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Directed Path 3-Arc-Connectivity of Cartesian Product Digraphs

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Abstract: Let $D = (V(D), A(D))$ be a digraph of order n and let $r \in S \subseteq V(D)$ with $2 \le |S| \le n$. A directed (S,r)-Steiner path (or an (S,r)-path for short) is a directed path P beginning at r such that $S \subseteq V(P)$. Arc-disjoint between two (S,r)-paths is characterized by the absence of common arcs. Let $\lambda^p_{S,r}(D)$ be the maximum number of arc-disjoint (S,r)-paths in D. The directed path k-arc-connectivity of D is defined as $\lambda^p_k(D) = \min\{\lambda^p_{S,r}(D) \mid S \subseteq V(D), |S| = k, r \in S\}$. In this paper, we shall investigate the directed path 3-arc-connectivity of Cartesian product $\lambda^p_3(G\Box H)$ and prove that if G and H are two digraphs such that $\delta^0(G) \ge 4$, $\delta^0(H) \ge 4$, and $\kappa(G) \ge 2$, $\kappa(H) \ge 2$, then $\lambda^p_3(G\Box H) \ge \min\{2\kappa(G), 2\kappa(H)\}$; moreover, this bound is sharp. We also obtain exact values for $\lambda^p_3(G\Box H)$ for some digraph classes G and H, and most of these digraphs are symmetric.

Keywords: connectivity; directed path k-connectivity; Cartesian product

1. Introduction

For a detailed explanation of graph theoretical notation and terminology not provided here, readers are directed to reference [1]. It should be noted that all digraphs discussed in this paper do not contain parallel arcs or loops. The set of all natural numbers from 1 to n is denoted by $[n]$. If a directed graph D can be obtained from its underlying graph G by replacing each edge in G with corresponding arcs in both directions, then D is said to be symmetric, denoted as $D = \overleftrightarrow{G}$. The notation $\overleftrightarrow{T}_n$ is used for a symmetric digraph whose underlying graph forms a tree of order n. The notation $\overleftrightarrow{C}_n$ is used for a symmetric digraph whose underlying graph forms a cycle of order n. The cycle digraph of order n is denoted by $\overrightarrow{C}_n$. We denote the complete digraph of order n as $\overleftrightarrow{K}_n$.

The well-known Steiner tree packing problem is characterized as follows. Given a graph G and a set of terminal vertices $S \subseteq V(G)$, the goal is to identify as many edge-disjoint S-Steiner trees (i.e., trees T in G with $S \subseteq V(T)$) as feasible. This particular problem, along with its associated topics, garners significant interest from researchers due to its extensive applications in VLSI circuit design [2–4] and Internet Domain [5]. In practical applications, the construction of vertex-disjoint or arc-disjoint paths in graphs holds significance, as they play a crucial role in improving transmission reliability and boosting network transmission rates [6]. This paper will specifically delve into a variant of the directed Steiner tree packing problem, termed the directed Steiner path packing problem, closely interconnected with the Steiner path problem and the Steiner path cover problem [7,8].

We now consider two types of directed Steiner path packing problems and related parameters. Let $D = (V(D), A(D))$ be a digraph of order n and let $r \in S \subseteq V(D)$ with $2 \le |S| \le n$. A directed (S,r)-Steiner path, or simply an (S,r)-path, refers to a directed path P originating from r such that S is a subset of the vertices in P. Arc-disjoint between two (S,r)-paths implies that they share no common arcs, while two arc-disjoint (S,r)-paths are internally disjoint when their common vertex set is precisely S. Let $\lambda^p_{S,r}(D)$ (and $\kappa^p_{S,r}(D)$) represent the maximum number of arc-disjoint (and internally disjoint) (S,r)-paths in D, respectively. The Arc-disjoint (or Internally disjoint) Directed Steiner Path Packing problem is formulated as follows. Given a digraph D and letting $r \in S \subseteq V(D)$, the objective is

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to maximize the count of arc-disjoint (or internally disjoint) (S,r)-paths. The notion of directed path connectivity, which is a derivative of path connectivity in undirected graphs, is intricately linked to the directed Steiner path packing problem and serves as a logical progression from path connectivity in directed graphs (refer to [5] for the initial presentation of path connectivity). The directed path k-connectivity [9] of D is defined as

$$\kappa_k^p(D) = \min\{\kappa_{S,r}^p(D) \mid S \subseteq V(D), |S| = k, r \in S\}$$

Similarly, the directed path k-arc-connectivity [9] of D is defined as

$$\lambda_k^p(D) = \min\{\lambda_{S,r}^p(D) \mid S \subseteq V(D), |S| = k, r \in S\}$$

The concepts of directed path k-connectivity and directed path k-arc-connectivity are synonymous with directed path connectivity. In the context of $k = 2$, $\kappa_2^p(D)$ equates to $\kappa(D)$ and $\lambda_2^p(D)$ equates to $\lambda(D)$, where $\kappa(D)$ and $\lambda(D)$ denote vertex-strong connectivity and arc-strong connectivity of digraphs, respectively. Hence, these parameters can be viewed as extensions of the classical connectivity measures in a digraph. It is pertinent to emphasize the close relationship between strong subgraph connectivity and directed path connectivity; refer to [10–12] for further insights on this interconnected topic.

It is a widely recognized fact that Cartesian products of digraphs are of great interest in graph theory and its applications. For a comprehensive overview of various findings on Cartesian products of digraphs, one may refer to a recent survey chapter by Hammack [13]. In this paper, we continue research on directed path connectivity and focus on the directed path 3-arc-connectivity of Cartesian products of digraphs.

In Section 2, we introduce terminology and notation on Cartesian products of digraphs. In Section 3, we prove that if G and H are two digraphs such that $\delta^0(G) \geq 4$, $\delta^0(H) \geq 4$, and $\kappa(G) \geq 2$, $\kappa(H) \geq 2$, then

$$\lambda_3^p(G \Box H) \geq \min\{2\kappa(G), 2\kappa(H)\};$$

moreover, this bound is sharp. Finally, we obtain exact values of $\lambda_3^p(G \Box H)$ for some digraph classes G and H in Section 4.

2. Cartesian Product of Digraphs

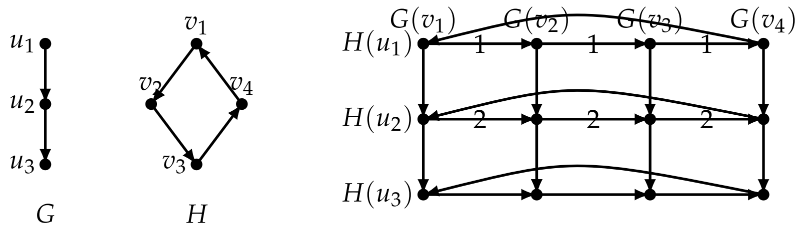
Consider two digraphs G and H with vertex sets $V(G) = \{u_i \mid i \in [n]\}$ and $V(H) = \{v_j \mid j \in [m]\}$. The Cartesian product of G and H, denoted by $G \Box H$, is a digraph with vertex set

$$V(G \Box H) = V(G) \times V(H) = \{(x, x') \mid x \in V(G), x' \in V(H)\}.$$

The arc set of $G \Box H$, denoted by $A(G \Box H)$, is given by $\{(x,x')(y,y') \mid xy \in A(G), x' = y', \text{or } x = y, x'y' \in A(H)\}$. It is worth noting that Cartesian product is an associative and commutative operation. Furthermore, $G \Box H$ is strongly connected if and only if both G and H are strongly connected, as shown in a recent survey chapter by Hammack [13].

In the rest of the paper, we will use $u_{i,j}$ to denote (u_i, v_j). Additionally, $G(v_j)$ will refer to the subgraph of $G \Box H$ induced by the vertex set $\{u_{i,j} \mid i \in [n]\}$ with $j \in [m]$, while $H(u_i)$ will denote the subgraph of $G \Box H$ induced by the vertex set $\{u_{i,j} \mid j \in [m]\}$ with $i \in [n]$. It is evident that $G(v_j)$ is isomorphic to G and $H(u_i)$ is isomorphic to H. To illustrate this, refer to Figure 1 (this figure comes from [14]), where it can be observed that $G(v_j)$ is isomorphic to G for $1 \leq j \leq 4$, and $H(u_i)$ is isomorphic to H for $1 \leq i \leq 3$.

For distinct indices j_1 and j_2 with $1 \leq j_1 \neq j_2 \leq m$, the vertices u_{i,j_1} and u_{i,j_2} belong to the same digraph $H(u_i)$, where u_i is an element of $V(G)$. u_{i,j_2} is referred to as the vertex corresponding to u_{i,j_1} in $G(v_{j_2})$. Similarly, for distinct indices i_1 and i_2 with $1 \leq i_1 \neq i_2 \leq n$, $u_{i_2,j}$ is the vertex corresponding to $u_{i_1,j}$ in $H(u_{i_2})$. Analogously, the subgraph corresponding to a given subgraph can also be defined. For instance, in the digraph (c) depicted in Figure 1, if we label the path 1 as P_1 (and the path 2 as P_2) in $H(u_1)$ ($H(u_2)$), then P_2 is identified as the path that corresponds to P_1 in $H(u_2)$.



(**a**) (**b**) (**c**)

Figure 1. *G*, *H* and their Cartesian product [14] (1 denotes arc $u_{1,1}u_{1,2}$, $u_{1,2}u_{1,3}$ and arc $u_{1,3}u_{1,4}$; 2 denotes arc $u_{2,1}u_{2,2}$, $u_{2,2}u_{2,3}$ and arc $u_{2,3}u_{2,4}$).

Sun and Zhang proved some results of directed path connectivity, that is, the following lemma.

Lemma 1 ([9])**.** *Let D be a digraph of order n, and let k be an integer satisfying* $2 \le k \le n$. *The following statements are valid:*
(1)*:* $\lambda^p_{k+1}(D) \le \lambda^p_k(D)$ *when* $k \le n-1$.
(2)*:* $\kappa^p_k(D) \le \lambda^p_k(D) \le \delta^0(D) = \min\{\delta^+(D), \delta^-(D)\}$.

Lemma 2 ([15])**.** $\kappa(\overleftrightarrow{K}_n) = n - 1$.

3. A General Lower Bound

Now we will provide a lower bound for $\lambda^p_3(G\Box H)$.

Theorem 1. *Let G and H be two digraphs such that* $\delta^0(G) \ge 4$, $\delta^0(H) \ge 4$, *and* $\kappa(G) \ge 2$, $\kappa(H) \ge 2$. *We have*

$$\lambda^p_3(G\Box H) \ge \min\{2\kappa(G), 2\kappa(H)\}.$$

Furthermore, this bound is sharp.

Proof. It suffices to show that there are at least $2\kappa(G)$ or $2\kappa(H)$ arc-disjoint (S,r)-paths for any $S \subseteq V(G\Box H)$ with $|S| = 3$, $r \in S$. Let $S = \{x, y, z\}$ and let $r = x$. Without loss of generality, we may assume $\kappa(G) \le \kappa(H)$ and consider the following six cases.

