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On Generalized t-Transformation of Free Convolution

Ayed. R. A. Alanzi [1,2], Raouf Fakhfakh [2,3,*] and Fatimah Alshahrani [4]

[1] Department of Mathematics, College of Science, Jouf University, Sakaka P.O. Box 2014, Saudi Arabia; aralenzi@ju.edu.sa
[2] Department of Mathematics, College of Science and Arts in Gurayat, Jouf University, Gurayat 77454, Saudi Arabia
[3] Laboratory of Probability and Statistics, Faculty of Sciences of Sfax, University of Sfax, Sfax 3000, Tunisia
[4] Department of Mathematical Sciences, College of Science, Princess Nourah bint Abdulrahman University, P.O. Box 84428, Riyadh 11671, Saudi Arabia; fmalshahrani@pnu.edu.sa
* Correspondence: rfakhfakh@ju.edu.sa

Abstract: A notion of generalized (two-parameterized) t-transformation of free convolution, also called $(\mathbf{t} = (a, b))$-deformed free convolution, is introduced for $a \in \mathbb{R}$ and $b > 0$. In this article, some results of $\mathbf{t}$-deformed free convolution are given within the theory of Cauchy-Stieltjes Kernel (CSK) families. The variance function is a fundamental concept in CSK families. An expression is provided for the variance function under $\mathbf{t}$-deformed free convolution power. In addition, through the use of the variance function, an approximation is provided for members of the $\mathbf{t}$-deformed free Gaussian CSK family and members of the $\mathbf{t}$-deformed free Poisson CSK family respectively. Furthermore, by involving the free multiplicative convolution, a new limit theorem is provided with respect to $\mathbf{t}$-deformed free convolution.

Keywords: variance function; Cauchy–Stieltjes transform; deformation of measures

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1. Introduction

The notion of the t-deformation of a measure and of a convolution was introduced by Bożejko and Wysoczański [1,2]. The definition of t-transformation of measure is based on the Cauchy–Stieltjes transform $G_\sigma(.)$, defined by

$$G_\sigma(w) = \int \frac{1}{w-y}\sigma(dy) \quad \text{for } w \in \mathbb{C}\backslash supp(\sigma), \tag{1}$$

where σ is a real probability measure. The t-transformation of a measure σ is introduced in the following way: Let $t > 0$, based on the Nevanlinna theorem, the function $G_{\sigma_t}(w)$, provided by:

$$\frac{1}{G_{\sigma_t}(w)} = \frac{t}{G_\sigma(w)} + (1-t)w, \tag{2}$$

is the Cauchy–Stieltjes transform of a probability measure denoted by $U_t(\sigma) := \sigma_t$.

The t-transformation of σ is nothing but the t-th Boolean additive convolution power of σ.

The t-transformation of any probability measure σ (with all finite moments) can be interpreted as a multiplication of the two Jacobi coefficients α_1 and λ_1 of the first level in the continued fraction notation of the Cauchy–Stieltjes transform. That is, if

$$G_\sigma(w) = \cfrac{1}{w - \alpha_1 - \cfrac{\lambda_1}{w - \alpha_2 - \cfrac{\lambda_2}{w - \alpha_3 - \cfrac{\lambda_3}{w - \alpha_4 - \ldots}}}},$$

then the Cauchy–Stieltjes transform of the deformed measure $U_t(\sigma)$ is

$$G_{U_t(\sigma)}(w) = \cfrac{1}{w - t\alpha_1 - \cfrac{t\lambda_1}{w - \alpha_2 - \cfrac{\lambda_2}{w - \alpha_3 - \cfrac{\lambda_3}{w - \alpha_4 - \ldots}}}}.$$

Based on the t-transformation of measures, a new type of convolution, called t-deformed free convolution (or t-free convolution), denoted as $\boxed{t}$-convolution, is defined in [1,2]: The t-deformed free convolution $\boxed{t}$ is introduced by

$$\rho \boxed{t} \sigma = U_{1/t}(U_t(\rho) \boxplus U_t(\sigma)), \tag{3}$$

where ρ and σ are the real probability measures. However, the central limit theorem with respect to $\boxed{t}$-convolution is established. The limit law is called t-deformed free Gaussian law. The Poisson limit theorem with respect to $\boxed{t}$-convolution is proven. The limit law is called the t-deformed free Poisson law. Families of free random variables associated with these central limit measures are constructed, see [1,2] for more details. Further studies related to $\boxed{t}$-convolution are presented in [3,4].

This topic was further studied and extended in many ways in a number of papers. Krystek and Yoshida [5] introduced a generalized (two-parameterized) t-transformation, whereby the t-transformation of Bożejko and Wysoczański was reduced to a special case. The corresponding transformed convolutions were also defined. They considered a deformation of the Cauchy–Stieltjes transform of σ (with all finite moments) as follows: Let $a \in \mathbb{R}$ and $b > 0$, denote $\mathbf{t} = (a, b)$ and consider the $(\mathbf{t} = (a, b))$-transformation defined by

$$\frac{1}{G_{\widetilde{U}^{\mathbf{t}}(\sigma)}(w)} = \frac{b}{G_\sigma(w)} + (1-b)w + (b-a)m_0(\sigma), \tag{4}$$

where $m_0(\sigma)$ denotes the moment of order 1 of the measure σ.

If $a = b = t$, the transformation $\widetilde{U}^{\mathbf{t}}(.)$ is the t-transformation $U_t(.)$ introduced in [1,2]. The $(\mathbf{t} = (a, b))$-transformation can be interpreted by means of continued fractions. The coefficients α_1 and λ_1 in the continued fraction representation of the original probability measure σ (with finite all moments) are multiplied by a and b, respectively: that is

$$G_{\widetilde{U}^{\mathbf{t}}(\sigma)}(w) = \cfrac{1}{w - a\alpha_1 - \cfrac{b\lambda_1}{w - \alpha_2 - \cfrac{\lambda_2}{w - \alpha_3 - \cfrac{\lambda_3}{w - \alpha_4 - \ldots}}}}.$$

Based on $\widetilde{U}^{\mathbf{t}}$-transformation of measures, the $\mathbf{t}$-deformed free and classical convolutions is introduced. From [5] (Proposition 1.4), for $a \neq 0$ and $b > 0$, one can see that the (a, b)-transformation is invertible. That is, if we write $\mathbf{t}^{-1} = (a^{-1}, b^{-1})$, then $\widetilde{U}^{\mathbf{t}}$ and $\widetilde{U}^{\mathbf{t}^{-1}}$ are inverse of the other. The $\mathbf{t}$-transformation of free convolution $\boxplus_{(\mathbf{t})}$ is

$$\rho \boxplus_{(\mathbf{t})} \sigma = \widetilde{U}^{\mathbf{t}^{-1}}(\widetilde{U}^{\mathbf{t}}(\rho) \boxplus \widetilde{U}^{\mathbf{t}}(\sigma)), \tag{5}$$

where ρ and σ are real measures with all finite moments. The $\mathbf{t}$-transformation of classical convolution, denoted by $*_{(\mathbf{t})}$, is obtained in the same way by replacing in (5) the free convolution $\boxplus$ by the classical convolution $*$. It has been shown that the central limit measures associated with $\mathbf{t}$-deformed classical and free convolutions is exactly the same for the original t-deformations, but the Poisson limit is different and depends on two parameters. A calculation is made for the $\mathbf{t}$-deformed classical and free Poisson limits. The orthogonal polynomials that correspond to the limit measures are provided explicitly.

