Remark 3. *Unlike the inertial methods used in [4,19,20,22], the inertial method used in this article does not impose any tight conditions on θ_n. Furthermore, we stress that the inertial approach in* (1) *is original, as defined by Polyak [24], and is neither relaxed nor modified. To the best of our knowledge, no one has used a Halpern approach to accomplish this in the framework of p-uniformly convex real Banach space that is also uniformly smooth.*

Remark 4. *Based on the description of $\mathcal{C}'$ and $\mathcal{C}_n'$, the fact that $\mathcal{C} \subset \mathcal{C}_n$ is apparent. Specifically, the subdifferential inequality yields, for each $i = 1, 2, \ldots, m$ and $v \in \mathcal{C}'$,*

$$g_i(u_n) + \langle g_i'(u_n), v - u_n \rangle \leq g_i(v) \leq 0.$$

The notion of $\mathcal{C}_n'$ implies that $v \in \mathcal{C}_n'$. We may then conclude that $\mathcal{C} \subset \mathcal{C}_n$ for all $n \geq 1$ since $\mathcal{C}^i \subset \mathcal{C}_n^i$ for all i.

Lemma 7. *If we assume that $\{\lambda_n\}$ is the sequence defined in* (26), *then $\lim_{n\to\infty} \lambda_n = \lambda$ as well as $\lambda \in \left[\min\{\frac{\varepsilon}{L}, \lambda_0\}, \lambda_0 + \delta\right]$. In this case, $\delta = \sum_{n=0}^{\infty} \delta_n$.*

Proof. For any bounded subset of $\mathcal{E}$ with constant $L > 0$, $\mathcal{F}$ is Lipschitz-continuous. Consequently, in the case of $\mathcal{F}z_n - \mathcal{F}u_n \neq 0$, we obtain

$$\frac{\varepsilon\|z_n - u_n\|}{\|\mathcal{F}z_n - \mathcal{F}u_n\|} \geq \frac{\varepsilon\|z_n - u_n\|}{L\|z_n - u_n\|} = \frac{\varepsilon}{L}.$$

The sequence $\{\lambda_n\}$ has an upper bound of $\lambda_0 + \delta$ and a lower bound of $\min\{\frac{\varepsilon}{L}, \lambda_0\}$ since λ_{n+1} is defined and mathematical induction is used. Similar to Lemma 3.1 in [35], the remainder of the argument is presented. □

Lemma 8. *Assume that (L1) through (L5) are true. Then, the sequences produced by* (1), *$\{a_n\}$, $\{u_n\}$, $\{z_n\}$, and $\{w_n\}$, are bounded.*

Proof. If $a^* \in \Psi$, then (1) implies that

$$\begin{aligned}\Delta_p(a^*, w_n) &= \Delta_p(a^*, J_{\mathcal{E}^*}^q[J_{\mathcal{E}}^p(z_n) - \lambda_n(\mathcal{F}(z_n) - \mathcal{F}(u_n))]) \\ &= \frac{1}{p}\|a^*\|^p - \langle a^*, J_{\mathcal{E}}^p(z_n) - \lambda_n(\mathcal{F}(z_n) - \mathcal{F}(u_n))\rangle \\ &+ \frac{1}{q}\|J_{\mathcal{E}}^p(z_n) - \lambda_n(\mathcal{F}(z_n) - \mathcal{F}(u_n))\|^q. \end{aligned} \tag{28}$$

On applying Lemma 1, we obtain

$$\begin{aligned}\|J_{\mathcal{E}}^p(z_n) - \lambda_n(\mathcal{F}(z_n) - \mathcal{F}(u_n))\|^q &\leq \|J_{\mathcal{E}}^p(z_n)\|^q - q\lambda_n\langle z_n, \mathcal{F}(z_n) - \mathcal{F}(u_n)\rangle \\ &+ C_q\lambda_n^q\|\mathcal{F}(z_n) - \mathcal{F}(u_n)\|^q. \end{aligned} \tag{29}$$

By substituting (29) into (28), we obtain

$$\begin{aligned}\Delta_p(a^*, w_n) &= \frac{1}{p}\|a^*\|^p - \langle a^*, J^p_{\mathcal{E}}(z_n) - \lambda_n(\mathcal{F}(z_n) - \mathcal{F}(u_n))\rangle + \frac{1}{q}\|z_n\|^q \\ &\quad - \lambda_n\langle z_n, \mathcal{F}(z_n) - \mathcal{F}(u_n)\rangle + \frac{\mathcal{C}_q\lambda_n^q}{q}\|\mathcal{F}(z_n) - \mathcal{F}(u_n)\|^q \\ &= \Delta_p(a^*, z_n) + \lambda_n\langle a^* - z_n, \mathcal{F}(z_n) - \mathcal{F}(u_n)\rangle + \frac{\mathcal{C}_q\lambda_n^q}{q}\|\mathcal{F}(z_n) - \mathcal{F}(u_n)\|^q. \end{aligned} \tag{30}$$

By applying (12), we obtain

$$\begin{aligned}\Delta_p(a^*, w_n) &= \Delta_p(a^*, u_n) + \Delta_p(u_n, z_n) + \langle a^* - u_n, J^p_{\mathcal{E}}u_n - J^p_{\mathcal{E}}z_n\rangle \\ &\quad + \lambda_n\langle a^* - z_n, \mathcal{F}(z_n) - \mathcal{F}(u_n)\rangle + \frac{\mathcal{C}_q\lambda_n^q}{q}\|\mathcal{F}(z_n) - \mathcal{F}(u_n)\|^q.\end{aligned}$$

