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Some New Fractional Inequalities Defined Using cr-Log-h-Convex Functions and Applications

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Abstract: There is a strong correlation between the concept of convexity and symmetry. One of these is the class of interval-valued cr-log-h-convex functions, which is closely related to the theory of symmetry. In this paper, we obtain Hermite–Hadamard and its weighted version inequalities that are related to interval-valued cr-log-h-convex functions, and some known results are recaptured. To support our main results, we offer three examples and two applications related to modified Bessel functions and special means as well.

Keywords: Hermite–Hadamard inequality; Fejér-type inequality; cr-log-h-convex functions; modified Bessel function of first kind

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1. Introduction

In recent years, there has been a notable surge in the exploration of various extensions of convex functions, unveiling a rich landscape beyond traditional convexity. Convexity, a fundamental concept with far-reaching implications in fields such as optimal control and game theory, has long been a cornerstone of mathematical analysis. However, real-world applications often present functions that exhibit properties falling within a broader spectrum than strict convexity. This realization has sparked considerable interest in the study of generalized convexity, an area of research that continues to captivate scholars. The quest to understand and leverage generalized convexity has led to the development of numerous novel frameworks tailored to address practical challenges. Among these, the Hermite–Hadamard inequality stands out as a bridge between convex function theory and integral inequalities, finding relevance across diverse scientific domains. Moreover, the intricate interplay between convexity and symmetry concepts has given rise to intriguing classes of functions, such as interval-valued cr-log-h-convex functions, with profound implications in symmetry theory.

These inequalities serve as powerful tools with practical utility spanning optimization, numerical analysis, and statistics. Notably, the Hermite–Hadamard inequality, recognized as an analog of convexity, necessitates the presence of generalized convexity for its establishment. In engineering, particularly in the realm of 3D printing technology, both the Hermite–Hadamard inequality and He Chengtian's inequality are frequently employed to approximate printing speeds, addressing the challenge of forecasting speeds with precision (see e.g., [1–3]).

The field of inequalities research encompasses a broad spectrum of theoretical and applied mathematics, attracting contributions from renowned scholars such as Josip E. Pecaric and Dragoslav S. Mitrinović. Pioneering works like "Convex Functions, Partial Orderings, and Statistical Applications" by Pecarič in [4] and "Analytic Inequalities" by Mitrinović in [5] shed light on the intricate relationships between analytic, convex, and probabilistic aspects of inequalities. These works, along with seminal contributions like the Hermite–Hadamard inequality, enrich our understanding of these mathematical tools and their applications, broadening the horizon of mathematical knowledge.

Let W be convex subset of $\mathbb{R}$ and $\mathrm{F} : W \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function with $c_1, c_2 \in W$ and $c_1 < c_2$, then

$$\mathrm{F}\left(\frac{c_1 + c_2}{2}\right) \leq \frac{1}{c_2 - c_1} \int_{c_1}^{c_2} \mathrm{F}(v)dv \leq \frac{\mathrm{F}(c_1) + \mathrm{F}(c_2)}{2}. \tag{1}$$

Fejér [6], gave the generalized form of inequality Equation (1), as follows:

$$\mathrm{F}\left(\frac{c_1 + c_2}{2}\right) \int_{c_1}^{c_2} \tilde{H}(v)dv \quad \leq \quad \int_{c_1}^{c_2} \tilde{H}(v)\mathrm{F}(v)dv \leq \frac{\mathrm{F}(c_1) + \mathrm{F}(c_2)}{2} \int_{c_1}^{c_2} \tilde{H}(v)dv, \tag{2}$$

holds, where $\tilde{H} : [c_1; c_2] \to \mathbb{R}$ is integrable, symmetric and nonnegative function about $v = \frac{c_1+c_2}{2}$.

Definition 1. *In reference [4] A function* F *is said to be log-convex if*

$$\mathrm{F}(tc_1 + (1-t)c_2) \leq [\mathrm{F}(c_1)]^t [\mathrm{F}(c_2)]^{(1-t)},$$

holds for all $c_1, c_2 \in W$ with $t \in [0, 1]$.

Many generalizations and improvements related to log-convex functions can be found (see, e.g., [7–9]).

Definition 2. *In reference [10] Let W, V be convex subset of $\mathbb{R}$ and $h : W \to \mathbb{R}$ be a nonnegative function. A function $\mathrm{F} : V \to (0, \infty)$ is said to be h-convex if for all $c_1, c_2 \in W$ and $t \in [0, 1]$, one has*

$$\mathrm{F}(tc_1 + (1-t)c_2) \leq h(t)\mathrm{F}(c_1) + h(1-t)\mathrm{F}(c_2).$$

The class of h-convex functions generalizes several other known classes of convexity, see [10].

In [11], Noor et al., mentioned log-h-convex functions as follows:

Definition 3. *Let $h : W \to \mathbb{R}$ be a non-negative function. A function $\mathrm{F} : V \to (0, \infty)$ is said to be* log-*h-convex for all $c_1, c_2 \in W$ and $t \in [0, 1]$, if*

$$\mathrm{F}(tc_1 + (1-t)c_2) \leq [\mathrm{F}(c_1)]^{h(t)} [\mathrm{F}(c_2)]^{h(1-t)},$$

holds.

Due to broad utility of Hermite–Hadamard inequalities and fractional calculus, and across various scientific disciplines, researchers are actively exploring these type of inequalities. This research direction has gained momentum, as evidenced by recent developments in the field (see e.g., [12–17]).