The theory of Cauchy–Stieltjes Kernel (CSK) families in non-commutative probability has been introduced recently. It is defined analogously to the theory of natural exponential families in classical probability. The variance function is an important concept in CSK families. In this article, some properties of $\boxplus_{(\mathbf{t})}$-convolution are provided within the framework of CSK families. For the clarity of the results provided in this article, some facts about CSK families are presented in Section 2. In Section 3, an expression is provided for the variance function under $\boxplus_{(\mathbf{t})}$-convolution power. This expression for the variance function together with the notion of $\boxplus_{(\mathbf{t})}$-convolution are used in Section 4 to approximate the elements of the $\mathbf{t}$-deformed free Gaussian CSK family and the elements of the $\mathbf{t}$-deformed free Poisson CSK family. Furthermore, by involving the free multiplicative convolution and based on the variance function, a limit theorem is presented in Section 5 for the $\boxplus_{(\mathbf{t})}$-convolution.

2. Cauchy–Stieltjes Kernel Families

A concept of family generated by the measure μ is introduced in [6] for any kernel $\mathcal{N}(y,\vartheta)$, such that

$$\mathcal{L}(\vartheta) = \int \mathcal{N}(y,\vartheta)\mu(dy)$$

converges in a open set Ω. It is the family of probability measures

$$\{(\mathcal{N}(y,\vartheta)/\mathcal{L}(\vartheta))\mu(dy)\ :\ \vartheta \in \Omega\}.$$

Bryc and Ismail [7] introduced some properties of q-exponential families. In particular, the case $q = 0$ has been connected to the free probability using the Cauchy–Stieltjes kernel $1/(1-\vartheta y)$. If $q = 1$, we can recover the exponential families. Some results for the CSK families are provided in [8], where the generating measure μ is compactly supported. Extended results are provided in [9] to involve measures μ with support bounded from one side (say from above). Further studies on CSK families are presented in [10–12]. In the following, we review some basic concepts on CSK families.

Let μ be a probability measure that is non-degenerate and has support bounded from above. Then

$$M_\mu(\vartheta) = \int \frac{1}{1-\vartheta y}\mu(dy) \tag{6}$$

converges $\forall\vartheta \in [0,\vartheta_+)$ with $1/\vartheta_+ = \max\{0, \sup \operatorname{supp}(\mu)\}$. For $\vartheta \in [0,\vartheta_+)$, we set

$$P_{(\vartheta,\mu)}(dy) = \frac{1}{M_\mu(\vartheta)(1-\vartheta y)}\nu(dy).$$

The (one-sided) CSK family generated by μ is the set of probability measures

$$\mathcal{K}_+(\mu) = \{P_{(\vartheta,\mu)}(dy);\ \ \vartheta \in (0,\vartheta_+)\}.$$

The mean function $\vartheta \mapsto k_\mu(\vartheta) = \int y P_{(\vartheta,\mu)}(dy)$ is strictly increasing on $(0,\vartheta_+)$, see [9], and

$$k_\mu(\vartheta) = \frac{M_\mu(\vartheta)-1}{\vartheta M_\mu(\vartheta)}. \tag{7}$$

For $\mathcal{K}_+(\mu)$, the (one-sided) mean domain is the interval $(m_0(\mu), m_+(\mu)) = k_\mu((0, \vartheta_+))$. This provides a mean parametrization for $\mathcal{K}_+(\mu)$: The inverse of $k_\mu(\cdot)$ is denoted $\psi_\mu(\cdot)$. For $s \in (m_0(\mu), m_+(\mu))$, consider $Q_{(s,\mu)}(dy) = P_{(\psi_\mu(s),\mu)}(dy)$. We get

$$\mathcal{K}_+(\mu) = \{Q_{(s,\mu)}(dy);\ \ s \in (m_0(\mu), m_+(\mu))\}.$$

Denote

$$B = B(\mu) = \max\{0, \sup \operatorname{supp}(\mu)\} = 1/\vartheta_+ \in [0, \infty) \tag{8}$$

and

$$A = A(\mu) = \min\{0, \inf supp(\mu)\}.$$

Then it is shown in [9] that

$$m_0(\mu) = \lim_{\vartheta \to 0^+} k_\mu(\vartheta) \quad \text{ and } \quad m_+(\mu) = B - \lim_{w \to B^+} \frac{1}{G_\mu(w)}. \tag{9}$$

If the measure μ has support bounded from below, the corresponding CSK family is denoted by $\mathcal{K}_-(\mu)$ and $\vartheta_- < \vartheta < 0$, where ϑ_- is either $1/A(\mu)$ or $-\infty$. For $\mathcal{K}_-(\mu)$, the mean domain is $(m_-(\mu), m_0(\mu))$ with $m_-(\mu) = A - 1/G_\mu(A)$. If μ is compactly supported, the (two-sided) CSK family is $\mathcal{K}(\mu) = \{\mu\} \cup \mathcal{K}_-(\mu) \cup \mathcal{K}_+(\mu)$ and $\vartheta \in (\vartheta_-, \vartheta_+)$.

The variance function (see [8]) is

$$s \mapsto \mathcal{V}_\mu(s) = \int (y - s)^2 Q_{(s,\mu)}(dy). \tag{10}$$

If μ does not have a moment of order 1, all members of $\mathcal{K}_+(\mu)$ have infinite variance. A concept of pseudo-variance function $\mathbb{V}_\mu(\cdot)$ is introduced in [9] by

$$\mathbb{V}_\mu(s) = s\left(\frac{1}{\psi_\mu(s)} - s\right), \tag{11}$$

If $m_0(\mu) = \int y\mu(dy)$ is finite, then (see [9]) $\mathcal{V}_\mu(\cdot)$ exists and

$$\mathbb{V}_\mu(s) = \frac{s}{s - m_0(\mu)} \mathcal{V}_\mu(s). \tag{12}$$

Let $\phi(\mu)$ be the image of μ by $\phi : y \longmapsto \xi y + \lambda$ where $\xi \neq 0$ and $\lambda \in \mathbb{R}$. Then, $\forall\, s$ close enough to $m_0(\phi(\mu)) = \phi(m_0(\mu)) = \xi m_0(\mu) + \lambda$,

$$\mathbb{V}_{\phi(\mu)}(s) = \frac{\xi^2 s}{s - \lambda} \mathbb{V}_\mu\left(\frac{s - \lambda}{\xi}\right). \tag{13}$$

If $\mathcal{V}_\mu(.)$ exists, then

$$\mathcal{V}_{\phi(\mu)}(s) = \xi^2 \mathcal{V}_\mu\left(\frac{s - \lambda}{\xi}\right). \tag{14}$$

3. $\boxplus_{(\mathbf{t})}$-Convolution and Variance Function

In this section, we establish the expression of the variance function under $\boxplus_{(\mathbf{t})}$-convolution power. To do so, we begin by presenting some results concerning the **t**-transformation of measures defined by (4). In the following, we assume that the considered measures are compactly supported. $\mathcal{M}_c$ will denote the set of compactly supported real probability measures. The next result concerns the mean function.