Case 1. Let x, y and z be in the same $H(u_i)$ or $G(v_j)$ for some $i \in [n]$, $j \in [m]$. Without loss of generality, we may assume that $x = u_{1,1}$, $y = u_{2,1}$, $z = u_{3,1}$. In this case, our overall goal is that we will use arc-disjoint paths between x and y in $G(v_1)$, y and z in $G(v_1)$, x and its out-neighbors in $H(u_1)$, y and its in-neighbors in $H(u_2)$, z and its in-neighbors in $H(u_3)$, and combine them together to form the required arc-disjoint paths. The general idea of the proof process is briefly described in Figure 2. The vertices and paths contained in Figure 2 are explained below.

Let $S_1 = \{x, y\}$, $r_1 = x$. It is known that there are at least $\kappa(G)$ internally disjoint (S_1, r_1)-paths in $G(v_1)$, denoted as $\widetilde{P}_{1i}$ ($i \in [\kappa(G)]$). Considering $S'_1 = \{y, z\}$, $r'_1 = y$, there are at least $\kappa(G)$ internally disjoint (S'_1, r'_1)-paths in $G(v_1)$, denoted as $\overline{P}_{2j}$ ($j \in [\kappa(G)]$). For each $j \in [\kappa(G)]$, let $u_{s_j,1}$ be the out-neighbor of y in $\overline{P}_{2j}$; clearly these out-neighbors are distinct. Similarly, an in-neighbor $u_{k_j,1}$ ($j \in [\kappa(G)]$) of z in $\overline{P}_{2j}$ can be chosen such that these in-neighbors are distinct. In $H(u_1)$, if there is a vertex that is not an out-neighbor of x, then choose such a vertex as $u_{1,a}$, where $a \ne 1$. If there is no such vertex, that is, all vertices are out-neighbours of x, then choose any vertex as $u_{1,a}$, where $a \ne 1$. In $H(u_1)$, let $S'_2 = \{x, u_{1,a}\}$, $r'_2 = x$, and it is established that there exist at least $\kappa(G)$ internally disjoint (S'_2, r'_2)-paths, say $\widetilde{P}_{2j}$ ($j \in [\kappa(G)]$). In $G(v_a)$, let $S'_3 = \{u_{1,a}, u_{2,a}\}$, $r'_3 = u_{1,a}$, and it is established that there exist at least $\kappa(G)$ internally disjoint (S_3', r_3')-paths, say $\widehat{P}_{2j}$ $(j \in [\kappa(G)])$. In $H(u_2)$, let $S_4' = \{y, u_{2,a}\}$, $r_4' = u_{2,a}$, and it is established that there exist at least $\kappa(G)$ internally disjoint (S_4', r_4')-paths, say $\overset{\frown}{P}_{2j}$ $(j \in [\kappa(G)])$.

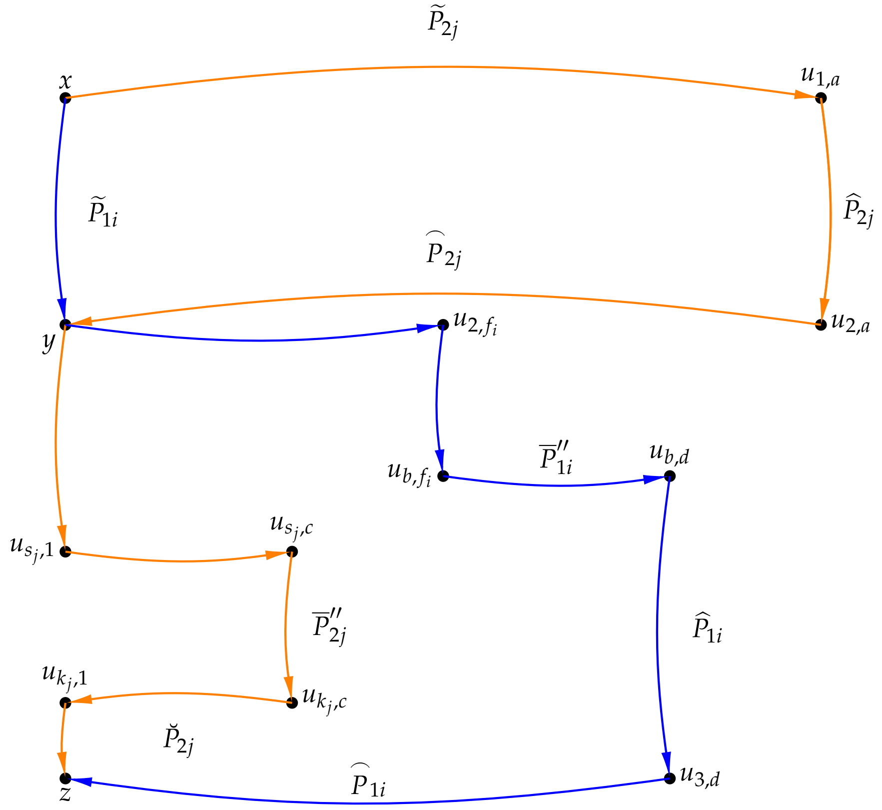


Figure 2. Depiction of the arc-disjoint paths found in Case 1 of the proof of Theorem 1.

In $H(u_1)$, if there is a vertex that is not an out-neighbor of x in $\widetilde{P}_{2j}$, then choose such a vertex as $u_{1,d}$, with $d \notin \{1, a\}$. If there is no such vertex, then choose any vertex as $u_{1,d}$, with $d \notin \{1, a\}$. In $H(u_2)$, with $S_2 = \{y, u_{2,d}\}$ and $r_2 = y$, it is known that there are at least $\kappa(G)$ internally disjoint (S_2, r_2)-paths, denoted as $\overline{P}_{1i}$ $(i \in [\kappa(G)])$. For each $i \in [\kappa(G)]$, let u_{2,f_i} be the out-neighbor of y in $\overline{P}_{1i}$; clearly these out-neighbors are distinct. For each $i \in [\kappa(G)]$, since $\delta^0(G) \geq 4$, an out-neighbor of u_{2,f_i} in $G(v_{f_i})$, denoted by u_{b,f_i} $(b \in [n])$, can be chosen, with $b \notin \{1, 3\}$. If there exists a vertex $u_{s_j,1} \notin \{u_{1,1}, u_{3,1}\}$, let $b = s_j$. If there is no such vertex, then let $b \neq k_j$. In $H(u_b)$, $\overline{P}'_{1i}$ is the (S_3, r_3)-path corresponding to $\overline{P}_{1i}$, where $S_3 = \{u_{b,1}, u_{b,d}\}$, and $r_3 = u_{b,1}$. In $\overline{P}'_{1i}$, the path from vertex u_{b,f_i} to $u_{b,d}$ is denoted as $\overline{P}''_{1i}$. Let $S_4 = \{u_{b,d}, u_{3,d}\}$, $r_4 = u_{b,d}$, and it is established that there exist at least $\kappa(G)$ internally disjoint (S_4, r_4)-paths, say $\widehat{P}_{1i}$ $(i \in [\kappa(G)])$. If $u_{2,f_t} = u_{2,d}$ $(t \in [\kappa(G)])$, then let $u_{2,d} \notin \widehat{P}_{1t}$ in $\widehat{P}_{1t}$. In $H(u_3)$, let $S_5 = \{u_{3,d}, z\}$, $r_5 = u_{3,d}$, and it is established that there exist at least $\kappa(G)$ internally disjoint (S_5, r_5)-paths, say $\overset{\frown}{P}_{1i}$ $(i \in [\kappa(G)])$.

In $H(u_1)$, if there is an out-neighbor of x that is not an out-neighbor of x in $\widetilde{P}_{2j}$, then choose such a vertex as $u_{1,c}$, with $c \notin \{a, d\}$. If there is no such vertex, then choose any out-neighbor of x as $u_{1,c}$, with $c \notin \{a, d\}$. And $u_{s_j,c}$ is an out-neighbor of $u_{s_j,1}$ in $H(u_{s_j})$. In $G(v_c)$, $\overline{P}'_{2j}$ is the (S_5', r_5')-path corresponding to $\overline{P}_{2j}$, where $S_5' = \{u_{2,c}, u_{3,c}\}$ and $r_5' = u_{2,c}$. In $\overline{P}'_{2j}$, the path from vertex $u_{s_j,c}$ to $u_{k_j,c}$ is denoted as $\overline{P}''_{2j}$. If $u_{s_t,1} = u_{k_t,1}$ $(t \in [\kappa(G)])$, then $\overline{P}''_{2t} = \{y u_{s_t,1}, u_{s_t,1} z\}$. If $u_{s_l,1} = z$ $(l \in [\kappa(G)])$, then $\overline{P}''_{2l} = \{yz\}$. If $u_{1,c} \in \overline{P}''_{2h}$ $(h \in [\kappa(G)])$, then $u_{1,c} \notin \widetilde{P}_{2h}$. In $H(u_{k_j})$, with $S_{6_j}' = \{u_{k_j,c}, u_{k_j,1}\}$, and $r_{6_j}' = u_{k_j,c}$, it is known that there exist at least $\kappa(G)$ internally disjoint (S'_{6_j}, r'_{6_j})-paths. Then in these paths, one of the paths $\check{P}_{2j}$ $(j \in [\kappa(G)])$ is chosen, with $u_{k_j,a} \notin \check{P}_{2j}$.

Subcase 1.1. In the set $\{u_{s_j,1}, u_{k_j,1}\}$, there is no vertex such that $u_{s_j,1} = x$ or $u_{k_j,1} = x$, and the vertex z is not in path $\widetilde{P}_{1i}$. We now construct the arc-disjoint (S,r)-paths by letting

$P_{1i} = \widetilde{P}_{1i} \cup \overline{P}''_{1i} \cup \widehat{P}_{1i} \cup \overset{\frown}{P}_{1i} \cup \{yu_{2,f_i}, u_{2,f_i}u_{b,f_i}\}, i \in [\kappa(G)]$,

$P_{2j} = \widetilde{P}_{2j} \cup \widehat{P}_{2j} \cup \overset{\frown}{P}_{2j} \cup \overline{P}''_{2j} \cup \check{P}_{2j} \cup \{yu_{s_j,1}, u_{s_j,1}u_{s_j,c}, u_{k_j,1}z\}, j \in [\kappa(G)] \setminus \{t,l\}$,

$P_{2t} = \widetilde{P}_{2t} \cup \widehat{P}_{2t} \cup \overline{P}''_{2t} \cup \overset{\frown}{P}_{2t}$,

$P_{2l} = \widetilde{P}_{2l} \cup \widehat{P}_{2l} \cup \overline{P}''_{2l} \cup \overset{\frown}{P}_{2l}$.

Then we obtain $2\kappa(G)$ arc-disjoint (S,r)-paths.

Subcase 1.2. In the set $\{u_{s_j,1}, u_{k_j,1}\}$, there is no vertex such that $u_{s_j,1} = x$ or $u_{k_j,1} = x$, and there exist $z \in \widetilde{P}_{1h}$ $(h \in [\kappa(G)])$, but there is no arc $u_{k_j,1}z$ in path $\widetilde{P}_{1h}$. Let $P_{1h} = \widetilde{P}_{1h}$. The other paths are the same as Subcase 1.1.

