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Fixed Point Dynamics in a New Type of Contraction in b-Metric Spaces

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Abstract: Since the topological properties of a b-metric space (which generalizes the concept of a metric space) seem sometimes counterintuitive due to the fact that the "open" balls may not be open sets, we review some aspects of these spaces concerning compactness, metrizability, continuity and fixed points. After doing so, we introduce new types of contractivities that extend the concept of Banach contraction. We study some of their properties, giving sufficient conditions for the existence of fixed points and common fixed points. Afterwards, we consider some iterative schemes in quasi-normed spaces for the approximation of these critical points, analyzing their convergence and stability. We apply these concepts to the resolution of a model of integral equation of Urysohn type. In the last part of the paper, we refine some results about partial contractivities in the case where the underlying set is a strong b-metric space, and we establish some relations between mutual weak contractions and quasi-contractions and the new type of contractivity.

Keywords: iteration; fixed-point theorems; contractions; b-metric spaces; quasi-normed spaces; Urysohn integral equations

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1. Introduction

Since the existence and approximation of fixed points plays a central role in the implementation of numerical methods for the resolution of all kinds of equations, a great number of authors are focusing on the generalizations of previous theories related to this topic in order to extend them to wider contexts. In particular, many different distance spaces are being considered. There are extensive works dealing with the structure of b-metric space. This kind of framework was formerly called quasi-metric (see, for instance, references [1,2]). However, there are a lot of different generalizations of the concept of metric space with this terminology (see articles [2–4]). Lately, the name b-metric space is predominantly used to describe a set X endowed with a kind of distance, where the triangular inequality is substituted by the general expression

$$d(x,y) \leq s(d(x,z) + d(z,y)),$$

for $x, y, z \in X$ and some fixed real number $s \geq 1$.

In this paper, we review first the topological background of b-metric spaces, regarding compactness, metrizability, continuity and existence of fixed points of self-maps (Section 2).

In Section 3, we define new types of contractions and mutual contractions on b-metric spaces. We prove that the novel mappings include many others present in the literature when the associated constants satisfy certain restrictions. That is true for the quasi-contractions defined by Ćirić, which include, in turn, the Kannan [5–7], Chatterjea [8], and other self-maps. We also give sufficient conditions for the existence of common fixed points for mappings related by an inequality of partial contractivity type.

Section 4 is devoted to the analysis of the Noor algorithm and the so-called SP iterative procedure for the approximation of fixed points, in the context of quasi-normed spaces (the distance associated with a quasi-norm is a particular case of b-metric space). We study the convergence and stability of these iterative methods, regarding the values of their parameters.

Section 5 presents an application to obtain the solution of an integral equation of Urysohn type, when the integrand satisfies certain conditions. Urysohn's equation [9] is a prototypical example of a nonlinear integral equation. Depending on the integrand, the equation may not have an explicit exact solution. In this section, we tackle the problem of finding solutions of this type of equation by means of iterative methods of fixed-point approximation in the context of functional b-metric spaces.

Section 6 is devoted to refining some results about partial contractivities in the case where the underlying set is a strong b-metric space.

In the paper's last part, some relations between mutual weak, partial and quasi-contractions are established, and a collage theorem for weak contractions is proved. The article finishes with a section of Conclusions.

2. Topology of b-Metric Spaces and Continuous Maps

The metric, normed and quasi-normed spaces are particular cases of b-metric structures. According to the exhaustive historical review on the introduction of these spaces carried out in reference [10], the first authors to define and use the concept of a b-metric were Coifman and Guzmán in an article of 1970 [11]. This fact is quoted in references [12,13], for instance. In general, it is considered that the papers by Bakhtin [14] and Czerwik [15,16] are foundational for this topic as well. The readers are encouraged to consult the quoted article by Berinde and Pacurar [10] for historical details and clarifications.

Since the topological properties of a b-metric space seem sometimes counterintuitive due to the fact that the "open" balls may not be open sets, we mention in this section a review of some aspects of a space of this type, concerning compactness, metrizability and fixed points of mappings. These results have been collected from references [1,2,13,17–21] and the authors. For new insights on the topic, see, for instance, [22–24].

Definition 1. *A b-metric space X is a set endowed with a mapping $d : X \times X \to \mathbb{R}^+$ with the following properties:*

1. $d(x,y) \geq 0$, $d(x,y) = 0$ *if and only if* $x = y$.
2. $d(x,y) = d(y,x)$ *for any* $x, y \in X$.
3. *There exists* $s \geq 1$ *such that* $d(x,y) \leq s(d(x,z) + d(z,y))$ *for any* $x, y, z \in X$.

The constant s is the index of the b-metric space, and d is called a b-metric.

Remark 1. *For $s = 1$, this structure agrees with the usual metric space.*

The definition of ball $B(x,r)$, for $x \in X$ and $r > 0$, is similar to the metric case.

Definition 2. *A subset A of a b-metric space X is open if $\forall x \in X$, there exists $r > 0$ such that $B(x,r) \subseteq A$. A subset $F \subset X$ is closed if its complementary subset is open.*

The previous definition of open sets endows X with a topology τ. The definition of compact set agrees with the given by this topology.

Proposition 1. *A subset F of a b-metric space X is closed if and only if for any sequence $(x_n) \subset X$ with $\lim_{n\to\infty} x_n = x$, the limit $x \in F$.*

Let $\bar{A}$ be the intersection of all closed subsets of X that contain a subset A, then $x \in \bar{A}$ if for any $\epsilon > 0$, $B(x,\epsilon) \cap A \neq \emptyset$.

Proof. See Prop 2 of [20]. □

$\bar{A}$ is the (topological) closure of A.

Proposition 2. *$\bar{A}$ is a closed set.*

Proof. See Proposition 3.2 of reference [2]. □

The convergence of sequences in this type of spaces can be easily characterized. We mention the following result.

Proposition 3. *A sequence $(x_n) \subset X$ is convergent to a limit $x \in X$ if and only if* $\lim_{n\to\infty} d(x_n, x) = 0$.

Proof. See Corollary 2.3 of the reference [13]. □

For sequences $(x_n), (y_n) \subseteq X$ such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, one has

$$\frac{1}{s^2} d(x,y) \leq \liminf_{n\to\infty} d(x_n, y_n) \leq \limsup_{n\to\infty} d(x_n, y_n) \leq s^2 d(x,y) \tag{1}$$

(see Lemma 3.1 of [2]).

Thus, in general, $\lim_{n\to\infty} d(x_n, y_n)$ does not agree with $d(x,y)$, and d is not continuous. For a metric space ($s = 1$), the former chain of inequalities gives the equality $\lim_{n\to\infty} d(x_n, y_n) = d(x,y)$, and consequently, d is continuous.

Proposition 4. *If a sequence (x_n) of a b-metric space is convergent, its limit is unique.*

Proof. It is a consequence of (1) taking $x_n = y_n$. □

The definition of a Cauchy sequence is similar to the metric case.

Definition 3. *X is a complete b-metric space if every Cauchy sequence is convergent.*

The next result is similar to the metric case.

Proposition 5. *Let X be a complete b-metric space. A subset $Y \subseteq X$ is complete if and only if Y is closed.*

Definition 4. *Let (X,d) be a b-metric space, then $A \subseteq X$ is totally bounded if for any $\epsilon > 0$ there exist $x_1, x_2, \ldots x_n \in A$ such that*

$$A \subseteq (\cup_{i=1}^{n} B(x_i, \epsilon)).$$

Definition 5. *Let (X,d) be a b-metric space, then $A \subseteq X$ is sequentially compact if any sequence contained in A has a convergent subsequence whose limit belongs to A.*

Definition 6. *A subset A is bounded if $\sup_{x,y\in A} d(x,y) < \infty$. In this case the diameter of A is defined as $diam(A) := \sup_{x,y\in A} d(x,y)$.*

Proposition 6. *A is bounded if and only if there exist $x \in X$ and a real number $R > 0$ such that $A \subseteq B(x,R)$.*

Proof. Let $\epsilon > 0$; if A is bounded, let us consider any $x \in A$; then, for any $y \in A$, $d(y,x) \leq diam(A) < diam(A) + \epsilon$ and consequently, $y \in B(x, diam(A) + \epsilon)$. Then, we can take $R = diam(A) + \epsilon$ and $A \subseteq B(x,R)$.