Proposition 1. *Let $\nu \in \mathcal{M}_c$ be non degenerate. Then, $\forall \vartheta$ is close enough to* 0,

$$k_{\widetilde{U}^t(\nu)}(\vartheta) = bk_\nu(\vartheta) - (b - a)m_0(\nu). \tag{15}$$

Proof. From the fact that $M_\nu(\vartheta) = \frac{1}{\vartheta} G_\nu(\frac{1}{\vartheta})$, we see from (4), that

$$M_{\widetilde{U}^{\mathbf{t}}(\nu)}(\vartheta) = \frac{M_\nu(\vartheta)}{b + ((1-b) + (b-a)\vartheta m_0(\nu))M_\nu(\vartheta)}. \tag{16}$$

We have that $M_{\widetilde{U}^{\mathbf{t}}(\nu)}(0) = 1$. The function $M_{\widetilde{U}^{\mathbf{t}}(\nu)}(.)$ is well defined in a small neighborhood of 0. Combining (7) with (16), we obtain

$$k_{\widetilde{U}^{\mathbf{t}}(\nu)}(\vartheta) = \frac{M_{\widetilde{U}^{\mathbf{t}}(\nu)}(\vartheta) - 1}{\vartheta M_{\widetilde{U}^{\mathbf{t}}(\nu)}(\vartheta)} = \frac{b(M_\nu(\vartheta) - 1)}{\vartheta M_\nu(\vartheta)} - (b-a)m_0(\nu) = bk_\nu(\vartheta) - (b-a)m_0(\nu).$$

□

Next, we establish the affect on $\mathcal{V}_\nu(\cdot)$ by applying $\mathbf{t}$-transformation to ν.

Theorem 1. *Let $\nu \in \mathcal{M}_c$ be non degenerate. Then, $\forall\ \widetilde{m}$ is close enough to $m_0(\widetilde{U}^{\mathbf{t}}(\nu)) = am_0(\nu)$,*

$$\mathbb{V}_{\widetilde{U}^t(\nu)}(\widetilde{m}) = \frac{b\widetilde{m}}{\widetilde{m} + (b-a)m_0(\nu)} \mathbb{V}_\nu\left(\frac{\widetilde{m} + (b-a)m_0(\nu)}{b}\right) + \widetilde{m}\left(\frac{\widetilde{m} + (b-a)m_0(\nu)}{b} - \widetilde{m}\right). \tag{17}$$

Furthermore,

$$\mathcal{V}_{\widetilde{U}^t(\nu)}(\widetilde{m}) = b\mathcal{V}_\nu\left(\frac{\widetilde{m} + (b-a)m_0(\nu)}{b}\right) + (\widetilde{m} - am_0(\nu))\left(\frac{\widetilde{m} + (b-a)m_0(\nu)}{b} - \widetilde{m}\right). \tag{18}$$

Proof. $\forall \vartheta$ is close enough to 0, which is denoted by $m = \int y P_{(\theta,\nu)}(dy)$ and $\widetilde{m} = \int y P_{(\theta,\widetilde{U}^{\mathbf{t}}(\nu))}(dy)$. From (15), one see that

$$\widetilde{m} = bm - (b-a)m_0(\nu), \tag{19}$$

and

$$m_0(\widetilde{U}^{\mathbf{t}}(\nu)) = k_{\widetilde{U}^{\mathbf{t}}(\nu)}(0) = bm_0(\nu) - (b-a)m_0(\nu) = am_0(\nu).$$

One see that $\forall\ \vartheta$ is close enough to 0,

$$\psi_{\widetilde{U}^{\mathbf{t}}(\nu)}(\widetilde{m}) = \vartheta = \psi_\nu(m). \tag{20}$$

In terms of pseudo-variance functions, relation (20) can be written as

$$\frac{\mathbb{V}_{\widetilde{U}^{\mathbf{t}}(\nu)}(\widetilde{m})}{\widetilde{m}} + \widetilde{m} = \frac{\mathbb{V}_\nu(m)}{m} + m. \tag{21}$$

From (19), we express m as a function of $\widetilde{m}$. Inserting it in (21), we obtain (17). Furthermore, as $m_0(\nu)$ is finite, then $\mathcal{V}_\nu(.)$ and $\mathcal{V}_{\widetilde{U}^{\mathbf{t}}(\nu)}(.)$ exists. Equation (18) follows from (17) and (12). □

Remark 1. *Note that Proposition 1 and Theorem 2 can be proven for the measure of ν with support bounded from one side and with the finite first moment.*

For $\nu \in \mathcal{M}_c$, consider the $\mathcal{R}^{(\mathbf{t})}$-transform introduced in [5], by

$$\mathcal{R}^{(\mathbf{t})}_\nu(w) := \frac{1}{b}\Big(\mathcal{R}_{\widetilde{U}^{\mathbf{t}}(\nu)}(w) + (b-a)m_0(\nu)\Big). \tag{22}$$

where

$$\mathcal{R}_\nu(G_\nu(w)) = w - \frac{1}{G_\nu(w)}, \qquad \forall\ w \text{ in an appropriate domain}, \tag{23}$$

see [13] for more details about $\mathcal{R}_\nu(\cdot)$. For $\mu,\ \nu \in \mathcal{M}_c$,

$$\mathcal{R}^{(\mathbf{t})}_{\mu \boxplus_{(\mathbf{t})} \nu}(w) = \mathcal{R}^{(\mathbf{t})}_{\mu}(w) + \mathcal{R}^{(\mathbf{t})}_{\nu}(w). \tag{24}$$

$\sigma \in \mathcal{M}_c$ is $\boxplus_{(\mathbf{t})}$-infinitely divisible, if for each $q \in \mathbb{N}$, $\sigma_q \in \mathcal{M}_c$ exists, so that

$$\sigma = \underbrace{\sigma_q \boxplus_{(\mathbf{t})} \ldots\ldots \boxplus_{(\mathbf{t})} \sigma_q}_{q \text{ times}}.$$

The r-fold $\boxplus_{(\mathbf{t})}$-convolution of $\sigma \in \mathcal{M}_c$ with itself is denoted $\sigma^{\boxplus_{(\mathbf{t})} r}$. This operation is well defined for $r \geq 1$, (see [14]) and

$$\mathcal{R}^{(\mathbf{t})}_{\sigma^{\boxplus_{(\mathbf{t})} r}}(w) = r\mathcal{R}^{(\mathbf{t})}_{\sigma}(w). \tag{25}$$

Proposition 2. *Let $\nu \in \mathcal{M}_c$ be non degenerate. Then,*

(i) $\mathcal{R}^{(t)}_\nu$ *is increasing strictly on* $(G_{\widetilde{U}^{\mathbf{t}}(\nu)}(A(\widetilde{U}^{\mathbf{t}}(\nu))), G_{\widetilde{U}^{\mathbf{t}}(\nu)}(B(\widetilde{U}^{\mathbf{t}}(\nu))))$.

