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Review

The Fox Trapezoidal Conjecture for Alternating Knots

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Abstract: A long-standing conjecture due to R. Fox states that the coefficients of the Alexander polynomial of an alternating knot exhibit a trapezoidal pattern. In other words, these coefficients increase, stabilize, then decrease in a symmetric way. A stronger version of this conjecture states that these coefficients form a log-concave sequence. This conjecture has been recently highlighted by J. Huh as one of the most interesting problems on log-concavity of sequences. In this expository paper, we shall review the various versions of the conjecture, highlight settled cases and outline some future directions.

Keywords: alternating knot; Alexander polynomial; trapezoidal conjecture; log-concave

1. Introduction

Let n be a positive integer. An n-component link is defined as the embedding of n disjoint circles $\coprod S^1$ within the three-dimensional sphere S^3. A one-component link is called a knot. Links are represented by their projections on S^2 (equivalently in the Euclidean plane $\mathbb{R}^2$). A link diagram is a regular planar projection of the link, augmented with height information indicating the overpass and the underpass of each double point where two strands intersect. A classical result in knot theory asserts that the study of links up to natural deformations (isotopies) is equivalent to the study of link diagrams up to local transformations known as Reidemeister moves [1]. The main purpose of knot theory is the classification of these objects up to isotopy. Indeed, the central question in knot theory is to decide whether two given knots or links are isotopic. While no simple algorithm solving this problem exists, the study and the classification of certain particular classes of links is an important step towards this primary goal of classification. This paper is concerned with one of the most important classes of links.

An *alternating* link is defined as a link which can be represented by an alternating diagram, a diagram where the overpass and the underpass alternate as one follows any strand. Extensive research has been dedicated to studying alternating links. In particular, it was shown that certain of their polynomial invariants exhibit the alternating character of the diagram in a remarkable way. Investigating the Jones polynomials of these links has resulted in the resolution of important conjectures in classical knot theory [2–4]. A topological characterization of alternating links is obtained in [5,6]. It can be easily verified that all knots with at most seven crossings are alternating. The first non-alternating knot in Rolfsen's knot table [7] is 8_{19}, which is indeed the $(3,4)$-torus knot. Diagrams of an alternating knot and a non-alternating one are displayed in Figure 1 below.

The Alexander polynomial [8] is a topological invariant of oriented links which assigns to each link L a Laurent polynomial with integral coefficients $\Delta_L(t) \in \mathbb{Z}[t^{\pm 1/2}]$. Conway introduced a simple recursive way to define this polynomial [9]. Additionally, the Alexander polynomial can be derived from the reduced Burau representation of the braid group [10]. More recently, Ozsváth and Szabó [11] showed that this polynomial, up to the multiplication by some factor, can be defined as the Euler characteristic of the link Floer homology. The Alexander polynomial of a knot K is symmetric. More precisely, for any knot K, we

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have $\Delta_K(t) = \Delta_K(t^{-1})$. Hence, it is always possible to write $\Delta_K(t) = \sum_{i=-g}^{g} a_i t^i$ with $a_g \neq 0$ and $a_i = a_{-i}$ for all $-g \leq i \leq g$. We will refer to the integer g as the degree of $\Delta_K(t)$. It is worth mentioning that since $\Delta_K(t)$ is defined up to the multiplication by a unit of $\mathbb{Z}[t^{1/2}, t^{-1/2}]$, we find it sometimes more convenient to write the polynomial in the form $\Delta_K(t) = \sum_{i=0}^{2g} \alpha_i t^i$.

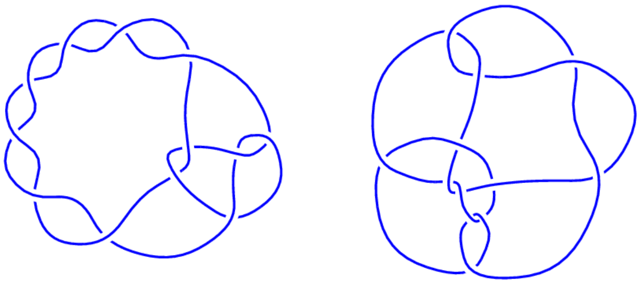

Figure 1. An alternating diagram of the knot $12a146$ (**left**) and a non-alternating knot digram of the knot $12n146$ (**right**).

Murasugi [12,13] proved that the coefficients of the Alexander polynomial of an alternating knot satisfy the condition $a_i a_{i+1} < 0$ for all $-g \leq i < g$. Consequently, the Alexander polynomial of an alternating knot can be represented in the form $\Delta_K(t) = \pm \sum_{i=-g}^{g} a_i (-t)^i$, with $a_i > 0$. Furthermore, Murasugi showed that if K is alternating, then the degree of $\Delta_K(t)$ is equal to the genus of the knot [12,13]. Fox [14] posed the question of whether it is possible to characterize polynomials that appear as the Alexander polynomial of an alternating knot, and conjectured that the coefficients of the Alexander polynomial of such a knot form a trapezoidal pattern. This means that these coefficients increase, stabilize, and then decrease symmetrically.

