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The Hamilton–Waterloo Problem with C_{16}-Factors and C_m-Factors for Odd m

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Abstract: The Hamilton–Waterloo problem is a problem of graph factorization. The Hamilton–Waterloo problem HWP$(H; m, n; \alpha, \beta)$ asks for a two-factorization of a graph H containing α C_m-factors and β C_n-factors. Let K_v^* denote the complete graph K_v if v is odd and K_v minus a one-factor if v is even. In this paper, we completely solve the Hamilton–Waterloo problem HWP$(K_v^*; m, 16; \alpha, \beta)$ for odd $m \geq 9$ and $\alpha \geq 15$.

Keywords: Hamilton–Waterloo problem; two-factorization; cycle decomposition

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1. Introduction

A central theme in combinatorics and related areas is the decomposition of large discrete objects into simpler or smaller ones. Usually, these simpler or smaller objects are given in advance as needed and have some special properties such as symmetry and uniformity. In this paper, we will focus on a problem of graph factorization. We assume that the reader is familiar with basic concepts in graph theory and design theory, and refers to [1,2] for further details. In this paper, every graph will be simple. In general, the vertex-set and the edge-set of a graph H are denoted by $V(H)$ and $E(H)$, respectively. We denote the cycle of length k by C_k and the complete graph on n vertices by K_n. We use $K_u[g]$ to denote the complete u-partite graph with u parts of size g. In fact, $K_u[1]$ is a complete graph K_u and the graph $K_u[2]$ is K_{2u} minus a one-factor. These graphs are all regular graphs and each of them possesses highly symmetric properties. A *factor* of H is a spanning subgraph of H whose vertex-set is exactly $V(H)$. We call it a *G-factor* if its connected components are isomorphic to G. A *G-factorization* of H is a set of edge-disjoint G-factors of H whose edge-sets partition $E(H)$. A C_k-factorization of H is a partition of $E(H)$ into C_k-factors.

For the existence of a C_k-factorization of K_u, Ray-Chadhuri, Wilson [3], and Lu [4] independently proved the existence for the case of $k = 3$. For the other cases, the necessary conditions of the existence of a C_k-factorization of K_u are also sufficient, see [5–7]. The existence problem for a C_k-factorization of $K_u[2]$ has been solved, see [8,9]. Finally, Liu [10,11] completely solved the existence of a C_k-factorization of $K_u[g]$.

Theorem 1. *There exists a C_k-factorization of $K_u[g]$ if and only if $g(u-1) \equiv 0 \pmod 2$, $gu \equiv 0 \pmod k$, k is even when $u = 2$, and $(k, u, g) \notin \{(3,3,2), (3,6,2), (3,3,6), (6,2,6)\}$.*

As usual, an r-regular factor is called an *r-factor*. In particular, a two-factor is a collection of vertex-disjoint cycles. A *two-factorization* of a graph H is a partition of $E(H)$ into two-factors. The Hamilton–Waterloo problem HWP$(H; m, n; \alpha, \beta)$ asks for a two-factorization of a specified graph H containing α C_m-factors and β C_n-factors. Let K_v^* denote the complete graph K_v if v is odd and K_v minus a one-factor if v is even. We denote a solution to HWP$(K_v^*; m, n; \alpha, \beta)$ by HW$(v; m, n; \alpha, \beta)$. Also, we use HWP$(v; m, n)$ to denote the set of (α, β) for which an HW$(v; m, n; \alpha, \beta)$ exists. The necessary conditions for the existence of an HW$(v; m, n; \alpha, \beta)$ are shown so that $m|v$ when $\alpha > 0$, $n|v$ when $\beta > 0$ and

$\alpha + \beta = \lfloor \frac{v-1}{2} \rfloor$. Theorem 1 indicates that the existence of an $\mathrm{HW}(v; m, n; \alpha, \beta)$ has been completely solved when $\alpha\beta = 0$.

For small values of m and n, the known results of the Hamilton–Waterloo problem are as follows. A complete solution for the existence of an $\mathrm{HW}(v; 3, n; \alpha, \beta)$ in the cases $n \in \{4, 5, 7\}$ is given in [12–17]. For the cases $(m, n) \in \{(3, 15), (5, 15), (4, 6), (4, 8), (4, 16), (8, 16)\}$, see [12]. Kamin [18] showed that the necessary conditions for the existence of an $\mathrm{HW}(v; 3, 9; \alpha, \beta)$ are also sufficient, apart from the exceptional case $\beta = 1$. Asplund et al. [19] constructed many infinite classes of $\mathrm{HW}(v; 3, 3x; \alpha, \beta)$s.

The existence of an $\mathrm{HW}(v; 4, m; \alpha, \beta)$ for odd $m \geq 3$ has been solved with some possible exceptions, see [16,17,20]. Fu and Huang [21] give a complete solution for an $\mathrm{HW}(v; 4, m; \alpha, \beta)$ for even $m \geq 4$.

Theorem 2 ([16,17,20,21])**.** *$(\alpha, \beta) \in HWP(v; 4, m)$ for $m \geq 3$ if and only if $\alpha, \beta \geq 0$ and $\alpha + \beta = \lfloor \frac{v-1}{2} \rfloor$, except possibly when $m \geq 5$ is odd, $v = 8m$, and $\alpha = 2$.*

Wang and Cao [22] considered the Hamilton–Waterloo problem with C_8-factors and C_m-factors and gave the following results.

Theorem 3 ([22])**.** *$(\alpha, \beta) \in HWP(8mt; 8, m)$ for $m \geq 3$ and $t \geq 1$ if and only if $\alpha, \beta \geq 0$ and $\alpha + \beta = 4mt - 1$, except possibly when $\alpha \in \{1, 2\}$ and mt is odd or $\alpha \in \{1, 2, 4, 5, 6\}$ and mt is even.*

Bryant et al. [23,24] completely solved the Hamilton–Waterloo problem for bipartite two-factors. Buratti and Danziger [25] as well as Merola and Traetta [26] focused on infinitely many cyclic solutions to the Hamilton–Waterloo problem with odd length cycles. Dinitz and Ling [27] as well as Lei and Shen [28] gave an analysis of the Hamilton–Waterloo problem for Hamilton cycles and triangle factors. Wang, Lu, and Cao [29] considered the existence of an $\mathrm{HW}(k(2kt+1); k, 2kt+1; \alpha, \beta)$ for $t \geq 1$ and odd $k \geq 3$. For the case of two cycles sizes m, n of different parity on the Hamilton–Waterloo problem, Keranen and Pastine [30] mainly focused on the case $m|n$ and $(m, n) = (2^l x, n)$ with odd x, n, and $\gcd(x, n) \geq 3$. Burgess, Danziger, and Traetta studied the Hamilton–Waterloo problem in detail, see [31–34]. In 2022, Burgess et al. [35] made further progress when m and n are not coprime in two regards. In 2023, [36] presented a survey of constructive methods for the Hamilton–Waterloo problem which have allowed recent progress. The readers can have a comprehensive understanding of this problem.