Thus, we obtain from (13) that

$$\begin{aligned}\Delta_p(a^*, w_n) &= \Delta_p(a^*, u_n) - \Delta_p(z_n, u_n) + \langle a^* - u_n, J^p_{\mathcal{E}}u_n - J^p_{\mathcal{E}}z_n\rangle \\ &\quad + \langle z_n - u_n, J^p_{\mathcal{E}}z_n - J^p_{\mathcal{E}}u_n\rangle + \lambda_n\langle a^* - z_n, \mathcal{F}(z_n) - \mathcal{F}(u_n)\rangle + \frac{\mathcal{C}_q\lambda_n^q}{q}\|\mathcal{F}(z_n) - \mathcal{F}(u_n)\|^q \\ &= \Delta_p(a^*, u_n) - \Delta_p(z_n, u_n) + \langle a^* - z_n, J^p_{\mathcal{E}}u_n - J^p_{\mathcal{E}}z_n\rangle + \lambda_n\langle a^* - z_n, \mathcal{F}(z_n) - \mathcal{F}(u_n)\rangle \\ &\quad + \frac{\mathcal{C}_q\lambda_n^q}{q}\|\mathcal{F}(z_n) - \mathcal{F}(u_n)\|^q. \end{aligned} \tag{31}$$

As a result of (17) and the definition of z_n, it can be concluded that

$$\langle a^* - z_n, J^p_{\mathcal{E}}u_n - \lambda_n\mathcal{F}(u_n) - J^p_{\mathcal{E}}z_n\rangle \le 0,$$

which implies that

$$\langle a^* - z_n, J^p_{\mathcal{E}}u_n - J^p_{\mathcal{E}}z_n\rangle \le \lambda_n\langle a^* - z_n, \mathcal{F}(u_n)\rangle.$$

We obtain (31) after substituting the previous inequality.

$$\begin{aligned}\Delta_p(a^*, w_n) &\le \Delta_p(a^*, u_n) - \Delta_p(z_n, u_n) + \lambda_n\langle a^* - z_n, \mathcal{F}(u_n)\rangle \\ &\quad + \lambda_n\langle a^* - z_n, \mathcal{F}(z_n) - \mathcal{F}(u_n)\rangle + \frac{\mathcal{C}_q\lambda_n^q}{q}\|\mathcal{F}(z_n) - \mathcal{F}(u_n)\|^q \\ &= \Delta_p(a^*, u_n) - \Delta_p(z_n, u_n) + \lambda_n\langle a^* - z_n, \mathcal{F}(z_n)\rangle \\ &\quad + \frac{\mathcal{C}_q\lambda_n^q}{q}\|\mathcal{F}(z_n) - \mathcal{F}(u_n)\|^q. \end{aligned} \tag{32}$$

Given that $a^* \in VIP(\mathcal{C}, \mathcal{F})$, $\langle \mathcal{F}(a^*), z_n - a^*\rangle \ge 0$; $\mathcal{F}$'s pseudomonotonicity property implies that $\langle \mathcal{F}(a^*), z_n - a^*\rangle \ge 0$. Therefore, (32) generates

$$\Delta_p(a^*, w_n) \le \Delta_p(a^*, u_n) - \Delta(z_n, u_n) + \frac{\mathcal{C}_q\lambda_n^q}{q}\|\mathcal{F}(z_n) - \mathcal{F}(u_n)\|^q.$$

By applying (26), we have

$$\Delta_p(a^*, w_n) \le \Delta_p(a^*, u_n) - \Delta_p(z_n, u_n) + \frac{\lambda_n^q\mathcal{C}_q\varepsilon^q}{q\lambda_{n+1}^q}\|z_n - u_n\|^q.$$

Using Lemma 2, we obtain

$$\Delta_p(a^*, w_n) \leq \Delta_p(a^*, u_n) - \Delta_p(z_n, u_n) + \frac{\lambda_n^q \mathcal{C}_q \varepsilon^q}{q\lambda_{n+1}^q}\|z_n - u_n\|^q$$
$$\leq \Delta_p(a^*, u_n) - (1 - \frac{\lambda_n^q \mathcal{C}_q \varepsilon^q}{q\lambda_{n+1}^q})\Delta_p(z_n, u_n). \tag{33}$$

Given that $\lim_{n\to\infty}(1 - \frac{\lambda_n^q \mathcal{C}_q \varepsilon^q}{q\lambda_{n+1}^q}) = 1 - \mu^q > 0$, there is a $N \geq 0 \; \forall \; n \geq N$ such that $1 - \frac{\lambda_n^q \mathcal{C}_q \varepsilon^q}{q\lambda_{n+1}^q} > 0$. As a result, we obtain from (33) that

$$\Delta_p(a^*, w_n) \leq \Delta_p(a^*, u_n). \tag{34}$$

From (12), then

$$\Delta_p(a^*, u_n) = \Delta_p(a^*, a_n) - \Delta_p(u_n, a_n) + \langle a^* - u_n, J_{\mathcal{E}}^p u_n - J_{\mathcal{E}}^p a_n\rangle. \tag{35}$$