Conjecture 1 ([14])**.** *Let K be an alternating knot and $\Delta_K(t) = \pm \sum_{i=-g}^{g} a_i (-t)^i$, with $a_i > 0$, its Alexander polynomial. Then there exists an integer $0 \leq l \leq g$ such that:*

$$a_{-g} < \cdots < a_{-l/2} = \cdots = a_{l/2} > \cdots > a_g.$$

Over the past fifty years, Conjecture 1 has been confirmed for various families of alternating knots. Notably, Hartley confirmed the conjecture for the class of two-bridge knots [15]. Murasugi proved that the conjecture holds for a large family of alternating algebraic knots [16]. Using knot Floer homology, Ozsváth and Szabó [11] confirmed the validity of the conjecture for alternating knots of genus 2, a result independently proved by Jong using combinatorial methods [17,18]. Hirasawa and Murasugi [19] confirmed that the conjecture holds for stable alternating knots. Furthermore, they proposed a refinement of Conjecture 1, suggesting that the length of the stable part should not exceed $|\sigma(K)|$, where $\sigma(K)$ denotes the knot's signature. This refined version of the conjecture was confirmed for two-bridge knots [20] and for certain classes of closed alternating 3-braids and 4-braids [21–23].

On the other hand, Stoimenow [24] proposed a strong version of the conjecture. Indeed, he conjectured that the coefficients of the Alexander polynomial of an alternating knot are logarithmically concave. In other words, they satisfy the condition $a_i^2 \geq |a_{i+1}||a_{i-1}|$ for $0 < i < 2g$. In [25], Huh highlighted this conjecture as one of the most interesting open problems about log-concavity of sequences. The strong Fox conjecture was proved for two-bridge knots [26] and for special alternating knots [27].

Let us consider the alternaing knots $12a146$ in Figure 1 (left) and the weaving knot $W(3,5)$ in Figure 2. Recall that these knots have signatures 6 and 0, respectively. According to [28], their Alexander polynomials are given by:

$$\begin{aligned} \Delta_{12a146}(t) = &\quad 1 - 3t + 6t^2 - 7t^3 + 7t^4 - 7t^5 + 7t^6 - 7t^7 + 6t^8 - 3t^9 + t^{10}. \\ \Delta_{W(3,5)}(t) = &\quad 1 - 6t + 15t^2 - 24t^3 + 29t^4 - 24t^5 + 15t^6 - 6t^7 + t^8. \end{aligned}$$

These polynomials are trapezoidal and the lengths of the stable parts are 4 and 0, respectively.

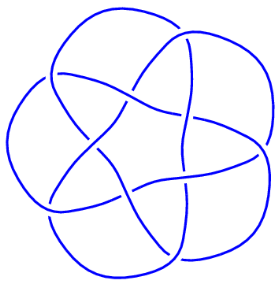

Figure 2. The weaving knot $W(3,5)$ which is the knot 10_{123} in Rolfsen's Table.

Here is an outline of this paper. In Section 2, we shall define the Alexander polynomial and recall its properties, which are relevant to our context. In Section 3, we shall survey the literature on the progress made toward solving the Fox trapezoidal conjecture. In Section 4, we consider the case of closed alternating braids. Finally, in Section 5, we shall discuss the generalization of the conjecture to other classes of links.

2. The Alexander Polynomial

The Alexander polynomial [8] is an invariant of oriented links which plays a fundamental role in classical knot theory. This topological invariant is a one-variable Laurent polynomial with integral coefficients $\Delta_L(t)$. Several distinct yet equivalent methods exist for defining this polynomial. Notably, it can be defined recursively using Conway skein relations [9]:

$$\begin{aligned} &\Delta_U(t) = 1, \\ &\Delta_{L_+}(t) - \Delta_{L_-}(t) = (\sqrt{t} - \frac{1}{\sqrt{t}})\Delta_{L_0}(t), \end{aligned}$$

where U denotes the unknot and L_+, L_- and L_0 depict three oriented link diagrams that are identical except in a small region where they are as displayed in Figure 3.

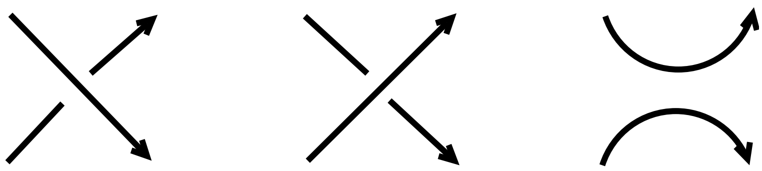

Figure 3. The three oriented links L_+, L_- and L_0, respectively.

It is worth noting that this polynomial is symmetric, in the sense that it satisfies $\Delta_L(t) = \pm\Delta_L(t^{-1})$ for any link L.

For $n \geq 2$, let B_n denote the group of braids on n strands. This group is generated by the elementary braids $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$, see Figure 4 subject to the following relations:

$$\begin{gathered} \sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j| \geq 2 \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1},\ \forall\, 1 \leq i \leq n-2. \end{gathered}$$

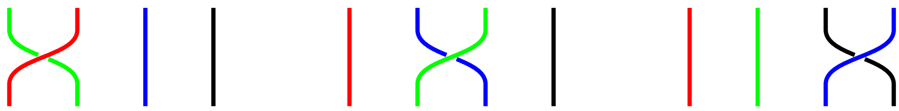

Figure 4. The 3 generators σ_1, σ_2 and σ_3 of the braid group B_4.