In this paper, we consider the remaining situation of the Hamilton–Waterloo problem. We will focus on the existence of an $\mathrm{HW}(16mt; 16, m; \alpha, \beta)$ for odd m and give the following main result.

Theorem 4. *For any odd $m \geq 9$, $(\alpha, \beta) \in$* HWP$(16mt; 16, m)$ if and only if $\alpha + \beta = 8mt - 1$, where $\alpha \geq 0$, $\beta \geq 0$, and $t \geq 1$, except possibly when $\alpha \in [1, 6]$ and t is odd or $\alpha \in [1, 6] \cup [8, 14]$ and t is even.

2. Preliminary

In this section, we introduce some necessary definitions, notations, and known results which will be used later.

To begin with, we introduce the definition of a Cayley graph. Let Γ be a finite additive group and let S be a subset of $\Gamma \backslash \{0\}$ closed under taking additive inverses. The *Cayley graph* over Γ with connection set S, denoted by $\mathrm{Cay}(\Gamma, S)$, is the graph with vertex-set Γ and edge-set $E(\mathrm{Cay}(\Gamma, S)) = \{(a, b) | a, b \in \Gamma, a - b \in S\}$. For our constructions, we need the following results on a C_m-factorization or a C_n-factorization of $\mathrm{Cay}(Z_m \times Z_n, S)$.

Lemma 1 ([17,22])**.** (1) *Let $m, n \geq 3$, let $a \in Z_n$ satisfying $|\pm\{0, a, 2a\}| = 5$, and let $\gcd(i, m) = 1$. There exist five C_m-factors which form a C_m-factorization of* $\mathrm{Cay}(Z_m \times Z_n, \{\pm i\} \times (\pm\{0, a, 2a\}))$.

(2) *Let $m \geq 3$ be odd, let $n \geq 4$ be even, and let $a, b \in Z_n$ with $| \pm \{a, b, a+b\}| = 6$. There exist six C_m-factors which form a C_m-factorization of* $\mathrm{Cay}(Z_m \times Z_n, \{\pm i\} \times (\pm\{a, b, a+b\}))$ *with* $\gcd(i, m) = 1$.

(3) *Let $m \geq 3$ be odd, let $n \geq 4$ be even, and let $1 \leq d < n$. There exist three C_m-factors which form a C_m-factorization of* $\mathrm{Cay}(Z_m \times Z_n, \{\pm i\} \times \{0, \pm d\})$ *with* $\gcd(i, m) = 1$.

(4) *Let $n \geq 4$ be even and let $d = 0$ ($m \geq 3$) or $d = n/2$ ($m \geq 4$ is even). There exists a C_m-factor which forms a C_m-factorization of* $\mathrm{Cay}(Z_m \times Z_n, \{\pm i\} \times \{d\})$ *with* $\gcd(i, m) = 1$.

(5) *Let $m \geq 3$, let $n \geq 4$ be even and let $0 < d < n$ be coprime to n. There exist two C_n-factors which form a C_n-factorization of* $\mathrm{Cay}(Z_m \times Z_n, \{\pm i\} \times \{\pm d\})$ *with* $\gcd(i, m) = 1$.

Next, we introduce the concept of the wreath product of two graphs. If both G and H are graphs, the *wreath product* $G \wr H$ of G and H has a vertex-set $V(G) \times V(H)$ in which $(u_1, v_1)(u_2, v_2) \in E(G \wr H)$ whenever $u_1u_2 \in E(G)$ or, $u_1 = u_2$ and $v_1v_2 \in E(H)$. For brevity, we denote $C_m \wr \overline{K_n}$ by $C_m[n]$, where $\overline{K_n}$ is the complement of K_n. We will give some known results and constructions that will be used later.

Theorem 5 ([6,37])**.** *For $m \geq 3$ and $n \geq 1$, a C_m-factorization of $C_m[n]$ exists, except for $(m, n) = (3, 6)$ and $(m, n) \in \{(l, 2) \mid l \geq 3 \text{ is odd}\}$.*

Theorem 6 ([38])**.** *For $m \geq 3$ and $n \geq 1$, there exists a C_{mn}-factorization of $C_m[n]$.*

Theorem 7 ([16])**.** *The graph $C_m[4]$ can be decomposed into α C_4-factors and $4 - \alpha$ C_m-factors for $m \geq 3$ and $\alpha \in \{0, 2, 4\}$.*

Construction 1 ([22])**.** *If $(\alpha, \beta) \in \mathrm{HWP}(C_m[n]; m, n)$, then $(\alpha, \beta + \lfloor \frac{n-1}{2} \rfloor) \in \mathrm{HWP}(C_m \wr K_n; m, n)$.*

Construction 2 ([17])**.** *If there exists an $\mathrm{HW}(K_u[g]; m, n; \alpha, \beta)$ and an $\mathrm{HW}(g; m, n; \alpha', \beta')$, then an $\mathrm{HW}(gu; m, n; \alpha + \alpha', \beta + \beta')$ exists.*

Construction 3. *If $C_m[n]$ can be decomposed into α C_n-factors and $n - \alpha$ C_m-factors, and there exists a C_{nw}-factorization of $C_n[w]$ and a C_m-factorization of $C_m[w]$, then $C_m[nw]$ can be decomposed into $w\alpha$ C_{nw}-factors and $w(n - \alpha)$ C_m-factors.*

Proof. The graph $C_m[n]$ can be decomposed into α C_n-factors and $n - \alpha$ C_m-factors. Then, we give each vertex weight w to obtain α $C_n[w]$-factors and $n - \alpha$ $C_m[w]$-factors. Each $C_n[w]$ ($C_m[w]$) can be partitioned into w C_{nw}-factors (C_m-factors). Finally, each $C_m[nw]$-factor can be decomposed into $w\alpha$ C_{nw}-factors and $w(n - \alpha)$ C_m-factors. □

3. Decompositions of Some Cayley Graphs

In this section, we will give some new decompositions of Cayley graphs. For brevity, we will denote the vertex (x, y) by x_y.