(ii) *For* $m \in (m_-(\nu), m_+(\nu))$

$$\mathcal{R}^{(t)}_\nu\left(\frac{bm-(b-a)m_0(\nu)}{\mathbb{V}_{\widetilde{U}^{\mathbf{t}}(\nu)}(bm-(b-a)m_0(\nu))}\right) = m. \tag{26}$$

(iii) $\lim\limits_{w \searrow 0} \mathcal{R}^{(t)}_\nu(w) = m_0(\nu)$.

(iv) $\lim\limits_{w \searrow 0} z\mathcal{R}^{(t)}_\nu(w) = 0$.

Proof. The proof is based on the properties of $\mathcal{R}_{\widetilde{U}^{\mathbf{t}}(\nu)}(.)$, which are provided by considering measure $\widetilde{U}^{\mathbf{t}}(\nu)$ instead of measure μ in [9] (Proposition 3.8).

(i) One see from [9] (Proposition 3.8(i)), that $\mathcal{R}_{\widetilde{U}^{\mathbf{t}}(\nu)}(.)$ is increasing strictly on

$$(G_{\widetilde{U}^{\mathbf{t}}(\nu)}(A(\widetilde{U}^{\mathbf{t}}(\nu))), G_{\widetilde{U}^{\mathbf{t}}(\nu)}(B(\widetilde{U}^{\mathbf{t}}(\nu)))).$$

So, the proof of (i) follows easily from relation (22).

(ii) Combining (22), (19) and [9] (Proposition 3.8(ii)), we obtain

$$\begin{aligned}
\mathcal{R}^{(\mathbf{t})}_\nu\left(\frac{bm-(b-a)m_0(\nu)}{\mathbb{V}_{\widetilde{U}^{\mathbf{t}}(\nu)}(bm-(b-a)m_0(\nu))}\right) &= \mathcal{R}^{(\mathbf{t})}_\nu\left(\frac{\widetilde{m}}{\mathbb{V}_{\widetilde{U}^{\mathbf{t}}(\nu)}(\widetilde{m})}\right) \\
&= \frac{1}{b}\left(\mathcal{R}_{\widetilde{U}^{\mathbf{t}}(\nu)}\left(\frac{\widetilde{m}}{\mathbb{V}_{\widetilde{U}^{\mathbf{t}}(\nu)}(\widetilde{m})}\right) + (b-a)m_0(\nu)\right) \\
&= \frac{1}{b}(\widetilde{m} + (b-a)m_0(\nu)) = m.
\end{aligned}$$

(iii) Using (22) and [9] (Proposition 3.8(iii)), we have that

$$\begin{aligned}
\lim_{z \searrow 0} \mathcal{R}^{(\mathbf{t})}_\nu(z) &= \lim_{z \searrow 0} \frac{1}{b}\left(\mathcal{R}_{\widetilde{U}^{\mathbf{t}}(\nu)}(z) + (b-a)m_0(\nu)\right) \\
&= \frac{1}{b}\left(m_0(\widetilde{U}^{\mathbf{t}}(\nu)) + (b-a)m_0(\nu)\right) \\
&= \frac{1}{b}(am_0(\nu) + (b-a)m_0(\nu)) = m_0(\nu).
\end{aligned}$$

(iv) From (22) and [9] (Proposition 3.8(iv)), one see that

$$\lim_{z\searrow 0} z\mathcal{R}_{\nu}^{(\mathbf{t})}(z) = \lim_{z\searrow 0} \frac{1}{b}\Big(z\mathcal{R}_{\widetilde{\mathbb{U}}^{\mathbf{t}}(\nu)}(z) + z(b-a)m_0(\nu)\Big) = 0.$$

□

Next, the main result of this section is stated and demonstrated.

Theorem 2. *Let $\nu \in \mathcal{M}_c$ be non degenerate. Then, for $r > 0$ so that $\nu^{\boxplus_{(\mathbf{t})} r}$ is defined,*

(i) $\nu^{\boxplus_{(\mathbf{t})} r} \in \mathcal{M}_c$.

(ii) $\forall m$ *close enough to* $m_0\left(\nu^{\boxplus_{(\mathbf{t})} r}\right) = rm_0(\nu)$,

$$\mathbb{V}_{\nu^{\boxplus_{(\mathbf{t})} r}}(m) = r\mathbb{V}_{\nu}(m/r) + (1/r-1)m(m(1-b) + r(b-a)m_0(\nu)). \tag{27}$$

Furthermore,

$$\mathcal{V}_{\nu^{\boxplus_{(\mathbf{t})} r}}(m) = r\mathcal{V}_{\nu}(m/r) + (1/r-1)(m - rm_0(\nu))(m(1-b) + r(b-a)m_0(\nu)). \tag{28}$$

Proof.

(i) As $\nu \in \mathcal{M}_c$, then the measure $\widetilde{\mathbb{U}}^{\mathbf{t}}(\nu)$ is in $\mathcal{M}_c$. Thus, in a domain containing some open interval $(-\delta, \delta)$ for $\delta > 0$, the function

$$\mathcal{R}_{\nu}^{(\mathbf{t})}(.) = \frac{1}{b}\Big(\mathcal{R}_{\widetilde{\mathbb{U}}^{\mathbf{t}}(\nu)}(.) + (b-a)m_0(\nu)\Big)$$

is univalent. Therefore, in the same domain, the function

$$r\mathcal{R}_{\nu}^{(\mathbf{t})}(.) = \mathcal{R}^{(\mathbf{t})}{}_{\nu^{\boxplus_{(\mathbf{t})} r}}(.) = \frac{1}{b}\left(\mathcal{R}_{\widetilde{\mathbb{U}}^{\mathbf{t}}\left(\nu^{\boxplus_{(\mathbf{t})} r}\right)}(.) + (b-a)m_0\left(\nu^{\boxplus_{(\mathbf{t})} r}\right)\right)$$

is univalent. This implies that that $G_{\widetilde{\mathbb{U}}^{\mathbf{t}}\left(\nu^{\boxplus_{(\mathbf{t})} r}\right)}(w)$ and so $G_{\nu^{\boxplus_{(\mathbf{t})} r}}(w)$ is analytic for $|w| > c$, with $c = G^{-1}_{\widetilde{\mathbb{U}}^{\mathbf{t}}\left(\nu^{\boxplus_{(\mathbf{t})} r}\right)}(\delta)$. Then, $\nu^{\boxplus_{(\mathbf{t})} r} \in \mathcal{M}_c$, (see [15] (Proposition 6.1)).