By Alexander's theorem [29], we know that any link L can be represented as the closure of a braid $b \in B_n$; $L = \mathrm{cl}(b)$. The Alexander polynomial can be defined via the reduced Burau representation of the braid group [10]. This representation is defined as follows. Let b be a given n-braid and e_b the exponent sum of b as a word in the elementary braids $\sigma_1 \ldots \sigma_{n-1}$. Let $\psi_{n,t} : B_n \longrightarrow GL(n-1, \mathbb{Z}[t, t^{-1}])$ be the reduced Burau representation defined on the generators of B_n by:

$$\psi_{n,t}(\sigma_1) = \left(\begin{array}{cc|c} -t & 1 & 0 \\ 0 & 1 & 0 \\ ine0 & 0 & I_{n-3} \end{array} \right),$$

$$\psi_{n,t}(\sigma_i) = \left(\begin{array}{c|ccc|c} I_{n-2} & 0 & 0 & 0 & 0 \\ ine0 & 1 & 0 & 0 & 0 \\ 0 & t & -t & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ ine0 & 0 & 0 & 0 & I_{n-i+2} \end{array} \right) \text{ for } 2 \leq i \leq n-2,$$

$$\psi_{n,t}(\sigma_{n-1}) = \left(\begin{array}{c|cc} I_{n-3} & 0 & 0 \\ ine0 & 1 & 0 \\ 0 & t & -t \end{array} \right)$$

where I_k denotes the $k \times k$-identity matrix. The Alexander polynomial, up to the multiplication by a unit of $\mathbb{Z}[t^{1/2}, t^{-1/2}]$ of the link $L = \mathrm{cl}(b)$ is obtained from the Burau representation by the following formula:

$$\Delta_L(t) = \left(\frac{-1}{\sqrt{t}}\right)^{e_b - n + 1} \frac{1-t}{\det(1 - \psi_{n,t}(\sigma_1 \ldots \sigma_{n-1}))} \det(I_{n-1} - \psi_{n,t}(b)).$$

A categorification of the Alexander polynomial has been introduced by Ozsváth and Szabó in [30] and independently by Rasmussen in [31], who defined a bi-graded link homology theory $\widehat{HFK}^{*,*}$ called link Floer homology. This categorification provides an alternative view on the Alexander polynomial. Indeed, for any l-component link L, the graded Euler characteristic of $\widehat{HFK}^{*,*}(L)$ is, up to the multiplication by a factor, the Alexander–Conway polynomial of L. More precisely, we have

$$(t^{-1/2} - t^{1/2})^{l-1} \Delta_L(t) = \sum_{j \in \mathbb{Z}, i \in \mathbb{Z} + \frac{l-1}{2}} (-1)^{i + \frac{l-1}{2}} t^j rank(\widehat{HFK}^{i,j}(L)).$$

It is well-known that for any Laurent polynomial $f(t)$ satisfying the conditions $f(t) = f(t^{-1})$ and $f(1) = 1$, there exists a knot K whose Alexander polynomial is $f(t)$. Given an oriented knot K, the genus of K, denoted hereafter $g(K)$, is the minimal genus of a Seifert surface of K. Recall that $g(K)$ is an upper bound for the degree of the Alexander polynomial of K. Moreover, for alternating knots, we have:

Theorem 1 ([12,13])**.** *Suppose that K is an alternating knot, then:*

1. *The genus of the knot $g(K)$ is equal to the degree of $\Delta_K(t)$.*
2. *For all i, $a_i \neq 0$ and $a_i a_{i+1} < 0$, for all $-g \leq i \leq g-1$.*

Similar results for alternating links can be found in [32]. Notice that the coefficients of the Alexander polynomial of an alternating knot alternate in sign with no internal zero coefficients, as stated in the theorem above. Given an oriented link L, the signature of L denoted here as $\sigma(L)$ is a numerical topological invariant of links which is derived from the Seifert matrix of L. It is well-known that the signature of a knot is always an even integer. The determinant of a link L, denoted hereafter as $\det(L)$, is another link invariant obtained from the Seifert matrix of the link and is expressed as $\det(L) = |\Delta_L(-1)|$.

An interesting subclass of alternating links that will appear in our discussions in the following sections is the class of special alternating links. Recall that a link is said to be *positive* if it can be represented by a diagram in which all crossings are positive. A link is said to be *special alternating* if it is both positive and alternating.

3. Fox Trapezoidal Conjecture

A polynomial with real positive coefficients $f(x) = \sum_{k=0}^{n} \alpha_k x^k$ is said to be *logarithmically concave* , or log-concave for short, if the sequence (α_k) satisfies the condition $\alpha_k^2 \geq \alpha_{k-1}\alpha_{k+1}$ for all $0 < k < n$. On the other hand, f is said to be *unimodal* if for some $0 \leq j \leq n$, we have $\alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_j \geq \alpha_{j+1} \geq \cdots \geq \alpha_n$. If there exist integers j and l such that $\alpha_0 < \alpha_1 < \cdots < \alpha_j = \cdots = \alpha_{j+l} > \cdots > \alpha_{n-1} > \alpha_n$, then f is said to be *trapezoidal*. Obviously, every trapezoidal polynomial is unimodal. It can also be checked easily that every log-concave polynomial with no internal zero coefficients is trapezoidal. Sequences and polynomilas with such properties have been subject to extensive studies. We refer the reader to [25,33] and the references therein for examples of sequences with such properties and their important applications in combinatorics, algebra and geometry.