Lemma 2. *Let $m \geq 3$ be odd and $n \equiv 0 \pmod 4$. The graph* $\mathrm{Cay}(Z_m \times Z_n, \{\pm 1\} \times \{\pm\frac{n}{4}, \frac{n}{2}\})$ *can be decomposed into three C_m-factors.*

Proof. Let

$$
\begin{aligned}
C_1^1 &= (0_0, 1_{\frac{n}{4}}, 2_{\frac{n}{2}}, 3_{\frac{n}{4}}, 4_{\frac{n}{2}}, \ldots, (m-2)_{\frac{n}{4}}, (m-1)_{\frac{n}{2}}),\\
C_1^2 &= (0_{\frac{n}{4}}, 1_0, 2_{-\frac{n}{4}}, 3_0, 4_{-\frac{n}{4}}, \ldots, (m-2)_0, (m-1)_{-\frac{n}{4}}),\\
C_2^1 &= (0_0, 1_{-\frac{n}{4}}, 2_{\frac{n}{4}}, 3_{-\frac{n}{4}}, 4_{\frac{n}{4}}, \ldots, (m-2)_{-\frac{n}{4}}, (m-1)_{\frac{n}{4}}),\\
C_2^2 &= (0_{-\frac{n}{4}}, 1_0, 2_{\frac{n}{2}}, 3_0, 4_{\frac{n}{2}}, \ldots, (m-2)_0, (m-1)_{\frac{n}{2}}),\\
C_3^1 &= (0_0, 1_{\frac{n}{2}}, 2_{-\frac{n}{4}}, 3_{\frac{n}{2}}, 4_{-\frac{n}{4}}, \ldots, (m-2)_{\frac{n}{2}}, (m-1)_{-\frac{n}{4}}),\\
C_3^2 &= (0_{-\frac{n}{4}}, 1_{\frac{n}{4}}, 2_0, 3_{\frac{n}{4}}, 4_0, \ldots, (m-2)_{\frac{n}{4}}, (m-1)_0).
\end{aligned}
$$

Let $\mathcal{B}_i = \{C_i^j + (0,s), C_i^j + (0, s + \frac{n}{2}) \mid 1 \le j \le 2,\ 0 \le s \le \frac{n}{4} - 1\}, 1 \le i \le 3$. Each $\mathcal{B}_i$ has $2 \times 2 \times \frac{n}{4} = n$ cycles with length m and $V(\mathcal{B}_i) = Z_m \times Z_n$, thus it is a C_m-factor. By counting the edges of $\cup_{i=1}^3 \mathcal{B}_i$, we obtain that $\cup_{i=1}^3 E(\mathcal{B}_i)$ coincides with the edge-set of the Cayley graph $\mathrm{Cay}(Z_m \times Z_n, \{\pm 1\} \times \{\pm\frac{n}{4}, \frac{n}{2}\})$. So, this Cayley graph can be decomposed into three C_m-factors. □

Lemma 3. *Let $l \ge 3$ and $m \ge 2^{l-1} + 1$ be odd. The graph* $\mathrm{Cay}(Z_m \times Z_{2^l}, \{\pm 1\} \times \{\pm 1, 2^{l-1}\})$ *can be partitioned into two C_{2^l}-factors and a C_m-factor.*

Proof. Let

$C_1 = (0_0, (m-1)_1, 0_2, (m-1)_3, \ldots, 0_{2^l-2}, (m-1)_{2^l-1})$,

$C_2 = (0_1, 1_{1+2^{l-1}}, 2_1, 3_{1+2^{l-1}}, \ldots, (2^{l-1}-2)_1, (2^{l-1}-1)_{1+2^{l-1}}, (2^{l-1})_{2^{l-1}}, (2^{l-1}-1)_0, (2^{l-1}-2)_{2^{l-1}}, (2^{l-1}-3)_0, \ldots, 2_{2^{l-1}}, 1_0)$,

$C_3 = ((m-1)_0, 0_1, (m-1)_2, 0_3, \ldots, (m-1)_{2^l-2}, 0_{2^l-1})$,

$C_4 = (0_0, 1_{2^{l-1}}, 2_{2^{l-1}-1}, 3_{2^{l-1}-2}, \ldots, (2^{l-1}-2)_3, (2^{l-1}-1)_2, (2^{l-1})_1, (2^{l-1}-1)_{2^{l-1}+1}, (2^{l-1}-2)_{2^{l-1}+2}, \ldots, 2_{2^l-2}, 1_{2^l-1})$.

(1) For $m = 2^{l-1}+1$, let $\mathcal{B}_1 = \{(0_0, 1_1, 2_2, \ldots, (2^{l-1})_{2^{l-1}}) + (0,h) \mid h \in Z_{2^l}\}$. It contains 2^l cycles with a length of m and $V(\mathcal{B}_1) = Z_m \times Z_{2^l}$, then $\mathcal{B}_1$ is a C_m-factor.

Let $\mathcal{B}_2 = \{C_1, C_2 + (0, 2i) \mid 0 \le i \le 2^{l-1} - 1\}$ and $\mathcal{B}_3 = \{C_3, C_4 + (0, 2i) \mid 0 \le i \le 2^{l-1} - 1\}$. Each of them has m cycles with a length of 2^l and its vertex-set is $Z_m \times Z_{2^l}$, then $\mathcal{B}_2$ and $\mathcal{B}_3$ are two C_{2^l}-factors. By counting $\cup_{i=1}^3 E(\mathcal{B}_i)$, we obtain that it coincides with the edge-set of the Cayley graph.

(2) For $m \ge 2^{l-1} + 3$, let

$\mathcal{B}_1 = \{(0_0, 1_1, \ldots, (2^{l-1})_{2^{l-1}}, (2^{l-1}+1)_0, (2^{l-1}+2)_{2^{l-1}}, (2^{l-1}+3)_0, (2^{l-1}+4)_{2^{l-1}}, \cdots, (m-2)_0, (m-1)_{2^{l-1}}) + (0,h) \mid h \in Z_{2^l}\}$,

$\mathcal{B}_2 = \{C_1, C_2 + (0,2i), ((j+1)_0, j_1, (j+1)_2, j_3, \ldots, (j+1)_{2^l-2}, j_{2^l-1}) \mid 0 \le i \le 2^{l-1} - 1, 2^{l-1} \le j \le m-2\}$,

$\mathcal{B}_8 = \{C_3, C_4 + (0,2i), (j_0, (j+1)_1, j_2, (j+1)_3, \ldots, j_{2^l-2}, (j+1)_{2^l-1}) \mid 0 \le i \le 2^{l-1} - 1, 2^{l-1} \le j \le m-2\}$.