For the reverse implication, if $A \subseteq B(x,R)$ then for any $y, z \in A$,

$$d(y,z) \leq sd(x,y) + sd(x,z) \leq 2sR < \infty.$$

□

Proposition 7. *Let A be a nonempty subset of a b-metric space X; then,*

- *A is compact if and only if A is sequentially compact.*
- *If A is compact then A is totally bounded.*

Proof. See Theorem 3.1 of the reference [20]. □

Proposition 8. *If A is compact, then A is bounded.*

Proof. Since A is totally bounded by Proposition 7, for a fixed $\epsilon > 0$, there exist $x_1, x_2, \ldots x_n \in A$ such that

$$A \subseteq (\cup_{i=1}^{n} B(x_i, \epsilon)).$$

Let us define $K = \sup_{i,j} d(x_i, x_j)$. For any $x \in A$, $x \in B(x_i, \epsilon)$, for some $i = 1, 2, \ldots n$ and

$$d(x, x_1) \leq sd(x, x_i) + sd(x_i, x_1) < s\epsilon + sK = K'.$$

Consequently $A \subseteq B(x_1, K')$, and A is bounded. □

Proposition 9. *If A is compact, then A is closed.*

Proof. Let $(x_n) \subseteq A$ be a sequence such that $\lim_{n\to\infty} x_n = x$. Then, A is sequentially compact according to Proposition 7, and consequently, there exists a subsequence $(x_{n_j}) \subseteq A$ such that $\lim_{n\to\infty} x_{n_j} = x' \in A$. Since the limit is unique, $x = x' \in A$, and A is closed. □

2.1. Continuous Functions and Compactness

In this subsection, we review some properties of the continuous maps defined on compact sets of a b-metric space.

We will consider the definition of a continuous function $T : X \to Y$, where X, Y are b-metric spaces, corresponding to the topologies defined in both spaces. An important fact is the following.

Proposition 10. *A map $T : X \to Y$, where X, Y are b-metric spaces, is continuous at $x \in X$ if and only if T is sequentially continuous at x, that is to say, if $\lim_{n\to\infty} x_n = x$ then $\lim_{n\to\infty} Tx_n = Tx$.*

Proof. It is similar to the metric case. □

The image of a compact by a continuous function is compact. As a consequence:

Proposition 11. *If the map $T : X \to Y$, where X, Y are b-metric spaces, is continuous, and K is a compact subset of X, then $T(K)$ is bounded.*

Proof. It is a consequence of Proposition 8. □

Proposition 12. *If the map $T : X \to \mathbb{R}$, where X is a b-metric space, is continuous, then T attains its extreme values, that it is to say, T has a maximum and a minimum.*

Proof. By Propositions 8 and 9, $T(X) \subseteq \mathbb{R}$ is closed and bounded' consequently, $\inf T(X)$, $\sup T(X) \in T(X)$. □

It is well known that a b-metric need not be continuous. The following important result can be read from reference [13].

Proposition 13. *Let X be a b-metric space; then, a ball $B(x, r)$, and $r > 0$ is open for any $x \in X$ if and only if $d(x, \cdot)$ is upper semi-continuous.*

Proof. See Remark 2.5 of reference [13]. □

Remark 2. *In particular, if d is continuous, all the "open" balls are open sets.*

2.2. Metrizability of a b-Metric Space

In this section, we consider some results concerning the existence of a metric equivalent to a b-metric.

Let X be a b-metric space and $p \in \mathbb{R}$ such that $0 < p \leq 1$. Let us define

$$d_p(x,y) := \inf\{\sum_{i=1}^{n} d^p(x_{i-1}, x_i)\}, \tag{2}$$

where $x_i \in X$, for $i = 0, 1, \ldots, n$, are arbitrary such that $x = x_0$ and $y = x_n$. The function d_p is a pseudo-metric and the following inequality holds:

$$d_p(x,y) \leq d^p(x,y) \tag{3}$$

for any $x, y \in X$.

If p is suitably chosen, d_p is a metric, and we have a kind of equivalence between d_p and d. Paluszyinski and Stempak [1] proved that if the relation between the index s of the b-metric d and p is given by the equation

$$(2s)^p = 2$$

then d_p is a metric on X and

$$d_p(x,y) \leq d^p(x,y) \leq 2d_p(x,y), \tag{4}$$

for any $x, y \in X$. This implies that the topology τ of X is such that $\tau = \tau_{d_p}$, that is to say, τ is metrizable. As a consequence, the convergence of a sequence (x_n) to $x \in X$ is given if and only if $d(x_n, x) \to 0$ as stated in Proposition 3.

Remark 3. *Paluszyinski and Stempak use in reference [1] the term quasi-metric space instead of b-metric space, as noted in the Introduction.*

2.3. Fixed Points of Self-Maps in b-Metric Spaces

Let X be a b-metric space with index s. We consider functions $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the following conditions:

1. ϕ is non-decreasing.
2. $\lim_{n\to\infty} \phi^n(t) = 0$.
3. $\phi(t) < ts^{-1}$, for any $t > 0$.

The first and second conditions imply that $\phi(t) < t$ for any $t > 0$. The next theorem is proved in [13] (Theorem 4.2).

Theorem 1. *Let X be a complete b-metric space and let $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfy the conditions (1)–(3). If a self-map $T : X \to X$ is such that*

$$d(Tx, Ty) \leq \phi(d(x,y))$$

for any $x, y \in X$, then T has a unique fixed point $x^* \in X$ and $\lim_{n\to\infty} T^n x = x^*$ for any $x \in X$ (T is called a Picard operator).

Corollary 1. *If X is a complete b-metric space and $T : X \to X$ is a Banach contraction, that is to say, there exists $k \in \mathbb{R}$, $0 < k < s^{-1}$, and*

$$d(Tx, Ty) \leq kd(x,y)$$

for any $x, y \in X$, then T has a unique fixed point $x^* \in X$ and $\lim_{n\to\infty} T^n x = x^*$ for any $x \in X$ (T is a Picard operator).

However, in reference [18] (Theorem 3.1), for instance, it is proved that the condition $k < s^{-1}$ is not necessary, and $k < 1$ is sufficient to have the same result about the existence of fixed point and convergence.

2.4. Strong b-Metric Spaces and Metrizability

In this subsection, we consider a particular case of a b-metric, the so-called strong b-metric.

Definition 7. *A strong b-metric space X is a set endowed with a mapping $d : X \times X \to \mathbb{R}^+$ with the following properties:*

1. *$d(x,y) \geq 0$, $d(x,y) = 0$ if and only if $x = y$.*
2. *$d(x,y) = d(y,x)$ for any $x, y \in X$.*
3. *There exists $s \geq 1$ such that, for any $x, y, z \in X$,*

$$d(x,y) \leq d(x,z) + sd(z,y). \tag{5}$$

The constant s is the index of the strong b-metric space, and d is called a strong b-metric.

References [22,24,25], for instance, contain several interesting examples of strong b-metric spaces.

The third property implies the "s-relaxed polygonal inequality" or "s-polygonal inequality":

$$d(x_0, x_n) \leq s(\sum_{j=1}^{n} d(x_{j-1}, x_j)),$$

for any $n \geq 1$ and $x_0, x_1, \ldots x_n \in X$.

Remark 4. *A strong b-metric space is a b-metric space, but the converse may not be true [21].*

The main topological difference between a b-metric and a strong b-metric concerns the openness of the balls. In a strong b-metric space, the "open" balls are open sets [21]. The second difference is the continuity of d. A strong b-metric is always continuous (see, for instance, Lemma 2 of [26]).

There is also an important difference regarding the metrizability. According to Theorem 4.4 of reference [19], a strong b-metric d is Lipschitz equivalent to a metric ρ in X, that is to say, there exist constants $c, C \geq 0$ such that

$$c\rho(x,y) \leq d(x,y) \leq C\rho(x,y), \tag{6}$$

for any $x, y \in X$. The Lipschitz equivalence implies the topological equivalence.

The property (6) implies the preservation of the completeness as well. Thus, (X,d) is complete if and only if (X,ρ) is complete. Another important feature of the strong b-metric spaces is the existence of completion [17]. That is to say, for any strong b-metric space (X,d), there exists a complete strong b-metric space $(\bar{X}, \bar{d})$ and a mapping $h : X \to \bar{X}$ such that $\bar{d}(j(x), j(y)) = d(x,y)$ for all $x, y \in X$. The completion is unique up to an isometry.

3. Some Properties of a Partial Contractivity

In this section, we propose a new type of contraction on a b-metric space, and prove some of its properties.

Definition 8. *Given a b-metric space* (X, d), *a self-map* $T : X \to X$ *is a partial contractivity if there exist real constants* a, b *such that* $0 < a < 1$ *and* $B \geq 0$ *such that for all* $x, y \in X$

$$d(Tx, Ty) \leq ad(x, y) + Bd(x, Tx). \tag{7}$$

Due to the symmetry of the b-metric, inequality (7) is equivalent to

$$d(Tx, Ty) \leq ad(x, y) + Bd(y, Ty), \tag{8}$$

for any $x, y \in X$.