Since we have $u_n = J_{\mathcal{E}^*}^q(J_{\mathcal{E}}^p a_n + \theta_n(J_{\mathcal{E}}^p a_n - J_{\mathcal{E}}^p a_{n-1}))$, it follows from (22), (23), and (1) that

$$\begin{aligned}
\langle u_n - a^*, J_{\mathcal{E}}^p u_n - J_{\mathcal{E}}^p a_n\rangle &\leq \|u_n - a^*\| \|J_{\mathcal{E}}^p u_n - J_{\mathcal{E}}^p a_n\| \\
&= \theta_n \|J_{\mathcal{E}}^p a_n - J_{\mathcal{E}}^p a_{n-1}\| \; \|u_n - a^*\| \\
&\leq \theta_n \|J_{\mathcal{E}}^p a_n - J_{\mathcal{E}}^p a_{n-1}\| [\frac{1}{p}\|u_n - a^*\|^p + \frac{1}{q}] \\
&\leq \frac{\theta_n}{p}\|J_{\mathcal{E}}^p a_n - J_{\mathcal{E}}^p a_{n-1}\| [2^{p-1}(\|a_n - u_n\|^p + \|a_n - a^*\|^p)] \\
&+ \frac{\theta_n}{q}\|J_{\mathcal{E}}^p a_n - J_{\mathcal{E}}^p a_{n-1}\| \\
&\leq \frac{2^{p-1}\mu_n}{p\pi_p}(\Delta_p(a_n, u_n) + \Delta_p(a_n, a^*)) + \frac{\mu_n}{q}.
\end{aligned} \tag{36}$$

By combining (35) and (36), we can determine that

$$\Delta_p(a^*, u_n) \leq (1 + \frac{2^{p-1}\mu_n}{p\pi_p})\Delta_p(a^*, a_n) - (1 - \frac{2^{p-1}\mu_n}{p\pi_p})\Delta_p(a_n, u_n) + \frac{\mu_n}{q}. \tag{37}$$

We can see from (32) and (37) that

$$\Delta_p(a^*, w_n) \leq (1 + \frac{2^{p-1}\mu_n}{p\pi_p})\Delta_p(a^*, a_n) - (1 - \frac{2^{p-1}\mu_n}{p\pi_p})\Delta_p(a_n, u_n) + \frac{\mu_n}{q}$$
$$- (1 - \frac{\lambda_n^q \mathcal{C}_q \varepsilon^q}{q\lambda_{n+1}^q})\Delta_p(z_n, u_n). \tag{38}$$

Let $\xi \in (0, \frac{p\pi_p}{2^{p-1}})$. Based on (L4), there exists $n \in \mathbb{N}$ such that, for any $n \geq \mathbb{N}$,

$$\frac{\mu_n 2^{p-1}}{p\pi_p} < \varrho_n \xi.$$

Therefore, for some constant $M = \frac{\mu_n}{q} > 0$, we obtain from (38) that

$$\Delta_p(a^*, w_n) \le (1+\varrho_n\xi)\Delta_p(a^*, a_n) - (1-\varrho_n\xi)\Delta_p(a_n, u_n) + \varrho_n M$$
$$- (1 - \frac{\lambda_n^q \mathcal{C}_q \varepsilon^q}{q\lambda_{n+1}^q})\Delta_p(z_n, u_n). \tag{39}$$
$$\le (1+\varrho_n\xi)\Delta_p(a^*, a_n) + \varrho_n M. \tag{40}$$

Thus, using (1) and (39), we arrive at

$$\begin{aligned}
\Delta_p(a^*, a_{n+1}) &= \Delta_p(a^*, J^q_{\mathcal{E}^*}(\varrho_n J^p_{\mathcal{E}}(u) + \eta_n J^p_{\mathcal{E}}(w_n) + \sigma_n J^p_{\mathcal{E}}(Tw_n))) \\
&\le \varrho_n \Delta_p(a^*, u) + \eta_n \Delta_p(a^*, w_n) + \sigma_n \Delta_p(a^*, Tw_n) \\
&\le \varrho_n \Delta_p(a^*, u) + \eta_n \Delta_p(a^*, w_n) + \sigma_n \Delta_p(a^*, w_n) \\
&= \varrho_n \Delta_p(a^*, u) + (1-\varrho_n)\Delta_p(a^*, w_n) \\
&\le \varrho_n \Delta_p(a^*, u) + (1-\varrho_n)\big[(1+\varrho_n\xi)\Delta_p(a^*, a_n) + \varrho_n M\big] \\
&\le \varrho_n \Delta_p(a^*, u) + (1-\varrho_n(1-\xi))\Delta_p(a^*, a_n) + \varrho_n M \\
&= (1-\varrho_n(1-\xi))\Delta_p(a^*, a_n) + \varrho_n(1-\xi)\frac{\Delta_p(a^*, u) + M}{1-\xi} \\
&\le \max\left\{\Delta_p(a^*, a_n), \frac{\Delta_p(a^*, u) + M}{1-\xi}\right\} \\
&\vdots \\
&\max\left\{\Delta_p(a^*, a_N), \frac{\Delta_p(a^*, u) + M}{1-\xi}\right\}.
\end{aligned}$$

By induction, we obtain

$$\Delta_p(a^*, a_n) \le \max\{\Delta_p(a^*, a_N), \frac{\Delta_p(a^*, u) + M}{1-\xi}\}, \ \forall\, n \ge N.$$

As a result, $\{\Delta_p(a^*, a_n)\}$ has a limit. $\Delta_p(a^*, u_n)$, $\Delta_p(a^*, z_n)$, and $\Delta_p(a^*, w_n)$ are therefore restricted by $\mathcal{C}$. We conclude that $\{a_n\}$, $\{u_n\}$, $\{z_n\}$, and $\{w_n\}$ are bounded in light of Lemma 4. □

Lemma 9. *Assume that Algorithm 1 produces the sequence $\{u_{n_{k_j}}\}$, whose subsequence $\{u_{n_{k_j}}\}$ converges weakly to $y \in \mathcal{E}$, and that Assumption 1 holds. Therefore, $y \in VI(\mathcal{C}, \mathcal{F})$ if $\lim_{n\to\infty} \|u_{n_{k_j}} - z_{n_{k_j}}\| = 0$.*