Sarikaya et al., in [18] established the Hermite–Hadamard type inequalities for fractional integrals:

Theorem 1. *Let* $\mathrm{F} : [c_1, c_2] \subseteq \mathbb{R} \to \mathbb{R}$ *be positive function with* $0 \leq c_1 < c_2$ *and* $\mathrm{F} \in L[c_1, c_2]$*. If* F *is positive on* $[c_1, c_2]$*, then*

$$\mathrm{F}\left(\frac{c_1 + c_2}{2}\right) \leq \frac{\Gamma(\varsigma + 1)}{2(c_2 - c_1)^{\varsigma}} \left[J^{\varsigma}_{c_1-}\mathrm{F}(c_1) + J^{\varsigma}_{c_2+}\mathrm{F}(c_2)\right] \leq \frac{\mathrm{F}(c_1) + \mathrm{F}(c_2)}{2},$$

with $\varsigma > 0$*. Here* $L[c_1, c_2]$ *is the set of all Lebesgue integrable functions on* $[c_1, c_2]$*. The symbols* $J^{\varsigma}_{c_1^+}$ *and* $J^{\varsigma}_{c_2^-}$ *represent the left-sided and right-sided Riemann—Liouville fractional integrals of the order* $\varsigma \in \mathbb{R}^+$ *that are defined in [15]*

$$J^{\varsigma}_{c_1^+}\mathrm{F}(c) = \frac{1}{\Gamma(\varsigma)} \int_{c_1}^{c} (c - \varphi)^{\varsigma - 1}\mathrm{F}(\varphi)d\varphi, \quad (0 \leq c_1 < c \leq c_2),$$

and

$$J^{\varsigma}_{c_2^-}\mathrm{F}(c) = \frac{1}{\Gamma(\varsigma)} \int_{c}^{c_2} (\varphi - c)^{\varsigma - 1}\mathrm{F}(\varphi)d\varphi, \quad (0 \leq c_1 \leq c < c_2).$$

The set $\mathbb{R}_I$ contains all closed intervals on $\mathbb{R}$. For $[\underline{\vartheta}, \overline{\vartheta}] \in \mathbb{R}_I$, if $\underline{\vartheta} > 0$, then $[\underline{\vartheta}, \overline{\vartheta}]$ is a positive interval. The set of all positive intervals is denoted by $\mathbb{R}_I^+$.

Definition 4. *In reference [19] For any* $\lambda \in \mathbb{R}$*,* $\vartheta = [\underline{\vartheta}, \overline{\vartheta}]$*,* $\sigma = [\underline{\sigma}, \overline{\sigma}] \in \mathbb{R}_I$*, we have*

$$\vartheta + \sigma = [\underline{\vartheta}, \overline{\vartheta}] + [\underline{\sigma}, \overline{\sigma}] = [\underline{\vartheta} + \underline{\sigma}, \overline{\vartheta} + \overline{\sigma}],$$

and

$$\lambda\vartheta = \lambda[\underline{\vartheta}, \overline{\vartheta}] = \begin{cases} [\lambda\underline{\vartheta}, \lambda\overline{\vartheta}] & , \lambda > 0; \\ [0, 0] & , \lambda = 0; \\ [\lambda\overline{\vartheta}, \lambda\underline{\vartheta}] & , \lambda < 0. \end{cases}$$

Let $\vartheta = [\underline{\vartheta}, \overline{\vartheta}] \in \mathbb{R}_I$*, the centre of* ϑ *is defined as* $\vartheta_c = \frac{\underline{\vartheta} + \overline{\vartheta}}{2}$ *while radius of* ϑ *is given as* $\vartheta_r = \frac{\overline{\vartheta} - \underline{\vartheta}}{2}$*. Then* $\vartheta = [\underline{\vartheta}, \overline{\vartheta}]$ *can also be presented in the form of centre-radius as:*

$$\vartheta = \left\langle \frac{\underline{\vartheta} + \overline{\vartheta}}{2}, \frac{\overline{\vartheta} - \underline{\vartheta}}{2} \right\rangle = \langle \vartheta_c, \vartheta_r \rangle.$$

Definition 5. *In reference [20] Let* $\vartheta = [\underline{\vartheta}, \overline{\vartheta}] = \langle \vartheta_c, \vartheta_r \rangle$*,* $\sigma = [\underline{\sigma}, \overline{\sigma}] = \langle \sigma_c, \sigma_r \rangle \in \mathbb{R}_I$*, then the center-radius order relation is defined by:*

$$\vartheta \leq_{cr} \sigma \Longleftrightarrow \begin{cases} \vartheta_c < \sigma_c, & \text{if } \vartheta_c \neq \sigma_c, \\ \vartheta_c \leq \sigma_c, & \text{if } \vartheta_c = \sigma_c. \end{cases}$$

Obviously, for $\vartheta, \sigma \in \mathbb{R}_I$*, either* $\vartheta \leq_{cr} \sigma$ *or* $\sigma \leq_{cr} \vartheta$*.*

From log-h-convexity can be derived some known convexity classes.
In [11], Noor et al. proposed following inequality for log-h-convex functions:

Theorem 2. *Suppose that* F *be a* log-h*-convex function with* $h(\frac{1}{2}) \neq 0$*, then*

$$\begin{aligned} \mathrm{F}\left(\frac{c_1 + c_2}{2}\right)^{\frac{1}{2h(\frac{1}{2})}} &\leq \exp\left[\frac{1}{c_2 - c_1} \int_{c_1}^{c_2} \ln \mathrm{F}(\mu)d\mu\right] \\ &\leq (\mathrm{F}(c_1)\mathrm{F}(c_2))^{\int_0^1 h(\varphi)d\varphi}. \end{aligned}$$

Liu et al. [21] generalized this concept of log-h-convex function to interval-valued functions.

Definition 6. *Assume that* $\mathrm{F} : [c_1, c_2] \to \mathbb{R}_I^+$ *be an interval valued function with* $\mathrm{F} = [\underline{\mathrm{F}}, \overline{\mathrm{F}}]$ *and* $\mathrm{F} \in I\mathbb{R}_{([c_1,c_2])}$. *A function* F *is said to be cr*-$\log - h$*-convex on* $[c_1, c_2]$ *where* $h : [0,1] \to \mathbb{R}^+$ *is a nonnegative function if*

$$\mathrm{F}(tc_1 + (1-t)c_2) \leq_{cr} [\mathrm{F}(c_1)]^{h(t)}[\mathrm{F}(c_2)]^{h(1-t)}.$$

The set of all Riemann integrable interval-valued functions on $[c_1, c_2]$ *is denoted by* $I\mathbb{R}_{([c_1,c_2])}$.