(ii) From Proposition 2(iii), we see that

$$m_0\left(\nu^{\boxplus_{(\mathbf{t})} r}\right) = \lim_{z\longrightarrow 0} \mathcal{R}^{(\mathbf{t})}_{\nu^{\boxplus_{(\mathbf{t})} r}}(z) = r\lim_{z\longrightarrow 0} \mathcal{R}_{\nu}^{(\mathbf{t})}(z) = rm_0(\nu).$$

$\forall m$ is close enough to $rm_0(\nu)$, such that $m/r \in (m_-(\nu), m_+(\nu))$ and

$$\frac{bm - (b-a)m_0\left(\nu^{\boxplus_{(\mathbf{t})} r}\right)}{\mathbb{V}_{\widetilde{\mathbb{U}}^{\mathbf{t}}\left(\nu^{\boxplus_{(\mathbf{t})} r}\right)}\left(bm - (b-a)m_0\left(\nu^{\boxplus_{(\mathbf{t})} r}\right)\right)} \in (G_{\widetilde{\mathbb{U}}^{\mathbf{t}}(\nu)}(A(\widetilde{\mathbb{U}}^{\mathbf{t}}(\nu))), G_{\widetilde{\mathbb{U}}^{\mathbf{t}}(\nu)}(B(\widetilde{\mathbb{U}}^{\mathbf{t}}(\nu)))),$$

we have

$$\begin{aligned}
\mathcal{R}_{\nu}^{(\mathbf{t})}\left(\frac{bm - (b-a)m_0\left(\nu^{\boxplus_{(\mathbf{t})} r}\right)}{\mathbb{V}_{\widetilde{\mathbb{U}}^{\mathbf{t}}\left(\nu^{\boxplus_{(\mathbf{t})} r}\right)}\left(bm - (b-a)m_0\left(\nu^{\boxplus_{(\mathbf{t})} r}\right)\right)}\right) &= \frac{1}{r}\mathcal{R}^{(\mathbf{t})}_{\nu^{\boxplus_{(\mathbf{t})} r}}\left(\frac{bm - (b-a)m_0\left(\nu^{\boxplus_{(\mathbf{t})} r}\right)}{\mathbb{V}_{\widetilde{\mathbb{U}}^{\mathbf{t}}\left(\nu^{\boxplus_{(\mathbf{t})} r}\right)}\left(bm - (b-a)m_0\left(\nu^{\boxplus_{(\mathbf{t})} r}\right)\right)}\right) \\
&= \frac{m}{r} \\
&= \mathcal{R}_{\nu}^{(\mathbf{t})}\left(\frac{bm/r - (b-a)m_0(\nu)}{\mathbb{V}_{\widetilde{\mathbb{U}}^{\mathbf{t}}(\nu)}(bm/r - (b-a)m_0(\nu))}\right).
\end{aligned}$$

Recall Proposition 2(i), $\mathcal{R}_\nu^{(\mathbf{t})}(\cdot)$ is one-to-one on $(G_{\widetilde{U}^{\mathbf{t}}(\nu)}(A(\widetilde{U}^{\mathbf{t}}(\nu))), G_{\widetilde{U}^{\mathbf{t}}(\nu)}(B(\widetilde{U}^{\mathbf{t}}(\nu))))$. So

$$\frac{bm-(b-a)m_0\left(\nu^{\boxplus_{(\mathbf{t})}r}\right)}{\mathbb{V}_{\widetilde{U}^{\mathbf{t}}\left(\nu^{\boxplus_{(\mathbf{t})}r}\right)}\left(bm-(b-a)m_0\left(\nu^{\boxplus_{(\mathbf{t})}r}\right)\right)}=\frac{bm/r-(b-a)m_0(\nu)}{\mathbb{V}_{\widetilde{U}^{\mathbf{t}}(\nu)}(bm/r-(b-a)m_0(\nu))},$$

or equivalently

$$\mathbb{V}_{\widetilde{U}^{\mathbf{t}}\left(\nu^{\boxplus_{(\mathbf{t})}r}\right)}(bm-r(b-a)m_0(\nu))=r\mathbb{V}_{\widetilde{U}^{\mathbf{t}}(\nu)}(bm/r-(b-a)m_0(\nu)). \tag{29}$$

However, from (17), one can see that

$$\begin{aligned}\mathbb{V}_{\widetilde{U}^{\mathbf{t}}\left(\nu^{\boxplus_{(\mathbf{t})}r}\right)}&(bm-r(b-a)m_0(\nu))=\\ &\frac{b(bm-r(b-a)m_0(\nu))}{(bm-r(b-a)m_0(\nu))+(b-a)m_0\left(\nu^{\boxplus_{(\mathbf{t})}r}\right)}\mathbb{V}_{\nu^{\boxplus_{(\mathbf{t})}r}}\left(\frac{bm-r(b-a)m_0(\nu)+(b-a)m_0\left(\nu^{\boxplus_{(\mathbf{t})}r}\right)}{b}\right)\\ &+(bm-r(b-a)m_0(\nu))\left(\frac{bm-r(b-a)m_0(\nu)+(b-a)m_0\left(\nu^{\boxplus_{(\mathbf{t})}r}\right)}{b}-(bm-r(b-a)m_0(\nu))\right)\end{aligned}$$

That is

$$\begin{aligned}&\mathbb{V}_{\widetilde{U}^{\mathbf{t}}\left(\nu^{\boxplus_{(\mathbf{t})}r}\right)}(bm-r(b-a)m_0(\nu))=\\ &=\frac{bm-r(b-a)m_0(\nu)}{m}\mathbb{V}_{\nu^{\boxplus_{(\mathbf{t})}r}}(m)+(bm-r(b-a)m_0(\nu))(m(1-b)+r(b-a)m_0(\nu)).\end{aligned} \tag{30}$$

We also have, from (17)

$$\begin{aligned}&\mathbb{V}_{\widetilde{U}^{\mathbf{t}}(\nu)}(bm/r-(b-a)m_0(\nu))=\\ &\frac{bm-r(b-a)m_0(\nu)}{m}\mathbb{V}_\nu(m/r)+\frac{1}{r^2}(bm-r(b-a)m_0(\nu))(m(1-b)+r(b-a)m_0(\nu)).\end{aligned} \tag{31}$$

Combining (30) and (31) with (29), we obtain

$$\begin{aligned}\frac{bm-r(b-a)m_0(\nu)}{m}\mathbb{V}_{\nu^{\boxplus_{(\mathbf{t})}r}}(m)=&\frac{bm-r(b-a)m_0(\nu)}{m}r\mathbb{V}_\nu(m/r)\\ &+(1/r-1)(bm-r(b-a)m_0(\nu))(m(1-b)+r(b-a)m_0(\nu)),\end{aligned}$$

which is nothing but relation (27).

