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Pinching Results for Doubly Warped Products' Pointwise Bi-Slant Submanifolds in Locally Conformal Almost Cosymplectic Manifolds with a Quarter-Symmetric Connection

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Abstract: In this research paper, we establish geometric inequalities that characterize the relationship between the squared mean curvature and the warping functions of a doubly warped product pointwise bi-slant submanifold. Our investigation takes place in the context of locally conformal almost cosymplectic manifolds, which are equipped with a quarter-symmetric metric connection. We also consider the cases of equality in these inequalities. Additionally, we derive some geometric applications of our obtained results.

Keywords: doubly warped product; pointwise bi-slant submanifolds; locally conformal almost cosymplectic manifolds; quarter-symmetric metric connection; inequalities



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1. Introduction

In 2000, B. Unal [1] introduced the concept of doubly warped products as an extension of warped products [2]. According to Unal, given two Riemannian manifolds N_1 and N_2 with Riemannian metrics g_1 and g_2 , respectively, and positive differentiable functions f_1 on N_1 and f_2 on N_2 , the doubly warped product $N = {}_{f_2}N_1 \times_{f_1} N_2$ of dimension n is defined on the product manifold $N_1 \times N_2$ equipped with the warped metric $g = f_2^2 g_1 + f_1^2 g_2$. The metric g is given by [1]

$$g(\chi_1, \chi_2) = (f_2 \circ t_2)^2 g_1(t_1^* \chi_1, t_1^* \chi_2) + (f_1 \circ t_1)^2 g_2(t_2^* \chi_1, t_2^* \chi_2), \quad (1)$$

where $t_1 : N_1 \times N_2 \rightarrow N_1$ and $t_2 : N_1 \times N_2 \rightarrow N_2$ are the natural projections, $*$ denotes the tangent maps, and f_1 and f_2 are the warping functions on N_1 and N_2 , respectively.

It is worth noting that if either f_1 or f_2 is constant on N (but not both), then N reduces to a single warped product. Similarly, if both f_1 and f_2 are constant functions on N , then N becomes locally a Riemannian product. A doubly warped product manifold is considered proper if both f_1 and f_2 are non-constant functions on N .

Shifting the focus, the question of whether a Riemannian manifold can be immersed in a space form is a crucial matter in submanifold theory, tracing its roots back to Nash's renowned embedding theorem [3]. However, Nash's original objective could not be realized due to the constraints imposed by intrinsic invariants in governing extrinsic properties of submanifolds. In order to surmount these obstacles, Chen introduced novel Riemannian invariants and established optimal connections between intrinsic and extrinsic invariants on submanifolds.

The research conducted by Chen has sparked the interest of geometers, resulting in the derivation of several geometric inequalities for warped products and doubly warped products [4–14]. These investigations have taken place in different scenarios, incorporating various ambient manifolds [15–18]. In particular, number of articles has been published by considering locally conformal almost cosymplectic manifold as an ambient space [19–27].

In this paper, we embark on an investigation concerning the isometric immersion of doubly warped products into locally conformal almost cosymplectic manifolds endowed with a quarter symmetric metric connection. We obtained inequalities possessing a remarkable character, as they establish upper bounds for the warping functions in relation to mean curvature, scalar curvature, and pointwise constant φ -sectional curvature c . These results not only generalize but also encompass other inequalities as specific cases, which we obtain as a geometric application of the results.

2. Preliminaries

Consider \tilde{N} , a Riemannian manifold equipped with the Riemannian metric g . Let $\tilde{\nabla}$ denote the Levi-Civita connection on \tilde{N} . We also introduce $\bar{\nabla}$, a linear connection defined by [28], given as follows:

$$\bar{\nabla}_{\chi_1}\chi_2 = \tilde{\nabla}_{\chi_1}\chi_2 + \mu_1\pi(\chi_2)\chi_1 - \mu_2g(\chi_1, \chi_2)\mathcal{Q}. \quad (2)$$

Here, χ_1 and χ_2 are arbitrary elements of \tilde{N} , μ_1 and μ_2 are real constants, and \mathcal{Q} is a vector field on \tilde{N} such that $\pi(\chi_1) = g(\chi_1, \mathcal{Q})$, where π represents a one-form. If $\bar{\nabla}g = 0$, the connection $\bar{\nabla}$ is referred to as a quarter-symmetric metric connection. Conversely, if $\bar{\nabla}g \neq 0$, it is known as a quarter-symmetric non-metric connection. A quarter-symmetric connection (generalization of semi-symmetric metric connection and semi-symmetric non-metric connection) plays a crucial role in understanding the curvature properties of Riemannian manifolds. It possesses certain symmetry properties, and studying this connection helps in understanding the underlying symmetries of the manifold.