Remark 1. *Taking* $\underline{\mathrm{F}} = \overline{\mathrm{F}}$, *the function* F *reduces to* $\log$-h *convex.*

Definition 7. *In reference [21] assume* $\mathrm{F} : [c_1, c_2] \to \mathbb{R}_I^+$ *be an interval-valued function with* $\mathrm{F} = [\underline{\mathrm{F}}, \overline{\mathrm{F}}]$ *and* $h : [0,1] \to \mathbb{R}^+$, *then* F *is called cr*-$\log$-h*-convex on* $[c_1, c_2]$ *if*

$$\mathrm{F}(tx + (1-t)y) \leq_{cr} [\mathrm{F}(x)]^{h(t)}[\mathrm{F}(y)]^{h(1-t)},$$

holds $\forall t \in (0,1)$ *with* $x, y \in [c_1, c_2]$.

Theorem 3. *In reference [22] let* $\mathrm{F} = [\underline{\mathrm{F}}, \overline{\mathrm{F}}]$ *is an interval-valued function where* $\mathrm{F} : [c_1, c_2] \to \mathbb{R}_I$. *Then the function* F *is called Riemann integrable on* $[c_1, c_2]$, *provided* $\underline{\mathrm{F}}$ *and* $\overline{\mathrm{F}}$ *are Riemann integrable on* $[c_1, c_2]$ *and*

$$\int_{c_1}^{c_2} \mathrm{F}(\mu) d\mu = \left[\int_{c_1}^{c_2} \underline{\mathrm{F}}(\mu) d\mu, \int_{c_1}^{c_2} \overline{\mathrm{F}}(\mu) d\mu\right].$$

Theorem 4. *In reference [23] the functions* $\digamma_1$, $\tilde{H} : [c_1, c_2] \to \mathbb{R}_I^+$ *are interval-valued functions where* $\digamma_1 = [\underline{\digamma}, \overline{\digamma_1}]$ *and* $\tilde{H} = [\underline{G}, \overline{\tilde{H}}]$. *If* $\digamma_1, \tilde{H} \in I\mathbb{R}_{([c_1,c_2])}$, *and* $\digamma_1(\mu) \leq_{cr} \tilde{H}(\mu)$ *for all* $\mu \in [c_1, c_2]$, *then*

$$\int_{c_1}^{c_2} \digamma_1(\mu) d\mu \leq_{cr} \int_{c_1}^{c_2} \tilde{H}(\mu) d\mu.$$

Utilizing cr-log-h-convex interval-valued functions introduces an innovative framework for comprehending and enhancing intricate mathematical systems. This interdisciplinary approach, blending concepts from convex analysis, interval mathematics, and fractional calculus, offers a more nuanced and comprehensive method for addressing mathematical problems. It has the potential to generate novel insights and solutions across various domains, including optimization, control theory, and other fields reliant on mathematical modeling.

In the main section, we give various type inequalities of Hermite–Hadamard and its weighted version specifically for functions that are cr-log-h-convex. To further illustrate the validity of our findings, we present two applications and three examples. Finally, in the last section, we wrap up the paper by summarizing our conclusions and offering suggestions for potential avenues of future research.

2. Main Results

Throughout the discussion cr-log-h-convex functions on $[c_1, c_2]$ are denoted by $SX(cr -$ log- $h, [c_1, c_2], \mathbb{R}_I^+)$. The set of all Riemann integrable interval-valued functions on $[c_1, c_2]$ is denoted by $I\mathbb{R}_{([c_1,c_2])}$.

We are ready to prove the Hermite—Hadamard-type inequality for cr-log-h-convex functions.

Theorem 5. *Suppose that* F *be an interval valued function with* $\mathrm{F}:[c_1,c_2]\to\mathbb{R}_I^+$, $\mathrm{F}=[\underline{\mathrm{F}},\overline{\mathrm{F}}]$ *and* $\mathrm{F}\in I\mathbb{R}_{([c_1,c_2])}$, *where* $h:[0,1]\to\mathbb{R}^+$ *with* $h(\frac{1}{2})\neq 0$. *If* $\mathrm{F}\in SX(cr-\log-h,[c_1,c_2],\mathbb{R}_I^+)$, *then we have*

$$\begin{aligned}\mathrm{F}\left(\frac{c_1+c_2}{2}\right)^{\frac{1}{h(\frac{1}{2})}} &\leq_{cr} \exp\left[\frac{\Gamma(\varsigma)}{(c_2-c_1)^{\varsigma}}\left(J^{\varsigma}_{c_2+}\ln\mathrm{F}(\mu)+J^{\varsigma}_{c_1-}\ln\mathrm{F}(\mu)\right)\right] \\ &\leq_{cr} (\mathrm{F}(c_1)\mathrm{F}(c_2))^{\int_0^1(\varphi^{\varsigma-1}+(1-\varphi)^{\varsigma-1})h(\varphi)d\varphi}.\end{aligned} \tag{3}$$

Proof. As $\mathrm{F}\in SX(cr-\log-h,[c_1,c_2],\mathbb{R}_I^+)$, then

$$\mathrm{F}\left(\frac{u_1+u_2}{2}\right)\leq_{cr}[\mathrm{F}(u_1)\mathrm{F}(u_2)]^{h(\frac{1}{2})}. \tag{4}$$

We can write Equation (4) as

$$\frac{1}{h(\frac{1}{2})}\ln\mathrm{F}\left(\frac{u_1+u_2}{2}\right)\leq_{cr}\ln\mathrm{F}(u_1)+\ln\mathrm{F}(u_2). \tag{5}$$

On substituting $u_1=\varphi c_1+(1-\varphi)c_2$ and $u_2=\varphi c_2+(1-\varphi)c_1$ in Equation (5), we get

$$\frac{1}{h(\frac{1}{2})}\ln\mathrm{F}\left(\frac{c_1+c_2}{2}\right)\leq_{cr}\ln\mathrm{F}(\varphi c_1+(1-\varphi)c_2)+\ln\mathrm{F}(\varphi c_2+(1-\varphi)c_1). \tag{6}$$