Example 1. *The polynomial $f(t) = 1 + 2t + 4t^2 + 4t^3 + 4t^4 + 2t^5 + t^6$ is log-concave with no internal zero coefficients, hence trapezoidal and unimodal, while the polynomial $g(t) = 1 + 2t + 3t^2 + 5t^3 + 3t^4 + 2t^5 + t^6$ is trapezoidal but not log-concave.*

3.1. Original Fox Conjecture

In 1962, Fox published a list of open problems in knot theory [14]. Problem 12 in Fox's list reads as follows:

Problem 1 ([14])**.** *Characterize, among the knot polynomials, those that are polynomials of alternating knots, of special alternating knots.*

In addition to the conditions in Theorem 1, Fox conjectured that the coefficients of the Alexander polynomial of alternating knots are trapezoidal. Fox checked this property for alternating knots with fewer than 12 crossings. The first cases where the conjecture has been settled appeared no earlier than 15 years later. Indeed, in 1979, Hartley confirmed the conjecture for the class of two-bridge knots [15]. Around the same time, Parris [34] showed that the conjecture is true for alternating pretzel knots. In 1985, Murasugi [16] proved the conjecture for a large class of alternating algebraic (arborescent) knots. This class includes in particular two-bridge knots. It is worth mentioning here that Murasugi's proof is inductive and is based on the study of the Conway polynomial.

In 2003, Ozsváth and Szabó proved that the knot Floer homology of an alternating knot is determined by the Alexander polynomial and the signature of the knot. As a byproduct, they obtained that Conjecture 1 is true for alternating knots of genus 2. Let us recall the following results from [11].

Let K be a knot and r an integer. We denote by $t_r(K)$ the torsion coefficients defined by

$$t_r(K) = \sum_{j=1}^{\infty} j a_{|r|+j},$$

where here the a_r are the coefficients of the symmetrized Alexander polynomial of K. For $\sigma \in 2\mathbb{Z}$, we let $\delta(\sigma, r)$ be the rth torsion coefficient of the torus knot of type $(2, |\sigma| + 1)$. This integer is defined by the following formula:

$$\delta(\sigma, r) = \max(0, \lceil \frac{|\sigma| - 2|r|}{4} \rceil).$$

In [11], Ozsváth and Szabó proved the following result.

Theorem 2. *Let K be an alternating knot. Then, for all $r \in \mathbb{Z}$, we have that $(-1)^{r+\frac{\sigma}{2}}(t_r(K) - \delta(\sigma, r)) \leq 0$.*

Conjecture 1 holds for knots of genus 2 as a consequence of Theorem 2; see [11]. The trapezoidal property of the Alexander polynomial of alternating knots of genus 2 has also been obtained by Jong using combinatorial methods; see [17,18].

The study of the roots of the Alexander polynomial is closely related to the trapezoidal structure of its coefficients. For instance, we know that if the roots of the polynomial are all real or are all in the sector $\{z \in \mathbb{C}; \frac{2\pi}{3} \leq \arg(z) \leq \frac{4\pi}{3}\}$, then the polynomial is log-concave, hence trapezoidal [33]. A knot is said to be real stable (respectively, circular stable) if the roots of its Alexander polynomial are real (respectively, unit complex). Hirasawa and Murasugi [19] studied the distribution of the roots of the Alexander polynomial of alternating knots and showed that the trapezoidal conjecture holds for alternating stable knots.

3.2. A Refined Version of Fox Trapezoidal Conjecture

The study of the Alexander polynomial of stable alternating knots [19] led to the following refined version of Fox's trapezoidal conjecture.

Conjecture 2 ([19])**.** *Let K be an alternating knot and $\Delta_K(t) = \pm \sum_{i=-g}^{g} a_i(-t)^i$, with $a_i > 0$ be its Alexander polynomial. Then there exists an integer $0 \leq l \leq g$ such that:*

$$a_{-g} < \cdots < a_{-l/2} = \cdots = a_{l/2} > \cdots > a_g.$$

Moreover, $l \leq |\sigma(K)|$.

Conjecture 2 has been verified for two-bridge knots [20]. It has been also checked for some classes of alternating knots which admit an alternating 3-braid or an alternating 4-braid representation [21–23]. These results will be discussed in the next section. We shall now verify that Conjecture 2 holds for alternating knots of genus 1 and genus 2.