A Banach contraction is a particular case of partial contractivity (for $B = 0$).

Next, we provide examples and counterexamples of partial contractivities.

Example 1. *Let us consider the set* $X = [0, 1] \cup [9/4, 11/4]$ *with the b-metric of index* $s = 2$ *defined as* $d_2(x, y) = (x - y)^2$ *for any* $x, y \in X$ *(see, for instance, reference [2]). Let* $T : X \to X$ *be defined as* $Tx = 0$ *for all* $x \in [0, 1]$ *and* $Tx = 1$ *for all* $x \in [9/4, 11/4]$. *Let us check that* T *is a partial contractivity:*

1. *If* $x, y \in [0, 1]$ *or* $x, y \in [9/4, 11/4]$ *then* $d_2(Tx, Ty) = 0$ *and inequality (7) is satisfied.*
2. *If* $x \in [0, 1]$ *and* $y \in [9/4, 11/4]$ *then* $d_2(Tx, Ty) = 1$ *and*

$$d_2(x, y) = (y - x)^2,$$

$$d_2(x, Tx) = d_2(x, 0) = x^2,$$

$$d_2(y, Ty) = d_2(y, 1) = (y - 1)^2.$$

Taking $a = B = 4/5$ *we have*

$$d_2(Tx, Ty) = 1 \leq \frac{4}{5}(y - x)^2 + \frac{4}{5}x^2 = \frac{4}{5}d_2(x, y) + \frac{4}{5}d_2(x, Tx),$$

$$d_2(Tx, Ty) = 1 \leq \frac{4}{5}(y - x)^2 + \frac{4}{5}(y - 1)^2 = \frac{4}{5}d_2(x, y) + \frac{4}{5}d_2(y, Ty).$$

Consequently T *is a partial contractivity.*

Example 2. *Let us consider the set* $X = [0, 1] \cup [9/4, 11/4]$ *with the usual metric and* $T : X \to X$ *defined as in Example 1. It is easy to check that* T *is also a partial contractivity with the same constants.*

Example 3. *The identity map is not a partial contractivity (condition (7) is incompatible with this type of mappings).*

The next example proves that a partial contractivity need not be continuous. It also shows that there exist partial contractivities that are not Banach contractions.

Example 4. *Let* X *be the interval* $[0, 1]$ *with the b-metric* d_2 *defined in Example 1, and* $T : X \to X$ *be defined as* $Tx = x/2$ *if* $x \in [0, 1)$ *and* $T(1) = 0$. *Let us see that* T *is a partial contractivity.*

1. *If* $x, y \in [0, 1)$ *then* $d_2(Tx, Ty) = (x - y)^2/4$.
2. *If* $x \in [0, 1)$ *and* $y = 1$ *then* $d_2(Tx, Ty) = d_2(x/2, 0) = x^2/4$, $d_2(x, Tx) = d_2(x, x/2)^2 = x^2/4$, *then*

$$d_2(Tx, Ty) = \frac{x^2}{4} \leq ad_2(x, y) + Bd_2(x, Tx) = a(x - y)^2 + B\frac{x^2}{4},$$

for $B = 1$.

Inequality (7) holds for any $x, y \in X$ taking $a = 1/4$ and $B = 1$. Consequently, T is a partial contractivity and T is discontinuous. T is not a Banach contraction since these types of maps are continuous.

In the reference [27], L. Ćirić proposed the concept of a quasi-contraction in a metric space. Next, we generalize it to a b-metric space.

Definition 9. *For a b-metric space (X, d), a self-map $T : X \to X$ is a quasi-contraction if there exists a real constant λ such that $0 < \lambda < 1$ such that for all $x, y \in X$*

$$d(Tx, Ty) \leq \lambda \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}. \tag{9}$$

The quasi-contraction contains some other types of contractivities as particular cases, like Kannan, Chatterjea, and Reich self-maps. The next results states that a quasi-contraction is a partial contractivity if there is some relation between the ratio λ and the index s of the b-metric.

Proposition 14. *If (X, d) is a b-metric space and $T : X \to X$ is a quasi-contraction with ratio $\lambda > 0$ such that $\lambda < 1/(s(s+1))$, then T is a partial contractivity with*

$$a = \frac{\lambda s}{1 - \lambda s^2}$$

and

$$B = \frac{\lambda s^2}{1 - \lambda s^2}.$$

Proof. For $x, y \in X$ let us denote

$$K = \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}.$$

If $K = d(x, y)$, the expression (9) takes the form of (7) with $a = \lambda$ and $B = 0$.
If $K = d(x, Tx)$, we have the same result with $a = 0$ and $B = \lambda$.
In the case $K = d(y, Ty)$, we have

$$d(Tx, Ty) \leq \lambda s d(x, y) + \lambda s^2 d(x, Tx) + \lambda s^2 d(Tx, Ty),$$

and thus

$$d(Tx, Ty) \leq \frac{\lambda s}{1 - \lambda s^2} d(x, y) + \frac{\lambda s^2}{1 - \lambda s^2} d(x, Tx),$$

For $K = d(x, Ty)$, then

$$d(Tx, Ty) \leq \lambda s d(x, Tx) + \lambda s d(Tx, Ty),$$

and

$$d(Tx, Ty) \leq \frac{\lambda s}{1 - \lambda s} d(x, Tx).$$

For $K = d(y, Tx)$ then

$$d(Tx, Ty) \leq \lambda s d(x, y) + \lambda s d(x, Tx).$$

Since $\lambda \leq \lambda s \leq \frac{\lambda s}{1 - \lambda s} \leq \frac{\lambda s}{1 - \lambda s^2} \leq \frac{\lambda s^2}{1 - \lambda s^2}$, in all the cases we have the expression (7) with the constants a, B given in the statement. □

Corollary 2. *If X is a metric space, any quasi-contraction with $0 < \lambda < 1/2$ is a partial contractivity.*

In the following example, we prove that the condition given for λ in Proposition 14 is sufficient, but it is not necessary.

Example 5. *Let Q_1, Q_2 be two subsets of natural numbers defined as*

$$Q_1 = \{\frac{p}{q} : p = \dot{3}, q = \dot{3} + 1\}$$

$$Q_2 = \{\frac{p}{q} : p = \dot{3}, q = \dot{3} + 2\},$$

and let us consider $X = Q_1 \cup Q_2$ with the metric d_2 defined in Example 1. Let $T : X \to X$ be defined as $Tx = 3x/5$ if $x \in Q_1$ and $Tx = x/8$ if $x \in Q_2$. Then,

1. *If $x, y \in Q_1$ or $x, y \in Q_2$, then $d_2(Tx, Ty) \leq \frac{9}{25} d_2(x, y)$.*
2. *If $x \in Q_1$ and $y \in Q_2$, then*
   If $x < \frac{5}{24} y$,

$$d_2(Tx, Ty) \leq \frac{9}{25}(\frac{5}{24}y - x)^2 \leq \frac{9}{25}(y - x)^2 = \frac{9}{25} d_2(x, y). \tag{10}$$

If $x > \frac{5}{24} y$,

$$d_2(Tx, Ty) \leq \frac{9}{25}(x - \frac{5}{24}y)^2 \leq \frac{9}{25}(x - \frac{y}{8})^2 = \frac{9}{25} d_2(x, Ty). \tag{11}$$

Then,

$$d_2(Tx, Ty) \leq \frac{18}{25} d_2(x, Tx) + \frac{18}{25} d_2(Tx, Ty),$$

and

$$d_2(Tx, Ty) \leq \frac{18}{7} d_2(x, Tx).$$

As a consequence, T is a partial contractivity with constants $a = \frac{9}{25}$ and $B = \frac{18}{7}$. The inequalities (10) and (11) prove that T is also a quasi-contraction for $\lambda = \frac{9}{25}$. However, the ratio λ does not satisfy the condition $\lambda < 1/(s(s+1))$ given in Proposition 14.

In the following, we give a theorem of collage type for partial contractivities.