Proof. Using (17) and the concept of $\{z_{n_k}\}$, we obtain

$$\langle J^p_{\mathcal{E}} u_{n_k} - \lambda_{n_k}\mathcal{F}(u_{n_k}) - J^p_{\mathcal{E}} z_{n_k}, v - z_{n_k}\rangle \le 0 \ \forall\, v \in \mathcal{C}_{n_k}$$

or, equivalently,

$$\langle J^p_{\mathcal{E}} u_{n_k} - J^p_{\mathcal{E}} z_{n_k}, v - z_{n_k}\rangle \le \lambda_{n_k}\langle \mathcal{F}(u_{n_k}), v - z_{n_k}\rangle \ \forall\, v \in \mathcal{C}_{n_k}.$$

Therefore, we obtain

$$\frac{1}{\lambda_{n_k}}\langle J^p_{\mathcal{E}} u_{n_k} - J^p_{\mathcal{E}} z_{n_k}, v - z_{n_k}\rangle + \langle \mathcal{F}(u_{n_k}), z_{n_k} - u_{n_k}\rangle \le \langle \mathcal{F}(u_{n_k}), v - u_{n_k}\rangle, \ \forall\, v \in \mathcal{C}_{n_k}. \tag{41}$$

Using the facts that $\lim_{n\to\infty} \|u_{n_k} - z_{n_k}\| = 0$ and $J^p_{\mathcal{E}}$ is norm-to-norm uniformly continuous on bounded subsets of $\mathcal{E}$ and fixing $v \in \mathcal{C}_{n_k}$, permitting $k \to \infty$, we obtain that

$$\|J^p_{\mathcal{E}} u_{n_k} - J^p_{\mathcal{E}} z_{n_k}\| \to 0, \ k \to \infty. \tag{42}$$

By considering the limit in (41) as $k \to \infty$, we obtain

$$\liminf_{k\to\infty}\langle \mathcal{F}(u_{n_k}), v - u_{n_k}\rangle \geq 0,\ \forall\ v \in \mathcal{C}_{n_k}.$$

Now, using the facts that $u_{n_k} \in \mathcal{C}_{n_k}$ and $\mathcal{C} \subset \mathcal{C}_{n_k}$, we see that

$$\liminf_{k\to\infty}\langle \mathcal{F}(u_{n_k}), v - u_{n_k}\rangle \geq 0,\ \forall\ v \in \mathcal{C}.$$

Then, we demonstrate that $v \in \mathcal{C}$. Moreover, $z_{n_k} \subset \mathcal{C}_{n_k}$ implies that

$$g_i(u_{n_k}) + \langle g_i^{'}(u_{n_k}), z_{n_k} - u_{n_k}\rangle \leq 0,$$

and then

$$\begin{aligned} g_i(u_{n_k}) &\leq \langle g_i^{'}(u_{n_k}), u_{n_k} - z_{n_k}\rangle \\ &\leq \|g_i^{'}(u_{n_k})\| \cdot \|u_{n_k} - z_{n_k}\|. \end{aligned}$$

It follows that $\{g_i^{'}(u_{n_k})\}$ is bounded since $g_i^{'}$ is Lipschitz continuous and $\{u_{n_k}\}$ is bounded. Consequently, for any i, there exists $N_i > 0$ such that $\|g_i^{'}(u_{n_k})\| \leq N_i$. Thus, we obtain

$$g_i(u_{n_k}) \leq N \cdot \|u_{n_k} - z_{n_k}\|,$$

where $N = \max\limits_{1\leq i\leq m}\{N_i\}$. Thus, using the weak continuity of g_i, we obtain that

$$g_i(v) \leq \liminf_{k\to\infty} g_i(u_{n_k}) \leq \lim_{k\to\infty} \|u_{n_k} - z_{n_k}\| = 0.$$

Therefore, $v \in \mathcal{C}$.

Assuming that $\{\varphi_k\}$ of positive numbers is such that $\{\varphi_k\}$ is declining and $\varphi_k \to 0$ as $k \to \infty$, we indicate N_{n_k} is the lowest positive integer, such that, for each $k \geq 1$,

$$\langle \mathcal{F}(u_{n_k}), v - u_{n_k}\rangle + \varphi_k \geq 0\ \forall\ k \in N_{n_k}. \tag{43}$$

Observe that $\{N_{n_k}\}$ rises as $\{\varphi_k\}$ falls. Select a point in $\mathcal{E}$, $H_{N_{n_k}}$ such that $\langle H_{n_k}, \mathcal{F}(u_{n_k})\rangle$ = 1. Consequently, (43) becomes

$$\langle \mathcal{F}(u_{N_{n_k}}), v + \varphi_k H_{N_{n_k}} - u_{N_{n_k}}\rangle \geq 0.$$

By utilizing the pseudomonotone nature of $\mathcal{F}$, we obtain

$$\langle \mathcal{F}(v + \varphi_k H_{N_{n_k}}), v + \varphi_k H_{N_{n_k}} - u_{N_{n_k}}\rangle \geq 0.$$

Therefore,

$$\langle \mathcal{F}(v), v - u_{N_{n_k}}\rangle \geq \langle \mathcal{F}(v) - \mathcal{F}(v + \varphi_k H_{N_{n_k}}), v + \varphi_k H_{N_{n_k}} - u_{N_{n_k}}\rangle - \varphi_k\langle \mathcal{F}(v), H_{N_{n_k}}\rangle. \tag{44}$$