On multiplying Equation (6) with $\varphi^{\varsigma-1}$ and after integrating between 0 to 1 w.r.t. φ, we obtain

$$\begin{aligned}&\frac{1}{h(\frac{1}{2})}\ln\mathrm{F}\left(\frac{c_1+c_2}{2}\right)\\ \leq_{cr}&\int_0^1\varphi^{\varsigma-1}\ln\mathrm{F}(\varphi c_1+(1-\varphi)c_2)d\varphi+\int_0^1\varphi^{\varsigma-1}\ln\mathrm{F}(\varphi c_2+(1-\varphi)c_1)d\varphi.\end{aligned} \tag{7}$$

From Equation (7), we have

$$\begin{aligned}&\frac{1}{h(\frac{1}{2})}\ln\mathrm{F}\left(\frac{c_1+c_2}{2}\right)\\ =&\left[\int_0^1\varphi^{\varsigma-1}\ln\overline{\mathrm{F}(}\varphi c_1+(1-\varphi)c_2)d\varphi,\int_0^1\varphi^{\varsigma-1}\ln\underline{\mathrm{F}}(\varphi c_1+(1-\varphi)c_2)d\varphi\right]\\ &+\left[\int_0^1\varphi^{\varsigma-1}\ln\overline{\mathrm{F}(}\varphi c_2+(1-\varphi)c_1)d\varphi,\int_0^1\varphi^{\varsigma-1}\ln\underline{\mathrm{F}}(\varphi c_1+(1-\varphi)c_2)d\varphi\right].\end{aligned}$$

After suitable substitution, we obtain

$$\begin{aligned}&\frac{1}{h(\frac{1}{2})}\ln\mathrm{F}\left(\frac{c_1+c_2}{2}\right)\\ =&\frac{1}{(c_2-c_1)^{\varsigma}}\left[\int_{c_1}^{c_2}(c_2-\mu)^{\varsigma-1}\ln\overline{\mathrm{F}}(\mu)d\mu,\int_{c_1}^{c_2}(c_2-\mu)^{\varsigma-1}\ln\underline{\mathrm{F}}(\mu)d\mu\right]\\ &+\frac{1}{(c_2-c_1)^{\varsigma}}\left[\int_{c_1}^{c_2}(\mu-c_1)^{\varsigma-1}\ln\overline{\mathrm{F}}(\mu)d\mu,\int_{c_1}^{c_2}(\mu-c_1)^{\varsigma-1}\ln\underline{\mathrm{F}}(\mu)d\mu\right]\\ =&\frac{1}{(c_2-c_1)^{\varsigma}}\left[\int_{c_1}^{c_2}(c_2-\mu)^{\varsigma-1}\ln\mathrm{F}(\mu)d\mu+\int_{c_1}^{c_2}(\mu-c_1)^{\varsigma-1}\ln\mathrm{F}(\mu)d\mu\right]\\ =&\frac{\Gamma(\varsigma)}{(c_2-c_1)^{\varsigma}}\left[J^{\varsigma}_{c_2+}\ln\mathrm{F}(\mu)+J^{\varsigma}_{c_1-}\ln\mathrm{F}(\mu)\right].\end{aligned}$$

Similarly, as $\mathrm{F} \in SX(cr-\log-\mathrm{h}, [c_1, c_2], \mathbb{R}_I^+)$, we obtain

$$\mathrm{F}(\varphi c_1 + (1-\varphi)c_2) \leq_{cr} [\mathrm{F}(c_1)]^{h(\varphi)}[\mathrm{F}(c_2)]^{h(1-\varphi)},$$

and

$$\mathrm{F}(\varphi c_2 + (1-\varphi)c_1) \leq_{cr} [\mathrm{F}(c_2)]^{h(\varphi)}[\mathrm{F}(c_1)]^{h(1-\varphi)}.$$

So,

$$\ln \mathrm{F}(\varphi c_1 + (1-\varphi)c_2) \leq {}_{cr} h(\varphi)\ln \mathrm{F}(c_1) + h(1-\varphi)\ln \mathrm{F}(c_2). \tag{8}$$

$$\ln \mathrm{F}(\varphi c_2 + (1-\varphi)c_1) \leq {}_{cr} h(\varphi)\ln \mathrm{F}(c_2) + h(1-\varphi)\ln \mathrm{F}(c_1). \tag{9}$$

On multiplying Equations (8) and (9) with $\varphi^{\varsigma-1}$, and after integrating between 0 to 1 w.r.t. φ, we have

$$\begin{aligned} &\frac{\Gamma(\varsigma)}{(c_2-c_1)^{\varsigma}} J_{c_2}^{\varsigma} \ln \mathrm{F}(\mu) && (10) \\ = \;& \ln \mathrm{F}(c_1)\int_0^1 \varphi^{\varsigma-1}h(\varphi)d\varphi + \ln \mathrm{F}(c_2)\int_0^1 (1-\varphi)^{\varsigma-1}h(\varphi)d\varphi. \\ &\frac{\Gamma(\varsigma)}{(c_2-c_1)^{\varsigma}} J_{c_1}^{\varsigma} \ln \mathrm{F}(\mu) && (11) \\ = \;& \ln \mathrm{F}(c_2)\int_0^1 \varphi^{\varsigma-1}h(\varphi)d\varphi + \ln \mathrm{F}(c_1)\int_0^1 (1-\varphi)^{\varsigma-1}h(\varphi)d\varphi. \end{aligned}$$

On combining Equations (10) and (11), we obtain

$$\begin{aligned} &\frac{\Gamma(\varsigma)}{(c_2-c_1)^{\varsigma}}\left[J_{c_1-}^{\varsigma} \ln \mathrm{F}(\mu) + J_{c_2+}^{\varsigma} \ln \mathrm{F}(\mu)\right] \\ = \;& [\ln \mathrm{F}(c_1)\mathrm{F}(c_2)]\int_0^1 \left[(1-\varphi)^{\varsigma-1} + \varphi^{\varsigma-1}\right]h(\varphi)d\varphi. \end{aligned}$$