Furthermore, $\mathcal{V}_\nu(.)$ and $\mathcal{V}_{\nu^{\boxplus_{(\mathbf{t})}r}}(.)$ exists. Combining (27) with (12), we obtain (28). □

For $a=b=t>0$, the $\boxplus_{(\mathbf{t})}$-convolution is reduced to the $\boxed{t}$-convolution. We have the following corollary.

Corollary 1. *Let $\nu\in\mathcal{M}_c$ be non-degenerate. Then, for $r>0$, so that $\nu^{\boxed{t}\,r}$ is defined,*

(i) $\nu^{\boxed{t}\,r}\in\mathcal{M}_c$.

(ii) *$\forall m$ close enough to $m_0(\nu^{\boxed{t}\,r})=rm_0(\nu)$,*

$$\mathbb{V}_{\nu^{\boxed{t}\,r}}(m)=r\mathbb{V}_\nu(m/r)+m^2((1-t)/r+t-1). \tag{32}$$

In addition,

$$\mathcal{V}_{\nu^{\boxed{t}\,r}}(m) = r\mathcal{V}_{\nu}(m/r) + m(m - rm_0(\nu))((1-t)/r + t - 1). \tag{33}$$

4. Approximations in CSK Families Based on $\boxplus_{(\mathbf{t})}$-Convolution

4.1. Approximation of *t*-Deformed Free Gaussian CSK Family

As pointed in the introduction and according to [5] the $(\mathbf{t} = (a,b))$-deformed free Gaussian law is the same as the t-deformed free Gaussian law (or Kesten law), with $b = t$. According to [16], (see also [2]), the t-deformed free Gaussian law is provided by $\kappa_t = \widetilde{\kappa}_t + \widehat{\kappa}_t$ with

$$\widetilde{\kappa}_t(dy) = \frac{1}{2\pi}\frac{\sqrt{4t - y^2}}{1-(1-t)y^2}\mathbf{1}_{[-2\sqrt{t},2\sqrt{t}]}(y)dy$$

and for $t < 1/2$,

$$\widehat{\kappa}_t = \frac{1-2t}{2-2t}\left(\delta_{\frac{-1}{\sqrt{1-t}}} + \delta_{\frac{1}{\sqrt{1-t}}}\right).$$

Proposition 3. *$\forall m$ close enough to $m_0(\kappa_t) = 0$,*

$$\mathcal{V}_{\kappa_t}(m) = 1 + (t-1)m^2. \tag{34}$$

Proof. According to [16], (see also [1]), the Kesten distribution κ_t is related to the Wigner semi-circular distribution

$$SC(dy) = \frac{\sqrt{4-y^2}}{2\pi}\mathbf{1}_{(-2,2)}(y)dy,$$

by $\kappa_t = U_{1/t}(D_{\sqrt{t}}(SC))$. Note that $D_c(\nu)$ is the dilation of measure ν by $c \neq 0$. On the other hand, from [8] (Theorem 3.2), we have $\mathcal{V}_{SC}(m) = 1 = \mathbb{V}_{SC}(m)$, with $m_0(SC) = 0$.

Recall that the t-transformation of measures $U_t(\nu)$ is nothing but the t-th power of the Boolean additive convolution of ν. From [10] (Theorem 3.2), $\forall m$ in a neighborhood of $m_0(\kappa_t) = m_0(U_{1/t}(D_{\sqrt{t}}(SC))) = 0$, one see that

$$\mathcal{V}_{\kappa_t}(m) = \frac{1}{t}\mathcal{V}_{D_{\sqrt{t}}(SC)}(tm) + (t-1)m^2 = 1 + (t-1)m^2.$$

□

Next, an approximation is presented for elements of $\mathcal{K}(\kappa_t)$.

Theorem 3. *Let $\nu \in \mathcal{M}_c$ be non degenerate with a mean of 0. Then, there is $\varepsilon > 0$, such that if, for $r > 0$, the law of a random variable Y_r belonging to $\mathcal{K}(\nu_r)$ with $\nu_r = D_{1/r}\left(\nu^{\boxplus_{(t)}r}\right)$ and the mean of Y_r is equal to $m/\sqrt{r}$ with $|m| < \varepsilon$, then*

$$\sqrt{r}Y_r \xrightarrow{r\to+\infty} Q_{(m,\kappa_t)} \in \mathcal{K}(\kappa_t) \qquad \textit{in distribution.}$$

Proof. The law of the random variable Y_r is denoted by $\mathcal{L}(Y_r)$. As $\mathcal{L}(Y_r) \in \mathcal{K}(\nu_r)$ with

$$\mathcal{V}_{\nu_r}(m) = \mathcal{V}_{\nu^{\boxplus_{(\mathbf{t})}r}}(rm)/r^2 = \mathcal{V}_{\nu}(m)/r + (1/r-1)(1-t)m^2,$$

then $\mathcal{L}(\sqrt{r}Y_r)$ is in the CSK family with

$$\mathcal{V}_r(m) = r\mathcal{V}_{\nu_r}(m/\sqrt{r}) = \mathcal{V}_{\nu}(m/\sqrt{r}) + (1/r-1)(1-t)m^2.$$

We have

$$\mathcal{V}_r(m) \xrightarrow{r\to+\infty} \mathcal{V}_{\nu}(0) + (t-1)m^2.$$

Using [8] (Proposition 4.2), we conclude that there is $\varepsilon > 0$, such that if $|m| < \varepsilon$ and the mean of Y_α is equal to $m/\sqrt{r}$, then with $\mathcal{V}_v(0) = 1$, we have

$$\mathcal{L}(\sqrt{r}Y_r) \xrightarrow{r\to+\infty} Q_{(m,\kappa_t)} \in \mathcal{K}(\kappa_t) \text{ in distribution.}$$

For $m = 0$, we obtained the central limit theorem with respect to $\boxplus_{(\mathbf{t})}$-convolution. □

4.2. *Approximation of* **t***-Deformed Free Poisson CSK Family*

According to [5], the (a, b)-transformed free Poisson law p_α with a mean $m_0(p_\alpha) = \alpha > 0$ is provided by $p_\alpha = p_\alpha^C + p_\alpha^D$. The continuous part is

$$p_\alpha^C(dy) = \frac{\sqrt{4b\alpha - (y - a\alpha - 1)^2}}{2\pi f(y)} \mathbf{1}_{[1+a\alpha-2\sqrt{b\alpha},1+a\alpha+2\sqrt{b\alpha}]}(dy),$$

with $f(y) = (b-1)y^2 + (\alpha + a\alpha + 1 - 2b\alpha)y - (a-b)\alpha^2$. p_α^D is 0 except possibly for the following cases:

Case 1: $f(y)$ has two real roots, y_1 and y_2. Then,

$$p_\alpha^D(dy) = w_1\delta_{y_1} + w_2\delta_{y_2},$$

where

$$w_i = \frac{1}{\sqrt{(\alpha - a\alpha - 1)^2 - 4\alpha(b-1)}} \times \max\left\{0, \frac{\alpha}{|y_i - (\alpha - a\alpha - 1)|} - b|y_i - (\alpha - a\alpha - 1)|\right\}.$$