Following that, we demonstrate that $\lim_{k\to\infty}\varphi_k N_{N_k}$. Since $\{u_{N_k}\} \rightharpoonup y \in \mathcal{E}$ and $\mathcal{F}$ are weakly sequentially continuous on $\mathcal{C}$, $\{\mathcal{F}(u_{N_{n_k}})\} \rightharpoonup \mathcal{F}(y)$ follows. If $y \in VI(\mathcal{C}, \mathcal{F})$, then $\mathcal{F}(y) \neq 0$, else, given the progressively weakly lower semicontinuous nature of $\|.\|$, we obtain

$$\|\mathcal{F}(y)\| \leq \liminf_{k\to\infty}\|\mathcal{F}(u_{n_k})\|.$$

Since $u_{N_{n_k}} \subset u_{n_k}$, $\varphi_k \to 0$, and $k \to \infty$, we have

$$0 \leq \limsup_{k\to\infty} \|\varphi_k H_{N_{n_k}}\| = \limsup_{k\to\infty} \frac{\varphi_k}{\|\mathcal{F}(u_{n_k})\|} \leq \frac{\limsup_{k\to\infty} \varphi_k}{\liminf_{k\to\infty} \|\mathcal{F}(u_{n_k})\|} = 0,$$

and $\lim_{k\to\infty} \varphi_k H_{N_k} = 0$. Hence, it follows from (44) that

$$\liminf_{j\to\infty} \langle \mathcal{F}(v), v - u_{N_{n_k}} \rangle \geq 0.$$

Thus, for all $v \in \mathcal{C}$, we obtain that

$$\begin{aligned}\langle \mathcal{F}(v), v - y \rangle &= \lim_{k\to\infty} \langle \mathcal{F}(v), v - u_{N_{n_k}} \rangle \\ &= \liminf_{k\to\infty} \langle \mathcal{F}(v), v - u_{N_{n_k}} \rangle \\ &\geq 0.\end{aligned}$$

Therefore, applying Lemma 5, we conclude that $y \in VI(\mathcal{C}, \mathcal{F})$. □

Theorem 1. *If $\{a_n\}$ is a sequence produced by (1), then $\{a_n\}$ converges strongly to $\overline{x} \in \Psi$, where $\overline{x} = \Pi_\Psi u$.*

Proof. Using Lemma 3 (iii), (1), and (39), we derive

$$\begin{aligned}\Delta_p(a^*, a_{n+1}) &= \Delta_p(a^*, \varrho_n J^p_{\mathcal{E}}(u) + \eta_n J^p_{\mathcal{E}}(w_n) + \sigma_n J^p_{\mathcal{E}}(Tw_n)) \\ &= V_p(a^*, \varrho_n J^p_{\mathcal{E}}(u) + \eta_n J^p_{\mathcal{E}}(w_n) + \sigma_n J^p_{\mathcal{E}}(Tw_n)) \\ &\leq V_p(a^*, \varrho_n J^p_{\mathcal{E}}(u) + \eta_n J^p_{\mathcal{E}}(w_n) + \sigma_n J^p_{\mathcal{E}}(Tw_n)) - \varrho_n (J^p_{\mathcal{E}}(u) - J^p_{\mathcal{E}}(a^*)) \\ &- \langle -\varrho_n (J^p_{\mathcal{E}}(u) - J^p_{\mathcal{E}}(a^*)), J^q_{\mathcal{E}^*}(\varrho_n J^p_{\mathcal{E}}(a^*) + \eta_n J^p_{\mathcal{E}}(w_n) + \sigma_n J^p_{\mathcal{E}}(Tw_n)) - a^* \rangle \\ &= V_p(a^*, \varrho_n J^p_{\mathcal{E}}(a^*) + \eta_n J^p_{\mathcal{E}}(w_n) + \sigma_n J^p_{\mathcal{E}}(Tw_n)) \\ &+ \varrho_n \langle J^p_{\mathcal{E}}(u) - J^p_{\mathcal{E}}(a^*), a_{n+1} - a^* \rangle \\ &= \Delta_p(a^*, \varrho_n J^p_{\mathcal{E}}(a^*) + \eta_n J^p_{\mathcal{E}}(w_n) + \sigma_n J^p_{\mathcal{E}}(Tw_n)) \\ &+ \varrho_n \langle J^p_{\mathcal{E}}(u) - J^p_{\mathcal{E}}(a^*), a_{n+1} - a^* \rangle \\ &\leq \varrho_n \Delta_p(a^*, a^*) + \eta_n \Delta_p(a^*, w_n) + \sigma_n \Delta_p(a^*, Tw_n) \\ &+ \varrho_n \langle J^p_{\mathcal{E}}(u) - J^p_{\mathcal{E}}(a^*), a_{n+1} - a^* \rangle \\ &\leq (1 - \varrho_n) \Delta_p(a^*, w_n) + \varrho_n \langle J^p_{\mathcal{E}}(u) - J^p_{\mathcal{E}}(a^*), a_{n+1} - a^* \rangle \\ &\leq (1 - \varrho_n(1 - \xi)) \Delta_p(a^*, a_n) + \varrho_n (1 - \xi)\big((1 - \xi)^{-1} (\langle J^p_{\mathcal{E}}(u) - J^p_{\mathcal{E}}(a^*), a_{n+1} - a^* \rangle + \frac{\mu_n}{\varrho_n})\big).\end{aligned} \tag{45}$$

Likewise, we can determine from (1) and (39) that

$$\begin{aligned}\Delta_p(a^*, a_{n+1}) &\leq \varrho_n \Delta_p(a^*, u) + (1 + \varrho_n \xi) \Delta_p(a^*, a_n) + \varrho_n M \\ &- (1 - \varrho_n \xi) \Delta_p(a_n, u_n) - (1 - \frac{\lambda_n^q C_q \varepsilon^q}{q \lambda_{n+1}^q}) \Delta_p(z_n, u_n).\end{aligned} \tag{46}$$