Remark 1. *Let K be an alternating knot and write $\Delta_K(t) = a_0 + \sum_{i=1}^{g} a_i(t^i + t^{-i})$, where g is the genus of the knot. In [11], Ozsváth and Szabó proved that the last two coefficients of the Alexander polynomial of an alternating knot satisfy the following relation:*

$$|a_{g-1}| \geq 2|a_g| + \begin{cases} -1 & \text{if } |\sigma(K)| = 2g \\ 1 & \text{if } |\sigma(K)| = 2g - 2 \\ 0 & \text{otherwise} \end{cases}$$

This result was used by Ni [35] to prove that if, for an alternating knot K, we have $|a_g| = |a_{g-1}|$, then K is the torus knot of type $T(2g + 1, \pm 2)$. Consequently, the equality $l = 2g(K)$ holds only in the case of the torus knot $T(2g + 1, 2)$ or its mirror image.

Let us check that Conjecture 2 holds for alternating knots of genus less than or equal to 2. We shall first normalize the Alexander polynomial such that $\Delta_K(1) = 1$, [1]. The result is obvious for alternating knots of genus one. Indeed, since $\Delta_K(t) = a_0 + a_1(t + t^{-1})$ and $\Delta_K(1) = a_0 + 2a_1 = 1$, then $a_0 = -2a_1 + 1$. Hence, $|a_0| \geq |a_1|$ with equality holds only if $a_0 = a_1 = 1$ which corresponds to the case of the trefoil knot or its mirror image [35]. In both cases, the length of the stable part is less than $|\sigma(K)|$ and the refined version of the conjecture is satisfied.

If the knot K is alternating and has genus 2, then $\Delta_K(t) = a_0 + a_1(t + t^{-1}) + a_2(t^2 + t^{-2})$.

Note that in this case, the possible values of $|\sigma(K)|$ are 0, 2 and 4. This a consequence of the fact that the signature of a knot K is always even and that $|\sigma(K)| \leq 2g$; see ([1], [Chapter 6]). Assume that $|\sigma(K)| = 0$, then we have $\delta(\sigma, r) = \delta(0, r) = 0$ and Theorem 2 gives that $t_0(K) = a_1 + 2a_2 \leq 0$ and $(-1)^1 t_1(K) = -a_2 \leq 0$. Since the coefficients a_0, a_1 and a_2 are nonzero and alternate in sign, we conclude that $|a_1| > |a_2|$. Using that for a knot we have always $\Delta_K(1) = 1$, we obtain $a_0 + 2a_1 + 2a_2 = 1$. Thus,

$$a_0 = -2a_1 - 2a_2 + 1 = -a_1 - (a_1 + 2a_2) + 1 = -a_1 + (1 - (a_1 + 2a_1)).$$

Since $a_1 + 2a_2 \leq 0$, we obtain that $1 - (a_1 + 2a_1) > 0$ and $a_0 > -a_1$. Using the fact that a_0 and a_1 have opposite signs, we have $|a_0| > |a_1|$. Hence, $|a_0| > |a_1| > |a_2|$. Therefore, $l = 0 = \sigma(K)$ and Conjecture 2 holds.

If $|\sigma(K)| = 2$, then we have $\delta(\sigma, 0) = \delta(2, 0) = 1$ and $\delta(\sigma, r) = 0$ otherwise. Theorem 2 implies that $a_1 + 2a_2 - 1 \geq 0$ and $a_2 \leq 0$, and so $a_1 \geq -2a_2 + 1 > |a_2|$. Using the fact that $a_0 + 2a_1 + 2a_2 = 1$, it follows that $|a_0| \geq |a_1|$. In conclusion, $|a_0| \geq |a_1| > |a_2|$ and Conjecture 2 is verified as the length of the stable part is at most 2, so it is less than or equal to $|\sigma(K)|$.

Let us now examine the case $|\sigma(K)| = 4$. In this case, we have $\delta(\sigma, 0) = \delta(4, 0) = 1$, $\delta(\sigma, 1) = \delta(4, 1) = 1$ and $\delta(\sigma, 2) = \delta(4, 0) = 0$.

By Theorem 2, we obtain the following:

$$\begin{aligned} (-1)^{0+\frac{\sigma}{2}}(t_0(K) - \delta(\sigma, 0)) = &\quad a_1 + 2a_2 - 1 \leq 0 \\ (-1)^{1+\frac{\sigma}{2}}(t_1(K) - \delta(\sigma, 1)) = &\quad -(a_2 - 1) \leq 0. \end{aligned}$$

Thus, $a_1 \leq 1 - 2a_2$ and $a_2 \geq 1$. Since $a_2 > 0$ then $a_1 \leq 0$ and $a_0 \geq 0$. The inequality $a_1 \leq 1 - 2a_2$ implies that $|a_1| = -a_1 \geq 2a_2 - 1 > a_2 = |a_2|$. The identity $a_0 + 2a_1 + 2a_2 = 1$ implies that $|a_0| - |a_1| = a_0 + a_1 = 1 - a_1 - 2a_2 \geq 0$. In conclusion, we proved that $|a_0| \geq |a_1| \geq |a_2|$. Notice that if $|\sigma(K)| = 4$, then the condition on the length of the stable part is satisfied by default.