□

Corollary 1. *Taking* $\underline{\mathrm{F}} = \overline{\mathrm{F}}$ *and* $\varsigma = 1$, *we obtain ([11], Theorem 4.3).*

Corollary 2. *For* $h(\varphi) = 1$, *we have*

$$\begin{aligned} \sqrt{\mathrm{F}\left(\frac{c_1+c_2}{2}\right)} &\leq {}_{cr} \exp\left[\frac{\Gamma(\varsigma+1)}{2(c_2-c_1)^{\varsigma}}\left[J_{c_2}^{\varsigma} \ln \mathrm{F}(\mu) + J_{c_1}^{\varsigma} \ln \mathrm{F}(\mu)\right]\right] \\ &\leq {}_{cr} \mathrm{F}(c_1)\mathrm{F}(c_2). \end{aligned}$$

Corollary 3. *If* $h(\varphi) = \varphi^{\varsigma}$, *we obtain*

$$\begin{aligned} \ln \mathrm{F}\left(\frac{c_1+c_2}{2}\right)^{\frac{1}{h(\frac{1}{2})}} &\leq {}_{cr} \exp\left[\frac{\Gamma(\varsigma)}{(c_2-c_1)^{\varsigma}}\left[J_{c_2}^{\varsigma} \ln \mathrm{F}(\mu) + J_{c_1}^{\varsigma} \ln \mathrm{F}(\mu)\right]\right] \\ &\leq {}_{cr} \left(\frac{1}{1+\varsigma} + B(\varsigma+1, \varsigma)\right)\ln \mathrm{F}(c_1)\mathrm{F}(c_2), \end{aligned}$$

where B(x,y) is the Euler Beta function, it is defined as:

$$B(x,y) := \int_0^1 \varphi^{x-1}(1-\varphi)^{y-1}d\varphi.$$

Corollary 4. *For $\varsigma = 1$, we obtain ([2], Theorem 3.8):*

$$\begin{aligned} F\left(\frac{c_1+c_2}{2}\right)^{\frac{1}{2h(\frac{1}{2})}} &\leq_{cr} \exp\left[\frac{1}{c_2-c_1}\int_{c_1}^{c_2} \ln F(\mu)d\mu\right] \\ &\leq_{cr} (F(c_1)F(c_2))^{\int_0^1 h(\varphi)d\varphi}. \end{aligned} \tag{12}$$

Corollary 5. *If $h(\varphi) = 1$ and $\varsigma = 1$, we have*

$$\begin{aligned} \sqrt{F\left(\frac{c_1+c_2}{2}\right)} &\leq_{cr} \exp\left(\frac{1}{c_2-c_1}\int_{c_1}^{c_2} \ln F(\mu)d\mu\right) \\ &\leq_{cr} F(c_1)F(c_2). \end{aligned} \tag{13}$$

Now, we give weighted version inequality for cr-log-h-convexity.

Theorem 6. *Assume F be an interval valued function where $F : [c_1, c_2] \to \mathbb{R}_I^+$ and $F = [\underline{F}, \overline{F}]$ with $F \in I\mathbb{R}_{([c_1,c_2])}$, and $h : [0, 1] \to \mathbb{R}^+$ with $h(\frac{1}{2}) \neq 0$. Let $\tilde{H} : [c_1, c_2] \to \mathbb{R}^+$ be a function which is symmetric with respect to $\frac{c_1+c_2}{2}$. If $F \in SX(cr - \log -h, [c_1, c_2], \mathbb{R}_I^+)$, then*

$$\begin{aligned} &\frac{1}{2h(\frac{1}{2})} \ln F\left(\frac{c_1+c_2}{2}\right) \frac{\Gamma(\varsigma)}{(c_2-c_1)^{\varsigma}}\left[J^{\varsigma}_{c_1-}\tilde{H}(\mu) + J^{\varsigma}_{c_2+}\tilde{H}(\mu)\right] \\ \leq_{cr}\ & \frac{\Gamma(\varsigma)}{(c_2-c_1)^{\varsigma}}\left[J^{\varsigma}_{c_2+}[\ln F(\mu)]\tilde{H}(\mu) + J^{\varsigma}_{c_1-}[\ln F(\mu)]\tilde{H}(\mu)\right] \\ \leq_{cr}\ & (\ln F(c_1)F(c_2)) \int_0^1 \left[\varphi^{\varsigma-1} + (1-\varphi)^{\varsigma-1}\right]\tilde{H}(\varphi)\tilde{H}(\varphi c_1 + (1-\varphi)c_2)d\varphi. \end{aligned} \tag{14}$$

Proof. As $F \in SX(cr - \log -h, [c_1, c_2], \mathbb{R}_I^+)$, then

$$\ln F(\varphi c_1 + (1-\varphi)c_2) \leq_{cr} h(\varphi) \ln F(c_1) + h(1-\varphi) \ln F(c_2). \tag{15}$$

$$\ln F(\varphi c_2 + (1-\varphi)c_1) \leq_{cr} h(\varphi) \ln F(c_2) + h(1-\varphi) \ln F(c_1). \tag{16}$$

Multiplying Equation (15) by $\varphi^{\varsigma-1}\tilde{H}(\varphi c_1 + (1-\varphi)c_2)$ and integrating between 0 to 1 w.r.t. φ, we obtain

$$\begin{aligned} &\int_0^1 \varphi^{\varsigma-1}[\ln F(\varphi c_1 + (1-\varphi)c_2)]\tilde{H}(\varphi c_1 + (1-\varphi)c_2)d\varphi \\ \leq_{cr}\ & \int_0^1 \varphi^{\varsigma-1}\tilde{H}(\varphi)[\ln F(c_1)]\tilde{H}(\varphi c_1 + (1-\varphi)c_2)d\varphi \\ &+ \int_0^1 \varphi^{\varsigma-1}\tilde{H}(1-\varphi)\tilde{H}(\varphi c_1 + (1-\varphi)c_2)[\ln F(c_2)]d\varphi. \end{aligned}$$