Subcase 1.3. There is an arc $u_{k_r,1}z$ in path $\widetilde{P}_{1h}$ $(\{r,h\} \subseteq [\kappa(G)])$. In the set $\{u_{s_j,1}, u_{k_j,1}\}(j \neq r)$, there is no vertex x. We can find a path $\overset{\frown}{P}_{2r}$ such that $u_{2,f_h} \notin \overset{\frown}{P}_{2r}$. If $u_{b,a} \in \overline{P}''_{1h}$, then let $u_{b,a} \notin \widehat{P}_{2r}$. If $u_{1,d} \in \widehat{P}_{1h}$, then let $u_{1,d} \notin \widetilde{P}_{2r}$. In $\widehat{P}_{1h}$ and $\overset{\frown}{P}_{1h}$, let $u_{2,d} \notin \widehat{P}_{1h}$ and $u_{3,a} \notin \overset{\frown}{P}_{1h}$. Let

$P_{1h} = \widetilde{P}_{1h}$,

$P_{2r} = \widetilde{P}_{2r} \cup \widehat{P}_{2r} \cup \overset{\frown}{P}_{2r} \cup \overline{P}''_{1h} \cup \widehat{P}_{1h} \cup \overset{\frown}{P}_{1h} \cup \{yu_{2,f_h}, u_{2,f_h}u_{b,f_h}\}$.

The other paths are the same as Subcase 1.1.

Subcase 1.4. The set $\{u_{s_j,1}, u_{k_j,1}\}$ contains the vertex $u_{s_q,1} = x$ and $u_{k_j,1} \neq x$. There is no arc $u_{k_q,1}z$ in $\widetilde{P}_{1q}$. In $\overline{P}_{2q}$, there is an arc $xu_{g_1,1}$ $(q \in [\kappa(G)], g_1 \in [n])$. In $\widetilde{P}_{2j}$, there exists an out-neighbor u_{1,g_2} of x, where $g_2 \in [\kappa(G)] \setminus \{a,c,d\}$, and this path is denoted by $\widetilde{P}_{2q}$.

Subcase 1.4.1. There is no arc $xu_{g_1,1}$ in $\widetilde{P}_{1i}$.

In $\overline{P}''_{2q}$, the path from vertex $u_{g_1,c}$ to $u_{k_q,c}$ is denoted as $\overline{P}'''_{2q}$. In $G(v_{g_2})$, with $S'_7 = \{u_{3,g_2}, u_{1,g_2}\}$ and $r'_7 = u_{3,g_2}$, it is known that there exist at least $\kappa(G)$ internally disjoint (S'_7, r'_7)-paths. Then in these paths, one of the paths $\widetilde{P}_q$ is chosen, with $u_{k_q,g_2} \notin \widetilde{P}_q$. If $u_{2,g_2} \in \widetilde{P}_q$, then let $u_{2,g_2} \notin \overset{\frown}{P}_{2q}$. In $\widetilde{P}_{2q}$, the path from vertex u_{1,g_2} to $u_{1,a}$ is denoted as $\widetilde{P}'_{2q}$. Let

$P_{2q} = \overline{P}'''_{2q} \cup \check{P}_{2q} \cup \widetilde{P}_q \cup \widetilde{P}'_{2q} \cup \widehat{P}_{2q} \cup \overset{\frown}{P}_{2q} \cup \{xu_{g_1,1}, u_{g_1,1}u_{g_1,c}, u_{k_q,1}z, zu_{3,g_2}\}$.

If $u_{g_1,1} = u_{k_q,1}$, then $P_{2q} = \widetilde{P}_q \cup \widetilde{P}'_{2q} \cup \widehat{P}_{2q} \cup \overset{\frown}{P}_{2q} \cup \{xu_{g_1,1}, u_{k_q,1}z, zu_{3,g_2}\}$. The other paths are the same as Subcases 1.1–1.3.

Subcase 1.4.2. If there exists an arc $xu_{g_1,1}$ in $\widetilde{P}_{1g}$ $(g \in [\kappa(G)])$, then in $H(u_{g_1})$, with $S_6 = \{u_{g_1,1}, u_{g_1,g_2}\}$ and $r_6 = u_{g_1,g_2}$, it is known that there exist at least $\kappa(G)$ internally disjoint (S_6, r_6)-paths. Then in these paths, one of the paths $\widetilde{P}_g$ is chosen, with $u_{g_1,d} \notin \widetilde{P}_g$. In $\widetilde{P}_{1g}$, the path from vertex $u_{g_1,1}$ to y is denoted as $\widetilde{P}'_{1g}$. Let P_{2q} be the same as in Subcase 1.4.1. Let

$P_{1g} = \widetilde{P}_g \cup \widetilde{P}'_{1g} \cup \overline{P}''_{1g} \cup \widehat{P}_{1g} \cup \overset{\frown}{P}_{1g} \cup \{xu_{1,g_2}, u_{1,g_2}u_{g_1,g_2}, yu_{2,f_g}, u_{2,f_g}u_{b,f_g}\}$.

The other paths are the same as Subcases 1.1–1.3.

Subcase 1.5. In the set $\{u_{s_j,1}, u_{k_j,1}\}$, there exists vertex $u_{k_p,1} = x$. And there is no arc $u_{k_p,1}z$ in $\widetilde{P}_{1p}$.

In $\widetilde{P}_{2j}$, there is an out-neighbor $u_{1,g}$ of x such that $g \in [\kappa(G)] \setminus \{a,c,d\}$, and this path is denoted by $\widetilde{P}_{2p}$. In $G(v_g)$, let $S'_8 = \{u_{3,g}, u_{1,g}\}$, $r'_8 = u_{3,g}$, and we know there exist at least $\kappa(G)$ internally disjoint (S'_8, r'_8)-paths. Then in these paths, we choose one of the paths $\widetilde{P}_p$, and let $u_{2,g} \notin \widetilde{P}_p$. In $\widetilde{P}_{2p}$, we denote the path from vertex $u_{1,g}$ to $u_{1,a}$ as $\widetilde{P}'_{2p}$. Let

$P_{2p} = \widetilde{P}_p \cup \widetilde{P}'_{2p} \cup \widehat{P}_{2p} \cup \overset{\frown}{P}_{2p} \cup \{xz, zu_{3,g}\}$.

The other paths are the same as Subcases 1.1–1.3.

Case 2. Let x and y be in the same $G(v_j)$. Let x and z be in the same $H(u_i)$ for some $i \in [n]$, $j \in [m]$. Without loss of generality, we may assume that $x = u_{1,1}$, $y = u_{2,1}$, $z = u_{1,2}$. In this case, our overall goal is that we will use arc-disjoint paths between x and y in $G(v_1)$, y and its out-neighbors in $H(u_2)$, z and its in-neighbors in $G(v_2)$, and combine them together to form the required arc-disjoint paths. The general idea of the proof process is briefly described in Figure 3. The vertices and paths contained in Figure 3 are explained below.

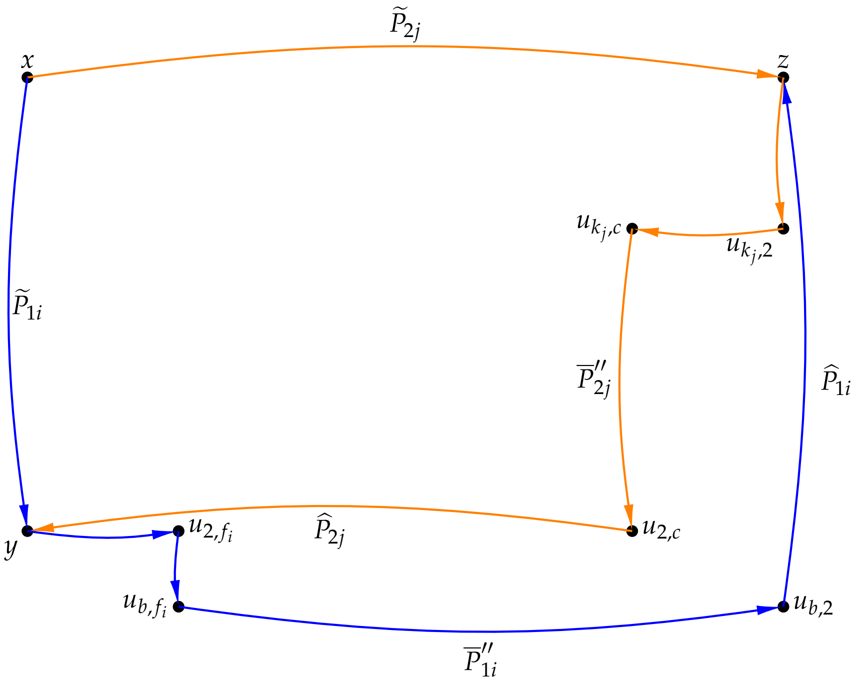


Figure 3. Depiction of the arc-disjoint paths found in Case 2 of the proof of Theorem 1.

Considering $S_1 = \{x, y\}$, $r_1 = x$, it is known that there exist at least $\kappa(G)$ internally disjoint (S_1, r_1)-paths in $G(v_1)$, denoted as $\widetilde{P}_{1i}$ $(i \in [\kappa(G)])$. Let $S_2 = \{y, u_{2,2}\}$, $r_2 = y$, and there exist at least $\kappa(G)$ internally disjoint (S_2, r_2)-paths in $H(u_2)$, denoted as $\overline{P}_{1i}$ $(i \in [\kappa(G)])$. For each $i \in [\kappa(G)]$, let u_{2,f_i} be the out-neighbor of y in $\overline{P}_{1i}$; clearly these out-neighbors are distinct. For each $i \in [\kappa(G)]$, an out-neighbor u_{b,f_i} of u_{2,f_i} in $G(v_{f_i})$ can be chosen, with $b \neq 1$. In $H(u_b)$, with $S_3 = \{u_{b,1}, u_{b,2}\}$ and $r_3 = u_{b,1}$. $\overline{P}'_{1i}$ is the (S_3, r_3)-path corresponding to $\overline{P}_{1i}$. In $\overline{P}'_{1i}$, the path from vertex u_{b,f_i} to $u_{b,2}$ is denoted $\overline{P}''_{1i}$. With $S_4 = \{u_{b,2}, z\}$ and $r_4 = u_{b,2}$, it is known that there exist at least $\kappa(G)$ internally disjoint (S_4, r_4)-paths in $G(v_2)$, denoted as $\widehat{P}_{1i}$ $(i \in [\kappa(G)])$. If $u_{2,f_k} = u_{2,2}$, then $u_{2,2} \notin \widehat{P}_{1k}$. The arc-disjoint (S, r)-paths can be constructed as

$P_{1i} = \widetilde{P}_{1i} \cup \overline{P}''_{1i} \cup \widehat{P}_{1i} \cup \{yu_{2,f_i}, u_{2,f_i}u_{b,f_i}\}$, $i \in [\kappa(G)]$.