Remark 2. *The inequality $l \leq |\sigma(K)|$ in Conjecture 2 is sharp. Indeed, given any $g > 1$ and $0 \leq l \leq g$, then there exists an alternating knot of genus g whose polynomial is trapezoidal and the length of the stable part is $l = |\sigma(K)|$. For $l = g$, we may consider the torus knot $T(2g+1, \pm 2)$. Otherwise, the knot $K = cl(\sigma_1^{p-1}\sigma_2^{-1}\sigma_1\sigma_2^{-q+1})$, where $p = \frac{g-l+2}{2}$ and $q = \frac{g+l+2}{2}$ has genus g and signature l. The Alexander polynomial of K is given by the following formula:*

$$\Delta_K(t) = -\frac{s^{-q}}{(1-s)^2}(s(1 - s^{p-1})(1 - s^{q-1}) + (1 - s^p)(1 - s^q)).$$

According to [21], this polynomial is trapezoidal and the length of its stable part is l.

3.3. The Strong Fox Conjecture

Stoimenow studied the roots of the Conway polynomial of special alternating links and proved that the coefficients of the Conway polynomial of such links satisfy certain interesting inequalities [24]. In light of these results, he suggested the following natural strengthening of Fox's trapezoidal conjecture.

Conjecture 3 ([24])**.** *If K is an alternating knot, then $\Delta_K(t)$ is log-concave.*

It is worth mentioning here that Conjecture 3 implies Conjecture 1. However, Conjecture 3 and Conjecture 2 are a priori independent because Conjecture 3 does not include information about the length of the stable part. In [26], Banfield proved Conjecture 3 for 2-bridge knots. In [27], Hafner, Mészáros, and Vidinas applied the theory of Lorentzian polynomials developed in [36] to certain multivariate generalization of the Alexander polynomial. Consequently, they confirmed that Conjecture 3 holds for special alternating links.

We shall here check that Conjecture 3 holds for alternating links of genus 2. Recall that if $\sigma(K) = 0$, then $t_0(K) = a_1 + 2a_2 \leq 0$, $a_2 \geq 0$ and $a_0 \geq 0$. Moreover, $a_0 + 2a_1 + 2a_2 = 1$. To prove the log-concavity, we need only to check that $a_1^2 \geq |a_0||a_1|$. Notice that:

$$4a_1^2 = (1 - a_0 - 2a_2)^2 = 1 + a_0^2 + 4a_2^2 - 2a_0 - 4a_2 + 4a_0a_2.$$

Thus,

$$\begin{aligned} 4a_1^2 - 4a_0a_2 &= (1 - a_0 - 2a_2)^2 - 4a_0a_2 \\ &= 1 + a_0^2 + 4a_2^2 - 2a_0 - 4a_2 \\ &= (1 + a_0^2 - 2a_0) + 4(a_2^2 - a_2) \geq 0. \end{aligned}$$

The proof in the cases $\sigma(K) = \pm 2$ and $\sigma(K) = \pm 4$ is similar.

Recall that the log-concavity of the Alexander polynomial of knots of genus 2 has also been verified in [17]. Moreover, Stoimenow [37] proved Conjecture 3 for alternating knots of genus at most 4.

Remark 3. *In [17], Jong gave examples of Alexander polynomials which are log-concave but cannot be realized by alternating knots.*

4. Closed Alternating Braids

In this section, we shall restrict our study to knots which can be obtained as the closures of alternating braids. Recall that any link with braid index 2 is alternating. Alternating links with braid index 3 are also classified [38,39].

Theorem 3 ([38])**.** *Let L be an alternating link of braid index 3. Then L is either:*

1. *the connected sum of two $(2, k)$-torus links (with parallel orientation);*
2. *the closure of an alternating 3-braid, including split unions of a $(2, k)$-torus link and an unknot and the 3 component unlink;*
3. *a pretzel link $P(1, c_1, c_2, c_3)$ with all $c_i \geq 1$ (oriented so that the twists corresponding to p, q, r are parallel).*

For $n \geq 3$, consider the alternating n-braid on m blocks

$$\beta_{n,m}((p_{i,j})) = \Pi_{j=1}^{m} \left(\sigma_1^{p_{1,j}} \sigma_2^{-p_{2,j}} \sigma_3^{p_{3,j}} \dots \sigma_{n-1}^{(-1)^{n-2} p_{n-1,j}} \right),$$

where all $p_{i,j} > 0$ or $p_{i,j} < 0$ for all i, j. Obviously, the closure of $\beta_{n,m}((p_{i,j}))$ is an alternating link. The signatures of these type of links have been computed in [23,40]. It is worth mentioning here that this family of links contains the classes of weaving links and the so-called generalized hybrid weaving links whose colored quantum invariants have been studied in [41,42].

One approach that can be followed in the case of alternating braids is to use the Burau representation defined in Section 2. For small values of n and m, it was possible to obtain explicit formulas for the Alexander polynomial of alternating closed braids and check that Conjecture 2 holds.

More precisely, the conjecture has been verified for closed alternating 3-braids $\mathrm{cl}(\beta_{3,m}((p_{i,j})))$ with $m \leq 3$ and for 4-braids $\mathrm{cl}(\beta_{4,m}((p_{i,j})))$ with $m \leq 2$. In the particular case of weaving knots of type $W(3, m)$, it was proved that the coefficients of the Alexander polynomial are related to Whitney numbers of Lucas lattices [22]. Thus, they are trapezoidal [43].