Let us now assume that there exists a subsequence $\{\Delta_p(a^*, a_{n_k})\}$ such that

$$\liminf_{k\to\infty} \{\Delta_p(a^*, a_{n_{k+1}}) - \Delta_p(a^*, a_{n_k})\} \geq 0.$$

Afterward, we obtain from (46) that

$$\begin{aligned}
&\limsup_{k\to\infty}\left\{(1-\varrho_{n_k}\xi)\Delta_p(a_{n_k},u_{n_k})+(1-\frac{\lambda_{n_k}^q\mathcal{C}_q\varepsilon^q}{q\lambda_{n_k}^q})\Delta_p(z_{n_k},u_{n_{k_j}})\right\}\\
&\leq \limsup_{k\to\infty}\left\{(1+\varrho_{n_k}\xi)\Delta_p(a^*,a_{n_k})-\Delta_p(a^*,a_{n_k})+\varrho_{n_k}\Delta_p(a^*,u)+\varrho_{n_k}M\right\}\\
&\leq \limsup_{k\to\infty}\left\{\Delta_p(a^*,a_{n_k})-\Delta_p(a^*,a_{n_{k+1}})\right\}+\limsup_{k\to\infty}\varrho_{n_k}M.\\
&\leq -\liminf_{k\to\infty}\left\{\Delta_p(a^*,a_{n_{k+1}})-\Delta_p(a^*,a_{n_k})\right\}\\
&\leq 0.
\end{aligned}$$

That is,

$$\lim_{k\to\infty}\Delta_p(z_{n_k},u_{n_k})=0=\lim_{k\to\infty}\Delta_p(a_{n_k},u_{n_k}). \tag{47}$$

With Lemma 4, we obtain

$$\lim_{k\to\infty}\|z_{n_k}-u_{n_k}\|=0=\lim_{k\to\infty}\|a_{n_k}-u_{n_k}\|. \tag{48}$$

It is clear that (48) yields

$$\lim_{k\to\infty}\|z_{n_k}-a_{n_k}\|=0. \tag{49}$$

More so, using (1) it follows that

$$\begin{aligned}
\|J_{\mathcal{E}}^p w_{n_k}-J_{\mathcal{E}}^p z_{n_k}\| &= \|J_{\mathcal{E}}^p z_{n_k}-\lambda_{n_k}(\mathcal{F}(z_{n_k})-\mathcal{F}(u_{n_k}))-J_{\mathcal{E}}^p z_{n_k}\|\\
&\leq \lambda_{n_k}\|\mathcal{F}(z_{n_k})-\mathcal{F}(u_{n_k})\|\\
&\leq \frac{\varepsilon\lambda_{n_k}}{\lambda_{n_{k+1}}}\|z_{n_k}-u_{n_k}\|.
\end{aligned}$$

Thus, it follows from (48) that

$$\lim_{k\to\infty}\|J_{\mathcal{E}}^p w_{n_k}-J_{\mathcal{E}}^p z_{n_k}\|=0. \tag{50}$$

This uniform continuity from norm to norm of $J_{\mathcal{E}^*}^q$ on bounded subsets of $\mathcal{E}^*$ now gives

$$\lim_{k\to\infty}\|w_{n_k}-z_{n_k}\|=0. \tag{51}$$

Let $b_n := J_{\mathcal{E}^*}^q\left(\frac{\eta_n}{1-\varrho_n}J_{\mathcal{E}}^p(w_n)+\frac{\sigma_n}{1-\varrho_n}J_{\mathcal{E}}^p(Tw_n)\right)$, then

$$\begin{aligned}
\Delta_p(a^*,b_n) &= \Delta_p\big(a^*,J_{\mathcal{E}^*}^q\frac{\eta_n}{1-\varrho_n}J_{\mathcal{E}}^p(w_n)+\frac{\sigma_n}{1-\varrho_n}J_{\mathcal{E}}^p(Tw_n)\big)\\
&\leq \frac{\eta_n}{1-\varrho_n}\Delta_p(a^*,w_n)+\frac{\sigma_n}{1-\varrho_n}\Delta_p(a^*,Tw_n)\\
&\leq \frac{\eta_n}{1-\varrho_n}\Delta_p(a^*,w_n)+\frac{\sigma_n}{1-\varrho_n}\Delta_p(a^*,w_n)\\
&= \frac{\eta_n+\sigma_n}{1-\varrho_n}\Delta_p(a^*,w_n)\\
&= \Delta_p(a^*,w_n).
\end{aligned}$$

Hence,

$$
\begin{aligned}
0 &\le \Delta_p(a^*, w_{n_k}) - \Delta_p(a^*, b_{n_k}) \\
&= \Delta_p(a^*, w_{n_k}) - \Delta_p(a^*, a_{n_{k+1}}) + \Delta_p(a^*, x_{n_{k+1}}) - \Delta_p(a^*, b_{n_k}) \\
&\le (1 + \varrho_{n_k}\xi)\Delta_p(a^*, a_{n_k}) + \varrho_{n_k}M - \Delta_p(a^*, a_{n_{k+1}}) + \varrho_{n_k}\Delta_p(a^*, u) \\
&- (1 - \varrho_{n_k})\Delta_p(a^*, b_{n_k}) - \Delta_p(a^*, b_{n_k}) \\
&= \Delta_p(a^*, a_{n_k}) - \Delta_p(a^*, a_{n_{k+1}}) + \varrho_{n_k}(\Delta_p(a^*, u) - \Delta_p(a^*, b_{n_k}) + M) \to \ 0 \text{ as } k \to \infty. \quad (52)
\end{aligned}
$$