Likewise, we can identify $\kappa(G)$ arc-disjoint (S, r)-paths from x to z and subsequently to y. Consequently, we can derive $2\kappa(G)$ arc-disjoint (S, r)-paths.

Case 3. Let x, y and z be in different $H(u_i)$ and $G(v_j)$ for some $i \in [n]$, $j \in [m]$. Without loss of generality, we can assume that $x = u_{1,1}$, $y = u_{2,2}$, $z = u_{3,3}$. In this case, our overall goal is that, we will use arc-disjoint paths between x and its out-neighbors in $H(u_1)$, y and its out-neighbors in $H(u_2)$, z and its in-neighbors in $G(v_3)$, x and its out-neighbors in $G(v_1)$, y and its out-neighbors in $G(v_2)$, z and its in-neighbors in $H(u_3)$, and combine them together to form the required arc-disjoint paths. The general idea of the proof process is briefly described in Figure 4. The vertices and paths contained in Figure 4 are explained below.

Considering $S_1 = \{x, u_{2,1}\}$, $r_1 = x$, it is known that there exist at least $\kappa(G)$ internally disjoint (S_1, r_1)-paths in $G(v_1)$, denoted as $\widetilde{P}_{1i}$ $(i \in [\kappa(G)])$. Let $S_2 = \{u_{2,1}, y\}$, $r_2 = u_{2,1}$, and there exist at least $\kappa(G)$ internally disjoint (S_2, r_2)-paths in $H(u_2)$, denoted as $\widehat{P}_{1i}$ ($i \in [\kappa(G)]$). Considering $S_1' = \{x, u_{1,2}\}$, $r_1' = x$, it is known that there exist at least $\kappa(G)$ internally disjoint (S_1', r_1')-paths in $H(u_1)$, denoted as $\widetilde{P}_{2j}$ ($j \in [\kappa(G)]$). Let $S_2' = \{u_{1,2}, y\}$, $r_2' = u_{1,2}$, and there exist at least $\kappa(G)$ internally disjoint (S_2', r_2')-paths in $G(v_2)$, denoted as $\widehat{P}_{2j}$ ($j \in [\kappa(G)]$). In $H(u_2)$, with $S_3' = \{y, u_{2,3}\}$, $r_3' = y$, it is known that there exist at least $\kappa(G)$ internally disjoint (S_3', r_3')-paths, denoted as $\overline{P}_{2j}$. For each $j \in [\kappa(G)]$, let u_{2,f_j} be the out-neighbor of y in $\overline{P}_{2j}$, clearly these out-neighbors are distinct.

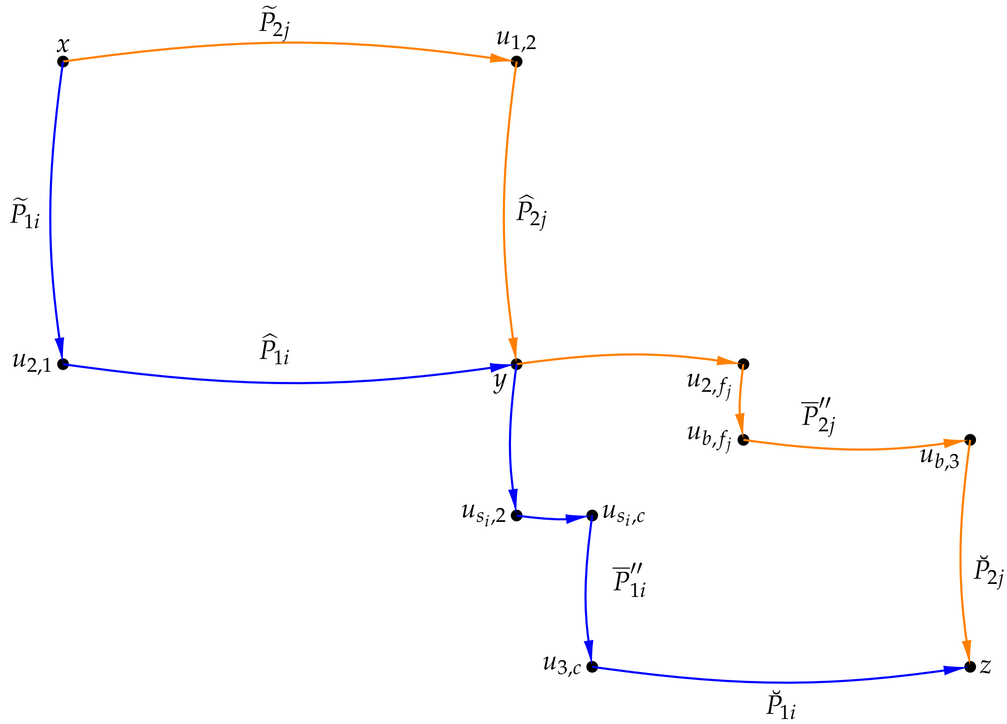


Figure 4. Depiction of the arc-disjoint paths found in Case 3 of the proof of Theorem 1.

In $G(v_2)$, with $S_3 = \{y, u_{3,2}\}$, $r_3 = y$, it is known that there exist at least $\kappa(G)$ internally disjoint (S_3, r_3)-paths in $G(v_2)$, denoted as $\overline{P}_{1i}$. For each $i \in [\kappa(G)]$, let $u_{s_i,2}$ be the out-neighbor of y in $\overline{P}_{1i}$, clearly these out-neighbors are distinct. For each $i \in [\kappa(G)]$, an out-neighbor of $u_{s_i,2}$ in $H(u_{s_i})$ can be chosen, denoted by $u_{s_i,c}$ ($c \in [m]$), with $c \notin \{1,3\}$. Similarly, an out-neighbor of u_{2,f_j} in $G(v_{f_j})$ can be chosen, denoted by u_{b,f_j} ($b \in [n]$), with $b \notin \{1,3\}$.

In $G(v_c)$, with $S_4 = \{u_{2,c}, u_{3,c}\}$, $r_4 = u_{2,c}$. $\overline{P}'_{1i}$ is the (S_4, r_4)-path corresponding to $\overline{P}_{1i}$. In $\overline{P}'_{1i}$, the path from vertex $u_{s_i,c}$ to $u_{3,c}$ is denoted as $\overline{P}''_{1i}$. In $H(u_3)$, with $S_5 = \{u_{3,c}, z\}$, $r_5 = u_{3,c}$, and it is known that there exist at least $\kappa(G)$ internally disjoint (S_5, r_5)-paths, say $\check{P}_{1i}$. In $H(v_b)$, with $S_4' = \{u_{b,2}, u_{b,3}\}$, $r_4' = u_{b,2}$, $\overline{P}'_{2j}$ is the (S_4', r_4')-path corresponding to $\overline{P}_{2j}$. In path $\overline{P}'_{2j}$, the path from vertex u_{b,f_j} to $u_{b,3}$ is denoted as $\overline{P}''_{2j}$. In $G(v_3)$, with $S_5' = \{u_{b,3}, z\}$, $r_5' = u_{b,3}$, and it is known that there exist at least $\kappa(G)$ internally disjoint (S_5', r_5')-paths in $G(v_3)$, say $\check{P}_{2j}$. If $u_{s_k,2} = u_{3,2}$, then $u_{3,2} \notin \check{P}_{1k}$ ($k \in [\kappa(G)]$). If $u_{3,1} \in \widetilde{P}_{1t}$, then $u_{3,1} \notin \check{P}_{1t}$ ($t \in [\kappa(G)]$). Similarly, if $u_{2,f_r} = u_{2,3}$, then $u_{2,3} \notin \check{P}_{2r}$ ($r \in [\kappa(G)]$). If $u_{1,3} \in \widetilde{P}_{2h}$, then $u_{1,3} \notin \check{P}_{2h}$ ($h \in [\kappa(G)]$). The arc-disjoint (S, r)-paths can be constructed as

$P_{1i} = \widetilde{P}_{1i} \cup \widehat{P}_{1i} \cup \overline{P}''_{1i} \cup \check{P}_{1i} \cup \{yu_{s_i,2}, u_{s_i,2}u_{s_i,c}\}$,

$P_{2j} = \widetilde{P}_{2j} \cup \widehat{P}_{2j} \cup \overline{P}''_{2j} \cup \check{P}_{2j} \cup \{yu_{2,f_j}, u_{2,f_j}u_{b,f_j}\}$.

Then we obtain $2\kappa(G)$ arc-disjoint (S, r)-paths.

Case 4. Let x and y be in the same $H(u_i)$. Let z, x, and y be in different $G(v_j)$ and let z, x be in different $H(u_i)$, for some $i \in [n]$, $j \in [m]$. Without loss of generality, we can assume that $x = u_{2,1}$, $y = u_{2,2}$, $z = u_{3,3}$. In this case, our overall goal is that we will use arc-disjoint paths between x and y in $H(u_2)$, y and its out-neighbors in $G(v_2)$, z and its in-neighbors in $H(u_3)$, x and its out-neighbors in $G(v_2)$, y and its out-neighbors in $H(u_2)$, z and its in-neighbors in $G(v_3)$, and combine them together to form the required arc-disjoint paths. The general idea of the proof process is briefly described in Figure 5. The vertices and paths contained in Figure 5 are explained below.

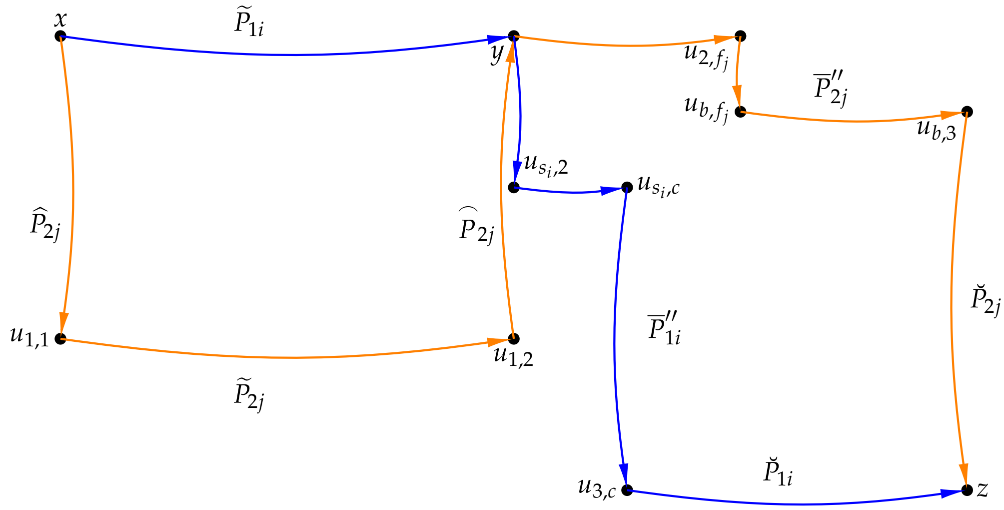


Figure 5. Depiction of the arc-disjoint paths found in Case 4 of the proof of Theorem 1.

