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On Darbo- and Sadovskii-Type Fixed Point Theorems in Banach Spaces

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Abstract: The paper aims to generalize several known Darbo- and Sadovskii-type fixed point theorems. These generalizations weaken the assumptions used so far. In addition, an example of an application is presented.

Keywords: convex-power mappings; Darbo- and Sadovskii-type fixed point theorem; measure of noncompactness

MSC: 47H08; 47H10

1. Introduction

The first and most widely used fixed point theorems, expressed in terms of measures of noncompactness, are the theorems of Darbo [1] and Sadovskii [2]. Recently, these theorems have lived to see generalizations formulated in terms of iterated or convex-power mappings. The value of these generalizations for studying the solvability of various types of equations is that there exist equations that generate operators that do not satisfy the assumptions of Darbo's or Sadowskii's theorems but do satisfy the assumptions of these generalized theorems. One of the main assumptions of these generalizations is the condition that for a continuous operator $T : \Omega \to \Omega$, there exist a number $k \in [0, 1)$ and a number $n \in \mathbb{N}$ such that for every nonempty subset $X \subset \Omega$, the condition for power condensing operators

$$\mu(\widetilde{T}^n(X)) \leq k\mu(X) \tag{1}$$

or the condition for convex-power condensing operators

$$\mu(T^{(n,x_0)}(X)) \leq k\mu(X), \tag{2}$$

where μ is a measure of noncompactness, is satisfied. The definitions of the above symbols are given in (5) and (7). In this paper, we show that these assumptions can be weakened by allowing in case (1) that the number $n = n_X$ can depend on the set X, and in case (2), even both numbers $n = n_X$ and $k = k_X$ can depend on subset $X \subset \Omega$.

This paper is organized as follows: in Section 2, we give some concepts and results about measures of noncompactness and Darbo and Sadovskii fixed point theorems; the main results are provided in Section 3; an example that illustrates our results is presented in Section 4.

2. Notation, Definitions and Auxiliary Facts

Let E be a real Banach space with a norm $\|\cdot\|$. If X is a subset of E, then the symbol $\overline{\text{co}}(X)$ stands for the closure of the convex hull of X. We will denote by $\mathfrak{M}_E$ the family of all nonempty and bounded subsets of E and by $\mathfrak{N}_E$ its subfamily consisting of all relatively compact sets. According to [3], we accept the following definition of a measure of noncompactness (some other definitions are also considered in [4]).

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Definition 1. *A mapping $\mu : \mathfrak{M}_E \to \mathbb{R}_+ := [0, \infty)$ is said to be a measure of noncompactness on a Banach space E if it satisfies the following properties:*

(i) *Regularity: the family $\ker(\mu) := \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ is nonempty and $\ker(\mu) \subseteq \mathfrak{N}_E$.*
(ii) *Monotonicity: if $X \subseteq Y$, then $\mu(X) \leq \mu(Y)$.*
(iii) *The weak maximum property: for every $x \in E$ and $X \in \mathfrak{M}_E$, $\mu(X \cup \{x\}) = \mu(X)$.*
(iv) *Invariant under the closed convex hull: $\mu(\overline{\mathrm{co}}(X)) = \mu(X)$.*

Remark 1. *It can be shown that conditions $(i) - (iii)$ imply (v).*

(v) *Generalized Cantor's intersection theorem: if $\{X_n\}_{n\in\mathbb{N}}$ is a sequence of closed sets from $\mathfrak{M}_E$ such that $X_{n+1} \subseteq X_n$ for all $n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} \mu(X_n) = 0$, then $X_\infty := \bigcap\limits_{n=1}^{\infty} X_n$ is nonempty and compact.*

The two simplest fixed point theorems, expressed in terms of measures of noncompactess, are Darbo's and Sadovskii's theorems.

Theorem 1 (Darbo [1])**.** *Let μ be a measures of noncompactness in a real Banach space E. Assume that Ω is a nonempty, closed, bounded and convex subset of E, a mapping $T : \Omega \to \Omega$ is continuous, and there is a constant $k \in [0, 1)$ such that for any nonempty subset $X \subset \Omega$*

$$\mu(T(X)) \leq k\mu(X). \tag{3}$$

Then T has at least one fixed point in Ω.

Theorem 2 (Sadovskii [2])**.** *Let μ be a measures of noncompactness in a real Banach space E. Assume that Ω is a nonempty, closed, bounded and convex subset of E, a mapping $T : \Omega \to \Omega$ is continuous, and for any nonempty subset $X \subset \Omega$ such that $\mu(X) > 0$*

$$\mu(T(X)) < \mu(X). \tag{4}$$

Then T has at least one fixed point in Ω.