More so,

$$
\begin{aligned}
\Delta_p(a^*, b_n) &\le \frac{\eta_n}{1 - \varrho_n}\Delta_p(a^*, w_n) + \frac{\sigma_n}{1 - \varrho_n}\Delta_p(a^*, Tw_n) \\
&= (1 - \frac{\sigma_n}{1 - \varrho_n})\Delta_p(a^*, w_n) + \frac{\sigma_n}{1 - \varrho_n}\Delta_p(a^*, Tw_n) \\
&\le \Delta_p(a^*, w_n) + \frac{\sigma_n}{1 - \varrho_n}\Big(\Delta_p(a^*, Tw_n) - \Delta_p(a^*, w_n)\Big)
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\Delta_p(a^*, w_{n_k}) - \Delta_p(a^*, Tw_{n_k}) &< \frac{\sigma_{n_k}}{1 - \varrho_{n_k}}\Big(\Delta_p(a^*, w_{n_k}) - \Delta_p(a^*, Tw_{n_k})\Big) \\
&\le \Delta_p(a^*, w_{n_k}) - \Delta_p(a^*, b_{n_k}) \ \to \ 0, \ k \to \infty.
\end{aligned}
$$

Consequently, by applying the knowledge that T is BSNE, we derive that

$$\Delta_p(w_{n_k}, Tw_{n_k}) = 0, \tag{53}$$

which implies from Lemma 4 that

$$\lim_{k\to\infty} \|w_{n_k} - Tw_{n_k}\| = 0. \tag{54}$$

From (49) and (51), we obtain

$$\lim_{k\to\infty} \|w_{n_k} - a_{n_k}\| = 0. \tag{55}$$

Using (1) and (53), we obtain

$$\Delta_p(w_{n_k}, a_{n_k+1}) \le \varrho_{n_k}\Delta_p(w_{n_k}, u) + \eta_{n_k}\Delta_p(w_{n_k}, w_{n_k}) + \sigma_{n_k}\Delta_p(w_{n_k}, Tw_{n_k}) \to 0, \ n \to \ \infty. \tag{56}$$

By applying Lemma 4, we obtain

$$\lim_{k\to\infty} \|w_{n_k} - a_{n_k+1}\| = 0. \tag{57}$$

Hence, we conclude from (55) and (57) that

$$\lim_{k\to\infty} \|a_{n_k+1} - a_{n_k}\| = 0. \tag{58}$$

Given that $\{a_{n_k}\}$ is bounded, $\{a_{n_{k_j}}\}$ has a subsequence that converges weakly to a^*. There exist subsequences $\{u_{n_{k_j}}\}$ of $\{u_{n_k}\}$ and $\{z_{n_{k_j}}\}$ of $\{z_{n_k}\}$ that converge weakly to a^*, respectively, by applying (48) and (49). In light of this, $a^* \in \mathcal{F}(T) = \hat{\mathcal{F}}(T)$ can be obtained by applying (54). Additionally, $a^* \in VI(\mathcal{C}, \mathcal{F})$ may be obtained by applying (48) and Lemma 9. Thus, we deduce that $a^* \in \Psi$.

In the event that $\overline{x} = \Pi_\Psi u$, we have from (45) that

$$\Delta_p(\overline{x}, a_{n+1}) \leq (1 - \varrho_n(1-\xi))\Delta_p(\overline{x}, a_n) + \varrho_n(1-\xi)\left((1-\xi)^{-1}\langle J^p_{\mathcal{E}}(u) - J^p_{\mathcal{E}}(\overline{x}), a_{n+1} - \overline{x}\rangle + \frac{\mu_n}{\varrho_n}\right). \quad (59)$$

Since $\{a_{n_{k_j}}\}$ is a subsequence of $\{a_{n_k}\}$, then we have from (17) that

$$\begin{aligned}\limsup_{k\to\infty}\langle J^p_{\mathcal{E}}(u) - J^p_{\mathcal{E}}(\overline{x}), a_{n_k} - \overline{x}\rangle &= \lim_{j\to\infty}\langle J^p_{\mathcal{E}}(u) - J^p_{\mathcal{E}}(\overline{x}), a_{n_{k_j}} - \overline{x}\rangle \\ &= \langle J^p_{\mathcal{E}}(u) - J^p_{\mathcal{E}}(\overline{x}), a^* - \overline{x}\rangle \leq 0.\end{aligned}$$

Hence, since (58) holds, then

$$\begin{aligned}\limsup_{j\to\infty}\langle J^p_{\mathcal{E}}(u) - J^p_{\mathcal{E}}(\overline{x}), a_{n_{k_j}+1} - \overline{x}\rangle &= \lim_{j\to\infty}\langle J^p_{\mathcal{E}}(u) - J^p_{\mathcal{E}}(\overline{x}), a_{n_{k_j}} - \overline{x}\rangle \\ &\leq 0. \end{aligned} \quad (60)$$

Therefore, using Lemma 6 and (60) in (59), we obtain that $\Delta_p(\overline{x}, a_n) \to 0$ as $n \to \infty$, and from Lemma 2, we know that $\pi_p\|a_n - \overline{x}\|^p \leq \Delta_p(\overline{x}, a_n) \to 0$. Hence, $\{a_n\} \to \overline{x}$, where $\overline{x} = \Pi_\Psi u$. □

4. Numerical Example

In this section, we give some numerical illustrations of our main result in the sequel.