Similarly to the above case, we obtain that $\mathcal{B}_1$ is a C_m-factor and $\mathcal{B}_2, \mathcal{B}_3$ are two C_{2^l}-factors. We check that $\cup_{i=1}^3 E(\mathcal{B}_i) = E(\mathrm{Cay}(Z_m \times Z_{2^l}, \{\pm 1\} \times \{\pm 1, 2^{l-1}\}))$. □

Lemma 4. *Let $m \ge 3$ and $l \ge 3$. The graph* $\mathrm{Cay}(Z_m \times Z_{2^l}, \{\pm 1\} \times \{2^{l-1}\}) \cup mK_{2^l}$ *can be partitioned into 2^{l-1} C_{2^l}-factors and a one-factor.*

Proof. Note that the graph mK_{2^l} is equivalent to the Caylay graph $\mathrm{Cay}(Z_m \times Z_{2^l}, \{0\} \times (Z_{2^l} \setminus \{0\}))$. By Theorem 1, there exists a C_{2^l}-factorization of the graph $K_{2^{l-1}}[2]$. Let $(e_1, e_2) = (0,1), (e_3, e_4) = (1 + 2^{l-1}, 2 + 2^{l-1}), (e_{2t+1}, e_{2t+2}) = (t, t+1+2^{l-1})$, $2 \le t \le 2^{l-1} - 2$, $(e_{2^l-1}, e_{2^l}) = (2^{l-1}-1, 2^{l-1})$. Without loss of generality, let $\{\{e_1, e_2\}, \{e_3, e_4\}, \ldots, \{e_{2^l-1}, e_{2^l}\}\}$ be the group set of $K_{2^{l-1}}[2]$. There are $2^{l-1} - 1$ C_{2^l}-factors of $K_{2^{l-1}}[2]$, denoted by $(b_{s1}, b_{s2}, \ldots, b_{s,2^l})$ for $1 \le s \le 2^{l-1} - 1$.

We first construct the required 2^{l-1} C_{2^l}-factors, each of which has exactly m cycles with a length of C_{2^l}. Let $C_s = (0_{b_{s1}}, 0_{b_{s2}}, \ldots, 0_{b_{s,2^l}})$, $1 \le s \le 2^{l-1} - 1$, and

$$C_{2^{l-1}} = (0_{e_1}, 0_{e_2}, 1_{e_3}, 1_{e_4}, 0_{e_5}, 0_{e_6}, 1_{e_7}, 1_{e_8}, \ldots, 0_{e_{2^l-3}}, 0_{e_{2^l-2}}, 1_{e_{2^l-1}}, 1_{e_{2^l}}).$$

For $1 \le i \le 2^{l-1}$, the set of the subscripts of C_i is actually Z_{2^l}, thus, each C_i can generate a C_{2^l}-factor by $(+1 \pmod m, -)$. In other words, $\{C_i + (l, 0) \mid l \in Z_m\}$ is a C_{2^l}-factor.

In the original graph, in addition to the edges in the cycles above, there are still some edges left, that is, $\{(1_{e_{4p-2}}, 0_{e_{4p-1}}) + (l,0), (0_{e_{4p}}, 1_{e_{4p+1}}) + (l,0) \mid l \in Z_m, 1 \le p \le 2^{l-2}, e_{2^l+1} = e_1\}$. The set of vertices on these edges is $Z_m \times Z_{2^l}$, so this set of edges forms a one-factor. □

Now, we construct two special one-factorizations of K_{16} with the vertex-set Z_{16} whose 15 one-factors are listed below for the following lemmas.

$I_1 = \{(0,1),(3,6),(4,5),(7,10),(8,9),(11,14),(12,13),(15,2)\}$,
$I_3 = \{(0,2),(6,1),(13,3),(7,9),(5,11),(15,12),(8,10),(14,4)\}$,
$I_5 = \{(0,4),(10,1),(11,3),(9,2),(12,8),(14,5),(15,7),(13,6)\}$,
$I_7 = \{(0,5),(3,7),(9,4),(2,10),(12,1),(15,11),(13,8),(6,14)\}$,
$I_9 = \{(0,6),(2,4),(8,3),(15,5),(9,11),(7,13),(1,14),(10,12)\}$,
$I_{11} = \{(0,15),(9,5),(11,12),(6,2),(8,7),(1,13),(3,4),(14,10)\}$,
$I_{13} = \{(0,9),(6,12),(5,2),(11,8),(14,7),(4,13),(15,1),(3,10)\}$,
$I_{14} = \{(9,6),(12,5),(2,11),(8,14),(7,4),(13,15),(1,3),(10,0)\}$,
$I_{15} = \{(0,11),(13,2),(12,7),(14,9),(3,5),(10,4),(15,6),(1,8)\}$,
$I_{2i} = \{(x+2i, y+2i) : (x,y) \in I_{2i-1}\}, i = 1,2,3$,
$I_{6+2i} = \{(x-2i, y-2i) : (x,y) \in I_{5+2i}\}, i = 1,2,3$.
Note that $I_{13} \cup I_{14}$ can form a 16-cycle.
$I'_1 = \{(0,2),(1,3),(5,4),(6,8),(7,9),(11,10),(15,13),(12,14)\}$,
$I'_2 = \{(2,1),(3,5),(4,6),(8,7),(9,11),(10,15),(13,12),(14,0)\}$,
$I'_3 = \{(0,4),(2,3),(6,1),(5,7),(10,8),(14,13),(9,15),(11,12)\}$,
$I'_4 = \{(4,2),(3,6),(1,5),(7,10),(8,14),(13,9),(15,11),(12,0)\}$,
$I'_5 = \{(0,5),(8,1),(7,3),(9,2),(15,4),(14,10),(12,6),(11,13)\}$,
$I'_6 = \{(5,8),(1,7),(3,9),(2,15),(4,14),(10,12),(6,11),(13,0)\}$,
$I'_7 = \{(0,6),(2,8),(3,12),(9,14),(5,10),(1,15),(7,13),(4,11)\}$,
$I'_8 = \{(6,2),(8,3),(12,9),(14,5),(10,1),(15,7),(13,4),(11,0)\}$,
$I'_9 = \{(0,7),(12,2),(13,3),(14,6),(10,4),(9,1),(11,5),(15,8)\}$,
$I'_{10} = \{(7,12),(2,13),(3,14),(6,10),(4,9),(1,11),(5,15),(8,0)\}$,
$I'_{11} = \{(0,10),(2,11),(3,15),(7,14),(6,13),(5,9),(1,12),(4,8)\}$,
$I'_{12} = \{(0,1),(7,2),(12,4),(10,3),(13,5),(11,14),(8,9),(15,6)\}$,
$I'_{13} = \{(6,7),(13,8),(2,10),(0,9),(3,11),(1,4),(14,15),(5,12)\}$,
$I'_{14} = \{(0,3),(13,1),(7,4),(14,2),(8,11),(5,6),(12,15),(9,10)\}$,
$I'_{15} = \{(10,13),(7,11),(1,14),(8,12),(2,5),(15,0),(6,9),(3,4)\}$.
Note that $I'_{2i-1} \cup I'_{2i}$ can form a 16-cycle for $1 \le i \le 5$.

For an integer $m \ge 2$, mH denotes m vertex-disjoint copies of a graph H. For brevity, we use mI_k (or mI'_k) to denote the graph with the vertex-set $Z_m \times Z_{16}$ and the edge-set $\{(j_a, j_b) \mid j \in Z_m, (a,b) \in I_k (\text{or } I'_k), a \neq b\}$ for $1 \le k \le 15$. Similarly, mK_n denotes the graph with the vertex-set $Z_m \times Z_n$ and the edge-set $\{(j_a, j_b) \mid j \in Z_m, (a,b) \in E(K_n)\}$.