For $n \geq 4$, the Burau representation can be used to compute the first coefficients of the Alexander polynomial. Assume that $\Delta_K(t) = \pm \sum_{i=0}^{2g} \alpha_i(-t)^i$. In [23], it was proved that for large values of p_i the first four coefficients of the Alexander polynomial of the closure of $\beta_{n,m}((p_{i,j}))$ are given by:

$$\begin{aligned}
\alpha_0 &= 1, \\
\alpha_1 &= (n-2)m+1, \\
\alpha_2 &= \frac{(n-2)^2m^2 + (3(n-2)+2)m}{2}, \\
\alpha_3 &= \frac{(n-2)^3m^3 + 6(n-1)(n-2)m^2 + (5(n-1)+1)m}{6}.
\end{aligned}$$

It can be easily checked that we have $\alpha_1^2 \geq |\alpha_0||\alpha_2|$ and $\alpha_2^2 \geq |\alpha_1||\alpha_3|$. Thus, the sequence made up of these four terms $(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$ is log-concave [23].

Remark 4. *It is worth mentioning that given an alternating link which has an alternating n-braid representation as described above, we can in certain cases determine the values of n and m from the coefficients α_1 and α_2. First, notice that we have*

$$\begin{aligned}
2\alpha_2 - \alpha_1^2 - \alpha_1 &= ((n-2)^2m^2 + (3n-4)m) - ((n-2)m+1)^2 - (n-2)m - 1 \\
&= (n-2)^2m^2 + 3nm - 4m - (n-2)^2m^2 - 2(n-1)m - 1 - nm + 2m - 1 \\
&= 2m - 2,
\end{aligned}$$

which implies that $m = \frac{2\alpha_2 - \alpha_1^2 - \alpha_1 + 2}{2}$. Once we obtain the value of m, we can find the value of n. Let us consider the knot 10_{79} whose Alexander polynomial is:

$$1 - 3t + 7t^2 - 12t^3 + 15t^4 - 12t^5 + 7t^6 - 3t^7 + t^8.$$

Here, $\alpha_1 = 3$ and $\alpha_2 = 7$, and hence, $2\alpha_2 - \alpha_1^2 - \alpha_1 + 2 = 14 - 9 - 3 + 2 = 4$, which means that $m = 2$. Consequently, we obtain $n = 3$ and a braid representation of the knot is of the form $\sigma_1^{p_{1,1}}\sigma_2^{-p_{2,1}}\sigma_1^{p_{1,2}}\sigma_2^{-p_{2,2}}$. According to [40], the signature of the closure of such a braid is $p_{2,1} + p_{2,2} - p_{1,1} - p_{1,2}$. By using the fact that the signature of the knot 10_{79} is zero, we conclude that $p_{1,1} + p_{1,2} = p_{2,1} + p_{2,2}$. Checking different possible values of $p_{i,j}$, we obtain that 10_{79} is the closure of $\sigma_1^3\sigma_2^{-2}\sigma_1^2\sigma_2^{-3}$; see Figure 5.

These formulas for the coefficients of the Alexander polynomial can also be applied to the pretzel knot $P(1, c_1, c_2, c_3)$, see Figure 6, which is alternating but has first coefficient $\alpha_0 = 2$. Hence, it is not a fibered knot. We conclude then that it cannot be represented as an alternating braid since the closure of an alternating braid is fibered [44]. Similar discussion for the Jones polynomials of 3-braid links can be found in [45].

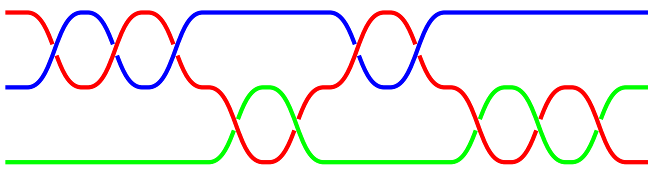

Figure 5. An alternating 3-braid representation of the knot 10_{79}.

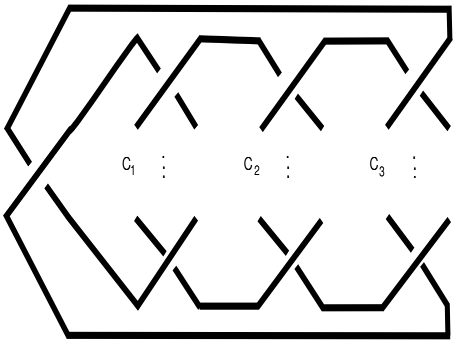

Figure 6. The pretzel link $P(1, c_1, c_2, c_3)$.

5. Quasi-Alternating Links

The class of alternating links has been generalized into several directions. For instance, a link is said to be *almost alternating* if it is non-alternating and it admits a diagram which can be turned into an alternating diagram by performing only one crossing change [46]. Another interesting generalization appeared through the study of the Heegaard Floer homology of branched double-covers of alternating links [11]. This new class of links, called *quasi-alternating*, has been defined recursively on planar diagrams. One of the basic feature of these links is that they have the same homological properties as alternating links [47].