Theorem 2. *If X is a b-metric space and $T : X \to X$ is a partial contractivity with a fixed point x^* and such that as <1, then, for any $x \in X$,*

$$\frac{1}{s(1+a)} d(x, Tx) \leq d(x, x^*) \leq \frac{s}{1 - sa} d(x, Tx).$$

Proof. For any $x \in X$, using (7),

$$d(x, Tx) \leq sd(x, x^*) + sd(x^*, Tx) \leq sd(x, x^*) + sad(x, x^*),$$

obtaining the left inequality. Now,

$$d(x, x^*) \leq sd(x, Tx) + sd(Tx, Tx^*) \leq sd(x, Tx) + sad(x, x^*).$$

Consequently,

$$d(x, x^*) \leq \frac{s}{1 - sa} d(x, Tx).$$

□

Corollary 3. *If X is a metric space and $T : X \to X$ is a partial contractivity with a fixed point x^*, then for any $x \in X$,*

$$\frac{1}{(1+a)} d(x, Tx) \leq d(x, x^*) \leq \frac{1}{(1-a)} d(x, Tx).$$

Remark 5. *Note that the collage theorem for a partial contractivity is identical to the corresponding result with a Banach contraction.*

Two Extensions of the Concept of Partial Contractivity

In this subsection, we propose two different ways of generalizing a partial contractivity. As a collateral result, we obtain sufficient conditions for the existence and uniqueness of fixed points for these type of mapping.

Definition 10. *Let us consider a map $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ (called comparison function) such that $\phi^n(t)$ tends to zero for any t when n tends to infinity, and ϕ is increasing. Let $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be such that $\psi(t) \geq 0$ for any t and $\psi(0) = 0$.*

Let X be a b-metric space and $T : X \to X$ such that

$$d(Tx, Ty) \leq \phi(d(x, y)) + \psi(d(x, Tx)), \tag{12}$$

for any $x, y \in X$. Then, T is a (ϕ, ψ)-partial contractivity.

Remark 6. *For $\phi(t) = at$ with $0 < a < 1$ and $\psi(t) = Bt$ with $B \geq 0$ a (ϕ, ψ)-partial contractivity fits Definition 8.*

Remark 7. *The hypotheses given for the comparison function ϕ imply that $\phi(0) = 0$ and $\phi(t) < t$ (see, for instance, [28]).*

We have the following result.

Proposition 15. *If T is a (ϕ, ψ)-partial contractivity and it has a fixed point $x^* \in X$, then it is unique and the Picard iterations of any point converge to $x^* \in X$.*

Proof. Let x^* be a fixed point of T, and define $x_n := T^n x$ for $x \in X$. Applying the contractivity condition, we obtain:

$$d(x_n, x^*) \leq \phi(d(x_{n-1}, x^*)) \leq \ldots \leq \phi^n(d(x_0, x^*)).$$

The limit condition on the mapping ϕ implies that $x_n \to x^*$. This fact ensures that the fixed point is unique and the Picard iterations of any point converge to it. □

Corollary 4. *If T is a partial contractivity on a b-metric space X and it has a fixed point $x^* \in X$, then it is unique.*

The next concept involves a relation of partial type between two operators, which may have a common fixed point.

Definition 11. *The operators $T_1, T_2 : X \to X$, where X is a b-metric space, are mutual partial contractivities if there exists a positive constant a with $0 < a < 1$ and $B \geq 0$ such that for all $x, y \in X$*

$$d(T_1 x, T_2 y) \leq ad(x, y) + B \min\{d(x, T_1 x), d(y, T_2 y)\}. \tag{13}$$

Example 6. *Let us consider the space $X = [0, 1] \cup [9/4, 11/4]$ with the b-metric d_2 defined in Example 1, and the maps $T_1 : X \to X$ defined as $T_1 x = 0$ for all $x \in [0, 1]$ and $T_1 x = 1$ for* *all $x \in [9/4, 11/4]$, $T_2 : X \to X$, defined as $T_2x = 0$ for all $x \in [0,1]$ and $T_2x = 1/2$ for all $x \in [9/4, 11/4]$. Let us check that T_1 and T_2 are mutual partial contractivities:*

For $x, y \in X$, we have the following:

1. *If $x, y \in [0,1]$, then $d_2(T_1x, T_2y) = 0$, and the inequality (13) holds.*
2. *If $x, y \in [\frac{9}{4}, \frac{11}{4}]$, then $d_2(T_1x, T_2y) = 1/4$. Then, $d_2(x, T_1x) = (x-1)^2$, $d_2(y, T_2y) = (y - 1/2)^2$. Taking $a = B = 2/5$,*

$$d_2(T_1x, T_2y) = \frac{1}{4} \le \frac{2}{5}|x-y|^2 + \frac{2}{5}(x-1)^2 = \frac{2}{5}d_2(x,y) + \frac{2}{5}d_2(x, T_1x)$$

and

$$d_2(T_1x, T_2y) = \frac{1}{4} \le \frac{2}{5}|x-y|^2 + \frac{2}{5}(y - \frac{1}{2})^2 = \frac{2}{5}d_2(x,y) + \frac{2}{5}d_2(y, T_2y)$$

3. *$x \in [0,1]$ and $y \in [\frac{9}{4}, \frac{11}{4}]$ then $d_2(T_1x, T_2y) = 1/4$, $d_2(x, T_1x) = x^2$, $d_2(y, T_2y) = (y - 1/2)^2$. Taking $a = B = 2/5$,*

$$d_2(T_1x, T_2y) = \frac{1}{4} \le \frac{2}{5}|x-y|^2 + \frac{2}{5}x^2 = \frac{2}{5}d_2(x,y) + \frac{2}{5}d_2(x, T_1x),$$

$$d_2(T_1x, T_2y) = \frac{1}{4} \le \frac{2}{5}|x-y|^2 + \frac{2}{5}(y - \frac{1}{2})^2 = \frac{2}{5}d_2(x,y) + \frac{2}{5}d_2(y, T_2y).$$

Consequently T_1, T_2 are mutual partial contractivities with constants $a = B = 2/5$.

Example 7. *Let us consider the space $X = [0,1] \cup [9/4, 11/4]$ with the usual metric and the maps T_1, T_2 defined as the previous example. It is easy to check that T_1 and T_2 are mutual partial contractivities as well, with the same constants.*

Let us obtain sufficient conditions for the existence of a common fixed point for mutual partial contractivities. For it, we will use a result given in [29].

Proposition 16. *Let (X, d) be a b-metric space. Any sequence $(x_n) \subseteq X$ such that there exists $\alpha \in [0,1)$ satisfying the inequality*

$$d(x_{n+1}, x_n) \le \alpha d(x_n, x_{n-1})$$

for all $n \ge 1$ is a Cauchy sequence.

Theorem 3. *Let (X, d) be a complete b-metric space and T_1, T_2 be two mutual partial contractivities with constants a and B. Then, if $a + B < 1$, T_1, T_2 have a unique common fixed point.*

Proof. Let us consider any element $x \in X$ and construct the sequence: $x_0 := x$, $x_1 := T_1x_0$, $x_2 := T_2x_1$, $x_3 := T_1x_2$, and so on. Using the contractivity condition, we have

$$d(x_1, x_2) = d(T_1(x_0), T_2(x_1)) \le ad(x_0, x_1) + Bd(x_0, x_1) = (a + B)d(x_0, x_1).$$

In general, we find that

$$d(x_n, x_{n+1}) \le (a + B)d(x_{n-1}, x_n).$$

This condition implies that the sequence is Cauchy (Proposition 16). Consequently, the sequence (x_n) is convergent and $x^* = \lim_{n\to\infty} x_n$. Let us see that the limit x^* is the common fixed point of T_1 and T_2. Considering an even $n \in \mathbb{N}$,

$$d(x^*, T_1(x^*)) \le sd(x^*, x_n) + sd(x_n, T_1(x^*)) = sd(x^*, x_n) + sd(T_2(x_{n-1}), T_1(x^*)).$$

Applying the condition of mutual partial contractivity in the last term,

$$d(x^*, T_1(x^*)) \le sd(x^*, x_n) + s(ad(x_{n-1}, x^*) + Bd(x_{n-1}, x_n)).$$

Since the right terms tend to zero, we obtain that $T_1(x^*) = x^*$. The equality $x^* = T_2(x^*)$ is proved in a similar way.

Let us assume that there exist two common fixed points x^*, x^+:

$$d(x^*, x^+) = d(T_1x^*, T_2x^+) \le ad(x^*, x^+).$$

This implies that $x^* = x^+$. □

Corollary 5. *Let (X, d) be a complete b-metric space and T be a partial contractivity such that $a + B < 1$; then, T has a unique fixed point. In particular, if T is a Banach contraction and the ratio is such that $a < 1$, then it has a unique fixed point.*

Remark 8. *Note that the relation $a + B < 1$ is a sufficient condition for the existence of fixed point, but it is not necessary. For instance, the contractivity of Example 1 has a fixed point $x^* = 0$, but $a + B = 8/5$.*

Corollary 6. *Let (X, d) be a complete b-metric space and T be such that there exists $B < 1$ satisfying*

$$d(Tx, Ty) \le Bd(x, Tx),$$

for all $x, y \in X$. Then, T has a unique fixed point.