Sadovskii's theorem is marginally broader then Darbo's. These theorems have been generalized and expressed using different versions of the power operators. Let us start by establishing the appropriate symbols. Firstly, we assume that $T : \Omega \to \Omega$, where $\Omega \subset E$. For $X \subset \Omega$, we put

$$\widetilde{T}^1(X) := T(X), \quad \widetilde{T}^n(X) := T(\overline{\mathrm{co}}(\widetilde{T}^{n-1}(X))), \quad n = 2, 3, \ldots . \tag{5}$$

We can now quote one of the most important generalizations of Darbo's theorem.

Theorem 3 ([5])**.** *Let μ be a measure of noncompactness in a real Banach space E. Assume that Ω is a nonempty, closed, bounded and convex subset of E, a mapping $T : \Omega \to \Omega$ is continuous, and there exist a constant $k \in [0, 1)$ and a number $n \in \mathbb{N}$ such that for any nonempty subset $X \subset \Omega$*

$$\mu(\widetilde{T}^n(X)) \leq k\mu(X). \tag{6}$$

Then T has at least one fixed point in Ω.

Remark 2. *This theorem is, of course, a generalization of Darbo's theorem 1, and its advantage is that there are operators that do not satisfy the condition (3) but satisfy the condition (6) for a suitably large n, and then, when examining the solvability of various equations, weaker assumptions guaranteeing solvability are possible.*

Generalizations in a slightly different direction are also possible. For $T : \Omega \to \Omega$, $\Omega \subset E$, $x_0 \in \Omega$, $X \subset \Omega$ let us denote the so-called *convex-power operator*.

$$T^{(1,x_0)}(X) := T(X), \quad T^{(n,x_0)}(X) := T(\overline{\mathrm{co}}(T^{(n-1,x_0)}(X) \cup \{x_0\})), \tag{7}$$

for $n = 2, 3, \ldots$

Theorem 4 ([6])**.** *Let μ be a measure of noncompactness in a real Banach space E. Assume that Ω is a nonempty, closed, bounded and convex subset of E, a mapping $T : \Omega \to \Omega$ is continuous and there exist $x_0 \in \Omega$, a constant $k \in [0, 1)$ and a number $n \in \mathbb{N}$ such that for any nonempty subset $X \subset \Omega$*

$$\mu(T^{(n,x_0)}(X)) \le k\mu(X). \tag{8}$$

Then T has at least one fixed point in Ω.

Obviously, Theorem 3 is slightly more general than Theorem 4. There is also a generalization of Sadovskii's theorem in these terms.

Theorem 5 ([6])**.** *Let μ be a measure of noncompactness in a real Banach space E. Assume that Ω is a nonempty, closed, bounded and convex subset of E, a mapping $T : \Omega \to \Omega$ is continuous and there exist $x_0 \in \Omega$ and a number $n \in \mathbb{N}$ such that for any nonempty subset $X \subset \Omega$ with $\mu(X) > 0$*

$$\mu(T^{(n,x_0)}(X)) < \mu(X). \tag{9}$$

Then T has at least one fixed point in Ω.

3. Main Results

In Theorem 3, it is possible to weaken the assumptions by assuming that the value n is not necessarily the same for all nonempty subsets of $X \subset \Omega$ but may be different for them. The first result of this paper presents this fact in more detail.

Theorem 6. *Let μ be a measure of noncompactness in a real Banach space E. Assume that Ω is a nonempty, closed, bounded and convex subset of E, a mapping $T : \Omega \to \Omega$ is continuous, and there exists a constant $k \in [0, 1)$ such that for any nonempty subset $X \subset \Omega$ there is a number $n = n_X \in \mathbb{N}$ such that*

$$\mu(\widetilde{T}^n(X)) \le k\mu(X). \tag{10}$$

Then T has at least one fixed point in Ω.

Proof. In the beginning, let us observe that

$$\widetilde{T}^{k+l}(X) = \widetilde{T}^k(\overline{\mathrm{co}}(\widetilde{T}^l(X))) \tag{11}$$

for $k, l \in \mathbb{N}$ and $X \subseteq \Omega$. Moreover, for $k < l$,

$$\widetilde{T}^l(\Omega) \subset \widetilde{T}^k(\Omega). \tag{12}$$

This is because $\widetilde{T}^1(\Omega) \subset \Omega$; hence, $\overline{\mathrm{co}}(\widetilde{T}^1(\Omega)) \subset \overline{\mathrm{co}}(\Omega) \subset \Omega$, and therefore, $T(\overline{\mathrm{co}}(\widetilde{T}^1(\Omega))) \subset T(\Omega)$, which means that $\widetilde{T}^2(\Omega) \subset \widetilde{T}^1(\Omega)$. Reasoning similarly, we get (12) by induction.