Definition 1. *The set $\mathcal{Q}$ of quasi-alternating links is the smallest set satisfying the following properties:*
1. *The unknot belongs to $\mathcal{Q}$;*
2. *If L is a link with a diagram D containing a crossing c such that:*
   - *(a)* *both smoothings of the diagram D at the crossing c, L_0 and L_∞ as in Figure 7 belong to $\mathcal{Q}$;*
   - *(b)* $\det(L_0), \det(L_\infty) \geq 1$;
   - *(c)* $\det(L) = \det(L_0) + \det(L_\infty)$;

then L is in $\mathcal{Q}$. In this case, we say that L is quasi-alternating with quasi-alternating diagram D at the crossing c.

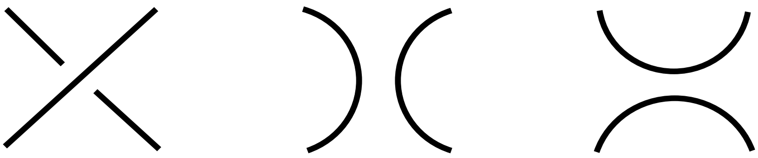

Figure 7. The diagram of the link L at the crossing c and its smoothing L_0 and L_∞, respectively.

It is noteworthy to highlight that many properties of the polynomial invariants of alternating links extend naturally to quasi-alternating links, as discussed in [48–50], for instance. On the other hand, Manolescu and Ozsváth proved that quasi-alternating links are thin in link Floer homology. By considering the Alexander polynomial as the graded Euler Characteristic of the link Floer homology, we conclude that the coefficients of the Alexander polynomial of a quasi-alternating knot alternate in sign; they satisfy $a_i a_{i+1} \leq 0$. Moreover, the degree of the Alexander polynomial of such a knot K is equal to its genus $g(K)$ [11]. For quasi-alternating knots, it is not known whether the coefficients of $\Delta_K(t)$ have internal zeros. On the other hand, Fox Trapezoidal Conjecture does not extend to quasi-alternating knots. For instance, the knot 10_{125} is a quasi-alternating non-alternating knot [51]. Its Alexander polynomial is $\Delta_{10_{125}}(t) = 1 - 2t + 2t^2 - t^3 + 2t^4 - 2t^5 + t^6$, hence not trapezoidal. The same knot is also known to be almost alternating and Fox's Conjecture also does not hold for this class of knots. The conjecture does not hold for special quasi-alternating knots (knots which are positive and quasi-alternating) either, as can be seen from the knot 10_{142}, which is positive and quasi-alternating. However, we have $\Delta_{10_{142}}(t) = 2 - 3t + 2t^2 - t^3 + 2t^4 - 3t^5 + 2t^6$, which is not trapezoidal. A natural question to ask is whether the Trapezoidal conjecture can be modified into a relaxed version that extends to quasi-alternating knots. This question is clearly related to the problem of internal zero coefficients mentioned above. Figure 8 displays the coefficients of the Alexander polynomials of the non-alternating knots $9_{43}, 9_{47}$, 10_{125} and $12n235$. These knots are proven to be quasi-alternating: 9_{43} and 9_{47} in [47], 10_{125} in [51], and $12n235$ in [52]. We can easily observe that for each of these knots, the coefficients of the Alexander polynomial form a bimodal sequence. This observation needs to be further investigated.

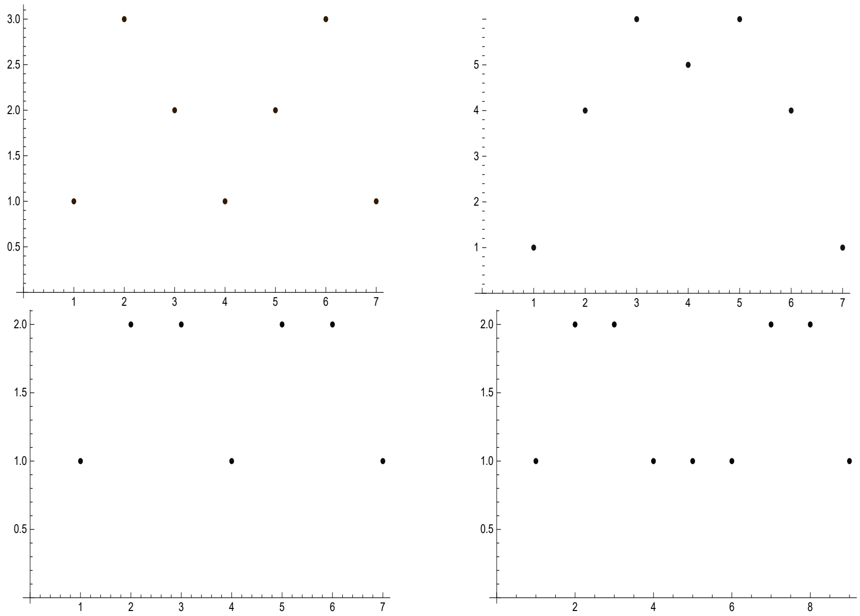

Figure 8. The coefficients of the Alexander polynomials of the quasi-alternating non-alternating knots $9_{43}, 9_{47}$, 10_{125} and $12n235$, respectively.

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