4. Iterative Procedures for the Approximation of Fixed Points of a Partial Contractivity

In this section, we study the convergence and stability of some types of iterations for the approximation of fixed points of partial contractivities.

We have seen in Proposition 15 that if X is a b-metric space, and $T : X \to X$ is a partial contractivity with a fixed point $x^* \in X$, then the Picard iterations $T^n(x)$ converge to it for any $x \in X$.

Moreover, let $x^* \in X$ be the fixed point of T. Applying the contractivity condition for any $x \in X$:

$$d(T^n(x), x*) \le ad(T^{n-1}x, x^*) \le \ldots \le a^n d(x, x^*),$$

and consequently, $d(T^n x, x^*) \to 0$ with a rate of convergence of $O(a^n)$.

In the following, we study two different iterative methods to approach a fixed point of a partial contractivity in the framework of a quasi-normed space (see, for instance, [30]).

Definition 12. *If X is a real linear space, the mapping $|\cdot|_s : X \times X \to \mathbb{R}^+$ is a quasi-norm of index s if*

1. $|x|_s \ge 0$; $x = 0$ *if and only if* $|x|_s = 0$.
2. $|\lambda x|_s = |\lambda||x|_s$.
3. *There exists $s \ge 1$ such that $|x + y|_s \le s(|x|_s + |y|_s)$ for any $x, y \in X$.*

Then, the space $(X, |\cdot|_s)$ is a quasi-normed space.

Remark 9. *A quasi-normed space is a particular case of b-metric space, concerning the distance induced by the quasi-norm. If X is complete for this b-metric, then X is a quasi-Banach space.*

The Noor algorithm (see, for instance, [31]) is a method for the approximation of fixed points for self-maps defined on normed spaces. This is based on the three-step iterative procedure given for $n \ge 0$ by

$$x_{n+1} = (1 - a_n)x_n + a_n Ty_n, \tag{14}$$

$$y_n = (1 - b_n)x_n + b_n T z_n, \tag{15}$$

$$z_n = (1 - c_n)x_n + c_n T x_n, \tag{16}$$

for $0 \leq a_n \leq 1$, $0 \leq b_n \leq 1$, $0 \leq c_n \leq 1$.

As particular cases of this method, we have:

- For $c_n = 0$ for all n, the procedure agrees with the Ishikawa iteration [32].
- In the case $b_n = c_n = 0$ for all n, the algorithm is a Mann iteration [33]. If, additionally, a_n is constant, we have the Krasnoselskii iteration [34].

Let us consider a partial contractivity $T : X \to X$ with a fixed point $x^* \in X$, where X is a quasi-normed space, and let us study the convergence of the Noor iterations under the hypothesis $as^2 < 1$. Let us note that for a metric space, this condition is satisfied by any partial contractivity.

For the sake of simplicity, we will denote as $|\cdot|$ the quasi-norm written before as $|\cdot|_s$. Applying (14) and the contractivity condition,

$$|x_{n+1} - x^*| = |(1 - a_n)(x_n - x^*) + a_n(Ty_n - x^*)| \leq s(1 - a_n)|x_n - x^*| + sa_n|Ty_n - x^*|. \tag{17}$$

and

$$|x_{n+1} - x^*| \leq s(1 - a_n)|x_n - x^*| + saa_n|y_n - x^*|. \tag{18}$$

By (15),

$$|y_n - x^*| = |(1 - b_n)(x_n - x^*) + b_n(Tz_n - x^*)| \leq s(1 - b_n)|x_n - x^*| + sb_n|Tz_n - x^*|. \tag{19}$$

Using step (16),

$$|Tz_n - x^*| \leq a|z_n - x^*| \leq a|(1 - c_n)(x_n - x^*) + c_n(Tx_n - x^*)|, \tag{20}$$

$$|Tz_n - x^*| \leq as(1 - c_n)|x_n - x^*| + a^2 sc_n|x_n - x^*| \tag{21}$$

and consequently,

$$|Tz_n - x^*| \leq as|x_n - x^*|. \tag{22}$$

Applying (22) in (19),

$$|y_n - x^*| \leq (s(1 - b_n) + as^2 b_n)|x_n - x^*|. \tag{23}$$

As a consequence, by (18),

$$|x_{n+1} - x^*| \leq (s(1 - a_n) + asa_n(s - sb_n(1 - as)))|x_n - x^*|. \tag{24}$$

Since $s - sb_n(1 - as) \leq s$, we have

$$|x_{n+1} - x^*| \leq (s(1 - a_n(1 - sa))|x_n - x^*|. \tag{25}$$

Let us consider a constant k such that $s(1 - a_n(1 - sa)) < k < 1$. Then,

$$|x_{n+1} - x^*| \leq k|x_n - x^*| \tag{26}$$

for

$$\frac{1 - s^{-1}k}{1 - as} < a_n \leq 1 \tag{27}$$

and $as^2 < k < 1$.

In this case, the convergence of x_n to x^* is ensured due to the bounding

$$|x_n - x^*| \leq k^n |x_0 - x^*|,$$

and the stability is asymptotic. This is limited to the values of a_n satisfying inequalities (27).

Now, we consider the so-called SP-algorithm (see, for instance, [35]) for the approximation of fixed points. We are going to apply this procedure for the case of a partial contractivity with a fixed point in a quasi-normed space. This is based on a different three-step iterative procedure given for $n \geq 0$ by:

$$x_{n+1} = (1-a_n)y_n + a_n T y_n, \tag{28}$$

$$y_n = (1-b_n)z_n + b_n T z_n, \tag{29}$$

$$z_n = (1-c_n)x_n + c_n T x_n, \tag{30}$$

for $0 \leq a_n \leq 1, 0 \leq b_n \leq 1, 0 \leq c_n \leq 1$.

For $a_n = c_n = 1$ for all n, one obtains the Karakaya method as a particular case of the SP-algorithm (see, for instance, [36,37]).

Let us consider $x^* \in Fix(T)$, where $Fix(T)$ represents the set of fixed points of a partial contractivity $T : X \to X$, where X is a quasi-normed space, and let us study the convergence of these iterations under the hypothesis $as < 1$.

Applying (30) and the contractivity condition,

$$|z_n - x^*| \leq s(1-c_n)|x_n - x^*| + sc_n a|x_n - x^*| \leq s(1 - c_n(1-a))|x_n - x^*|. \tag{31}$$

Using (29) and (31),

$$|y_n - x^*| \leq s(1-b_n)|z_n - x^*| + sb_n a|z_n - x^*| \leq s^2(1 - b_n(1-a))(1 - c_n(1-a))|x_n - x^*|. \tag{32}$$

By (28),

$$|x_{n+1} - x^*| \leq s^3(1 - a_n(1-a))(1 - b_n(1-a))(1 - c_n(1-a))|x_n - x^*|. \tag{33}$$

Let us consider a constant k such that

$$(1 - a_n(1-a)) < k < s^{-1},$$

$$(1 - b_n(1-a)) < k < s^{-1},$$

$$(1 - c_n(1-a)) < k < s^{-1}.$$

Then, by (33),

$$|x_{n+1} - x^*| \leq (ks)^3 |x_n - x^*|. \tag{34}$$

The values of a_n, b_n, c_n must be

$$\frac{1 - s^{-1}}{1-a} < \frac{1-k}{1-a} < a_n, b_n, c_n \leq 1. \tag{35}$$

Then, we must take $a < k < s^{-1}$.

In this case, the convergence of x_n to x^* is ensured and the stability is asymptotic. The rate of convergence is $O((ks)^{3n})$.

Remark 10. *Note that the condition on the ratio of the SP-algorithm ($as < 1$) is better than the one of Noor's method ($as^2 < 1$). If X is a normed space, both are reduced to the basic hypothesis $a < 1$. The condition on k ($k > a$) is also better in the SP-algorithm than that of the Noor procedure ($k > as^2$).*

For a normed space, the order of convergence is k^n for Noor and k^{3n} for SP. We can conclude that generally speaking, for convergence with asymptotic stability, the SP-algorithm is better for coefficients a_n, b_n, c_n satisfying the conditions prescribed.

5. An Application to the Solution of an Integral Equation

In this section, we consider an application to finding the solution of an integral equation of Urysohn type. This is expressed as

$$u(x) = f(x) + \int_0^1 F(x, y, u(y))dy, \tag{36}$$

where $x \in [0,1]$, and $f : [0,1] \to \mathbb{R}$, $F : [0,1]^2 \times \mathbb{R} \to \mathbb{R}$ are given continuous maps. If $F(x, y, u(y)) = K(x, y)G(y, u(y))$, we obtain Hammerstein's integral equation ([38,39]), but we will study here the general case.