In this case, the parameters should satisfy

$$(\alpha + 1)^2 + a\alpha(a\alpha - 2\alpha + 2) - 4\alpha b > 0,$$

and two real roots can be provided by

$$y_i = \frac{2b\alpha - \alpha - a\alpha - 1 \pm \sqrt{(\alpha+1)^2 + a\alpha(a\alpha - 2\alpha + 2) - 4\alpha b}}{2(b-1)}.$$

Case 2: $b = 1$ and $\alpha \neq a\alpha + 1$ so that $f(y)$ has one real root $y_3 = \alpha + \frac{\alpha}{\alpha - a\alpha - 1}$. Then,

$$p_\alpha^D(dx) = \max\left(0, 1 - \frac{b\alpha}{(\alpha - a\alpha - 1)^2}\right)\delta_{y_3}.$$

The **t**-deformed free Poisson law appears in the free probability as the limiting law of repeated $\boxplus_{(\mathbf{t})}$-convolution of measures of the form

$$\mu_K = \left(1 - \frac{\alpha}{K}\right)\delta_0 + \frac{\alpha}{K}\delta_1, \quad \text{for} \quad K = 1, 2, 3, \ldots. \text{ and } 0 < \alpha < K.$$

In other words,

$$\underbrace{\mu_K \boxplus_{(\mathbf{t})} \mu_K \ldots\ldots \boxplus_{(\mathbf{t})} \mu_K}_{K \text{ times}} \xrightarrow{K\to+\infty} p_\alpha \quad \text{in distribution.}$$

Proposition 4. *$\forall m$ close enough to $m_0(p_\alpha) = \alpha$,*

$$\mathcal{V}_{p_\alpha}(m) = m - (m - \alpha)(m(1-b) + (b-a)\alpha). \tag{35}$$

Proof. According to [5], we have

$$\mathcal{R}_{p_\alpha}^{(\mathbf{t})}(w) = \frac{\alpha}{1-w}. \tag{36}$$

Combining (36) with (26), $\forall m$ close enough to $m_0(p_\alpha) = \alpha$, we obtain

$$\frac{\alpha}{1 - \left(\frac{bm-(b-a)m_0(p_\alpha)}{\mathbb{V}_{\widetilde{U}^{\mathbf{t}}(p_\alpha)}(bm-(b-a)m_0(p_\alpha))} \right)} = m.$$

That is,

$$\mathbb{V}_{\widetilde{U}^{\mathbf{t}}(p_\alpha)}(bm-(b-a)\alpha) = \frac{m(bm-(b-a)\alpha)}{m-\alpha}. \tag{37}$$

One see from (17) and (37) that

$$\mathbb{V}_{p_\alpha}(m) = \frac{m}{m-\alpha}[m-(m-\alpha)(m(1-b)+(b-a)\alpha)]. \tag{38}$$

Combining (12) with (38), the expression of $\mathcal{V}_{p_\alpha}(.)$ is provided by (35). □

Now, an approximation is presented for members of $\mathcal{K}(p_\alpha)$.

Theorem 4. *For $K = 1, 2, 3, \ldots$ and $0 < \alpha < K$, let*

$$\mu_K = \left(1 - \frac{\alpha}{K}\right)\delta_0 + \frac{\alpha}{K}\delta_1.$$

Then, $\forall m$ in a neighborhood of α

$$Q_{\left(m, \mu_K^{\boxplus_{(\mathbf{t})} K}\right)} \xrightarrow{K \to +\infty} Q_{(m, p_\alpha)}, \quad \textit{in distribution},$$

Proof. We know from [11] (p. 878) that $\forall x$ in a small neighborhood of $m_0(\mu_K) = \alpha/K$,

$$\mathcal{V}_{\mu_K}(x) = x(1-x). \tag{39}$$

We have $m_0\left(\mu_K^{\boxplus_{(\mathbf{t})} K}\right) = \alpha = m_0(p_\alpha)$. Then, $\varepsilon > 0$ exists, so that

$$\left(m_-\left(\mu_K^{\boxplus_{(\mathbf{t})} K}\right), m_+\left(\mu_K^{\boxplus_{(\mathbf{t})} K}\right)\right) \cap (m_-(p_\alpha), m_+(p_\alpha)) = (\alpha - \varepsilon, \alpha + \varepsilon).$$

$\forall m \in (\alpha - \varepsilon, \alpha + \varepsilon)$

$$\int y Q_{\left(m, \mu_K^{\boxplus_{(\mathbf{t})} K}\right)}(dy) = m = \int y Q_{(m, p_\alpha)}(dy).$$

Using (28) and (39), $\forall m \in (\lambda - \varepsilon, \lambda + \varepsilon)$, we obtain

$$\begin{aligned}
\mathcal{V}_{\mu_K^{\boxplus_{(\mathbf{t})} K}}(m) &= K\mathcal{V}_{\mu_K}\left(\frac{m}{K}\right) + (1/K - 1)(m - Km_0(\mu_K))(m(1-b) + Km_0(\mu_K)(b-a)) \\
&= m\left(1 - \frac{m}{K}\right) + (1/K - 1)(m-\alpha)(m(1-b) + (b-a)\alpha). \\
&\xrightarrow{K \to +\infty} m - (m-\alpha)(m(1-b) + (b-a)\alpha) = \mathcal{V}_{p_\alpha}(m).
\end{aligned}$$

By using [8] (Proposition 4.2) applied to $Q_{\left(m,\mu_K^{\boxplus_{(\mathbf{t})}K}\right)}$, we obtain

$$Q_{\left(m,\mu_K^{\boxplus_{(\mathbf{t})}K}\right)} \xrightarrow{K\to+\infty} Q_{(m,p_\alpha)} \quad \text{in the distribution,} \qquad \forall\, m \in (\alpha-\varepsilon, \alpha+\varepsilon).$$

For $m = \alpha$, we recover the Poisson limit theorem with respect to $\boxplus_{(\mathbf{t})}$-convolution, i.e., $\mu_K^{\boxplus_{(\mathbf{t})}K} \xrightarrow{K\to+\infty} p_\alpha$, in distribution.