We put $Y_0 := \Omega$. Let $n_1 \in \mathbb{N}$ be such that $\mu(\widetilde{T}^{n_1}(Y_0)) \le k\mu(Y_0)$. We denote $Y_1 := \overline{\mathrm{co}}(\widetilde{T}^{n_1}(Y_0))$. Next, let $n_2 \in \mathbb{N}$ be such that $\mu(\widetilde{T}^{n_2}(Y_1)) \le k\mu(Y_1)$. Let us denote $Y_2 := \overline{\mathrm{co}}(\widetilde{T}^{n_2}(Y_1))$. Continuing this procedure, we obtain a sequence $\{n_i\}_{i=1}^{\infty} \subset \mathbb{N}$ and a second sequence $\{Y_i\}_{i=0}^{\infty}$ such that

$$\mu(\widetilde{T}^{n_i}(Y_{i-1})) \le k\mu(Y_{i-1}), \;\; i = 1, 2, \ldots \tag{13}$$

and

$$Y_i := \overline{\mathrm{co}}(\widetilde{T}^{n_i}(Y_{i-1})), \ \ i = 1, 2, \dots. \tag{14}$$

Let us put $s_i := n_1 + \ldots + n_i, \ \ i = 1, 2, \dots$. In view of (11), we get

$$Y_i = \overline{\mathrm{co}}(\widetilde{T}^{s_i}(Y_0)). \tag{15}$$

Condition (12) implies that $\{Y_i\}$ is the descending sequence of closed sets. Let us put

$$Y := \cap_{i=0}^{\infty} Y_i.$$

Linking (13) and (14), we yield

$$\mu(Y_i) = \mu(\widetilde{T}^{n_i}(Y_{i-1})) \leq k\mu(Y_{i-1}) = k\mu(\widetilde{T}^{n_{i-1}}(Y_{i-2}))$$
$$\leq k^2\mu(Y_{i-2}) \leq \ldots \leq k^i\mu(Y_0) \underset{i\to\infty}{\longrightarrow} 0.$$

Keeping in mind (v) from Remark 1, we get that Y is nonempty and compact. Finally, we need to prove that $T(Y) \subset Y$. First, we show the inclusion $T(Y_i) \subset Y_i$. From (12), we get $\widetilde{T}^{s_i}(Y_0) \subset \widetilde{T}^{s_i-1}(Y_0)$, and this together with (15) gives

$$T(Y_i) = T(\overline{\mathrm{co}}(\widetilde{T}^{s_i}(Y_0))) \subset T(\overline{\mathrm{co}}(\widetilde{T}^{s_i-1}(Y_0))) = \widetilde{T}^{s_i}(Y_0) \subset \overline{\mathrm{co}}(\widetilde{T}^{s_i}(Y_0)) = Y_i.$$

Furthermore, we have

$$T(Y) = T(\cap_{i=0}^{\infty} Y_i) \subset \cap_{i=0}^{\infty} T(Y_i) \subset \cap_{i=0}^{\infty} Y_i = Y.$$

In order to obtain the conclusion, Schauder's theorem must be applied to $T : Y \to Y$. □

Remark 3. *By reasoning similarly to as above, one can also show that Theorem 6 will also be true when* $\widetilde{T}^n(X)$ *is defined as follows:*

$$\widetilde{T}^1(X) := \overline{\mathrm{co}}(T(X)), \quad \widetilde{T}^n(X) := \overline{\mathrm{co}}(T(\widetilde{T}^{n-1}(X))), \quad n = 2, 3, \dots.$$

A natural question arises to which the answer is unknown: whether Theorem 6 above will be true under the weakened assumption that for every nonempty $X \subset \Omega$, there are both $k = k_X \in [0, 1)$ and $n = n_X \in \mathbb{N}$ such that (10) holds. It is not difficult to show that the question in this form is equivalent to the problem below.

Question 1. *Let μ be a measure of noncompactness in a real Banach space E. Assume that Ω is a nonempty, closed, bounded and convex subset of E, and $T : \Omega \to \Omega$ is a continuous mapping such that for any nonempty subset $X \subset \Omega$ with $\mu(X) > 0$ there is a number $n \in \mathbb{N}$ such that*

$$\mu(\widetilde{T}^n(X)) < \mu(X). \tag{16}$$

Then T has at least one fixed point in Ω.