We consider the space of continuous functions $C[0,1]$ endowed with the b-metric:

$$d(u, v) := |u - v|_2 := \sup\{(u(x) - v(x))^2 : x \in [0,1]\}.$$

for all $u, v \in C[0,1]$. Let us consider the operator

$$Tu(x) = f(x) + \int_0^1 F(x, y, u(y))dy. \tag{37}$$

It is obvious that finding the solution of Equation (36) is equivalent to the search for a fixed point of the operator T on $C[0,1]$. In the following, we define some sufficient conditions for the existence of such fixed point.

Let us assume that the map F is such that

$$|F(x, y, u(y)) - F(x, y, v(y))| \leq G(x, y, u(y), v(y))g(x, u), \tag{38}$$

where $G : [0,1]^2 \times \mathbb{R}^2 \to [0, \nu)$, $\nu < 1$, and $g(x, u) = |(u - f)(x) - \int_0^1 F(x, y, u(y))dy|$ for $x, y \in I$ and $u \in C[0,1]$. Then,

$$|Tu - Tv|_2 \leq \sup_{x \in I}(\int_0^1 |F(x, y, u(y) - F(x, y, v(y)|dy)^2.$$

Thus, applying condition (38),

$$|Tu - Tv|_2 \leq \nu^2 \sup_{x \in I} \int_0^1 (u(x) - Tu(x))^2 dy \leq \nu^2 |u - Tu|_2.$$

The operator T is a partial contractivity with $a = 0$ and $B = \nu^2$. Since the considered space is complete and $a + B < 1$, according to Corollary 5, there is a single fixed point, and Equation (36) has a unique solution. To obtain this solution, we can use the Picard iterative scheme, defining, for $v \in C[0,1]$, $v_0 := v$; and for $k \geq 0$

$$v_{k+1}(x) := f(x) + \int_0^1 F(x, y, v_k(y))dy.$$

SP and Noor algorithms may also be used, with the convergence values of a_n, b_n, c_n obtained in the previous section.

We consider now a different case. Let $C_w[0,1]$ denote the set of continuous real functions defined on the interval $I = [0,1]$ endowed with the general norm

$$||g||_w = \sup_{x \in I} w(x)|g(x)|,$$

for all $g \in C_w[0,1]$, and a fixed continuous function $w : I \to \mathbb{R}$ such that $w(x) > 0$ for any $x \in I$. Let us assume that $F(x, y, z) = j(x, y)(h(x) + M|z|)$, for $j \in C([0,1]^2)$, $h \in C[0,1]$

and $M \in \mathbb{R}$. Let us see that in this case, the Urysohn equation has also a solution for some conditions on M and j.

$$|F(x,y,u(y)) - F(x,y,v(y))| \leq |M||j(x,y)||u(y) - v(y)| \leq |M|||j||_\infty |u(y) - v(y)|.$$

Then

$$||Tu - Tv||_w \leq \sup_{x \in I} w(x) \int_0^1 |F(x,y,u(y) - F(x,y,v(y)|dy \leq |M|||j||_\infty ||u - v||_w.$$

If $|M|||j||_\infty < 1$, then T is a partial contractivity with $a = |M|||j||_\infty$ and $B = 0$. Hence, T has a fixed point and Equation (36) has a solution. The way to find it iteratively is similar to the previous case.

6. Partial Contractivities in Strong b-Metric Spaces

We refine here some results given in previous sections in the case where X is a strong b-metric space.

Proposition 17. *Let (X,d) be a strong b-metric space. If $T : X \to X$ is a quasi-contraction with $\mu < (1+s)^{-1}$, then T is a partial contractivity with constants $a = \frac{\mu}{1-\mu s} < 1$ and $B = \frac{\mu s^2}{1-\mu s}$.*

Proof. The definition of a quasi-contraction implies, for any $x, y \in X$,

$$d(Tx,Ty) \leq \mu M(x,y),$$

where $M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}$.

If $M(x,y) = d(x,Ty)$ then

$$d(Tx,Ty) \leq \mu d(x,Ty) \leq \mu d(x,Tx) + \mu s d(Tx,Ty).$$

Consequently,

$$d(Tx,Ty) \leq \frac{\mu}{1-\mu s} d(x,Tx).$$

If $M(x,y) = d(y,Ty)$ then

$$d(Tx,Ty) \leq \mu d(y,Ty) \leq \mu d(x,y) + \mu s d(x,Ty),$$

and

$$d(Tx,Ty) \leq \mu d(x,y) + \mu s(s d(x,Tx) + d(Tx,Ty)).$$

Then,

$$d(Tx,Ty) \leq \frac{\mu}{1-\mu s} d(x,y) + \frac{\mu s^2}{1-\mu s} d(x,Tx).$$

If $M(x,y) = d(y,Tx)$ then

$$d(Tx,Ty) \leq \mu d(y,Tx) \leq \mu d(x,y) + \mu s d(x,Tx).$$

For all the cases, we have

$$d(Tx,Ty) \leq \frac{\mu}{1-\mu s} d(x,y) + \frac{\mu s^2}{1-\mu s} d(x,Tx).$$

Thus, taking $a = \frac{\mu}{1-\mu s} < 1$ for the hypothesis given, and $B = \frac{\mu s^2}{1-\mu s}$, we obtain the inequality corresponding to a partial contractivity. □

In the following, we refine the collage theorem for partial contractivities in the case of strong b-metric spaces.

Theorem 4. *If X is a strong b-metric space and $T : X \to X$ is a partial contractivity with a fixed point x^*, then for any $x \in X$,*

$$\frac{1}{(s+a)} d(x, Tx) \leq d(x, x^*) \leq \frac{s}{1-a} d(x, Tx).$$

Proof. For any $x \in X$, using (5),

$$d(x, Tx) \leq sd(x, x^*) + d(x^*, Tx) \leq sd(x, x^*) + ad(x, x^*),$$

obtaining the left inequality. Now,

$$d(x, x^*) \leq sd(x, Tx) + d(Tx, Tx^*) \leq sd(x, Tx) + ad(x, x^*).$$

Consequently,

$$d(x, x^*) \leq \frac{s}{1-a} d(x, Tx).$$

□

7. Mutual Contractions

The concepts of weak contraction [28,40] and quasi-contraction [27] include in the metric case some other contractivities present in the current literature of fixed point theory. In this section, we prove some relations between mutual partial contractivities (Definition 11), mutual quasi-contractions (Definition 13) and mutual weak contractions (Definition 14).

As said previously, a quasi-contraction was defined by L. Ćirić [27] as a self-map $T : X \to X$ satisfying the inequality

$$d(Tx, Ty) \leq \mu \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}. \tag{39}$$

for any $x, y \in X$, with μ being a fixed real constant such that $0 < \mu < 1$. Let us consider now a b-metric space, and let us extend the concept of a quasi-contraction and weak contraction to two operators with a common condition.

Definition 13. *The operators $T_1, T_2 : X \to X$, where X is a b-metric space, are mutual quasi-contractions of if there exists a positive constant μ such that $\mu < 1$ and for all $x, y \in X$*

$$d(T_1x, T_2y) \leq \mu \max\{d(x, y), d(x, T_1x), d(y, T_2y), d(x, T_2y), d(y, T_1x)\}. \tag{40}$$

Definition 14. *The operators $T_1, T_2 : X \to X$, where X is a b-metric space, are mutually weak contractive (or mutual weak contractions) if there exist a constant c, such that $0 < c < 1$, and $L \geq 0$ satisfying for all $x, y \in X$*

$$d(T_1x, T_2y) \leq cd(x, y) + Ld(y, T_1x). \tag{41}$$

Remark 11. *Taking $T_1 = T_2$ in Definition 13 we obtain a quasi-contraction. Definition 14 for $T_1 = T_2$ provides a weak contraction. In this way, these definitions extend the concept of the corresponding contractivities given in the previous literature.*

Proposition 18. *Let X be a b-metric space and T_1, T_2 be mutual quasi-contractions with constant μ such that $\mu < s^{-1}(s+1)^{-1}$; then, T_1, T_2 are mutual weak contractions.*

Proof. For any $x, y \in X$, let us define

$$M(x, y) := \{d(x, y), d(x, T_1x), d(y, T_2y), d(x, T_2y), d(y, T_1x)\}.$$

If $\max M(x, y) = d(x, y)$, then obviously, inequality (40) takes the form (41) with $L = 0$.

If $\max M(x,y) = d(x, T_1x)$, then

$$d(T_1x, T_2y) \leq \mu d(x, T_1x) \leq \mu s d(x,y) + \mu s d(y, T_1x).$$

In the case where $\max M(x,y) = d(y, T_2y)$, we have

$$d(T_1x, T_2y) \leq \mu d(y, T_2y) \leq \mu s d(y, T_1x) + \mu s d(T_1x, T_2y)$$

and thus,

$$d(T_1x, T_2x) \leq \frac{\mu s}{1-\mu s} d(y, T_1x),$$

since $\mu s < (s+1)^{-1} < 1$.