We obtain

$$\begin{aligned} &\frac{1}{(c_2-c_1)^{\varsigma}}\int_{c_1}^{c_2} (c_2-\mu)^{\varsigma-1}[\ln F(\mu)]\tilde{H}(\mu)d\mu \\ \leq_{cr}\ & \ln F(c_1) \int_0^1 \varphi^{\varsigma-1}\tilde{H}(\varphi)\tilde{H}(\varphi c_1 + (1-\varphi)c_2)d\varphi \\ &+ \ln F(c_2)] \int_0^1 \varphi^{\varsigma-1}\tilde{H}(1-\varphi)\tilde{H}(\varphi c_1 + (1-\varphi)c_2)d\varphi. \end{aligned} \tag{17}$$

Multiplying Equation (16) by $\varphi^{\varsigma-1}\tilde{H}(\varphi c_2 + (1-\varphi)c_1)$ and integrating between 0 to 1 with respect to φ, we have

$$\begin{aligned} & \int_0^1 \varphi^{\varsigma-1}[\ln \mathrm{F}(\varphi c_2 + (1-\varphi)c_1)]\tilde{H}(\varphi c_1 + (1-\varphi)c_2)d\varphi \\ \leq & \ _{cr}\int_0^1 \varphi^{\varsigma-1}\tilde{H}(\varphi)[\ln \mathrm{F}(c_2)]\tilde{H}(\varphi c_2 + (1-\varphi)c_1)d\varphi \\ & + \int_0^1 \varphi^{\varsigma-1}\tilde{H}(1-\varphi)[\ln \mathrm{F}(c_1)]\tilde{H}(\varphi c_2 + (1-\varphi)c_1)d\varphi. \end{aligned}$$

We obtain

$$\begin{aligned} & \frac{1}{(c_2-c_1)^{\varsigma}}\int_{c_1}^{c_2}(\mu - c_1)^{\varsigma-1}[\ln \mathrm{F}(\mu)]\tilde{H}(\mu)d\mu \\ \leq & \ _{cr}\ln \mathrm{F}(c_2)\int_0^1 \varphi^{\varsigma-1}\tilde{H}(\varphi)\tilde{H}(\varphi c_2 + (1-\varphi)c_1)d\varphi \\ & + \ln \mathrm{F}(c_1)]\int_0^1 \varphi^{\varsigma-1}\tilde{H}(1-\varphi)\tilde{H}(\varphi c_2 + (1-\varphi)c_1)d\varphi. \end{aligned} \tag{18}$$

On combining Equations (17) and (18), we obtain

$$\begin{aligned} & \frac{\Gamma(\varsigma)}{(c_2-c_1)^{\varsigma}}\left[J_{c_2+}^{\varsigma}[\ln \mathrm{F}(\mu)]\tilde{H}(\mu) + J_{c_1-}^{\varsigma}[\ln \mathrm{F}(\mu)]\tilde{H}(\mu)\right] \\ = & \ (\ln \mathrm{F}(c_1)\mathrm{F}(c_2))\int_0^1\left[\varphi^{\varsigma-1} + (1-\varphi)^{\varsigma-1}\right]\tilde{H}(\varphi)\tilde{H}(\varphi c_1 + (1-\varphi)c_2)d\varphi. \end{aligned} \tag{19}$$

Now, multiplying Equation (6) by $\varphi^{\varsigma-1}\tilde{H}(\varphi c_1 + (1-\varphi)c_2)$ and after integrating between 0 to 1 w.r.t. φ, we have

$$\begin{aligned} & \frac{1}{h(\frac{1}{2})}\ln \mathrm{F}\left(\frac{c_1+c_2}{2}\right)\int_0^1 \varphi^{\varsigma-1}\tilde{H}(\varphi c_1 + (1-\varphi)c_2)d\varphi \\ \leq & \ _{cr}\int_0^1 \varphi^{\varsigma-1}\tilde{H}(\varphi c_1 + (1-\varphi)c_2)\ln \mathrm{F}(\varphi c_1 + (1-\varphi)c_2)d\varphi \\ & + \int_0^1 \varphi^{\varsigma-1}\tilde{H}(\varphi c_2 + (1-\varphi)c_1)\ln \mathrm{F}(\varphi c_2 + (1-\varphi)c_1)d\varphi. \end{aligned}$$

Utilizing definition of generalized fractional integral

$$\begin{aligned} & \frac{1}{h(\frac{1}{2})}\ln \mathrm{F}\left(\frac{c_1+c_2}{2}\right)\frac{\Gamma(\varsigma)}{(c_2-c_1)^{\varsigma}}J_{c_2}^{\varsigma}\tilde{H}(\mu) \\ = & \ \frac{1}{(c_2-c_1)^{\varsigma}}\int_{c_1}^{c_2}(c_2-\mu)^{\varsigma-1}\tilde{H}(\mu)\ln \mathrm{F}(\mu)d\mu \\ & + \frac{1}{(c_2-c_1)^{\varsigma}}\int_{c_1}^{c_2}(\mu-c_1)^{\varsigma-1}\tilde{H}(\mu)\ln \mathrm{F}(\mu)d\mu. \end{aligned} \tag{20}$$

Multiplying Equation (6) by $\varphi^{\varsigma-1}\tilde{H}(\varphi c_2 + (1-\varphi)c_1)$ and after integrating between 0 to 1 w.r.t. φ, we obtain

$$\begin{aligned} & \frac{1}{h(\frac{1}{2})}\ln \mathrm{F}\left(\frac{c_1+c_2}{2}\right)\int_0^1 \varphi^{\varsigma-1}\tilde{H}(\varphi c_2 + (1-\varphi)c_1)d\varphi \\ \leq & \ _{cr}\int_0^1 \varphi^{\varsigma-1}\tilde{H}(\varphi c_1 + (1-\varphi)c_2)\ln \mathrm{F}(\varphi c_1 + (1-\varphi)c_2)d\varphi \\ & + \int_0^1 \varphi^{\varsigma-1}\tilde{H}(\varphi c_2 + (1-\varphi)c_1)\ln \mathrm{F}(\varphi c_2 + (1-\varphi)c_1)d\varphi. \end{aligned}$$