The authors also do not known if the following is true.

Question 2. *Let μ be a measure of noncompactness in a real Banach space E. Assume that Ω is a nonempty, closed, bounded and convex subset of E, $T : \Omega \to \Omega$ is a continuous mapping, and there exist a constant $k \in [0, 1)$ and a number $n \in \mathbb{N}$ such that for any nonempty subset $X \subset \Omega$,*

$$\mu(T^n(X)) \leq k\mu(X). \tag{17}$$

Then T has at least one fixed point in Ω.

Darbo's theorem has also been generalized in another form in which condition (3) has been replaced with the inequality

$$\mu(T(X)) \leq g(\mu(X))\mu(X) \tag{18}$$

or others, in a similar form, where g is a function satisfying appropriate conditions. In these terms, one can also formulate generalizations of Darbo's theorem in a power mapping version.

We denote by $\mathcal{G}$ the set of all functions $g : \mathbb{R}_+ \to [0,1)$ such that if $\{t_n\}$ is a monotone decreasing sequence in $(0,\infty)$ and $g(t_n) \to 1$, then $t_n \to 0$. Note that this class of functions was introduced by Geraghty [7] in order to get an extension of Banach contraction principle. Obviously, the function $g(t) := k,\ k \in [0,1)$ belongs to $\mathcal{G}$. Using these concepts, we can formulate an "intermediate" theorem between Theorem 6 and the theorem from Question 1. Its generality in relation to known and similar statements of this type lies in the fact that the value of $n = n_X$ may depend on the nonempty subset $X \subset \Omega$, while in previously known theorems, it had to be a constant and invariant value for all nonempty subsets $X \subset \Omega$ (see, e.g., [8]).

Theorem 7. *Let μ be a measure of noncompactness in a real Banach space E. Assume that Ω is a nonempty, closed, bounded and convex subset of E, a mapping $T : \Omega \to \Omega$ is continuous, and there exists a function $g \in \mathcal{G}$ such that for any nonempty subset $X \subset \Omega$ there is a number $n = n_X \in \mathbb{N}$ such that*

$$\mu(\widetilde{T}^n(X)) \leq g(\mu(X))\mu(X), \tag{19}$$

then T has at least one fixed point in C.

Proof. Due to the fact that the proof is similar to the proof of Theorem 6, we will give it in an abbreviated form.

First, we put $Y_0 := \Omega$. Let $n_1 \in \mathbb{N}$ be such that $\mu(\widetilde{T}^{n_1}(Y_0)) \leq g(\mu(Y_0))\mu(Y_0)$. We put $Y_1 := \overline{\text{co}}(\widetilde{T}^{n_1}(Y_0))$. Let $n_2 \in \mathbb{N}$ satisfy the condition $\mu(\widetilde{T}^{n_2}(Y_1)) \leq g(\mu(Y_1))\mu(Y_1)$. Let us set $Y_2 := \overline{\text{co}}(\widetilde{T}^{n_2}(Y_1))$. Continuing this procedure, we get a sequence $\{n_i\}_{i=1}^{\infty} \subset \mathbb{N}$ and a second sequence $\{Y_i\}_{i=0}^{\infty}$ such that

$$Y_i := \overline{\text{co}}(\widetilde{T}^{n_i}(Y_{i-1})),\ \ i = 1, 2, \ldots \tag{20}$$

and

$$\mu(\widetilde{T}^{n_i}(Y_{i-1})) \leq g(\mu(Y_{i-1}))\mu(Y_{i-1}),\ \ i = 1, 2, \ldots$$

that is equivalently

$$\mu(Y_i) \leq g(\mu(Y_{i-1}))\mu(Y_{i-1}),\ \ i = 1, 2, \ldots \tag{21}$$

Condition (21) and the properties of the function g imply that the sequence $\{\mu(Y_i)\}$ is decreasing, so it has a limit $l \geq 0$. We will show that $l = 0$. If $l > 0$, then from (21), we would have the estimation

$$1 \underset{i\to\infty}{\longleftarrow} \frac{\mu(Y_i)}{\mu(Y_{i-1})} \leq g(\mu(Y_{i-1})) < 1,$$

and this together with the properties of g would imply the equality $l = 0$, so that $\lim_{i\to\infty} \mu(Y_i) = 0$. Let us put

$$Y := \cap_{i=0}^{\infty} Y_i.$$

From the equality $\lim_{i\to\infty} \mu(Y_i) = 0$ and (v) from Remark 1, we obtain that Y is nonempty and compact. The rest of the proof runs in the same way as the proof of Theorem 6. □

Remark 4. *A question arises whether the generalized version of Theorem 7 is true, in which we assume that for the subset $X \subset \Omega$ both $n = n_X$ and the function $g = g_X \in \mathcal{G}$ that satisfy* *the condition (19) depend on the subset X. It turns out that this hypothesis is also equivalent to Question 1.*

We will now generalize Theorem 4 using the concept of convex-power operator $T^{(n,x_0)}$. Contrary to Theorems 6 and 7, it turns out that in this case, the assumptions can be weakened even more because it is enough to demand that both values $k = k_X \in [0,1)$ and $n = n_X \in \mathbb{N}$ may depend on the set $X \subset \Omega$.