If $\max M(x,y) = d(x, T_2y)$ then

$$d(T_1x, T_2y) \leq \mu d(x, T_2y) \leq \mu s d(x,y) + \mu s d(y, T_2y)$$

and

$$d(T_1x, T_2y) \leq \mu s d(x,y) + \mu s (s d(y, T_1x) + s d(T_1x, T_2y)).$$

Thus, since $\mu < s^{-1}(s+1)^{-1} < s^{-2}$,

$$d(T_1x, T_2y) \leq \frac{\mu s}{1-\mu s^2} d(x,y) + \frac{\mu s^2}{1-\mu s^2} d(y, T_1x).$$

If $\max M(x,y) = d(y, T_1x)$ then obviously,

$$d(T_1x, T_2y) \leq \mu d(y, T_1x).$$

As

$$\mu \leq \mu s < \frac{\mu s}{1-\mu s^2}$$

and

$$\mu s < \frac{\mu s}{1-\mu s} \leq \frac{\mu s^2}{1-\mu s^2}$$

we obtain inequality (41) in all the cases for $c = \frac{\mu s}{1-\mu s^2}$ and $L = \frac{\mu s^2}{1-\mu s^2}$. The condition given for μ in the statement implies that $c < 1$. Consequently, T_1, T_2 are mutual weak contractions. □

Corollary 7. *If T is a quasi-contraction on a b-metric space and $\mu < s^{-1}(s+1)^{-1}$, then T is a weak contraction.*

The condition given in this corollary is sufficient, but it is not necessary. In reference [41], the author presents the following example: Let $X = [0,1]$ be endowed with the usual metric, and $T : X \to X$ be defined as $Tx = 2/3$ for $x \in [0,1)$ and $T(1) = 0$. Then, T is a quasi-contraction with $\mu \in [2/3, 1)$ and a weak contraction for $c \geq 2/3$ and $L \geq c$. However, μ does not satisfy the condition $\mu < s^{-1}(s+1)^{-1}$. The map T is not a partial contractivity (see Example 2 of Reference [41]). Consequently, a weak contraction need not be a partial contractivity.

The next result proves that two mutually weak contractive maps have always a common fixed point.

Theorem 5. *Let (X,d) be a complete b-metric space and T_1, T_2 two mutual weak contractions. Then, T_1, T_2 have a common fixed point. If $c + L < 1$, the fixed point is unique.*

Proof. Let us consider any element $x \in X$ and construct the sequence $x_0 := x$, $x_1 := T_1 x_0$, $x_2 := T_2 x_1$, $x_3 := T_1 x_2$, and so on. Using inequality (41), we have

$$d(x_1, x_2) = d(T_1(x_0), T_2(x_1)) \leq cd(x_0, x_1)$$

$$d(x_2, x_3) = d(T_2(x_1), T_1(x_2)) \leq cd(x_1, x_2) \leq c^2 d(x_0, x_1).$$

Thus, we obtain

$$d(x_n, x_{n+1}) \leq cd(x_{n-1}, x_n).$$

This condition implies that the sequence is Cauchy (see Proposition 16). Consequently, the sequence (x_n) is convergent to $x^* = \lim_{n\to\infty} x_n$. Let us see that the limit x^* is the common fixed point of T_1 and T_2. Considering an even $n \in \mathbb{N}$,

$$d(x^*, T_1(x^*)) \leq sd(x^*, x_n) + sd(x_n, T_1(x^*)) = sd(x^*, x_n) + sd(T_2(x_{n-1}), T_1(x^*)).$$

Applying the condition of weak contractivity in the last term,

$$d(x^*, T_1(x^*)) \leq sd(x^*, x_n) + s(cd(x_{n-1}, x^*) + Ld(x^*, x_n)).$$

Thus,

$$d(x^*, T_1(x^*)) \leq (Ls + s)d(x^*, x_n) + csd(x_{n-1}, x^*).$$

Since the right terms tend to zero, we obtain that $T_1(x^*) = x^*$. The equality $x^* = T_2(x^*)$ is proved in a similar way.

If $c + L < 1$, let us assume that there exist two common fixed points x^*, x^+:

$$d(x^*, x^+) = d(T_1 x^*, T_2 x^+) \leq cd(x^*, x^+) + Ld(x^*, T_2 x^+) = (c + L)d(x^*, x^+).$$

This implies that $x^* = x^+$. □

Corollary 8. *Let (X, d) be a complete b-metric space and T a weak contraction. Then T has a fixed point. If $c + L < 1$ then it is unique.*

Theorem 6. *Let (X, d) be a complete b-metric space and T_1, T_2 two mutual weak contractions and mutual quasi-contractions. Then, T_1, T_2 have a unique common fixed point.*

Proof. The existence of common fixed points has been proved in Theorem 5. For the uniqueness, let us assume that there are two common fixed points: x^*, x^+, and let us apply the condition of mutual quasi-contraction (40):

$$d(x^*, x^+) \leq \mu d(x^*, x^+).$$

Since $\mu < 1$, we have the equality $x^* = x^+$. □

Theorem 7. *Let (X, d) be a complete b-metric space and T_1, T_2 two mutual quasi-contractions with $\mu < s^{-1}(1 + s)^{-1}$; then, T_1, T_2 have a unique common fixed point.*

Proof. According to the Proposition 18, T_1, T_2 are also weak contractions, and we are in the conditions of the Theorem 6. □

Theorem 8. *Let (X, d) be a complete b-metric space, T_1, T_2 be mutual weak contractions and there exist $c, L' \in \mathbb{R}$ with $0 < c < 1$, $L \geq 0$ such that for any $x, y \in X$*

$$d(T_1 x, T_2 y) \leq cd(x, y) + Ld(x, T_1 x), \tag{42}$$

(that is to say, T_1 and T_2 are mutual partial contractivities); then, the common fixed point of T_1 and T_2 is unique.

Proof. According to Theorem 5, T_1, T_2 have a common fixed point. Let us assume that x^*, x^+ are two common fixed points, applying the inequality (42),

$$d(x^*, x^+) = d(T_1x^*, T_2x^+) \leq cd(x^*, x^+),$$

and consequently, $x^* = x^+$. □

Let us consider now a collage theorem for weak contractions, with some conditions on the constants. In this case, we obtain upper and lower bounds for the distance between an element and the fixed point of a weak contraction T.

Theorem 9. *Let X be a b-metric space and $T : X \to X$ be a weak contraction with a fixed point x^*. Then, for any $x \in X$,*

$$\frac{1}{(s + cs + Ls)} d(x, Tx) \leq d(x, x^*).$$

If the constants are such that $(c + L) < s^{-1}$ then, for any $x \in X$,

$$d(x, x^*) \leq \frac{s}{(1 - cs - Ls)} d(x, Tx).$$

Proof. We assume that T owns a fixed point x^*. For the lower bound of $d(x, x^*)$, let us consider that

$$d(x, Tx) \leq sd(x, x^*) + sd(Tx^*, Tx)$$

and thus,

$$d(x, Tx) \leq sd(x, x^*) + scd(x, x^*) + sLd(x, x^*).$$

$$\frac{1}{(s + cs + Ls)} d(x, Tx) \leq d(x, x^*).$$

For the second inequality, applying the third property of a b-metric and the contractivity condition (41), for any $x \in X$,

$$d(x, x^*) \leq sd(x, Tx) + sd(Tx, Tx^*) \leq sd(x, Tx) + scd(x, x^*) + sLd(x, x^*),$$

obtaining that

$$(1 - cs - Ls)d(x, x^*) \leq sd(x, Tx),$$

and the result is completed. □

Remark 12. *If X is complete and $c + L < s^{-1}$, then the fixed point x^* exists according to Corollary 8, and the inequalities of Theorem 9 hold.*

Corollary 9. *In particular, for a metric space X, we obtain that for for any $x \in X$, a fixed point x^* and a weak contraction T, one has*

$$\frac{1}{(1 + c + L)} d(x, Tx) \leq d(x, x^*) \leq \frac{1}{(1 - c - L)} d(x, Tx),$$

if $c + L < 1$. For a Banach contraction, we obtain the chain of inequalities

$$\frac{1}{(1 + c)} d(x, Tx) \leq d(x, x^*) \leq \frac{1}{(1 - c)} d(x, Tx).$$

8. Conclusions

In the first place, we have reviewed some topological aspects of b-metric spaces, in order to give a brief background for this type of structure. The results given establish, for instance, that a compact set is totally bounded in this framework. We have summarized important results related to continuous functions, the existence of extreme values and metrizability.