Example 1. *We consider the following fractional minimization problem, which was first given in [9].*

$$\min h(x) = \frac{x^T Bx + c^T x + d}{b^T x + e}$$
$$\text{, subject to } \{x \in \mathbb{R}^4 : b^T x + e > 0\},$$

where

$$B = \begin{pmatrix} 5 & -1 & 2 & 0 \\ -1 & 5 & -1 & 3 \\ 2 & -1 & 3 & 0 \\ 0 & 3 & 0 & 5 \end{pmatrix}, \quad c = \begin{pmatrix} 1 \\ -2 \\ -2 \\ 1 \end{pmatrix}, \quad b = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad d = -2 \text{ and } e = 4.$$

As observed in [36], h is pseudoconvex on $\mathcal{E}$ since B is symmetric and positive definite. Therefore,

$$F(x) = \nabla h(x) = \frac{(b^T x + e)(2Bx + c) - b(x^T Bx + c^T x + d)}{(b^T x + e)^2}$$

is pseudomonotone. We also define $T : \mathbb{R}^4 \to \mathbb{R}^4$ by $T = P_{\mathcal{C}}$, where $P_{\mathcal{C}}$ is the projection of $\mathcal{E}$ onto $\mathcal{C} := \{x \in \mathbb{R}^4 : 1 \leq x_i \leq 10,\ i = 1, \ldots, 4\}$ and $g_i(x) = x^2$ for all i. For this example, we choose the sequence $\varrho_n = \frac{1}{10n+1}, \eta_n = \frac{1}{2n+5}, \sigma_n = 1 - \varrho_n - \eta_n$, $\lambda = 2.97$, $\theta = 0.5$, $\epsilon_n = \frac{1}{n^{1.02}}$, and $\delta = \frac{n}{3n+1}$. We make a comparison of our method with the method in [37], where the inertial method was altered. The result of this experiment with the stopping criterion chosen as $E_n = \|x_{n+1} - x_n\| = 10^{-4}$ is reported below for different initial values if x_0 and x_1.

Case 1: $x_0 = [1, 1, 2, -1]'$ *and* $x_1 = [-1, 3, -1, 0]'$;
Case 2: $x_0 = [2, 1, 2, 3]'$ *and* $x_1 = [0, 2, 2, 0]'$;
Case 3: $x_0 = [0, 0, 2, 2]'$ *and* $x_1 = [1, 1, 2, 2]'$;
Case 4: $x_0 = [3, 1, 2, 3]'$ *and* $x_1 = [-1, -1, -1, 2]'$.

It can be seen from Figure 1 that our iterative method converges faster than the un-accelerated algorithm since the main advantage of introducing the inertial extrapolation method is to fasten the rate of convergence of our iterative method.

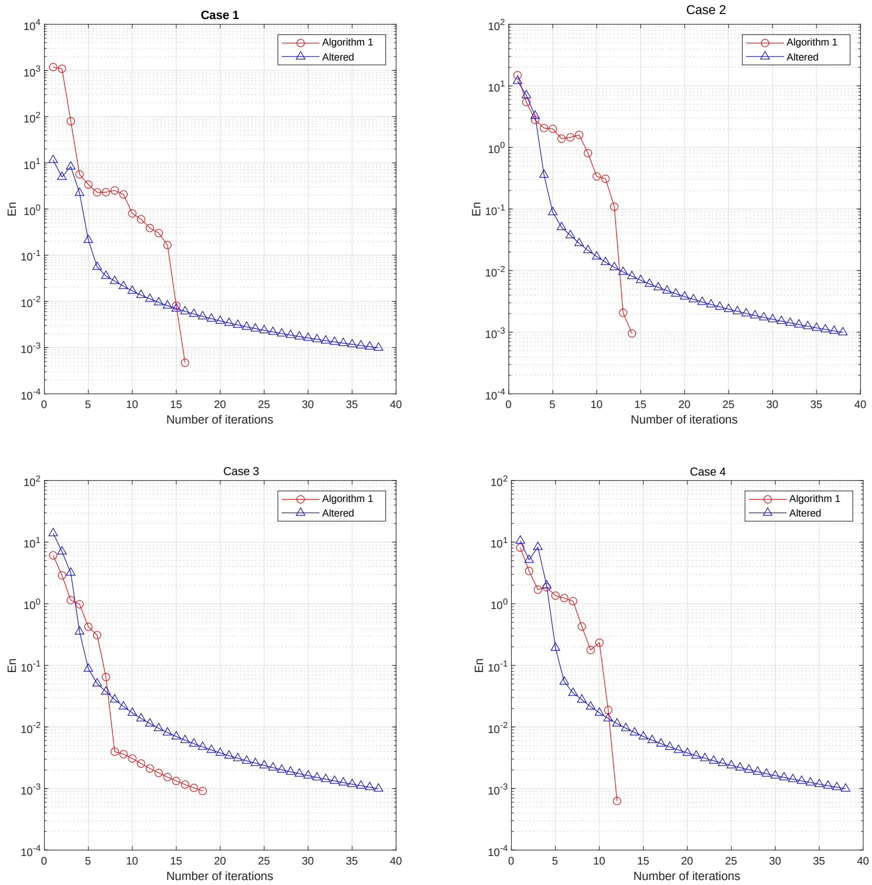

Figure 1. Example 1, (**Top left**): Case 1; (**Top right**): Case 2; (**Bottom left**): Case 3; (**Bottom right**): Case 4.

5. Conclusions

In this article, we propose a modified Halpern iterative method with combined inertial and Tseng techniques for finding a common solution to the fixed-point problem of Bregman strongly nonexpansive mapping and the pseudomonotone variational inequality problem in the settings of p-uniformly convex real Banach spaces that are uniformly smooth. To fasten the rate of convergence of our method, we introduce an inertial extrapolation method, as defined by Polyak, together with our proposed method. We present a numerical example to illustrate the performance of our proposed method. We will extend the result discussed in this article to the framework of a Hadamard manifold in our future research.

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