Theorem 8. *Let μ be a measure of noncompactness in a Banach space E. Assume that Ω is a nonempty, closed, bounded and convex subset of E, a mapping $T : \Omega \to \Omega$ is continuous, and there is $x_0 \in \Omega$ such that for any nonempty subset $X \subset \Omega$ there exist a natural number $n = n_X \in \mathbb{N}$ and a number $k = k_X \in [0,1)$ such that*

$$\mu(T^{(n,x_0)}(X)) \leq k\mu(X). \tag{22}$$

Then T has at least one fixed point in Ω.

Proof. Let $\mathcal{A}$ denote the set of all closed, convex subsets X of Ω for which $x_0 \in X$ and $T(X) \subseteq X$. Let us denote $X_\infty := \bigcap_{X\in\mathcal{A}} X$ and put $Y := \overline{\text{co}}(T(X_\infty) \cup \{x_0\})$. Obviously, $X_\infty \in \mathcal{A}$. Since $x_0 \in X_\infty$ and $T(X_\infty) \subseteq X_\infty$, then $Y \subseteq X_\infty$. On the other hand, $T(Y) \subseteq T(X_\infty) \subseteq Y$. Since $x_0 \in Y$, it follows that $Y \in \mathcal{A}$. Therefore, $X_\infty \subseteq Y$, which implies

$$X_\infty = \overline{\text{co}}(T(X_\infty) \cup \{x_0\}). \tag{23}$$

Applying the assumptions of the theorem to a set X_∞, we yield that there are $n \in \mathbb{N}$ and $k \in [0,1)$ such that

$$\mu(T^{(n,x_0)}(X_\infty)) \leq k\mu(X_\infty). \tag{24}$$

Next, in view of (23), we obtain

$$T^{(2,x_0)}(X_\infty) = T(\overline{\text{co}}(T(X_\infty) \cup \{x_0\})) = T(X_\infty) = T^{(1,x_0)}(X_\infty)$$

and hence,

$$\begin{aligned} T^{(3,x_0)}(X_\infty) &= T(\overline{\text{co}}(T^{(2,x_0)}(X_\infty) \cup \{x_0\})) \\ &= T(\overline{\text{co}}(T^{(1,x_0)}(X_\infty) \cup \{x_0\})) = T^{(2,x_0)}(X_\infty), \end{aligned}$$

and in a similar way, we obtain inductively

$$T^{(n,x_0)}(X_\infty) = T^{(1,x_0)}(X_\infty), \;\; n = 1, 2, \dots \; .$$

Applying the above, (23), and properties of the measure μ, we get

$$\begin{aligned} \mu(T^{(n,x_0)}(X_\infty)) &= \mu(T^{(1,x_0)}(X_\infty)) = \mu(T(X_\infty)) \\ &= \mu(\overline{\text{co}}(T(X_\infty) \cup \{x_0\})) = \mu(X_\infty). \end{aligned}$$

Linking this with (24), we obtain $\mu(X_\infty) \leq k\mu(X_\infty)$, and therefore, $\mu(X_\infty) = 0$. Thus, a set X_∞, due to its closure, is compact. Hence, we get the conclusion from the Schauder theorem for $T : X_\infty \to X_\infty$. □

We can also generalize Theorem 5.

Theorem 9. *Let μ be a measure of noncompactness in a real Banach space E. Assume that Ω is a nonempty, closed, bounded and convex subset of E, a mapping $T : \Omega \to \Omega$ is continuous, and there exists $x_0 \in \Omega$ such that for any nonempty subset $X \subset \Omega$ with $\mu(X) > 0$, there is $n \in \mathbb{N}$ satisfying*

$$\mu(T^{(n,x_0)}(X)) < \mu(X). \tag{25}$$

Then T has at least one fixed point in Ω.