In subsequent sections, we have described some properties of a partial contractivity (introduced by the first author in reference [37]. This concept has some similarities with a weak contraction as, for instance, the fact of containing several well-known contractions as particular cases. A difference lies in the existence of fixed points: a weak contraction defined on a complete metric space has always some fixed point; however, a partial contractivity may not have it. This lack does not prevent the self-map of having periodic points and a rich dynamics.

We have proved that a quasi-contraction is a partial contractivity if the constant associated is related to the index of the b-metric through a specific inequality. We have also extended the definition to a more general setting, and considered mutual partial contractivities, which relate two self-maps satisfying a joint inequality. In the latter case, we established sufficient conditions for the existence of a common fixed point.

In the case of quasi-normed spaces (particular cases of b-metric sets), we have studied the convergence and stability of two different iterative algorithms: the three-step Noor's procedure and the so-called SP-algorithm. We have obtained values on the parameters of these methods that assure convergence to a fixed point and stability of the computations. We have applied the concepts described to the resolution of an integral equation of Urysohn type, in the framework of b-metric spaces. We have given sufficient conditions on the integrand to ensure the existence and uniqueness of the solution, and some iterative methods for approximating it.

In subsequent sections, we have refined some results obtained for partial contractivities in the case of a strong b-metric space. We have established also some relations between weak and quasi-contractions and the new types of contractivity.

Through the results obtained, we generalize to some extent the modern theory of fixed points and common fixed points of self-maps defined on b-metric and quasi-normed spaces, and their approximation methods.

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References

1. Paluszyinski, M.; Stempak, K. On quasi-metric and metric spaces. *Proc. AMS* **2009**, *137*, 4307–4312. [CrossRef]
2. Rano, G.; Bag, T. Quasi-metric space and fixed point theorems. *Int. J. Math. Sci. Comp.* **2013**, *3*, 27–31.
3. Hitzler, P. Generalized Metrics and Topology in Logic Programming Semantics. Ph.D. Thesis, Department of Mathematics, National University of Ireland, University College Cork, Cork, Ireland, 2001.
4. Wilson, W.A. On quasi-metric spaces. *Am. J. Math.* **1931**, *53*, 675–684. [CrossRef]
5. Kannan, R. Some results on fixed points. *Bull. Calcutta Math. Soc.* **1968**, *60*, 71–76.
6. Kannan, R. Some results on fixed points. II. *Am. Math. Mon.* **1969**, *76*, 405–408.
7. Mohapatra, R.N.; Navascués, M.A.; Sebastián, M.V.; Verma, S. Iteration of operators with contractive mutual relations of Kannan type. *Mathematics* **2022**, *10*, 2632. [CrossRef]
8. Chatterjea, S.K. Fixed-point theorems. *CR Acad. Bulg. Sci.* **1972**, *25*, 727–730. [CrossRef]
9. Urysohn, P.S. Sur une classe d'equations integrales non lineaires. *Mat. Sb.* **1923**, *31*, 236–255.
10. Berinde, V.; Pacurar, M. The early developments in fixed point theory on b-metric spaces: A brief survey and some important related aspects. *Carpathian J. Math.* **2022**, *38*, 523–538. [CrossRef]
11. Coifman, R.R.; de Guzmán, M. Singular integrals and multipliers on homogeneous spaces. *Rev. Un. Mat. Argent.* **1970**, *25*, 137–143.

12. Aimar, H.; Iaffei, B.; Nitti, L. On the Macías-Segovia metrization of quasi-metric spaces. *Rev. Un. Mat. Argent.* **1998**, *41*, 67–75.
13. Cobzas, S.; Czerwik, S. The completion of generalized b-metric spaces and fixed points. *Fixed Point Theory* **2020**, *21*, 133–150. [CrossRef]
14. Bakhtin, I.A. Contraction mapping principle in an almost metric space. *Funkts. Anal.* **1989**, *30*, 26–37. (In Russian)
15. Czerwick, S. Contraction mappings in b-metric spaces. *Acta Math. Inform. Univ. Ostrav.* **1993**, *I* , 5–11.
16. Czerwick, S. Non-linear set-valued contraction mappings in b-metric spaces. *Atti. Sem. Math. Fiz. Univ. Modena* **1998**, *46*, 263–276.
17. An T.V.; Tuyen, L.Q.; Dung, N.V. Answers to Kirk-Shahzad's questions on strong b-metric spaces. *Taiwan. J. Math.* **2016**, *20*, 1175–1180. [CrossRef]
18. Dung, N.V.; An, T.V.; Le Hang, V.T. Remarks on Frinck's metrization technique and applications. *Fixed Point Theory Appl.* **2019**, *20*, 157–176. [CrossRef]
19. Fagin, R.; Kumar, R.; Sivakumar, D. Comparing top k lists. *SIAM J. Discret. Math.* **2003**, *17*, 134–160. [CrossRef]
20. Khamsi, M.A.; Hussain, N. KKM mappings in metric type spaces. *Nonlinear Anal.* **2010**, *73*, 3123–3129. [CrossRef]
21. Kirk, W.; Shahz, N. *Fixed Point Theory in Distance Spaces*; Springer: Cham, Switzerland, 2014.
22. Oner, T.; Sostak, A. On metric-type spaces based on extended t-conorms. *Mathematics* **2020**, *8*, 1097. [CrossRef]
23. Sostak, A. Some remarks on fuzzy k-pseudometric spaces. *Filomat* **2018**, *32*, 3567–3580. [CrossRef]
24. Sostak, A.; Oner, T.; Duman, I.C. On topological and metric properties of $\oplus$-sb-Metric Spaces. *Mathematics* **2023**, *11*, 4090. [CrossRef]
25. Doan, H. A new type of Kannan's fixed point theorem in strong b-metric spaces. *AIMS Math.* **2021**, *6*, 7895–7908. [CrossRef]
26. Navascués, M.A.; Jha, S.; Ch, A.K.B.; Mohapatra, R.N. Iterative schemes involving several mutual contractions. *Mathematics* **2023**, *11*, 2019. [CrossRef]
27. Ćirić, L.B. A generalization of Banach's contraction principle. *Proc. Am. Math. Soc.* **1974**, *45*, 267–273. [CrossRef]
28. Berinde, V. Iterative Approximation of Fixed Points. In *Lecture Notes in Mathematics*; Springer: Berlin-Heidelberg, Germany, 2007.
29. Miculescu, R.; Mihail, A. New fixed point theorems for set-valued contractions in b-metric spaces. *J. Fixed Point Theory Appl.* **2017**, *19*, 2153–2163. [CrossRef]
30. Köthe, G. *Topological Vector Spaces*; Springer: Berlin/Heidelberg, Germany, 1969.
31. Noor, M.A. New approximation schemes for general variational inequalities. *J. Math. Analysis Appl.* **2000**, *251*, 217–229. [CrossRef]
32. Ishikawa, S. Fixed points by a new iteration method. *Proc. Am. Math. Soc.* **1974**, *44*, 147–150. [CrossRef]
33. Mann, W.R. Mean value methods in iteration. *Proc. Am. Math. Soc.* **1953**, *44*, 506–510. [CrossRef]
34. Krasnoselskij, M.A. Two remarks on the method of successive approximations. *Usp. Mat. Nauk.* **1955**, *10*, 123–127. (In Russian)
35. Pheungrattana, W.; Suantai, S. On the rate of convergence of Mann, Ishikawa, Noor and SP iterations for continuous functions on an arbitrary interval. *J. Comput. Appl. Math.* **2011**, *235*, 3006–3014. [CrossRef]
36. Karakaya, V.; Atalan, Y.; Dogan, K.; Bouzara, N.E.H. Some fixed point results for a new three steps iteration process in Banach spaces. *Fixed Point Theory Appl.* **2017**, *18*, 625–640. [CrossRef]
37. Navascués, M.A. Approximation of fixed points and fractal functions by means of different iterative algorithms. *Chaos Solitons Fractals* **2024**, *180*, 114535. [CrossRef]
38. Hammerstein, A. Nichtlineare integralgleichungen nebst anwendungen. *Acta Math.* **1930**, *54*, 117–176. [CrossRef]
39. Banas, J. Integrable solutions of Hammerstein and Urysohn integral equations. *J. Austral. Math. Soc.* **1989**, *46*, 61–68. [CrossRef]
40. Berinde, V. On the approximation of fixed points of weak contractive mappings. *Carpathian J. Math.* **2003**, *19*, 7–22.
41. Berinde, V. Approximation of fixed points of weak contractions using the Picard iteration. *Nonlinear Anal. Forum* **2004**, *9*, 43–53.