Remark 1. We can obtain special cases of (2) as follows:

- (i) In the case where $\mu_1 = \mu_2 = 1$, the above connection reduces to a semi-symmetric metric connection.
- (ii) When $\mu_1 = 1$ and $\mu_2 = 0$, the above connection reduces to a semi-symmetric non-metric connection.

We can describe the curvature tensor with respect to $\bar{\nabla}$ as

$$\bar{R}(\chi_1, \chi_2)\chi_3 = \bar{\nabla}_{\chi_1}\bar{\nabla}_{\chi_2}\chi_3 - \bar{\nabla}_{\chi_2}\bar{\nabla}_{\chi_1}\chi_3 - \bar{\nabla}_{[\chi_1, \chi_2]}\chi_3. \quad (3)$$

Analogously, the curvature tensor can be defined in relation to $\tilde{\nabla}$. Utilizing (2), we find that the curvature tensor can be described as follows according to [28]:

$$\begin{aligned} \bar{R}(\chi_1, \chi_2, \chi_3, \chi_4) &= \tilde{R}(\chi_1, \chi_2, \chi_3, \chi_4) + \mu_1\alpha(\chi_1, \chi_3)g(\chi_2, \chi_4) - \mu_1\alpha(\chi_2, \chi_3)g(\chi_1, \chi_4) \\ &\quad + \mu_2\alpha(\chi_2, \chi_4)g(\chi_1, \chi_3) - \mu_2\alpha(\chi_1, \chi_4)g(\chi_2, \chi_3) \\ &\quad + \mu_2(\mu_1 - \mu_2)g(\chi_1, \chi_3)\beta(\chi_2, \chi_4) - \mu_2(\mu_1 - \mu_2)g(\chi_2, \chi_3)\beta(\chi_1, \chi_4), \end{aligned} \quad (4)$$

where

$$\begin{aligned} \alpha(\chi_1, \chi_2) &= (\tilde{\nabla}_{\chi_1}\pi)(\chi_2) - \mu_1\pi(\chi_1)\pi(\chi_2) + \frac{\mu_2}{2}g(\chi_1, \chi_2)\pi(\mathcal{Q}), \\ \beta(\chi_1, \chi_2) &= \frac{\pi(\mathcal{Q})}{2}g(\chi_1, \chi_2) + \pi(\chi_1)\pi(\chi_2) \end{aligned}$$

are $(0, 2)$ tensors, for any vector fields χ_1, χ_2, χ_3 , and χ_4 of \tilde{N} .

Let N denote an n -dimensional submanifold that resides within a $(2m + 1)$ -dimensional cosymplectic space form \tilde{N} . We examine the induced quarter-symmetric connection denoted by ∇ and the induced Levi-Civita connection denoted by $\tilde{\nabla}$ on N . By uniquely

decomposing the vector field \mathcal{Q} on N into its tangent component \mathcal{Q}^T and normal component \mathcal{Q}^\perp , we express \mathcal{Q} as $\mathcal{Q} = \mathcal{Q}^T + \mathcal{Q}^\perp$. The Gauss formula, with respect to ∇ and $\tilde{\nabla}$, can be represented as follows:

$$\bar{\nabla}_{\chi_1}\chi_2 = \nabla_{\chi_1}\chi_2 + \sigma(\chi_1, \chi_2), \quad (5)$$

$$\tilde{\nabla}_{\chi_1}\chi_2 = \tilde{\nabla}_{\chi_1}\chi_2 + \tilde{\sigma}(\chi_1, \chi_2), \quad (6)$$

for each $\chi_1, \chi_2 \in \Gamma(TN)$, where $\tilde{\sigma}$ is the second fundamental form of N in \tilde{N} and $\sigma(\chi_1, \chi_2) = \tilde{\sigma}(\chi_1, \chi_2) - \mu_2 g(\chi_1, \chi_2) \mathcal{Q}^\perp$.