Lemma 5. *Let $m \ge 3$ and $i \in \{2,4,6\}$. There exist two C_{16}-factors which form a C_{16}-factorization of* $\mathrm{Cay}(Z_m \times Z_{16}, \{\pm 1\} \times \{i\}) \cup mI_{i-1} \cup mI_i$.

Proof. Let
$$C_1^2 = (0_0, 0_1, 1_3, 1_6, 0_4, 0_5, 1_7, 1_{10}, 0_8, 0_9, 1_{11}, 1_{14}, 0_{12}, 0_{13}, 1_{15}, 1_2),$$
$$C_1^4 = (0_0, 0_2, 1_6, 1_1, 0_{13}, 0_3, 1_7, 1_9, 0_5, 0_{11}, 1_{15}, 1_{12}, 0_8, 0_{10}, 1_{14}, 1_4),$$
$$C_1^6 = (0_0, 0_4, 1_{10}, 1_1, 0_{11}, 0_3, 1_9, 1_2, 0_{12}, 0_8, 1_{14}, 1_5, 0_{15}, 0_7, 1_{13}, 1_6).$$
For each $i \in \{2,4,6\}$, let $C_2^i = C_1^i + (0,i)$. Because the set of the subscripts of C_t^i is Z_{16}, each $\mathcal{B}_t^i = \{C_t^i + (l,0) \mid l \in Z_m\}$ is a C_{16}-factor for $t = 1,2$. Since $E(\mathcal{B}_1^i) \cup E(\mathcal{B}_2^i) = E(\mathrm{Cay}(Z_m \times Z_{16}, \{\pm 1\} \times \{i\}) \cup mI_{i-1} \cup mI_i)$, these two C_{16}-factors can form a C_{16}-factorization of the graph. □

Lemma 6. *Let $m \ge 3$ and $i \in \{2,4,6\}$. There exist two C_{16}-factors which is a C_{16}-factorization of* $\mathrm{Cay}(Z_m \times Z_{16}, \{\pm 1\} \times \{-i\}) \cup mI_{6+i-1} \cup mI_{6+i}$.

Proof. Let
$$C_1^2 = (0_0, 0_5, 1_3, 1_7, 0_9, 0_4, 1_2, 1_{10}, 0_{12}, 0_1, 1_{15}, 1_{11}, 0_{13}, 0_8, 1_6, 1_{14}),$$
$$C_1^4 = (0_0, 0_6, 1_2, 1_4, 0_8, 0_3, 1_{15}, 1_5, 0_9, 0_{11}, 1_7, 1_{13}, 0_1, 0_{14}, 1_{10}, 1_{12}),$$

$$C_1^6 = (0_0, 0_{15}, 1_9, 1_5, 0_{11}, 0_{12}, 1_6, 1_2, 0_8, 0_7, 1_1, 1_{13}, 0_3, 0_4, 1_{14}, 1_{10}).$$

It is similar to the above lemma, let $C_2^i = C_1^i + (0, -i)$ for any $i \in \{2, 4, 6\}$. We have that each $\{C_t^i + (l, 0) \mid l \in Z_m\}$ is a C_{16}-factor for $t = 1, 2$, and they form a C_{16}-factorization. □

Lemma 7. *Let $m \geq 3$. The graph $\mathrm{Cay}(Z_m \times Z_{16}, \{\pm 1\} \times \{8\}) \cup m(\cup_{i=13}^{15} I_i)$ can be decomposed into two C_{16}-factors and a one-factor.*

Proof. Let
$$C_1 = (0_0, 0_9, 0_6, 0_{12}, 0_5, 0_2, 0_{11}, 0_8, 0_{14}, 0_7, 0_4, 0_{13}, 0_{15}, 0_1, 0_3, 0_{10}),$$
$$C_2 = (0_0, 0_{11}, 1_3, 1_5, 0_{13}, 0_2, 1_{10}, 1_4, 0_{12}, 0_7, 1_{15}, 1_6, 0_{14}, 0_9, 1_1, 1_8).$$

Similarly, each $\mathcal{B}_t = \{C_t + (l, 0) \mid l \in Z_m\}$ is a C_{16}-factor for $t = 1, 2$ because the set of the subscripts of C_t is Z_{16}. Let $I = \{((j+1)_{11}, j_3), (j_5, (j+1)_{13}), ((j+1)_2, j_{10}), (j_4, (j+1)_{12}), ((j+1)_7, j_{15}), (j_6, (j+1)_{14}), ((j+1)_9, j_1), (j_8, (j+1)_0) \mid j \in Z_m\}$. It is a one-factor since $V(I) = Z_m \times Z_{16}$. We check that $E(\mathrm{Cay}(Z_m \times Z_{16}, \{\pm 1\} \times \{8\}) \cup m(\cup_{i=13}^{15} I_i)) = E(\mathcal{B}_1) \cup E(\mathcal{B}_2) \cup E(I)$. Then, we obtain the conclusion. □

Lemma 8. *Let $m \geq 3$. The graph $\mathrm{Cay}(Z_m \times Z_{16}, \{\pm 1\} \times \{8\}) \cup mI'_{11}$ can be partitioned into a C_{16}-factor and a one-factor.*

Proof. Let $C = (0_0, 0_{10}, 1_2, 1_{11}, 0_3, 0_{15}, 1_7, 1_{14}, 0_6, 0_{13}, 1_5, 1_9, 0_1, 0_{12}, 1_4, 1_8)$. Since the set of the subscripts of C is Z_{16}, we obtain that $\mathcal{B} = \{C + (l, 0) \mid l \in Z_m\}$ is a C_{16}-factor of this graph. Let $I = \{((j+1)_{10}, j_2), (j_{11}, (j+1)_3), ((j+1)_{15}, j_7), (j_{14}, (j+1)_6), ((j+1)_{13}, j_5), (j_9, (j+1)_1), ((j+1)_{12}, j_4), (j_8, (j+1)_0) \mid j \in Z_m\}$. It is a set of edges and $V(I) = Z_m \times Z_{16}$, so it is a one-factor. We check that $E(\mathcal{B}) \cup E(I)$ coincides with the edge-set of the graph $\mathrm{Cay}(Z_m \times Z_{16}, \{\pm 1\} \times \{8\}) \cup mI'_{11}$ by counting the number of edges. □

Lemma 9. *Let $m \geq 3$. The graph $\mathrm{Cay}(Z_m \times Z_{16}, \{\pm 1\} \times \{\pm 6\}) \cup m(\cup_{i=12}^{15} I'_i)$ can be decomposed into four C_{16}-factors.*