Considering $S_1 = \{x, y\}$, $r_1 = x$, it is known that there exist at least $\kappa(G)$ internally disjoint (S_1, r_1)-paths in $H(u_2)$, denoted as $\widetilde{P}_{1i}$ $(i \in [\kappa(G)])$. In $G(v_1)$, with $S'_1 = \{x, u_{1,1}\}$, and $r'_1 = x$, it is known that there exist at least $\kappa(G)$ internally disjoint (S'_1, r'_1)-paths, denoted as $\widehat{P}_{2j}$. In $H(u_1)$, with $S'_2 = \{u_{1,1}, u_{1,2}\}$, and $r'_2 = u_{1,1}$, it is known that there exist at least $\kappa(G)$ internally disjoint (S'_2, r'_2)-paths, denoted as $\widetilde{P}_{2j}$. In $G(v_2)$, with $S'_3 = \{u_{1,2}, y\}$, and $r'_3 = u_{1,2}$, it is known that there exist at least $\kappa(G)$ internally disjoint (S'_3, r'_3)-paths, denoted as $\overset{\frown}{P}_{2j}$. Let $u_{s_i,2}$, $u_{s_i,c}$, u_{2,f_j}, u_{b,f_j}, $\check{P}_{1i}$, $\check{P}_{2j}$, $\overline{P}''_{1i}$ and $\overline{P}''_{2j}$ be the same as in Case 3.

If $u_{s_k,2} = u_{3,2}$, then $u_{3,2} \notin \check{P}_{1k}$ $(k \in [\kappa(G)])$. If $u_{2,f_r} = u_{2,3}$, then $u_{2,3} \notin \check{P}_{2r}$ $(r \in [\kappa(G)])$. If $u_{1,3} \in \widetilde{P}_{2h}$, then $u_{1,3} \notin \check{P}_{2h}$ $(h \in [\kappa(G)])$. If $u_{b,1} \in \overline{P}''_{2t}$, then $u_{b,1} \notin \widehat{P}_{2t}$ $(t \in [\kappa(G)])$. If $u_{1,3} \in \check{P}_{2l}$, then $u_{1,3} \notin \widetilde{P}_{2l}$ $(l \in [\kappa(G)])$.

Subcase 4.1. If there exists no vertex $u_{2,f_j} = x$. Let

$$P_{1i} = \widetilde{P}_{1i} \cup \overline{P}''_{1i} \cup \check{P}_{1i} \cup \{yu_{s_i,2}, u_{s_i,2}, u_{s_i,c}\},$$
$$P_{2j} = \widetilde{P}_{2j} \cup \overline{P}''_{2j} \cup \widehat{P}_{2j} \cup \check{P}_{2j} \cup \overset{\frown}{P}_{2j} \cup \{yu_{2,f_j}, u_{2,f_j}u_{b,f_j}\}.$$

Subcase 4.2. If there exists a vertex $u_{2,f_g} = x$ $(g \in [\kappa(G)])$, then in $G(v_1)$, there exists an out-neighbor $u_{b,1}$ of x. If $u_{b,1} \in \widehat{P}_{2j}$, this path is denoted by $\widehat{P}_{2g}$.

In $H(u_3)$, there exists an out-neighbor u_{3,g_1} of z such that $g_1 \in [m] \setminus \{c, 2, 1\}$. In $G(v_2)$, there exists an in-neighbor $u_{g_2,2}$ of y such that $g_2 \in [n] \setminus \{1, b, 3\}$. If $u_{g_2,2} \in \overset{\frown}{P}_{2j}$, this path is denoted by $\overset{\frown}{P}_{2g}$. Then in $H(u_{g_2})$, with $S'_4 = \{u_{g_2,g_1}, u_{g_2,2}\}$, and $r'_4 = u_{g_2,g_1}$, it is known that there are at least $\kappa(G)$ internally disjoint (S'_4, r'_4)-paths. One such (S'_4, r'_4)-path is chosen, denoted as $\widehat{P}_g$, with $u_{g_2,3} \notin \widehat{P}_g$. In $G(v_{g_1})$, with $S'_5 = \{u_{3,g_1}, u_{g_2,g_1}\}$, and $r'_5 = u_{3,g_1}$, it is known that there are at least $\kappa(G)$ internally disjoint (S'_5, r'_5)-paths. One such (S'_5, r'_5)-path is chosen, denoted as $\overline{P}_g$, with $u_{b,g_1} \notin \overline{P}_g$. Then, P_{2g} is constructed as

$$P_{2g} = \overline{P}''_{2g} \cup \check{P}_{2g} \cup \overline{P}_g \cup \widehat{P}_g \cup \{xu_{b,1}, zu_{3,g_1}, u_{g_2,2}y\}.$$

The other paths are the same as Subcase 4.1. Then we obtain $2\kappa(G)$ arc-disjoint (S, r)-paths.

Case 5. Let x and y be in the same $H(u_i)$. Let y and z be in the same $G(v_j)$, for some $i \in [n]$, $j \in [m]$. Without loss of generality, we can assume that $x = u_{1,1}$, $y = u_{1,2}$, $z = u_{2,2}$. In this case, our overall goal is that we will use arc-disjoint paths between x and y in $H(u_1)$, y and z in $G(v_2)$, x and its out-neighbors in $G(v_1)$, x and its out-neighbors in $G(v_1)$, z and y in $G(v_2)$, and combine them together to form the required arc-disjoint paths. The general idea of the proof process is briefly described in Figure 6. The vertices and paths contained in Figure 6 are explained below.

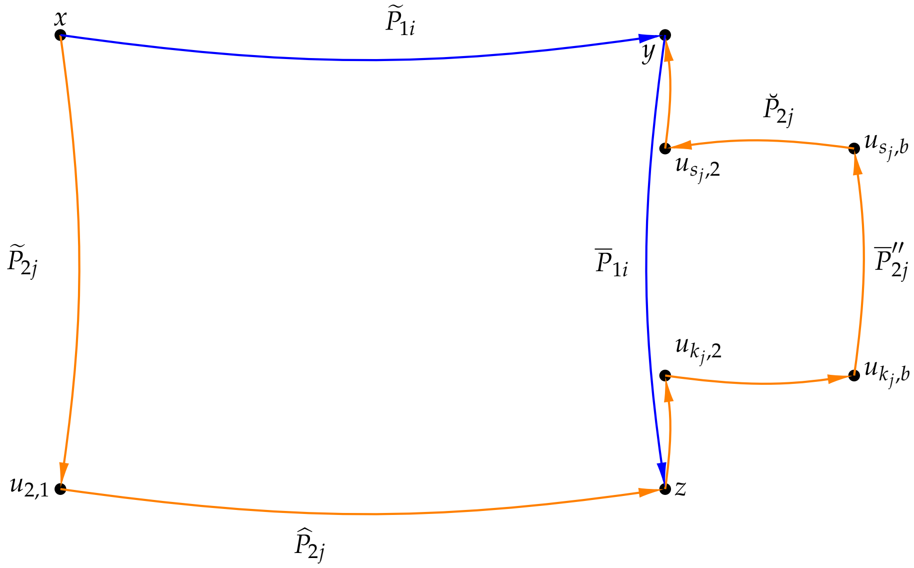


Figure 6. Depiction of the arc-disjoint paths found in Case 5 of the proof of Theorem 1.

It is known that there exist at least $\kappa(G)$ internally disjoint (S_1, r_1)-paths in $H(u_1)$, denoted as $\widetilde{P}_{1i}$ ($i \in [\kappa(G)]$), where $S_1 = \{x, y\}$ and $r_1 = x$. In $G(v_2)$, there exist at least $\kappa(G)$ internally disjoint (S_2, r_2)-paths, denoted as $\overline{P}_{1i}$ ($i \in [\kappa(G)]$), where $S_2 = \{y, z\}$ and $r_2 = y$. Similarly, in $G(v_1)$, there exist at least $\kappa(G)$ internally disjoint (S'_1, r'_1)-paths, denoted as $\widetilde{P}_{2j}$ ($j \in [\kappa(G)]$), where $S'_1 = \{x, u_{2,1}\}$ and $r'_1 = x$. In $H(u_2)$, there exist at least $\kappa(G)$ internally disjoint (S'_2, r'_2)-paths, denoted as $\widehat{P}_{2j}$ ($j \in [\kappa(G)]$), where $S'_2 = \{u_{2,1}, z\}$ and $r_2 = u_{2,1}$. In $G(v_2)$, there exist at least $\kappa(G)$ internally disjoint (S'_3, r'_3)-paths, denoted as $\overline{P}_{2j}$ ($j \in [\kappa(G)]$), where $S'_3 = \{z, y\}$ and $r'_3 = z$. For each $j \in [\kappa(G)]$, let $u_{s_j,2}$ be the in-neighbor of y in $\overline{P}_{2j}$, and clearly these in-neighbors are distinct. Similarly, let $u_{k_j,2}$ ($j \in [\kappa(G)]$) be the out-neighbor of z in $\overline{P}_{2j}$. For each $j \in [\kappa(G)]$, an out-neighbor $u_{k_j,b}$ of $u_{k_j,2}$ is chosen in $H(u_{k_j})$, where $b \neq 1$.

In $G(v_b)$, with $S'_4 = \{u_{2,b}, u_{1,b}\}$ and $r'_4 = u_{2,b}$. $\overline{P}'_{2j}$ is the (S'_4, r'_4)-path corresponding to $\overline{P}_{2j}$. In $\overline{P}'_{2j}$, the path from vertex $u_{k_j,b}$ to $u_{s_j,b}$ is denoted as $\overline{P}''_{2j}$. Then, in $H(u_{s_j})$, with $S'_{5_j} = \{u_{s_j,b}, u_{s_j,2}\}$ and $r'_{5_j} = u_{s_j,b}$, it is known that there exist at least $\kappa(G)$ internally disjoint (S'_{5_j}, r'_{5_j})-paths. One such (S'_{5_j}, r'_{5_j})-path, denoted as $\breve{P}_{2j}$ ($j \in [\kappa(G)]$), is chosen, with $u_{s_j,1} \notin \breve{P}_{2j}$. The arc-disjoint (S, r)-paths can be constructed as

$P_{1i} = \widetilde{P}_{1i} \cup \overline{P}_{1i}$,

$P_{2j} = \widetilde{P}_{2j} \cup \widehat{P}_{2j} \cup \overline{P}''_{2j} \cup \breve{P}_{2j} \cup \{zu_{k_j,2}, u_{s_j,2}y, u_{k_j,2}u_{k_j,b}\}$.

If $u_{s_t,2} = u_{k_t,2}$ ($t \in [\kappa(G)]$), then $P_{2t} = \widetilde{P}_{2t} \cup \widehat{P}_{2t} \cup \{zu_{k_t,2}, u_{s_t,2}y\}$. And if $u_{k_l,2} = y$ ($l \in [\kappa(G)]$), then $P_{2l} = \widetilde{P}_{2l} \cup \widehat{P}_{2l} \cup \{zy\}$. This results in obtaining $2\kappa(G)$ arc-disjoint (S, r)-paths.