If a smooth manifold \tilde{N} has a dimension of $(2m + 1)$ and possesses an endomorphism φ of its tangent bundle $T\tilde{N}$, along with a structure vector field ξ and a 1-form η , then it is termed as a locally conformal almost cosymplectic manifold. The conditions specified below establish the necessary requirements for this characterization:

$$\begin{cases} \varphi^2 = -I + \eta \otimes \xi, & \eta(\xi) = 1, & \eta \circ \varphi = 0 \\ g(\varphi\chi_1, \varphi\chi_2) = g(\chi_1, \chi_2) - \eta(\chi_1)\eta(\chi_2), & \eta(\chi_1) = g(\chi_1, \xi) \\ (\tilde{\nabla}_{\chi_1}\varphi)\chi_2 = u\{g(\varphi\chi_1, \chi_2) - \eta(\chi_2)\varphi\chi_1\} \\ \tilde{\nabla}_{\chi_1}\xi = u\{\chi_1 - \eta(\chi_1)\xi\} \end{cases}$$

where χ_1, χ_2 tangent to N and u is the conformal function such that $\omega = u\eta$ (see [22]). Consider the cases where the function u takes on the values $u = 0$ and $u = 1$. In the former case, \tilde{N} is identified as a cosymplectic manifold, while in the latter case, it is recognized as a Kenmotsu manifold (refer to [29,30] for more details).

For an almost contact metric manifold \tilde{N} , a plane section σ in $T_p\tilde{N}$ is referred to as a φ -section if σ is orthogonal to the structural vector field ξ and $\varphi(\sigma) = \sigma$. If the sectional curvature $\tilde{K}(\sigma)$ remains constant regardless of the choice of φ -section σ at each point $p \in \tilde{N}$, then \tilde{N} is said to have a pointwise constant φ -sectional curvature.

Let us assume that N is a submanifold within an almost contact metric manifold \tilde{N} , equipped with an induced metric g . If ∇ and ∇^\perp represent the induced connections on the tangent bundle TN and the normal bundle $T^\perp N$ of N , respectively, then the Weingarten map is defined by

$$\tilde{\nabla}_{\chi_1}N = -A_N\chi_1 + \nabla_{\chi_1}^\perp N \quad (7)$$

for every $\chi_1, \chi_2 \in TN$ and $N \in T^\perp N$. Here, h and A_N denote the second fundamental form and the shape operator (associated with the normal vector field N), respectively, characterizing the embedding of N into \tilde{N} . They are related as follows:

$$g(\sigma(\chi_1, \chi_2), N) = g(A_N\chi_1, \chi_2), \quad (8)$$

where g represents the Riemannian metric on \tilde{N} as well as the metric induced on N .

In the context of \tilde{N}^{2m+1} , we make the choice of $\{v_1, \dots, v_n\}$ as an orthonormal tangent frame and $\{v_{n+1}, \dots, v_{2m+1}\}$ as an orthonormal normal frame on N . For any $p \in \tilde{N}$ and for a φ -section σ of $T_p\tilde{N}$, the function c defined by $c(p) = \tilde{K}(p)$ is termed as the φ -sectional curvature of \tilde{N} . In other words, in the case of a locally conformal almost cosymplectic manifold \tilde{N} with a dimension of at least 5 and possessing pointwise φ -sectional curvature c , the curvature tensor \tilde{R} with respect to the Levi-Civita connection $\tilde{\nabla}$ on \tilde{N} can be represented as follows:

$$\begin{aligned}
\bar{\bar{R}}(\chi_1, \chi_2, \chi_3, \chi_4) &= \frac{c - 3u^2}{4} \{g(\chi_1, \chi_4)g(\chi_2, \chi_3) - g(\chi_1, \chi_3)g(\chi_2, \chi_4)\} \\
&+ \frac{c + u^2}{4} \{g(\chi_1, \varphi\chi_4)g(\chi_2, \varphi\chi_3) - g(\chi_1, \varphi\chi_3)g(\chi_2, \varphi\chi_4) \\
&- 2g(\chi_1, \varphi\chi_2)g(\chi_3, \varphi\chi_4)\} \\
&- \left\{ \frac{c + u^2}{4} + u' \right\} \{g(\chi_1, \chi_4)\eta(\chi_2)\eta(\chi_3) - g(\chi_1, \chi_3)\eta(\chi_1)\eta(\chi_4) \\
&+ g(\chi_2, \chi_3)\eta(\chi_1)\eta(\chi_4)g(\chi_2, \chi_4)(\chi_1)\eta(\chi_3)\}.
\end{aligned} \tag{9}$$

Then, from (2) and (9), we have

$$\begin{aligned}
\bar{R}(\chi_1, \chi_2, \chi_3, \chi_4) &= \frac{c - 3u^2}{4} \{g(\chi_1, \chi_4)g(\chi_2, \chi_3) - g(\chi_1, \chi_3)g(\chi_2, \chi_4)\} \\
&+ \frac{c + u^2}{4} \{g(\chi_1, \varphi\chi_4)g(\chi_2, \varphi\chi_3) - g(\chi_1, \varphi\chi_3)g(\chi_2, \varphi\chi_4) \\
&- 2g(\chi_1, \varphi\chi_2)g(\chi_3, \varphi\chi_4)\} \\
&- \left\{ \frac{c + u^2}{4} + u' \right\} \{g(\chi_1, \chi_4)\eta(\chi_2)\eta(\chi_3) - g(\chi_1, \chi_3)\eta(\chi_1)\eta(\chi_4) \\
&+ g(\chi_2, \chi_3)\eta(\chi_1)\eta(\chi_4)g(\chi_2, \chi_4)(\chi_1)\eta(\chi_3)\} + \mu_1\alpha(\chi_1, \chi_3)g(\chi_2, \chi_4) \\
&- \mu_1\alpha(\chi_2, \chi_3)g(\chi_3, \chi_4) + \mu_2g(\chi_1, \chi_3)\alpha(\chi_2, \chi_4) - \mu_2g(\chi_2, \chi_3)\alpha(\chi_1, \chi_4) \\
&+ \mu_2(\mu_1 - \mu_2)g(\chi_1, \chi_3)\beta(\chi_2, \chi_4) - \mu_2(\mu_1 - \mu_2)g(\chi_2, \chi_3)\beta(\chi_1, \chi_4).
\end{aligned} \tag{10}$$

Similarly, we have

$$\begin{aligned}
\bar{R}(\chi_1, \chi_2, \chi_3, \chi_4) &= R(\chi_1, \chi_2, \chi_3, \chi_4) - g(\sigma(\chi_1, \chi_4), \sigma(\chi_2, \chi_3)) + g(\sigma(\chi_2, \chi_4), \sigma(\chi_1, \chi_3)) \\
&+ (\mu_1 - \mu_2)g(\sigma(\chi_2, \chi_3), \mathcal{Q}^\perp)g(\chi_1, \chi_4) + (\mu_2 - \mu_1)g(\sigma(\chi_1, \chi_3), \mathcal{Q}^\perp)g(\chi_2, \chi_4).
\end{aligned} \tag{11}$$

Consider a vector field χ_1 tangent to the submanifold N . We can express $J\chi_1$ as the sum of its tangential component $T\chi_1$ and its normal component $F\chi_1$. In the case where $T = 0$, the submanifold is classified as totally real, while a submanifold is considered holomorphic when $F = 0$.

To calculate the squared norm of T at a point $p \in N$, we can utilize the equation

$$\|T\|^2 = \sum_{i,j=1}^n g^2(Jv_i, v_j), \tag{12}$$

where $\{v_1, \dots, v_n\}$ denotes any orthonormal basis of the tangent space TN of N .

In a research conducted by Chen [31], it was demonstrated that a submanifold N of an almost Hermitian manifold (\tilde{N}, J, g) is classified as pointwise slant if and only if the equation can be expressed as follows:

$$T^2 = -\cos^2 \theta(p)I, \quad \forall p \in N, \tag{13}$$

where $\theta(p)$ represents a real-valued function on N . A pointwise slant submanifold is considered proper if it does not contain any totally real points or complex points.