Proof. Reasoning similarly to the previous proof, we get the existence of a set X_∞ such that $X_\infty = \overline{\text{co}}(T(X_\infty) \cup \{x_0\})$ and $\mu(T^{(n,x_0)}(X_\infty)) = \mu(X_\infty)$, $n = 1, 2, \ldots$ If $\mu(X_\infty) > 0$, then for some $n \in \mathbb{N}$

$$\mu(X_\infty) = \mu(T^{(n,x_0)}(X_\infty)) < \mu(X_\infty),$$

and this implies $\mu(X_\infty) = 0$. The rest of the proof runs exactly as in the previous one. □

Remark 5. *By reasoning similarly to as above, it can also be shown that Theorems 8 and 9 will also be true when $T^{(n,x_0)}(X)$ is defined as follows:*

$$T^{(1,x_0)}(X) := \overline{\text{co}}(T(X)), \quad T^{(n,x_0)}(X) := \overline{\text{co}}(T(T^{(n-1,x_0)}(X) \cup \{x_0\}))$$

for $n = 2, 3, \ldots$.

4. An Application

We present a simple example of an application of the fixed point theorem from the previous chapter. We place our considerations in Banach space $C(J, E)$ with the supremum norm $\|x\|_\infty := \sup\{\|x(t)\| : t \in J\}$, $J := [0, a]$. In the space $C(J, E)$, we use a measure of noncompactness μ given by the formula

$$\mu(X) := \omega_0(X) + \overline{\beta}(X),$$

where

$$\omega_0(X) := \lim_{\varepsilon \to 0} \sup\{\|x(t_1) - x(t_2)\| : x \in X, t_1, t_2 \in [0, a], \ |t_1 - t_2| \le \varepsilon\} :$$

$$\overline{\beta}(X) := \sup\{\beta(X(t)) : t \in J\} \tag{26}$$

for nonempty bounded $X \subset C(J, E)$, where $X(t) := \{x(t) : x \in X\}$, and β is the Hausdorff measure of noncompactness in the space E.

At the beginning of this section, we start by recalling some necessary facts.

Lemma 1 ([9]). *If $\{x_n\}_{n=1}^\infty \subset L^1([0, T], E)$ is uniformly integrable (i.e., there is $h \in L^1([0, T], \mathbb{R}_+)$ such that $\|x_n(t)\| \le h(t)$ for a.e. $t \in [0, T]$ and for all $n \in \mathbb{N}$), then the function $[0, T] \ni t \to \beta(\{x_n(t)\}_{n=1}^\infty) \in \mathbb{R}_+$ is measurable, and*

$$\beta\Big(\Big\{\int_0^t x_n(s)ds\Big\}_{n=1}^\infty\Big) \le 2\int_0^t \beta(\{x_n(s)\}_{n=1}^\infty)ds, \ \ t \in [0, T],$$

where β is the Hausdorff measure of noncompactness.

Lemma 2 ([10]). *Let E be a real Banach space. If $X \subset E$ is a nonempty and bounded set, then there exists a sequence $\{x_n\}_{n=1}^\infty \subset X$ such that $\beta(X) \le 2\beta(\{x_n\}_{n=1}^\infty)$.*

Lemma 3. *If $q \in L^1([0, T], \mathbb{R})$, $m \in \mathbb{N}$, then*

$$\int_0^t q(s) \int_0^s q(s_1) \ldots \int_0^{s_{m-2}} q(s_{m-1})ds_{m-1} \ldots ds_1 ds = \frac{\Big(\int_0^t q(s)ds\Big)^m}{m!}, \quad t \in [0, T].$$

Proof. We omit the simple inductive proof that uses basic properties of absolutely continuous functions. □

Now we show the applicability of the fixed point theorems. We study the following Volterra-type integral equation:

$$x(t) = \int_0^t v(t,s,x(s))ds, \quad t \in J. \tag{27}$$

We impose the following conditions on the functions appearing in this equation.

(H_1) The mapping $v : \{(t,s,x) : 0 \le s \le t \le a,\ x \in E\} \to E$ is uniformly continuous.
(H_2) There exist nondecreasing functions $p_1, p_2 : J \to \mathbb{R}_+$ such that

$$\|v(t,s,x)\| \le p_1(t)p_2(s)\|x\| \quad \text{for } 0 \le s \le t \le a,\ x \in E.$$

(H_3) For any nonempty and bounded $Y \subset E$, there exists a function $q = q_Y \in L^1(J, \mathbb{R}_+)$ such that

$$\beta(v(t,s,Y)) \le q(s)\beta(Y)$$

for almost every $0 \le s \le t \le a$.