Case 6. Let y and z be in the same $G(v_j)$. Let x, y be in different $G(v_j)$ and x, y, z be in different $H(u_i)$, for some $i \in [n]$, $j \in [m]$. Without loss of generality, we can assume that $x = u_{3,1}$, $y = u_{1,2}$, $z = u_{2,2}$. Let $u_{s_j,2}$ ($j \in [\kappa(G)]$), $u_{k_j,2}$, $\overline{P}_{1i}$, $\overline{P}_{2j}$, $\widehat{P}_{2j}$ be the same as in Case 5. In $G(v_1)$, with $S'_1 = \{x, u_{2,1}\}$ and $r'_1 = x$, it is known that there exist at least $\kappa(G)$ internally disjoint (S'_1, r'_1)-paths in $G(v_1)$, denoted as $\widetilde{P}_{2j}$. In this case, our overall goal is that we will use arc-disjoint paths between x and its out-neighbors in $H(u_3)$, y and its in-neighbors in $H(u_1)$, y and z in $G(v_2)$, x and its out-neighbors in $G(v_1)$, z and its in-neighbors in $H(u_2)$, z and y in $G(v_2)$, and combine them together to form the required arc-disjoint paths. The general idea of the proof process is briefly described in Figure 7. The vertices and paths contained in Figure 7 are explained below.

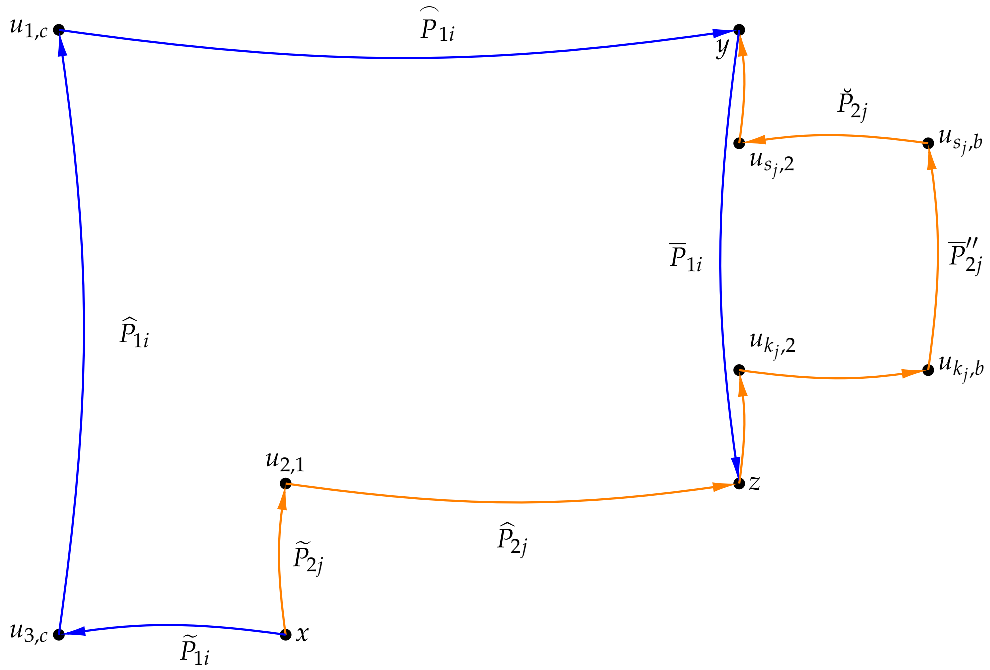


Figure 7. Depiction of the arc-disjoint paths found in Case 6 of the proof of Theorem 1.

Subcase 6.1. In the set $\{u_{s_j,2}, u_{k_j,2}\}$, there does not exist $u_{3,2} \in \{u_{s_j,2}, u_{k_j,2}\}$. Thus, $u_{s_j,b}$, $u_{k_j,b}$, $\check{P}_{2j}$, $\overline{P}''_{2j}$ remain the same as in Case 5.

In $H(u_3)$, with $S_1 = \{x, u_{3,c}\}$ $(c \in [m] \setminus \{1, 2, b\})$ and $r_1 = x$, it is known that there exist at least $\kappa(G)$ internally disjoint (S_1, r_1)-paths in $H(u_3)$, denoted as $\widetilde{P}_{1i}$. In $G(v_c)$, with $S_2 = \{u_{3,c}, u_{1,c}\}$ and $r_2 = u_{3,c}$, it is known that there exist at least $\kappa(G)$ internally disjoint (S_2, r_2)-paths in $G(v_c)$, denoted as $\widehat{P}_{1i}$. In $H(u_1)$, with $S_3 = \{u_{1,c}, y\}$ and $r_3 = u_{1,c}$, it is known that there exist at least $\kappa(G)$ internally disjoint (S_3, r_3)-paths in $H(u_1)$, denoted as $\overset{\frown}{P}_{1i}$. If $u_{3,2} \in \widetilde{P}_{1r}$, then $u_{3,2} \notin \overline{P}_{1r}$. Let

$$P_{1i} = \widetilde{P}_{1i} \cup \overline{P}_{1i} \cup \widehat{P}_{1i} \cup \overset{\frown}{P}_{1i},$$
$$P_{2j} = \widetilde{P}_{2j} \cup \overline{P}''_{2j} \cup \widehat{P}_{2j} \cup \check{P}_{2j} \cup \{zu_{k_j,2}, u_{k_j,2}u_{k_j,b}, u_{s_j,2}y\}.$$

If $u_{s_t,2} = u_{k_t,2}$ $(t \in [\kappa(G)])$, then $P_{2t} = \widetilde{P}_{2t} \cup \widehat{P}_{2t} \cup \{zu_{k_t,2}, u_{s_t,2}y\}$. And if $u_{k_l,2} = y$ $(l \in [\kappa(G)])$, then $P_{2l} = \widetilde{P}_{2l} \cup \widehat{P}_{2l} \cup \{zy\}$. Now we obtain $2\kappa(G)$ arc-disjoint (S, r)-paths.

Subcase 6.2. In the set $\{u_{s_j,2}, u_{k_j,2}\}$, only one vertex $u_{k_r,2} = u_{3,2}$ $(r \in [\kappa(G)])$ exists. Thus, $u_{s_j,b}$, $u_{k_j,b}$, $\check{P}_{2j}$, $\overline{P}''_{2j}$ remain the same as in Case 5.

If $u_{k_r,2}u_{k_r,b} \notin \widetilde{P}_{1i}$ in $\widetilde{P}_{1i}$, then P_{1i}, P_{2j} remain the same as in Subcase 6.1. If an arc $u_{k_r,2}u_{k_r,b}$ is in path $\widetilde{P}_{1i}$, since $\delta(G) \geq 4$, then an out-neighbor $u_{k_r,a}$ of $u_{k_r,2}$ can be found in $H(u_3)$ such that $u_{k_r,2}u_{k_r,a} \notin \widetilde{P}_{1i}$ and $a \in [m] \setminus \{c, 1\}$. In $G(v_a)$, $\overline{P}'''_{2r}$ is the (S'_3, r'_3)-path corresponding to $\overline{P}''_{2r}$, where $S'_3 = \{u_{k_r,a}, u_{s_r,a}\}$, $r'_3 = u_{k_r,a}$. In $H(u_{s_r})$, with $S'_4 = \{u_{s_r,a}, u_{s_r,2}\}$ and $r'_4 = u_{s_r,a}$, it is known that there exist at least $\kappa(G)$ internally disjoint (S'_4, r'_4)-paths. Then in these paths, one of the paths $\check{P}'_{2r}$ is chosen, with $u_{s_r,1} \notin \check{P}'_{2r}$. $P_{2j} (j \neq r)$ and P_{1i} remain the same as in Subcase 6.1. P_{2r} is constructed as

$$P_{2r} = \widetilde{P}_{2r} \cup \overline{P}'''_{2r} \cup \widehat{P}_{2r} \cup \check{P}'_{2r} \cup \{zu_{k_r,2}, u_{k_r,2}u_{k_r,a}, u_{s_r,2}y\}.$$

Subcase 6.3. In the set $\{u_{s_j,2}, u_{k_j,2}\}$, there is only one vertex $u_{s_g,2} = u_{3,2}$ ($g \in [\kappa(G)]$).

For each $j \in [\kappa(G)]$, an in-neighbor of $u_{s_j,1}$ in $H(u_{s_j})$ can be chosen, denoted by $u_{s_j,d}$ ($d \in [m]$), where $d \neq c, 1$. In $G(v_d)$, let $\overline{P}'_{2j}$ be the (S'_5, r'_5)-path corresponding to $\overline{P}_{2j}$, where $S'_5 = \{u_{2,d}, u_{1,d}\}$, $r'_5 = u_{2,d}$. The path from vertex $u_{k_j,d}$ to $u_{s_j,d}$ in path $\overline{P}'_{2j}$ is denoted as $\overline{P}''_{2j}$. In $H(u_{k_j})$, let $S'_{6_j} = \{u_{k_j,2}, u_{k_j,d}\}$, $r'_{6_j} = u_{k_j,2}$, and at least $\kappa(G)$ internally disjoint (S'_{6_j}, r'_{6_j})-paths are known to exist. Then, one of the paths $\breve{P}_{2j}$ ($j \in [\kappa(G)]$) is chosen, where $u_{k_j,1} \notin \breve{P}_{2j}$. If $u_{s_t,2} = u_{k_t,2}$ ($t \in [\kappa(G)]$), $P_{2t} = \widetilde{P}_{2t} \cup \widehat{P}_{2t} \cup \{zu_{k_t,2}, u_{s_t,2}y\}$. And if $u_{k_l,2} = y$ ($l \in [\kappa(G)]$), $P_{2l} = \widetilde{P}_{2l} \cup \widehat{P}_{2l} \cup \{zy\}$. If $u_{s_g,d}u_{s_g,2} \notin \widetilde{P}_{1i}$ in the path $\widetilde{P}_{1i}$. Let

$P_{1i} = \widetilde{P}_{1i} \cup \overline{P}_{1j} \cup \widehat{P}_{1i} \cup \overset{\frown}{P}_{1i}$,
$P_{2j} = \widetilde{P}_{2j} \cup \overline{P}''_{2j} \cup \widehat{P}_{2j} \cup \breve{P}_{2j} \cup \{zu_{k_j,2}, u_{s_j,d}u_{s_j,2}, u_{s_j,2}y\}$.

If an arc $u_{s_g,d}u_{s_g,2}$ is in path $\widetilde{P}_{1i}$, an in-neighbor $u_{s_g,f}$ of $u_{s_g,2}$ can be found in $H(u_3)$ such that $u_{s_g,f}u_{s_g,2} \notin \widetilde{P}_{1i}$ and $f \in [m] \setminus \{c, 1\}$. In $G(v_f)$, let $\overline{P}'''_{2g}$ be the (S'_7, r'_7)-path corresponding to $\overline{P}''_{2g}$, where $S'_7 = \{u_{k_g,f}, u_{s_g,f}\}$, $r'_7 = u_{k_g,f}$. In $H(u_{k_g})$, let $S'_8 = \{u_{k_g,2}, u_{k_g,f}\}$, $r'_8 = u_{k_g,2}$, and at least $\kappa(G)$ internally disjoint (S'_7, r'_7)-paths are known to exist. Then, one of the paths $\breve{P}'_{2g}$ is chosen, and let $u_{k_g,1} \notin \breve{P}'_{2g}$. Let

$P_{2g} = \widetilde{P}_{2g} \cup \overline{P}'''_{2g} \cup \widehat{P}_{2g} \cup \breve{P}'_{2g} \cup \{zu_{k_g,2}, u_{s_g,f}u_{s_g,2}, u_{s_g,2}y\}$.