We can easily verify the following relationships:

$$g(T\chi_1, T\chi_2) = \cos^2 \theta(p)g(\chi_1, \chi_2), \quad g(F\chi_1, F\chi_2) = \sin^2 \theta(p)g(\chi_1, \chi_2) \tag{14}$$

for any $\chi_1, \chi_2 \in \Gamma(TN)$.

Let us now introduce the concept of a pointwise bi-slant submanifold, as defined by Chen and Uddin in their work [9]: A submanifold N^n of an almost Hermitian manifold \tilde{N}^{4m} is referred to as a pointwise bi-slant submanifold if it possesses a pair of orthogonal distributions \mathfrak{L}_1 and \mathfrak{L}_2 that satisfy the following conditions:

- (i) $TN^n = \mathfrak{L}_1 \oplus \mathfrak{L}_2$;
- (ii) $J\mathfrak{L}_1 \perp \mathfrak{L}_2$ and $J\mathfrak{L}_2 \perp \mathfrak{L}_1$;
- (iii) Each distribution \mathfrak{L}_i is pointwise slant, with a slant function $\theta_i : TN - \{0\} \rightarrow \mathbb{R}$ for $i = 1, 2$.

Pointwise bi-slant submanifolds are a more general class of submanifolds, encompassing bi-slant, pointwise semi-slant, semi-slant, and CR-submanifolds as special cases.

Since N^n is a pointwise bi-slant submanifold, we can define an adapted orthonormal frame as $n = 2d_1 + 2d_2$, given by

$$\begin{aligned} \{v_1, v_2 = \sec \theta_1 T v_1, \dots, v_{2d_1-1}, v_{2d_1} \\ = \sec \theta_1 T v_{2d_1-1}, \dots, v_{2d_1+1}, v_{2d_1+2} \\ = \sec \theta_2 T v_{2d_1+1}, \dots, v_{2d_1+2d_2-1}, v_{2d_1+2d_2} = \sec \theta_2 T v_{2d_1+2d_2-1}\}. \end{aligned}$$

Consequently, we define it in such a way that

$$g(v_1, Jv_2) = -g(Jv_1, v_2) = -g(Jv_1, \sec \theta_1 T v_1),$$

which implies that

$$g(v_1, Jv_2) = -\sec \theta_1 g(Tv_1, Tv_2).$$

Based on Equation (14), we can observe that $g(v_1, Jv_2) = \cos \theta_1 g(v_1, v_2)$. As a result, we readily obtain the subsequent relation

$$g^2(v_i, Jv_j) = \begin{cases} \cos^2 \theta_1, & \forall i = 1, \dots, 2d_1 - 1, \\ \cos^2 \theta_2, & \forall j = 2d_1 + 1, \dots, 2d_1 + 2d_2 - 1. \end{cases}$$

Hence, we have

$$\|T\|^2 = \sum_{i,j=1}^n g^2(v_i, Jv_j) = (n_1 \cos^2 \theta_1 + n_2 \cos^2 \theta_2), \quad (15)$$

where $n_1 = \dim \mathfrak{D}_1$ and $n_2 = \dim \mathfrak{D}_2$.

When dealing with an almost contact metric manifold \tilde{N} , the totally umbilicity and total geodesicity of a submanifold N are established by the conditions $h(\chi_1, \chi_2) = g(\chi_1, \chi_2)H$ and $h(\chi_1, \chi_2) = 0$, respectively, where χ_1 and χ_2 belong to $\Gamma(TN)$. Here, H represents the mean curvature vector pertaining to N . Furthermore, if H is found to be zero, it signifies that N is a minimal submanifold in \tilde{N} .