Proof. Let
$$C_1 = (0_0, 0_1, 1_7, 1_2, 0_{12}, 0_4, 1_{10}, 1_3, 0_{13}, 0_5, 1_{11}, 1_{14}, 0_8, 0_9, 1_{15}, 1_6),$$
$$C_2 = (0_6, 0_7, 1_{13}, 1_8, 0_2, 0_{10}, 1_0, 1_9, 0_3, 0_{11}, 1_1, 1_4, 0_{14}, 0_{15}, 1_5, 1_{12}),$$
$$C_3 = (0_0, 0_3, 1_{13}, 1_1, 0_7, 0_4, 1_{14}, 1_2, 0_8, 0_{11}, 1_5, 1_6, 0_{12}, 0_{15}, 1_9, 1_{10}),$$
$$C_4 = (0_{10}, 0_{13}, 1_7, 1_{11}, 0_1, 0_{14}, 1_8, 1_{12}, 0_2, 0_5, 1_{15}, 1_0, 0_6, 0_9, 1_3, 1_4).$$

Let $\mathcal{B}_i = \{C_t + (l, 0) \mid l \in Z_m\}$ for $1 \leq t \leq 4$. Since the subscripts of C_t form the set Z_{16}, each $\mathcal{B}_i$ is a C_{16}-factor. By counting the edges of $\cup_{i=1}^{4} \mathcal{B}_i$, we obtain the required design. □

4. Main Results

In this section, we will prove our main results.

Lemma 10. *For odd $m \geq 9$ and $r \in \{0, 2, 4, 6, 8, 16\}$, $(r, 16 - r) \in \mathrm{HWP}(C_m[16]; 16, m)$.*

Proof. We consider the four following cases.

Case 1: $r = 0, 8, 16$.

By Theorem 7, the graph $C_m[4]$ can be decomposed into $\frac{r}{4}$ C_4-factors and $4 - \frac{r}{4}$ C_m-factors for $m \geq 3$. By Theorems 5 and 6, two graphs $C_m[4]$ and $C_4[4]$ can be partitioned into four C_m-factors and four C_{16}-factors, respectively. Then, we obtain the conclusion by applying Construction 3.

Case 2: $r = 2$.

We obtain two C_{16}-factors from a C_{16}-factorization of $\mathrm{Cay}(Z_m \times Z_{16}, \{\pm 1\} \times \{\pm 5\})$ by Lemma 1(5). The required fourteen C_m-factors can be obtained through three parts. The graph $\mathrm{Cay}(Z_m \times Z_{16}, \{\pm 1\} \times \{0, \pm 1, \pm 2\})$ can be decomposed into five C_m-factors by

Lemma 1(1). Similarly, we consider two Cayley graphs $\mathrm{Cay}(Z_m \times Z_{16}, \{\pm 1\} \times (\pm\{3,6,7\}))$ and $\mathrm{Cay}(Z_m \times Z_{16}, \{\pm 1\} \times \{\pm 4, 8\})$. They can be partitioned into six C_m-factors and three C_m-factors by Lemma 1(2) and Lemma 2, respectively.

Case 3: $r = 4$.

Four C_{16}-factors are given from a C_{16}-factorization of $\mathrm{Cay}(Z_m \times Z_{16}, \{\pm 1\} \times (\pm\{5,7\}))$ by Lemma 1(5). Six of the twelve required C_m-factors can be obtained from the decomposition of $\mathrm{Cay}(Z_m \times Z_{16}, \{\pm 1\} \times (\pm\{1,2,3\}))$ by Lemma 1(2). The graph $\mathrm{Cay}(Z_m \times Z_{16}, \{\pm 1\} \times \{0, \pm 6\})$ can be decomposed into three C_m-factors by Lemma 1(3). The last three C_m-factors come from the decomposition of $\mathrm{Cay}(Z_m \times Z_{16}, \{\pm 1\} \times \{\pm 4, 8\})$ by Lemma 2.

Case 4: $r = 6$.

A C_{16}-factorization of $\mathrm{Cay}(Z_m \times Z_{16}, \{\pm 1\} \times (\pm\{5,7\}))$ generates four C_{16}-factors by Lemma 1(5). The graph $\mathrm{Cay}(Z_m \times Z_{16}, \{\pm 1\} \times \{\pm 1, 8\})$ can be decomposed into two C_{16}-factors and a C_m-factor by Lemma 3. Now, we have the six required C_{16}-factors. The nine remaining C_m-factors are listed below. Six of them can be obtained from a C_m-factorization of $\mathrm{Cay}(Z_m \times Z_{16}, \{\pm 1\} \times (\pm\{2,4,6\}))$ by Lemma 1(2). The last three come from the decomposition of $\mathrm{Cay}(Z_m \times Z_{16}, \{\pm 1\} \times \{0, \pm 3\})$ by Lemma 1(3). □

Lemma 11. *For odd $m \geq 9$ and $7 \leq r \leq 23$, the graph $C_m \wr K_{16}$ can be partitioned into r C_{16}-factors, $23 - r$ C_m-factors and a one-factor.*

Proof. Let the vertex-set be $Z_m \times Z_{16}$. We distinguish 12 cases as shown below.

Case 1: $r \in \{7, 9, 11, 13, 15, 23\}$.

By Theorem 1 and Lemma 10, we have $(0,7) \in \mathrm{HWP}(16; m, 16)$ and $(16 - r_1, r_1) \in \mathrm{HWP}(C_m[16]; m, 16)$ for odd $m \geq 9$ and $r_1 \in \{0, 2, 4, 6, 8, 16\}$, respectively. Applying Construction 1, we obtain $(16 - r_1, r_1 + 7) \in \mathrm{HWP}(C_m \wr K_{16}; m, 16)$.

Case 2: $r = 8$.

$\mathrm{Cay}(Z_m \times Z_{16}, \{\pm 1\} \times \{8\}) \cup mK_{16}$ can be decomposed into eight C_{16}-factors and a one-factor by Lemma 4. $\mathrm{Cay}(Z_m \times Z_{16}, \{\pm 1\} \times (\pm\{1,5,6\}))$ and $\mathrm{Cay}(Z_m \times Z_{16}, \{\pm 1\} \times (\pm\{3,4,7\}))$ can be partitioned into twelve C_m-factors from Lemma 1(2). A C_m-factorization of $\mathrm{Cay}(Z_m \times Z_{16}, \{\pm 1\} \times \{0, \pm 2\})$ can generate three C_m-factors by Lemma 1(3).

Case 3: $r = 10$.

$\mathrm{Cay}(Z_m \times Z_{16}, \{\pm 1\} \times \{\pm 6, 8\}) \cup m(\cup_{i=11}^{15} I'_i)$ can be decomposed into five C_{16}-factors and a one-factor from Lemmas 8 and 9. Since $I'_{2j-1} \cup I'_{2j}$ can form a 16-cycle, we can obtain a C_{16}-factor for $1 \leq j \leq 5$ from the graph $m(I'_{2j-1} \cup I'_{2j})$. In other words, we obtained ten C_{16}-factors and a one-factor from $\mathrm{Cay}(Z_m \times Z_{16}, \{\pm 1\} \times \{\pm 6, 8\}) \cup mK_{16}$.