$\square$

5. A Limit Theorem Related to $\boxplus_{(\mathbf{t})}$-Convolution

$\mathcal{M}^+$ will denote the set of probability measures on $\mathbb{R}_+$. Let $\rho \in \mathcal{M}^+$, $(\rho \neq \delta_0)$. The $\mathbb{S}$-transform is introduced by

$$\mathcal{R}_\rho(\xi\mathbb{S}_\rho(\xi)) = \frac{1}{\mathbb{S}_\rho(\xi)} \qquad \forall\, \xi \text{ in a neighborhood of } 0.$$

Multiplication of $\mathbb{S}$-transforms remains an $\mathbb{S}$-transform. For $\rho_1, \rho_2 \in \mathcal{M}^+$, the multiplicative free convolution $\rho_1 \boxtimes \rho_2$ is defined by $\mathbb{S}_{\rho_1\boxtimes\rho_2}(\xi) = \mathbb{S}_{\rho_1}(\xi)\mathbb{S}_{\rho_2}(\xi)$. Powers of multiplicative free convolution $\rho^{\boxtimes p}$ are well defined, (at least) $\forall p \geq 1$, by $\mathbb{S}_{\rho^{\boxtimes p}}(\xi) = \mathbb{S}_\rho(\xi)^p$, see [17] (Theorem 2.17) for more details.

Next, involving the free multiplicative convolution, a limit theorem is provided for the $\boxplus_{(\mathbf{t})}$-convolution. $\mathcal{M}_c^+$ will denote the set of compactly supported measures on $\mathbb{R}_+$.

Theorem 5. *Let $\nu \in \mathcal{M}_c^+$ be non degenerate. Denoting $\gamma = \frac{Var(\nu)}{(m_0(\nu))^2} = \frac{\mathcal{V}_\nu(m_0)}{m_0^2}$, then*

$$D_{1/(qm_0^q)}\left(\nu^{\boxtimes q}\right)^{\boxplus_{(t)}q} \xrightarrow{q\to+\infty} \tau_\gamma \qquad \textit{in distribution,}$$

with

$$\mathcal{V}_{\tau_\gamma}(m) = \frac{\gamma m(m-1)}{\ln(m)} + (m-1)((b-1)m + (a-b)) \tag{40}$$

$\forall m$ in a small neighborhood of $m_0(\tau_\gamma) = 1$.

Proof. Using [12] (Theorem 2.4 (i)) and Theorem 2(ii) , we obtain

$$m_0\left(D_{1/(qm_0(\nu)^q)}\left(\nu^{\boxtimes q}\right)^{\boxplus_{(t)}q}\right) = \frac{m_0\left(\left(\nu^{\boxtimes q}\right)^{\boxplus_{(t)}q}\right)}{(qm_0(\nu)^q)} = \frac{(qm_0(\nu^{\boxtimes q}))}{(qm_0(\nu)^q)} = 1.$$

Combining [12] (Theorem 2.4 (ii)) and (28), $\forall m$ close enough to 1, we obtain

$$
\begin{aligned}
\mathcal{V}_{D_{1/(qm_0^q)}(\nu^{\boxtimes q})^{\boxplus_{(\mathbf{t})}q}}(m) &= \frac{1}{q^2 m_0^{2q}} \mathcal{V}_{(\nu^{\boxtimes q})^{\boxplus_{(\mathbf{t})}q}}\left(qmm_0^q\right) \\
&= \frac{1}{q^2 m_0^{2q}} \Big\{ q\mathcal{V}_{\nu^{\boxtimes q}}\left(mm_0^q\right) \\
&+ (1/q-1)(qmm_0^q - qm_0(\nu^{\boxtimes q}))\left(qmm_0^q(1-b) + q(b-a)m_0(\nu^{\boxtimes q})\right)\Big\} \\
&= \frac{1}{qm_0^{2q}} \mathcal{V}_{\nu^{\boxtimes q}}(mm_0^q) + (1/q-1)(m-1)(m(1-b)+b-a) \\
&= \frac{(mm_0^q - m_0^q)m^{1-1/q}m_0^{q-1}\mathcal{V}_\nu\left(m^{1/q}m_0\right)}{qm_0^{2q}[(mm_0^q)^{1/q} - m_0]} \\
&+ (1/q-1)(m-1)(m(1-b)+b-a). \\
&= \frac{(m-1)m^{1-1/q}}{m_0^2 \frac{m^{1/q}-1}{1/q}} \mathcal{V}_\nu\left(m^{1/q}m_0\right) + (1/q-1)(m-1)(m(1-b)+b-a). \\
&\xrightarrow{q\to+\infty} \frac{m(m-1)}{m_0^2 \ln(m)} \mathcal{V}_\nu(m_0) + (m-1)((b-1)m + (a-b)).
\end{aligned}
$$

Recall [8] (Proposition 4.2), the previous calculations implies that

$$
D_{1/(qm_0^q)}\left(\nu^{\boxtimes q}\right)^{\boxplus_{(\mathbf{t})}q} \xrightarrow{q\to+\infty} \tau_\gamma \qquad \text{in distribution,}
$$

with

$$
\mathcal{V}_{\tau_\gamma}(m) = \frac{m(m-1)}{m_0^2 \ln(m)} \mathcal{V}_\nu(m_0) + (m-1)((b-1)m + (a-b)) = \frac{\gamma m(m-1)}{\ln(m)} + (m-1)((b-1)m + (a-b)),
$$

and $m_0(\tau_\gamma) = m_0\left(D_{1/(qm_0(\nu)^q)}(\nu^{\boxtimes q})^{\boxplus_{(\mathbf{t})}q}\right) = 1.$ □

Remark 2. *The free cumulants $r_n = r_n(\tau_\gamma)$, $n = 1, 2, ...,$ of the measure τ_γ can be obtained from the expression of the variance function provided by* (40) *and* [8] *(formula (3.12)). That is $r_1(\tau_\gamma) = m_0(\tau_\gamma) = 1$ and for all $n \geq 1$*

$$
r_{n+1}(\tau_\gamma) = \frac{1}{n!} \frac{d^{n-1}}{dm^{n-1}} \left(\mathcal{V}_{\tau_\gamma}(m)\right)^n \Big|_{m=1}.
$$

One can see that the variance of the measure τ_γ is $r_2(\tau_\gamma) = \gamma$. Furthermore, after some calculations we obtain $r_3(\tau_\gamma) = \gamma(\frac{3}{2}\gamma + a - 1)$ and $r_4(\tau_\gamma) = \gamma(\frac{8}{3}\gamma^2 + (b + 3a - 4)\gamma + (a-1)^2)$.

6. Conclusions

The notion of $\boxplus_{(\mathbf{t})}$-convolution, is defined in [5] as a generalization of the original t-transformation of free convolution introduced in [1,2]. The central limit theorem with respect to $\boxplus_{(\mathbf{t})}$-convolution is provided and the **t**-deformed free Poisson measure is calculated in [5]. Further results related to $\boxplus_{(\mathbf{t})}$-convolution are presented in [5]. The goal of this article is to study of the notion of $\boxplus_{(\mathbf{t})}$-convolution from the perspective of CSK families, which has been recently introduced in [8,9]. A fundamental concept for CSK families is given by the variance function. An expression is provided for the variance function under $\boxplus_{(\mathbf{t})}$-convolution power. This expression is used to approximate elements of the **t**-deformed free Gaussian CSK family and elements of the **t**-deformed free Poisson CSK family. Furthermore, involving the free multiplicative convolution, a new limit theorem is proven with respect to $\boxplus_{(\mathbf{t})}$-convolution.

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