Remark 6. *Note that the condition (H_3) is weaker than its substitutes known from other applications because the function q is not the same for all subsets Y, but it may depend on the subset Y.*

Now we formulate our existence theorem as:

Theorem 10. *Under assumptions (H_1)–(H_3), Equation (27) has at least one solution in $C(J,E)$.*

Proof. First of all, we define operator V such that for any $x \in C(J,E)$,

$$(Vx)(t) := \int_0^t v(t,s,x(s))ds.$$

The assumptions (H_1) guarantee that $Vx \in C(J,E)$. Next, we define a function

$$r(t) := p_1(t)e^{\int_0^t p_1(s)p_2(s)ds}, \quad t \in J.$$

It is easy to check that the above-defined r is nondecreasing and satisfies the following inequality:

$$p_1(t)\int_0^t p_2(s)r(s)ds \le r(t). \tag{28}$$

Indeed, applying basic properties of absolutely continuous functions, for $\phi(s) := e^{\int_0^s p_1(\tau)p_2(\tau)d\tau}$, $s \in [0,t]$, we get $\phi'(s) = p_1(s)p_2(s)e^{\int_0^s p_1(\tau)p_2(\tau)d\tau}$ for a.e. $s \in [0,t]$, and therefore,

$$p_1(t)\int_0^t p_2(s)r(s)ds = p_1(t)\int_0^t p_2(s)p_1(s)e^{\int_0^s p_1(\tau)p_2(\tau)d\tau}ds = p_1(t)\int_0^t \phi'(s)ds$$

$$= p_1(t)(\phi(t)-1) \le p_1(t)\phi(t) = r(t).$$

Next, let us consider the set

$$\Omega := \{x \in C(J,E) : \|x(t)\| \le r(t), t \in J\}.$$

The set Ω is bounded, convex and closed. In virtue of (H_2) and (28), we obtain

$$\|(Vx)(t)\| \leq \int_0^t p_1(t)p_2(s)r(s)ds = p_1(t)\int_0^t p_2(s)r(s)ds \leq r(t)$$

for $x \in \Omega$. This shows that $V : \Omega \to \Omega$. We omit the standard proof that the operator V is continuous and is satisfied following equality $\omega_0(VX) = 0$ for any $X \subset \Omega$, and therefore,

$$\omega_0(\tilde{V}^n(X)) = 0. \tag{29}$$

Let us fix a nonempty set $X \subset \Omega$. In view of Lemma 2, there exists the sequence $\{x_n\} \subset X$ such that $\beta\Big(\int_0^t v(t,s,X(s))ds\Big) \leq 2\beta\Big(\int_0^t v(t,s,\{x_n(s)\}_{n=1}^{\infty})ds\Big)$.

This fact, assumptions (H_3), and Lemma 1 yield that for a fixed $t \in J$, we have

$$\begin{aligned}
\beta(\tilde{V}^1X(t)) = \beta(VX(t)) &= \beta\Big(\int_0^t v(t,s,X(s))ds\Big) \\
&\leq 2\beta\Big(\int_0^t v(t,s,\{x_n(s)\}_{n=1}^{\infty})ds\Big) \leq 4\int_0^t \beta\big(v(t,s,\{x_n(s)\}_{n=1}^{\infty})\big)ds \\
&\leq 4\int_0^t q(s)\beta(\{x_n(s)\}_{n=1}^{\infty})ds \\
&\leq 4\int_0^t q(s)ds \cdot \overline{\beta}(X).
\end{aligned} \tag{30}$$

In view of Lemma 2, there exists the sequence $\{y_n\} \subset \overline{\text{co}}(\tilde{V}^1X)$ such that

$$\beta\Big(\int_0^t v(t,s,\overline{\text{co}}(\tilde{V}^1X)(s))ds\Big) \leq 2\beta\Big(\int_0^t v(t,s,\{y_n(s)\}_{n=1}^{\infty})ds\Big).$$

Hence,

$$\begin{aligned}
\beta(\tilde{V}^2X(t)) = \beta\big(V(\overline{\text{co}}(\tilde{V}^1X))(t)\big) &= \beta\Big(\int_0^t v(t,s,\overline{\text{co}}(\tilde{V}^1X)(s))ds\Big) \\
&\leq 2\beta\Big(\int_0^t v(t,s,\{y_n(s)\}_{n=1}^{\infty})ds\Big) \leq 4\int_0^t \beta\big(v(t,s,\{y_n(s)\}_{n=1}^{\infty})\big)ds \\
&\leq 4\int_0^t q(s)\beta(\{y_n(s)\}_{n=1}^{\infty})ds.
\end{aligned} \tag{31}$$