Two Cayley graphs $\mathrm{Cay}(Z_m \times Z_{16}, \{\pm 1\} \times (\pm\{1,3,4\}))$ and $\mathrm{Cay}(Z_m \times Z_{16}, \{\pm 1\} \times (\pm\{2,5,7\}))$ can be partitioned into 12 C_m-factors by Lemma 1(2). The last C_m-factor comes from a C_m-factorization of $\mathrm{Cay}(Z_m \times Z_{16}, \{\pm 1\} \times \{0\})$ by Lemma 1(4).

Case 4: $r = 12$.

$\mathrm{Cay}(Z_m \times Z_{16}, \{\pm 1\} \times \{\pm 6, 8\}) \cup mK_{16}$ can be decomposed into ten C_{16}-factors and a one-factor from the above case. By Lemma 1(5), we can obtain the two remaining C_{16}-factors from the decomposition of $\mathrm{Cay}(Z_m \times Z_{16}, \{\pm 1\} \times \{\pm 5\})$. Five C_m-factors come from a C_m-factorization of $\mathrm{Cay}(Z_m \times Z_{16}, \{\pm 1\} \times \{0, \pm 1, \pm 2\})$ by Lemma 1(1). $\mathrm{Cay}(Z_m \times Z_{16}, \{\pm 1\} \times (\pm\{3,4,7\}))$ is precisely divided into six C_m-factors by Lemma 1(2).

Case 5: $r = 14$.

The Cayley graph $\mathrm{Cay}(Z_m \times Z_{16}, \{\pm 1\} \times \{8\}) \cup mK_{16}$ can be divided into eight C_{16}-factors and a one-factor by Lemma 4. The last six C_{16}-factors come from a C_{16}-factorization of $\mathrm{Cay}(Z_m \times Z_{16}, \{\pm 1\} \times (\pm\{3,5,7\}))$ by Lemma 1(5). In addition, $\mathrm{Cay}(Z_m \times Z_{16}, \{\pm 1\} \times (\pm\{2,4,6\}))$ can be partitioned into six C_m-factors by Lemma 1(2). A C_m-factorization of $\mathrm{Cay}(Z_m \times Z_{16}, \{\pm 1\} \times \{0, \pm 1\})$ contains exactly three C_m-factors by Lemma 1(3).

Case 6: $r = 16$.

By Lemma 4, $\mathrm{Cay}(Z_m \times Z_{16}, \{\pm 1\} \times \{8\}) \cup mK_{16}$ can be decomposed into eight C_{16}-factors and a one-factor. The last eight C_{16}-factors originate from a C_{16}-factorization of

$\mathrm{Cay}(Z_m \times Z_{16}, \{\pm 1\} \times (\pm\{1,3,5,7\}))$ by Lemma 1(5). $\mathrm{Cay}(Z_m \times Z_{16}, \{\pm 1\} \times (\pm\{2,4,6\}))$ can be divided into six C_m-factors from Lemma 1(2). The last C_m-factor comes from a C_m-factorization of $\mathrm{Cay}(Z_m \times Z_{16}, \{\pm 1\} \times \{0\})$ by Lemma 1(4).

Case 7: $r = 18$.

$\mathrm{Cay}(Z_m \times Z_{16}, \{\pm 1\} \times \{\pm 6, 8\}) \cup mK_{16}$ can be divided into ten C_{16}-factors and a one-factor from Case 3. Similarly, we obtain eight C_{16}-factors from $\mathrm{Cay}(Z_m \times Z_{16}, \{\pm 1\} \times (\pm\{1,3,5,7\}))$ by Lemma 1(5). The five required C_m-factors originate from a C_m-factorization of $\mathrm{Cay}(Z_m \times Z_{16}, \{\pm 1\} \times \{0, \pm 2, \pm 4\})$ by Lemma 1(1).

Case 8: $r = 20$.

$\mathrm{Cay}(Z_m \times Z_{16}, \{\pm 1\} \times (\pm\{2,4,6\})) \cup m(\cup_{i=1}^{12} I_i)$ can be partitioned into twelve C_{16}-factors by Lemmas 5 and 6. $\mathrm{Cay}(Z_m \times Z_{16}, \{\pm 1\} \times \{8\}) \cup m(\cup_{i=13}^{15} I_i)$ can be divided into two C_{16}-factors and a one-factor by Lemma 7. That is to say, $\mathrm{Cay}(Z_m \times Z_{16}, \{\pm 1\} \times \{\pm 2, \pm 4, \pm 6, 8\}) \cup mK_{16}$ can be decomposed into fourteen C_{16}-factors and a one-factor. We can obtain the other six C_{16}-factors from $\mathrm{Cay}(Z_m \times Z_{16}, \{\pm 1\} \times (\pm\{3,5,7\}))$ by Lemma 1(5) and obtain three C_m-factors from a C_m-factorization of $\mathrm{Cay}(Z_m \times Z_{16}, \{\pm 1\} \times \{0, \pm 1\})$ by Lemma 1(3).

Case 9: $r = 22$.

$\mathrm{Cay}(Z_m \times Z_{16}, \{\pm 1\} \times \{\pm 2, \pm 4, \pm 6, 8\}) \cup mK_{16}$ can be decomposed into fourteen C_{16}-factors and a one-factor from the above case. The other eight C_{16}-factors can be obtained from $\mathrm{Cay}(Z_m \times Z_{16}, \{\pm 1\} \times (\pm\{1,3,5,7\}))$ by Lemma 1(5). A C_m-factor originates from the decomposition of $\mathrm{Cay}(Z_m \times Z_{16}, \{\pm 1\} \times \{0\})$ by Lemma 1(4).

Case 10: $r = 17$.

Since $I'_{2j-1} \cup I'_{2j}$ form a 16-cycle, the graph $m(I'_{2j-1} \cup I'_{2j})$ is actually a C_{16}-factor for any $1 \leq j \leq 5$. Two graphs $\mathrm{Cay}(Z_m \times Z_{16}, \{\pm 1\} \times \{\pm 6\}) \cup m(\cup_{i=12}^{15} I'_i)$ and $\mathrm{Cay}(Z_m \times Z_{16}, \{\pm 1\} \times (\pm\{3,5,7\}))$ can be decomposed into four C_{16}-factors and six C_{16}-factors by Lemma 9 and Lemma 1(5), respectively. We can obtain two C_{16}-factors and a C_m-factor from $\mathrm{Cay}(Z_m \times Z_{16}, \{\pm 1\} \times \{\pm 1, 8\})$ by Lemma 3 and five C_m-factors from a C_m-factorization of $\mathrm{Cay}(Z_m \times Z_{16}, \{\pm 1\} \times \{0, \pm 2, \pm 4\})$ by Lemma 1(1). The one-factor is mI'_{11}.

Case 11: $r = 19$.