We consider the isometric immersion $\phi : N = {}_{f_2}N_1 \times_{f_1} N_2 \rightarrow \tilde{N}$, where ${}_{f_2}N_1 \times_{f_1} N_2 \rightarrow \tilde{N}$ is a doubly warped product, into a Riemannian manifold \tilde{N} characterized by a constant sectional curvature c . Let n_1, n_2 , and n represent the dimensions of N_1, N_2 , and $N_1 \times_f N_2$, respectively. In this context, for unit vector fields χ_1 and χ_3 that are tangent to N_1 and N_2 , respectively, we have

$$\begin{aligned} K(\chi_1 \wedge \chi_3) &= g(\nabla_{\chi_3} \nabla_{\chi_1} \chi_1 - \nabla_{\chi_1} \nabla_{\chi_3} \chi_1, \chi_3) \\ &= \frac{1}{f_1} \left\{ \left(\nabla_{\chi_1}^1 \chi_1 \right) f_1 - \chi_1^2 f_1 \right\} + \frac{1}{f_2} \left\{ \left(\nabla_{\chi_3}^2 Z \right) f_2 - \chi_3^2 f_2 \right\}. \end{aligned} \quad (16)$$

Let us define the sectional curvature of a general doubly warped product in terms of a local orthonormal frame $\{v_1, v_2, \dots, v_n\}$, where $\{v_1, v_2, \dots, v_{n_1}\}$ are tangent to N_1 and $\{v_{n_1+1}, \dots, v_n\}$ are tangent to N_2 . The sectional curvature can then be expressed as follows:

$$\sum_{1 \leq i \leq n_1} \sum_{n_1+1 \leq j \leq n} K(v_i \wedge v_j) = \frac{n_2 \Delta_1 f_1}{f_1} + \frac{n_1 \Delta_2 f_2}{f_2} \quad (17)$$

for each $j = n_1 + 1, \dots, n$.

Within this framework, we introduce another significant Riemannian intrinsic invariant known as the scalar curvature of \tilde{N}^{2m+1} , denoted as $\tilde{\tau}(T_p \tilde{N}^{2m+1})$. At a certain point p in \tilde{N}^{2m+1} , the scalar curvature can be expressed as follows:

$$\tilde{\tau}(T_p \tilde{N}^m) = \sum_{1 \leq i < j \leq 2m+1} \tilde{\kappa}_{ij}, \quad (18)$$

where $\tilde{\kappa}_{ij} = \tilde{\kappa}(v_i \wedge v_j)$. It is clear that the first equality (18) is congruent to the following equation, which is frequently used in subsequent proofs:

$$2\tilde{\tau}(T_p \tilde{N}^{2m+1}) = \sum_{1 \leq i \neq j \leq 2m+1} \tilde{\kappa}_{ij}. \quad (19)$$

Similarly, scalar curvature $\tilde{\tau}(L_p)$ of L -plan is given by

$$\tilde{\tau}(L_p) = \sum_{1 \leq i < j \leq 2m+1} \tilde{\kappa}_{ij}. \quad (20)$$

An orthonormal basis of the tangent space $T_p N$ is $\{v_1, \dots, v_n\}$ such that $v_r = \{v_{n+1}, \dots, v_{2m+1}\}$ belongs to the normal space $T^\perp N$. Then, we have

$$\begin{aligned} \sigma_{ij}^r &= g(\sigma(v_i, v_j), v_r), \quad \|\sigma\|^2 = \sum_{i,j=1}^n g(\sigma(v_i, v_j), \sigma(v_i, v_j)) = \sum_{r=n+1}^{2m+1} \sum_{i,j=1}^n (\sigma_{ij}^r)^2, \\ \|H\|^2 &= \frac{1}{n^2} \sum_{i=1}^n g(\sigma(v_i, v_i), \sigma(v_i, v_i)), \end{aligned}$$

where $\|H\|^2$ is the squared norm of the mean curvature vector H of N .