Applying (30), we get

$$\beta(\{y_n(s)\}_{n=1}^{\infty}) \leq \beta(\overline{\text{co}}(\tilde{V}^1X(s))) = \beta(\tilde{V}^1X(s)) \leq 4\int_0^s q(s_1)ds_1 \cdot \overline{\beta}(X).$$

Linking this with (31), we obtain

$$\beta\big(\tilde{V}^2X(t)\big) \le 4\int\limits_0^t q(s)4\int\limits_0^s q(s_1)ds_1 \cdot \overline{\beta}(X)ds = 4^2\int\limits_0^t q(s)\int\limits_0^s q(s_1)ds_1 ds \cdot \overline{\beta}(X).$$

Continuing this reasoning, we obtain

$$\beta\big(\tilde{V}^n(X)(t)\big) \le 4^n\int\limits_0^t q(s)\int\limits_0^s q(s_1)\dots\int\limits_0^{s_{n-2}} q(s_{n-1})ds_{n-1}\dots ds_1 ds \cdot \overline{\beta}(X).$$

Then, in virtue of (26), we have

$$\overline{\beta}(\tilde{V}^n(X)) \le 4^n\int\limits_0^a q(s)\int\limits_0^s q(s_1)\dots\int\limits_0^{s_{n-2}} q(s_{n-1})ds_{n-1}\dots ds_1 ds \cdot \overline{\beta}(X)$$

and applying Lemma 3, we get

$$\overline{\beta}(\tilde{V}^n(X)) \le \frac{\left(4\int_0^a q(t)dt\right)^n}{n!}\overline{\beta}(X). \tag{32}$$

Since $\lim\limits_{n\to\infty}\frac{\left(4\int_0^a q(t)dt\right)^n}{n!} = 0$, then there exists $n \in \mathbb{N}$ such that

$$k := \frac{\left(4\int_0^a q(t)dt\right)^n}{n!} < 1.$$

Keeping in mind (32) and (29), we obtain $\mu(\tilde{V}^n(X)) \le k\mu(X)$. This fact and Theorem 6 complete the proof. □

5. Discussion

We explain the new results of this study.

1°. The results of this paper strengthen the generalizations of the classic theorems of Darbo and Sadovski. These strengthenings do not consist in adopting new assumptions in a more enigmatic form, as sometimes happens, but in weakening the assumptions of these already classic theorems.

2°. The publication raises several questions that seem to be interesting and important in the theory of fixed points expressed in terms of measures of noncompactness.

3°. The obtained results, as shown by the example presented in this paper, allow us to obtain existential theorems for various equations with weaker assumptions than those that would be required using the previous theorems.

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References

1. Darbo, G. Punti uniti in transformazioni a condominio non compatto. *Rend. Semin. Mat. Univ. Padova* **1955**, *24*, 84–92.
2. Sadovskii, B.N. A fixed point principle. *Funktsional. Anal. Prilozhen.* **1967**, *1*, 74–76. [CrossRef]

3. Banaś, J.; Goebel, K. *Measures of Noncompactness in Banach Spaces*; Lecture Notes in Pure and Applied Mathematics; Marcel Dekker: New York, NY, USA, 1980; Volume 60.
4. Ayerbe-Toledano, J.M.; Dominguez-Benavides, T.; López-Aceda, G. *Measure of Noncompactness in Metric Fixed Point Theory*; Birkhauser Verlag: Basel, Switzerland; Boston, MA, USA; Berlin, Germany, 1997.
5. Liu, L.; Guo, G.; Wu, C.; Wu, Y. Existence theorems of global solutions for nonlinear Volterra type integral equations in Banach spaces. *J. Math. Anal. Appl.* **2005**, *309*, 638–649. [CrossRef]
6. Sun, J.X.; Zhang, X.Y. A fixed point theorem for convexpower condensing operators and its applications to abstract semilinear evolution equations. *Acta Math. Sin.* **2005**, *48*, 439–446.
7. Geraghty, M.A. On contractive mappings. *Proc. Amer. Math. Soc.* **1973**, *40*, 604–608. [CrossRef]
8. Dudek, S.; Olszowy, L. On generalization of Darbo–Sadovskii type fixed point theorems for iterated mappings in Frechet spaces. *J. Fixed Point Theory Appl.* **2018**, *20*, 146–157.
9. Mönch, H. Boundary value problems for nonlinear ordinary differential equations of second order in Banach spaces. *Nonlinear Anal. TMA* **1980**, *4*, 985–999. [CrossRef]
10. Bothe, D. Multivalued perturbation of m-accretive differential inclusions. *Isr. J. Math.* **1998**, *108*, 109–138. [CrossRef]