Utilizing definition of generalized fractional integral

$$\begin{aligned} &\frac{1}{h(\frac{1}{2})}\ln \mathrm{F}\left(\frac{c_1+c_2}{2}\right)\frac{\Gamma(\varsigma)}{(c_2-c_1)^{\varsigma}}J^{\varsigma}_{c_1}\tilde{H}(\mu) \\ = &\frac{1}{(c_2-c_1)^{\varsigma}}\int_{c_1}^{c_2}(c_2-\mu)^{\varsigma-1}\tilde{H}(\mu)\ln \mathrm{F}(\mu)d\mu \\ &+\frac{1}{(c_2-c_1)^{\varsigma}}\int_{c_1}^{c_2}(\mu-c_1)^{\varsigma-1}\tilde{H}(\mu)\ln \mathrm{F}(\mu)d\mu. \end{aligned} \tag{21}$$

On utilizing Equations (20) and (21), we obtain

$$\begin{aligned} &\frac{1}{h(\frac{1}{2})}\ln \mathrm{F}\left(\frac{c_1+c_2}{2}\right)\frac{\Gamma(\varsigma)}{(c_2-c_1)^{\varsigma}}\left[J^{\varsigma}_{c_1-}\tilde{H}(\mu)+J^{\varsigma}_{c_2+}\tilde{H}(\mu)\right] \\ = &\frac{2}{(c_2-c_1)^{\varsigma}}\int_{c_1}^{c_2}(c_2-\mu)^{\varsigma-1}\tilde{H}(\mu)\ln \mathrm{F}(\mu)d\mu \\ &+\frac{2}{(c_2-c_1)^{\varsigma}}\int_{c_1}^{c_2}(\mu-c_1)^{\varsigma-1}\tilde{H}(\mu)\ln \mathrm{F}(\mu)d\mu \\ = &\frac{2\Gamma(\varsigma)}{(c_2-c_1)^{\varsigma}}\left[J^{\varsigma}_{c_2+}[\ln \mathrm{F}(\mu)]\tilde{H}(\mu)+J^{\varsigma}_{c_1-}[\ln \mathrm{F}(\mu)]\tilde{H}(\mu)\right]. \end{aligned} \tag{22}$$

On combining Equations (19) and (22), we complete the proof. □

Corollary 6. *For $\varsigma = 1$, we obtain*

$$\begin{aligned} &\frac{1}{2h(\frac{1}{2})}\ln \mathrm{F}\left(\frac{c_1+c_2}{2}\right)\frac{1}{(c_2-c_1)}\int_{c_1}^{c_2}\ln \mathrm{F}(\mu)d\mu \\ \leq\ &_{cr}\frac{1}{(c_2-c_1)}\int_{c_1}^{c_2}\tilde{H}(\mu)\ln \mathrm{F}(\mu)d\mu \\ \leq\ &_{cr}(\ln \mathrm{F}(c_1)\mathrm{F}(c_2))\int_0^1\tilde{H}(\varphi)\tilde{H}(\varphi c_1+(1-\varphi)c_2)d\varphi. \end{aligned} \tag{23}$$

Corollary 7. *For $\tilde{H}(\mu) = 1$ in Equation (23), we obtain Equation (3).*

3. Examples

Example 1. *Let $F(\mu) = [0,2] \to \mathbb{R}^+_I$ is an interval-valued function given by $F(\mu) = [e^{\mu}, e^{2\mu}]$. Suppose $h(\varphi) = \varphi$ for all $\varphi \in [0,1]$, then for $\varsigma = 2$ we have*

$$\begin{aligned} \frac{c_1+c_2}{2} &= 1, \\ \mathrm{F}\left(\frac{c_1+c_2}{2}\right)^{\frac{1}{2h(\frac{1}{2})}} &= [2.72, 7.39], \\ \exp\left[\frac{\Gamma(\varsigma)}{(c_2-c_1)^{\varsigma}}\left(J^{\varsigma}_{c_2+}\ln \mathrm{F}(\mu)+J^{\varsigma}_{c_1-}\ln \mathrm{F}(\mu)\right)\right] & \\ &= [2.72, 7.39], \\ [\mathrm{F}(c_1)\mathrm{F}(c_2)]^{\int_0^1((1-\varphi)^{\varsigma-1}+\varphi^{\varsigma-1})h(\varphi)d\varphi} &= \left([e^2, e^4][1,1]\right)^{\frac{1}{2}} \\ &= [2.72, 7.39]. \end{aligned}$$

Since $[2.72, 7.39] \leq_{cr} [2.72, 7.39] \leq_{cr} [2.72, 7.39]$, then Theorem 5 is verified.

Example 2. *Let* $F(\mu) = [1,2] \to \mathbb{R}_I^+$ *is an interval-valued function given by* $F(\mu) = [\mu, 2\mu]$*. Suppose* $h(\varphi) = \cos\varphi$ *for all* $\varphi \in [0,1]$*, then for* $\varsigma = 2$ *we have*

$$\begin{aligned}
\frac{c_1 + c_2}{2} &= \frac{3}{2},\ h(\frac{1}{2}) = 0.87758,\\
F\left(\frac{c_1 + c_2}{2}\right)^{\frac{k}{\varsigma h(\frac{1}{2})}} &= [1.26, 1.87],\\
\exp\left[\frac{\Gamma(\varsigma)}{(c_2 - c_1)^{\zeta}}\left(J_{c_2+}^{\varsigma} \ln F(\mu) + J_{c_1-}^{\varsigma} \ln F(\mu)\right)\right]&\\
&= [1.47, 2.94],\\
[F(c_1)F(c_2)]^{\int_0^1 \left(\varphi^{\zeta-1} + (1-\varphi)^{\zeta-1}\right) h(\varphi) d\varphi} &= ([1,2][2,4])^{0.8415}\\
&= [1.79, 5.75].
\end{aligned}$$