By Lemmas 5 and 6, the graph $\mathrm{Cay}(Z_m \times Z_{16}, \{\pm 1\} \times (\pm\{2,4,6\})) \cup m(\cup_{i=1}^{12} I_i)$ can be divided into 12 C_{16}-factors. $m(I_{13} \cup I_{14})$ is a C_{16}-factor since $I_{13} \cup I_{14}$ can form a 16-cycle. The one-factor is mI_{15}. $\mathrm{Cay}(Z_m \times Z_{16}, \{\pm 1\} \times \{\pm 1, 8\})$ can be divided into two C_{16}-factors and a C_m-factor by Lemma 3. That is to say, we can obtain 15 C_{16}-factors, a C_m-factor, and a one-factor from $\mathrm{Cay}(Z_m \times Z_{16}, \{\pm 1\} \times (\pm\{1,2,4,6\} \cup \{8\})) \cup mK_{16}$.

The remaining four C_{16}-factors and three C_m-factors come from the factorization of $\mathrm{Cay}(Z_m \times Z_{16}, \{\pm 1\} \times (\pm\{5,7\}))$ and $\mathrm{Cay}(Z_m \times Z_{16}, \{\pm 1\} \times \{0, \pm 3\})$ by Lemma 1(5) and Lemma 1(3), respectively.

Case 12: $r = 21$.

Similarly to the above case, we obtain 15 C_{16}-factors, a C_m-factor, and a one-factor from $\mathrm{Cay}(Z_m \times Z_{16}, \{\pm 1\} \times (\pm\{1,2,4,6\} \cup \{8\})) \cup mK_{16}$. $\mathrm{Cay}(Z_m \times Z_{16}, \{\pm 1\} \times (\pm\{3,5,7\}))$ can be decomposed into six C_{16}-factors by Lemma 1(5). The last C_m-factor originates from a C_m-factorization of $\mathrm{Cay}(Z_m \times Z_{16}, \{\pm 1\} \times \{0\})$ by Lemma 1(4). □

Proof of Theorem 4. Let the vertex-set of K_{16mt} be $Z_{mt} \times Z_{16}$. We distinguish the two following cases.

Case 1: t is odd.

The complete graph K_{mt} on the vertex-set Z_{mt} can be decomposed into $\frac{mt-1}{2}$ C_m-factors by Theorem 1. We give each vertex of Z_{mt} a weight of 16 to obtain mtK_{16} and $\frac{mt-1}{2}$ $C_m[16]$-factors, which are denoted by P_i, $1 \leq i \leq \frac{mt-1}{2}$. Each P_i has t $C_m[16]$s, denoted by Q_{ij}, $1 \leq j \leq t$.

Let $0 \leq x \leq \frac{mt-3}{2}$, we replace each P_i $(1 \leq i \leq x)$ with an $\mathrm{HW}(C_m[16]; 16, m; 16, 0)$ and P_i $(x+1 \leq i \leq \frac{mt-3}{2})$ with an $\mathrm{HW}(C_m[16]; 16, m; 0, 16)$ by Lemma 10.

For $P_{\frac{mt-1}{2}}$ and $1 \le j \le t$, the graph $Q_{\frac{mt-1}{2},j} \cup mK_{16}$ can be partitioned into r C_{16}-factors, $23-r$ C_m-factors, and a one-factor for $7 \le r \le 23$ by Lemma 11. We put them together to obtain r C_{16}-factors, $23-r$ C_m-factors, and a one-factor on the vertex-set $Z_{mt} \times Z_{16}$.

We finally obtain $\alpha = 16x + r$ C_{16}-factors, $\beta = 16 \times (\frac{mt-3}{2} - x) + 23 - r = 8mt - 1 - (16x + r)$ C_m-factors and a one-factor for $0 \le x \le \frac{mt-3}{2}$ and $7 \le r \le 23$. Here, the range for α is 7 to $8mt - 1$.

Case 2: t is even.

(1) $\alpha = 7$.

We can obtain the conclusion by using Construction 2 with an $\mathrm{HW}(16; 16, m; 7, 0)$ and an $\mathrm{HW}(K_{mt}[16]; 16, m; 0, 8mt-8)$ from Theorem 1.

(2) $\alpha \ge 15$.

The graph $K_{\frac{mt}{2}}[2]$ can be partitioned into $\frac{mt-2}{2}$ C_m-factors by Theorem 1. In other words, the graph K_{mt} can be decomposed into $\frac{mt-2}{2}$ C_m-factors and a one-factor. Giving each vertex of the graph K_{mt} weight 16, we obtain $\frac{mt-2}{2}$ $C_m[16]$-factors which are denoted by P_i, $1 \le i \le \frac{mt-2}{2}$, mtK_{16}, and $\frac{mt}{2}K_2[16]$.

Let $0 \le x \le \frac{mt-4}{2}$. We replace any P_i $(1 \le i \le x)$ with an $\mathrm{HW}(C_m[16]; 16, m; 16, 0)$ and P_i $(x+1 \le i \le \frac{mt-4}{2})$ with an $\mathrm{HW}(C_m[16]; 16, m; 0, 16)$ from Lemma 10.

Similarly to the above case, $P_{\frac{mt-2}{2}} \cup mtK_{16}$ can be partitioned into r C_{16}-factors, $23-r$ C_m-factors, and a one-factor on the whole vertex-set $Z_{mt} \times Z_{16}$ for odd $m \ge 9$ and $7 \le r \le 23$. Furthermore, using Theorem 1, $\frac{mt}{2}K_2[16]$ can be decomposed into eight C_{16}-factors.

It is not difficult to calculate the number α of C_{16}-factors and the number β of C_m-factors. We obtained $\alpha = 16x + r + 8$ and $\beta = 16 \times (\frac{mt-4}{2} - x) + 23 - r = 8mt - 1 - (16x + r + 8)$ for $0 \le x \le \frac{mt-4}{2}$ and $7 \le r \le 23$. We check that the range for α is 15 to $8mt - 1$. $\square$

5. Concluding Remarks

We are working on the existence of an $\mathrm{HW}(v; 2^l, m; \alpha, \beta)$ for an odd m. Theorem 4 completes the proof of the case $l = 4$ and $\alpha \ge 15$. The construction method in this paper is still useful for other cases. We have some preliminary results and believe that the following conjecture is valid, but there is still a long way to go before the whole problem can be solved completely.

Conjecture 1. *For any odd $m \ge 2^{l-1} + 1$, $(\alpha, \beta) \in \mathrm{HWP}(2^l mt; 2^l, m)$ if and only if $\alpha + \beta = 2^{l-1}mt - 1$, where $\alpha \ge 0$, $\beta \ge 0$, and $t \ge 1$, except possibly when $\alpha \in [1, 2^{l-1} - 2]$ and t is odd or $\alpha \in [1, 2^{l-1} - 2] \cup [2^{l-1}, 2^l - 2]$ and t is even.*

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