Hence, we obtain $2\kappa(G)$ arc-disjoint (S, r)-paths.

Now we prove that this bound is sharp. By Proposition 1, $\lambda_3^p(\overleftrightarrow{K}_n \square \overleftrightarrow{K}_m) = n + m - 2$. By Lemma 2, $\kappa(\overleftrightarrow{K}_n) = n - 1$. So we have $\lambda_3^p(\overleftrightarrow{K}_n \square \overleftrightarrow{K}_n) = 2\kappa(\overleftrightarrow{K}_n) = 2n - 2$, with $n \geq 5$. Therefore, the lower bound holds and is sharp. □

4. Exact Values for Digraph Classes

In this section, we aim to determine precise values for the directed path 3-arc-connectivity of the Cartesian product of two digraphs within specific digraph classes.

Proposition 1. *We have* $\lambda_3^p(\overleftrightarrow{K}_n \square \overleftrightarrow{K}_m) = n + m - 2$.

Proof. Consider $S = \{x, y, z\}$ and $r = x$. We will focus solely on scenarios where x, y, and z do not all belong to the same $\overleftrightarrow{K}_m(u_i)$ or the same $\overleftrightarrow{K}_n(v_j)$ for any $i \in [n]$, $j \in [m]$. The rationale for the remaining cases follows a similar line of argument. Without loss of generality, let us assume $x = u_{1,1}$, $y = u_{2,2}$, $z = u_{3,3}$. It is feasible to derive $n + m - 2$ arc-disjoint (S, r)-paths in $\overleftrightarrow{K}_n \square \overleftrightarrow{K}_m$, say P_1 ,P_2, …, P_a ($a = \min\{i + 1, 3 < i \leq n\}$), P_{i+1} ($4 < i \leq n$), …, P_b ($b = \min\{n + j - 2, 3 < j \leq n\}$), P_{n+j-2} ($4 < j \leq m$) (as shown in Figure 8) such that

$P_1 : xu_{2,1}yu_{3,2}z$, $P_2 : xu_{1,2}yu_{2,3}z$, $P_3 : xu_{1,3}zu_{3,2}y$,
$P_4 : xu_{3,1}zu_{2,3}y$, $P_a : xu_{4,1}u_{4,3}zu_{1,3}u_{1,2}u_{4,2}y$, $P_b : xu_{1,4}u_{3,4}zu_{3,1}u_{2,1}u_{2,4}y$,
$P_{i+1} : xu_{i,1}u_{i,3}zu_{i-1,3}u_{i-1,2}u_{i,2}y$, $P_{n+j-2} : xu_{1,j}u_{3,j}zu_{3,j-1}u_{2,j-1}u_{2,j}y$.

Now, we add two cases to prove that the proposition holds, so as to show that the proposition has no constraint conditions.

First, let $n = m = 4$. We can assume that $x = u_{1,1}$, $y = u_{2,2}$, $z = u_{3,3}$. Let

$P_1 : xu_{2,1}yu_{3,2}z$, $P_2 : xu_{1,2}yu_{2,3}z$, $P_3 : xu_{3,1}u_{3,2}yu_{4,2}u_{4,3}z$,
$P_4 : xu_{4,1}u_{4,2}yu_{1,2}u_{1,3}z$, $P_5 : xu_{1,3}u_{2,3}yu_{2,4}u_{3,4}z$, $P_6 : xu_{1,4}u_{2,4}yu_{2,1}u_{3,1}z$.

Furthermore, let $n = 2$, $m = 4$. We can assume that $x = u_{1,1}$, $y = u_{1,2}$, $z = u_{1,3}$. Let

$P_1 : xyz$, $P_2 : xzy$, $P_3 : xu_{1,4}zu_{2,3}u_{2,2}y$, $P_4 : xu_{2,1}u_{2,3}zu_{1,4}y$.

Then we have $n + m - 2 = \min\{\delta^+(D), \delta^-(D)\} \geq \lambda_3^p(\overleftrightarrow{K}_n \square \overleftrightarrow{K}_m) \geq n + m - 2$. This concludes the proof. □

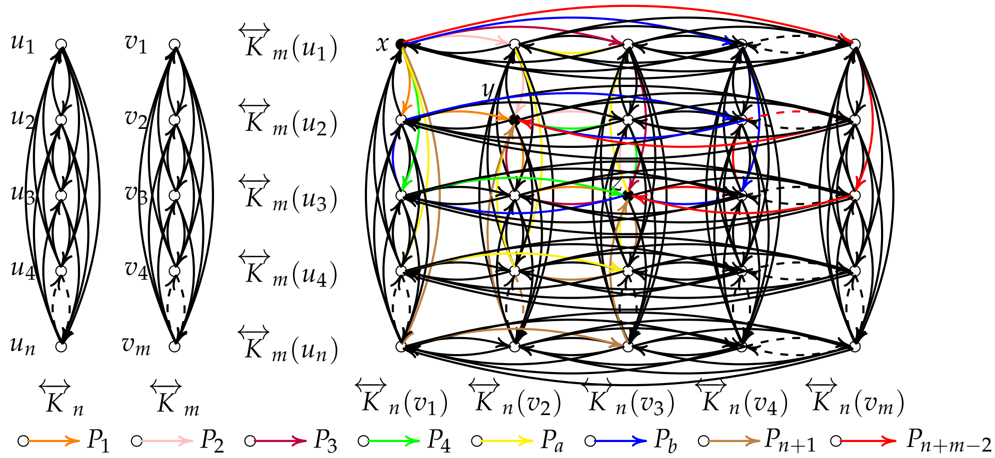


Figure 8. $\overleftrightarrow{K}_n \square \overleftrightarrow{K}_m$.

Proposition 2. *We have* $\lambda_3^p(\overleftrightarrow{C}_n \square \overleftrightarrow{K}_m) = m+1$*, with* $n \geq 3$.

Proof. Let $S = \{x, y, z\}$, $r = x$, and we only examine the case where x, y, and z are not all within the same $\overleftrightarrow{C}_n(u_i)$ or the same $\overleftrightarrow{K}_m(v_j)$ for any $i \in [n]$, $j \in [m]$. The rationale for the remaining cases follows a similar line of argument. Without loss of generality, let us assume $x = u_{1,1}$, $y = u_{2,2}$, $z = u_{3,3}$. We can obtain $m+1$ arc-disjoint (S, r)-paths in $\overleftrightarrow{C}_n \square \overleftrightarrow{K}_m$, say P_1 , $P_2, \ldots, P_{i+1}$ $(4 < i \leq m)$, P_{m-1}, P_m (as shown in Figure 9) such that

$P_1 : xu_{2,1}yu_{3,2}z$, $P_2 : xu_{1,2}yu_{2,3}z$, $P_3 : xu_{1,n}u_{3,n}u_{3,1}u_{3,2}yu_{1,2}u_{1,3}z$,
$P_4 : xu_{3,1}u_{3,n}...u_{3,j}...zu_{2,3}y$, $P_5 : xu_{4,1}u_{4,n} \ldots u_{4,j} \ldots u_{4,3}zu_{3,2}u_{4,2}y$,
$P_{i+1} : xu_{i,1}u_{i,n} \ldots u_{i,j} \ldots u_{i,3}zu_{i-1,3}u_{i-1,2}u_{i,2}y$.

Now, we add two cases to prove that the proposition holds, so as to show that the proposition has no constraint conditions.

First, let $n = 3$, $m = 4$. We can assume that $x = u_{1,1}$, $y = u_{2,1}$, $z = u_{3,1}$. Let
$P_1 : xyz$, $P_2 : xzy$, $P_3 : xu_{4,1}zu_{3,2}u_{2,2}y$, $P_4 : xu_{1,3}u_{2,3}yu_{2,2}u_{3,2}u_{3,3}z$.

Furthermore, let $n = 3$, $m = 2$. We can assume that $x = u_{1,1}$, $y = u_{1,2}$, $z = u_{1,3}$. Let
$P_1 : xyz$, $P_2 : xzu_{2,3}u_{2,2}y$, $P_3 : xu_{2,1}u_{2,3}zy$.

Then we have $m + 1 = \min\{\delta^+(D), \delta^-(D)\} \geq \lambda_3^p(\overleftrightarrow{C}_n \square \overleftrightarrow{K}_m) \geq m+1$. This concludes the proof. □

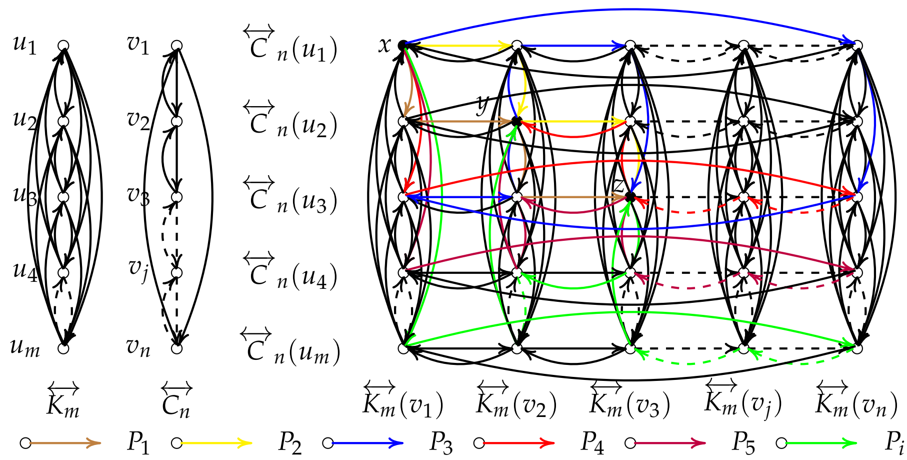


Figure 9. $\overleftrightarrow{C}_n \square \overleftrightarrow{K}_m$.

Proposition 3. *We have* $\lambda_3^p(\overrightarrow{C}_n \square \overleftrightarrow{K}_m) = m$.

Proof. Let $S = \{x, y, z\}$, $r = x$, and we only examine the case where x, y, and z are not all within the same $\overrightarrow{C}_n(u_i)$ or the same $\overleftrightarrow{K}_m(v_j)$ for any $i \in [n]$, $j \in [m]$. The rationale for the remaining cases follows a similar line of argument. Without loss of generality, let us assume $x = u_{1,1}$, $y = u_{2,2}$, $z = u_{3,3}$. We can obtain m arc-disjoint (S, r)-paths in $\overrightarrow{C}_n \square \overleftrightarrow{K}_m$.