Since $[1.26, 1.87] \leq_{cr} [1.47, 2.94] \leq_{cr} [1.79, 5.75]$*, then Theorem 5 is verified.*

Example 3. *Let* $F(\mu) = [1,e] \to \mathbb{R}_I^+$ *is an interval-valued function given by* $F(\mu) = [\mu, \mu^2]$*. Suppose* $h(\varphi) = 1 + \ln(\cos\varphi)$ *for all* $\varphi \in [0,1]$*, then for* $\varsigma = 2$ *we have*

$$\begin{aligned}
\frac{c_1 + c_2}{2} &= \frac{1+e}{2},\ h(\frac{1}{2}) = 0.8694,\\
F\left(\frac{c_1 + c_2}{2}\right)^{\frac{1}{\varsigma h(\frac{1}{2})}} &= [1.428, 2.041],\\
\exp\left[\frac{\Gamma(\varsigma)}{(c_2 - c_1)^{\zeta}}\left(J_{c_2+}^{\varsigma} \ln F(\mu) + J_{c_1-}^{\varsigma} \ln F(\mu)\right)\right]&\\
&= [1.789, 3.202],\\
[F(c_1)F(c_2)]^{\int_0^1 \left(\varphi^{\zeta-1} + (1-\varphi)^{\zeta-1}\right) h(\varphi) d\varphi} &= ([1,2][2,4])^{0.8415}\\
&= [2.253, 5.077].
\end{aligned}$$

Since $[1.428, 2.041] \leq_{cr} [1.789, 3.202] \leq_{cr} [2.253, 5.077]$*, then Theorem 5 is verified.*

4. Applications

4.1. Modified Bessel Functions

Let recall $\psi_{\overline{v}}(z) : \mathbb{R} \to [1, \infty]$ given by Watson's ([24], pp. 294, 480):

$$\psi_{\overline{v}}(z) = 2^{\overline{v}}\Gamma(\overline{v}+1)z^{-\overline{v}}\mathrm{F}_{\overline{v}}(z),\ z \in \mathbb{R},$$

where $\mathrm{F}_{\overline{v}}(z)$ is the modified Bessel function of the first kind:

$$\mathrm{F}_{\overline{v}}(z) = \sum_{\mathrm{n}=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{\overline{v}+2\mathrm{n}}}{\mathrm{n}!\Gamma(\overline{v}+\mathrm{n}+1)}.$$

Then, the relations for $\psi'_{\overline{v}}(z)$ and $\psi''_{\overline{v}}(z)$ are as follows:

$$\begin{aligned}
\psi'_{\overline{v}}(z) &:= \frac{z}{2(\overline{v}+1)}\psi_{\overline{v}+1}(z).\\
\psi''_{\overline{v}}(z) &:= \frac{1}{4(\overline{v}+1)}\left[\frac{z^2}{\overline{v}+2}\psi_{\overline{v}+2}(z) + 2\psi_{\overline{v}+1}(z)\right].
\end{aligned} \tag{24}$$

Let $\mathrm{F}_{\overline{v}}(z) = \psi'_{\overline{v}}(z)$ and $h(\varphi) = 1$. Then, from inequality Equation (13) and utilizing the identities in Equation (24), we can deduce

$$\begin{aligned}\sqrt{\frac{c_1+c_2}{4(\overline{v}+1)}\psi_{\overline{v}+1}\left(\frac{c_1+c_2}{2}\right)} &\leq_{cr} \exp\left[\frac{1}{c_2-c_1}\int_{c_1}^{c_2}\ln(\psi'_{\overline{v}}(\mu))d\mu\right] \\ &\leq_{cr} \frac{c_1c_2}{4(\overline{v}+1)^2}\psi_{\overline{v}+1}(c_1)\psi_{\overline{v}+1}(c_2).\end{aligned}$$

4.2. Special Means

1. Arithmetic mean:

$$\mathrm{A}(c_1,c_2) := \frac{c_1+c_2}{2}.$$

2. Geometric mean:

$$\mathrm{G}(c_1,c_2) := \sqrt{c_1c_2},\ 0 \leq c_1 < c_2.$$

Proposition 1. *Let $c_1, c_2 \in \mathbb{R}_+$ where $c_1 < c_2$ and $\mathrm{m} \in \mathbb{N}$, then*

$$\begin{aligned}&\sqrt{\mathrm{A}^{\mathrm{m}+1}(c_1,c_2)} \\ \leq_{cr}\ & \exp\left(\frac{\mathrm{m}+1}{(c_2-c_1)}[c_2(\ln c_2-1)-c_1(\ln c_1-1)]\right) \\ \leq_{cr}\ & \mathrm{G}^{2(\mathrm{m}+1)}(c_1,c_2).\end{aligned}$$

Proof. Obviously $\mathrm{F}(v) = v^{\mathrm{m}+1}$ is a convex function on $\mathbb{R}_+$. From Equation (13), we obtain the required inequality. □

Proposition 2. *For $c_1, c_2 \in \mathbb{R}_+$ where $c_1 < c_2$, then*

$$\sqrt{e^{A(c_1,c_2)}} \leq_{cr} e^{A(c_1,c_2)} \leq_{cr} e^{2A(c_1,c_2)}.$$

Proof. For $\mathrm{F}(v) = e^v$ is a convex function on $\mathbb{R}$ where $v \in \mathbb{R}$. From Equation (13), we obtain the required inequality. □

5. Conclusions

In this paper, using the notion of cr-log-h-convexity for interval-valued functions several types of Hermite–Hadamard and Fejér inequalities that are related to interval-valued cr-log-h-convex functions have been given. Moreover, several special cases are given and some known results are recaptured. To show the validity of our main results, we have offered three examples and two applications related to modified Bessel functions of the first kind, and special means. We believe that this class of convexity is a powerful type to find various type inequalities in the fields of fuzzy systems and real analysis, and with possible applications to optimization problems with convex shapes associated with them.

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