First assume that m is even number, let

$P_1 : xu_{2,1}yu_{3,2}z$, $P_2 : xu_{1,2}yu_{2,3}z$, $P_3 : xu_{3,1}u_{3,2}yu_{2,3}z$,

$P_4 : xu_{4,1}u_{4,2}yu_{1,2}u_{1,3}z$, $P_{i-1} : xu_{i-1,1}u_{i-1,2}yu_{i,2}u_{i,3}z$,

$P_i : xu_{i,1}u_{i,2}yu_{i-1,2}u_{i-1,3}z$, $4 < i \leq m$, and i is an even number.

Conversely assume that m is odd number, let

$P_1 : xu_{2,1}yu_{3,2}z$, $P_2 : xu_{1,2}yu_{2,3}z$, $P_3 : xu_{3,1}u_{3,2}yu_{1,2}u_{1,3}z$,

$P_{i-1} : xu_{i-1,1}u_{i-1,2}yu_{i,2}u_{i,3}z$,

$P_i : xu_{i,1}u_{i,2}yu_{i-1,2}u_{i-1,3}z$, $3 < i \leq m$, and i is an odd number. Then we have $m = \min\{\delta^+(D), \delta^-(D)\} \geq \lambda_3^p(\overrightarrow{C}_n \square \overleftrightarrow{K}_m) \geq m$. This completes the proof. □

Proposition 4. *We have* $\lambda_3^p(\overleftrightarrow{T}_n \square \overleftrightarrow{K}_m) = m$.

Proof. Let $S = \{x, y, z\}$, $r = x$, and we only examine the case where x, y, and z are not all within the same $\overleftrightarrow{T}_n(u_i)$ or the same $\overleftrightarrow{K}_m(v_j)$ for any $i \in [n]$, $j \in [m]$. The rationale for the remaining cases follows a similar line of argument. Without loss of generality, let us assume $x = u_{1,1}$, $y = u_{2,2}$, $z = u_{3,3}$. We can obtain m arc-disjoint (S, r)-paths in $\overleftrightarrow{T}_n \square \overleftrightarrow{K}_m$, say $P_1, P_2, \ldots, P_i$ $(4 < i \leq m)$, P_{m-1}, P_m such that

$P_1 : xu_{2,1}yu_{3,2}z$, $P_2 : xu_{1,2}yu_{2,3}z$, $P_3 : xu_{1,3}u_{2,3}yu_{2,1}u_{3,1}z$,

$P_4 : xu_{1,4}u_{2,4}u_{3,4}zu_{3,2}y$, $P_i : xu_{1,i}u_{2,i}u_{3,i}zu_{3,i-1}u_{2,i-1}y$.

Then we have $m = \min\{\delta^+(D), \delta^-(D)\} \geq \lambda_3^p(\overleftrightarrow{T}_n \square \overleftrightarrow{K}_m) \geq m$. This completes the proof. □

Proposition 5. *We have* $\lambda_3^p(\overrightarrow{C}_n \square \overrightarrow{C}_m) = 2$.

Proof. Let $S = \{x, y, z\}$, $r = x$, and we only examine the case where x, y, and z are not all within the same $\overrightarrow{C}_n(u_i)$ or the same $\overrightarrow{C}_m(v_j)$ for any $i \in [n]$, $j \in [m]$. The rationale for the remaining cases follows a similar line of argument. Without loss of generality, let us assume $x = u_{1,1}$, $y = u_{2,2}$, $z = u_{3,3}$. We can obtain two arc-disjoint (S, r)-paths in $\overrightarrow{C}_n \square \overrightarrow{C}_m$, say P_1 and P_2 such that

$P_1 : xu_{2,1}yu_{3,2}z$, $P_2 : xu_{1,2}yu_{2,3}z$.

Then we have $2 = \min\{\delta^+(D), \delta^-(D)\} \geq \lambda_3^p(\overrightarrow{C}_n \square \overrightarrow{C}_m) \geq 2$. This completes the proof. □

Proposition 6. *We have* $\lambda_3^p(\overrightarrow{C}_n \square \overleftrightarrow{C}_m) = 3$, *with* $m \geq 3$.

Proof. Let $S = \{x, y, z\}$, $r = x$, and we only examine the case where x, y, and z are not all within the same $\overrightarrow{C}_n(u_i)$ or the same $\overleftrightarrow{C}_m(v_j)$ for any $i \in [n]$, $j \in [m]$. The rationale for the remaining cases follows a similar line of argument. Without loss of generality, let us assume $x = u_{1,1}$, $y = u_{2,2}$, $z = u_{3,3}$. We can obtain three arc-disjoint (S, r)-paths in $\overrightarrow{C}_n \square \overleftrightarrow{C}_m$, say P_1, P_2, P_3 such that

$P_1 : xu_{2,1}yu_{3,2}z$, $P_2 : xu_{1,2}yu_{2,3}z$,

$P_3 : xu_{m,1}u_{m,2}u_{m-1,2} \ldots u_{3,2}yu_{1,2}u_{1,3}u_{m,3}u_{m-1,3} \ldots z$.

Then we have $3 = \min\{\delta^+(D), \delta^-(D)\} \geq \lambda_3^p(\overrightarrow{C}_n \square \overleftrightarrow{C}_m) \geq 3$. This completes the proof. □

Proposition 7. *We have* $\lambda_3^p(\overleftrightarrow{C}_n \square \overleftrightarrow{C}_m) = 4$, *with* $n \geq 3$, $m \geq 3$.

Proof. Let $S = \{x, y, z\}$, $r = x$, and we only examine the case where x, y, and z are not all within the same $\overleftrightarrow{C}_n(u_i)$ or the same $\overleftrightarrow{C}_m(v_j)$ for any $i \in [n]$, $j \in [m]$. The rationale for the remaining cases follows a similar line of argument. Without loss of generality, let us assume $x = u_{1,1}$, $y = u_{2,2}$, $z = u_{3,3}$. We can obtain four arc-disjoint (S, r)-paths in $\overleftrightarrow{C}_n \square \overleftrightarrow{C}_m$, say P_1, P_2, P_3, P_4 such that

$P_1 : xu_{2,1}yu_{3,2}z$, $P_2 : xu_{1,2}yu_{2,3}z$,

$P_3 : xu_{m,1}u_{m,2}u_{m,3}u_{m-1,3} \dots zu_{3,2}y$, $P_4 : xu_{1,n}u_{2,n}u_{3,n}zu_{2,3}y$.

Then we have $4 = \min\{\delta^+(D), \delta^-(D)\} \geq \lambda_3^p(\overleftrightarrow{C}_n \square \overleftrightarrow{C}_m) \geq 4$. This completes the proof. □

Proposition 8. *We have* $\lambda_3^p(\overrightarrow{C}_n \square \overleftrightarrow{T}_m) = 2$.

Proof. Let $S = \{x, y, z\}$, $r = x$, and we only examine the case where x, y, and z are not all within the same $\overrightarrow{C}_n(u_i)$ or the same $\overleftrightarrow{T}_m(v_j)$ for any $i \in [n]$, $j \in [m]$. The rationale for the remaining cases follows a similar line of argument. Without loss of generality, let us assume $x = u_{1,1}$, $y = u_{2,2}$, $z = u_{3,3}$. We can obtain three arc-disjoint (S, r)-paths in $\overrightarrow{C}_n \square \overleftrightarrow{T}_m$, say P_1 and P_2 such that

$P_1 : xu_{2,1}yu_{3,2}z$, $P_2 : xu_{1,2}yu_{2,3}z$.

Then we have $2 = \min\{\delta^+(D), \delta^-(D)\} \geq \lambda_3^p(\overrightarrow{C}_n \square \overleftrightarrow{T}_m) \geq 2$. This completes the proof. □

Proposition 9. *We have* $\lambda_3^p(\overleftrightarrow{C}_n \square \overleftrightarrow{T}_m) = 3$*, with* $n \geq 3$.

Proof. Let $S = \{x, y, z\}$, $r = x$, and we only examine the case where x, y, and z are not all within the same $\overleftrightarrow{C}_n(u_i)$ or the same $\overleftrightarrow{T}_m(v_j)$ for any $i \in [n]$, $j \in [m]$. The rationale for the remaining cases follows a similar line of argument. Without loss of generality, let us assume $x = u_{1,1}$, $y = u_{2,2}$, $z = u_{3,3}$. We can obtain three arc-disjoint (S, r)-paths in $\overleftrightarrow{C}_n \square \overleftrightarrow{T}_m$, say P_1, P_2, P_3 such that

$P_1 : xu_{2,1}yu_{3,2}z$, $P_2 : xu_{1,2}yu_{2,3}z$, $P_3 : xu_{m,1}u_{m,2}u_{m,3}u_{m-1,3} \dots zu_{3,2}y$.

Then we have $3 = \min\{\delta^+(D), \delta^-(D)\} \geq \lambda_3^p(\overleftrightarrow{C}_n \square \overleftrightarrow{T}_m) \geq 3$. This completes the proof. □

Proposition 10. *We have* $\lambda_3^p(\overleftrightarrow{T}_n \square \overleftrightarrow{T}_m) = 2$.

Proof. Let $S = \{x, y, z\}$, $r = x$, and we only examine the case where x, y, and z are not all within the same $\overleftrightarrow{T}_n(u_i)$ or the same $\overleftrightarrow{T}_m(v_j)$ for any $i \in [n]$, $j \in [m]$. The rationale for the remaining cases follows a similar line of argument. Without loss of generality, let us assume $x = u_{1,1}$, $y = u_{2,2}$, $z = u_{3,3}$. We can obtain three arc-disjoint (S, r)-paths in $\overleftrightarrow{T}_n \square \overleftrightarrow{T}_m$, say P_1 and P_2 such that

$P_1 : xu_{2,1}yu_{3,2}z$, $P_2 : xu_{1,2}yu_{2,3}z$.

Then we have $2 = \min\{\delta^+(D), \delta^-(D)\} \geq \lambda_3^p(\overleftrightarrow{T}_n \square \overleftrightarrow{T}_m) \geq 2$. This completes the proof. □

According to Propositions 1–9, we find that the directed path 3-arc-connectivity of some Cartesian products of digraphs is equal to the minimum semi-degrees. Based on this discovery, we can consider under what conditions the directed path 3-arc-connectivity of Cartesian products of digraphs can be equal to the minimum semi-degrees, which is a problem we can consider next.

5. Conclusions

In this paper, we prove that if G and H are two digraphs such that $\delta(G) \geq 4$, $\delta(H) \geq 4$, and $\kappa(G) \geq 2$, $\kappa(H) \geq 2$, then $\lambda_3^p(G \square H) \geq \min\{2\kappa(G), 2\kappa(H)\}$, and moreover, this bound is sharp. Finally, we obtain exact values of $\lambda_3^p(G \square H)$ for some digraph classes G and H. In practical terms, constructing vertex-disjoint or arc-disjoint paths in graphs is crucial. These paths play a significant role in improving transmission reliability and boosting network transmission speeds.

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