We define κ_{ij} and $\tilde{\kappa}_{ij}$ as the sectional curvatures of the plane section spanned by e_i and v_j at p in the submanifold N^n and the Riemannian manifold \tilde{N}^{2m+1} , respectively. Therefore, κ_{ij} and $\tilde{\kappa}_{ij}$ represent the intrinsic and extrinsic sectional curvatures of the span $\{v_i, v_j\}$ at p . Hence, from the Gauss equation, we obtain

$$\begin{aligned} 2\tau(T_p N^n) &= \kappa_{ij} = 2\tilde{\tau}(T_p N^n) - \sum_{i,j=1}^n \{(\mu_1 - \mu_2)g(\sigma(v_j, v_j), \mathcal{Q}^\perp)g(v_i, v_i) \\ &\quad + (\mu_2 - \mu_1)g(\sigma(v_i, v_j), \mathcal{Q}^\perp)g(v_j, v_i)\} + \sum_{r=n+1}^{2m+1} \sum_{i,j=1}^n (\sigma_{ii}^r h_{jj}^r - (\sigma_{ij}^r)^2) \\ &= \tilde{\kappa}_{ij} - \sum_{i,j=1}^n \{(\mu_1 - \mu_2)g(\sigma(v_j, v_j), \mathcal{Q}^\perp)g(v_i, v_i) \\ &\quad + (\mu_2 - \mu_1)g(\sigma(v_i, v_j), \mathcal{Q}^\perp)g(v_j, v_i)\} + \sum_{r=n+1}^{2m+1} \sum_{i,j=1}^n (\sigma_{ii}^r \sigma_{jj}^r - (\sigma_{ij}^r)^2). \quad (21) \end{aligned}$$

The subsequent implications arise from the Gauss equation and (21):

$$\begin{aligned} \tau(T_p N_1^{n_1}) &= \tilde{\tau}(T_p N_1^{n_1}) - \sum_{1 \leq j < k \leq n_1} \{(\mu_1 - \mu_2)g(\sigma(v_j, v_j), \mathcal{Q}^\perp)g(v_k, v_k) \\ &\quad + (\mu_2 - \mu_1)g(\sigma(v_j, v_k), \mathcal{Q}^\perp)g(v_k, v_j)\} + \sum_{r=n+1}^{2m+1} \sum_{1 \leq j < k \leq n_1} (\sigma_{jj}^r \sigma_{kk}^r - (\sigma_{jk}^r)^2), \end{aligned} \quad (22)$$

$$\begin{aligned} \tau(T_p N_2^{n_2}) &= \tilde{\tau}(T_p N_2^{n_2}) - \sum_{n_1+1 \leq s < t \leq n} \{(\mu_1 - \mu_2)g(\sigma(v_t, v_t), \mathcal{Q}^\perp)g(v_s, v_s) \\ &\quad + (\mu_2 - \mu_1)g(\sigma(v_s, v_t), \mathcal{Q}^\perp)g(v_t, v_s)\} + \sum_{r=n+1}^{2m+1} \sum_{n_1+1 \leq s < t \leq n} n(\sigma_{ss}^r \sigma_{tt}^r - (\sigma_{st}^r)^2). \end{aligned} \quad (23)$$

3. Main Inequalities

At the outset, we remind ourselves of an important result by B.-Y. Chen, which will come in handy at a later stage.

Lemma 1. [32] For $k \geq 2$ and real numbers w_1, w_1, \dots, w_n, b satisfying

$$\left(\sum_{i=1}^k w_i \right)^2 = (k-1) \left(\sum_{i=1}^k a_i^2 + b \right),$$

it follows that $2w_1 w_2 \geq b$. Furthermore, equality holds if and only if $w_1 + w_2 = w_3 = \dots = w_k$.

At this juncture, we demonstrate the principal outcome of this section by means of a formal proof.

Theorem 1. Let $\tilde{N}(c)$ be a $(2m+1)$ -dimensional locally conformal almost cosymplectic manifold, and let $\phi :_{f_2} N_1 \times_{f_1} N_2 \rightarrow \tilde{N}(c)$ denote an isometric immersion of an n -dimensional pointwise bi-slant doubly warped product submanifold into $\tilde{N}(c)$ with a quarter-symmetric connection. Then, the following statement holds true:

(i) The squared norm of the mean curvature can be related to warping functions through the following expression:

$$\begin{aligned} \frac{n_2 \Delta_1 f_1}{f_1} + \frac{n_1 \Delta_2 f_2}{f_2} &\leq \frac{n^2}{4} \|H\|^2 + \frac{c - 3u^2}{4} n_1 n_2 \\ &\quad - \frac{3(c + u^2)}{4} (n_1 \cos^2 \theta_1 + n_2 \cos^2 \theta_2) \\ &\quad - \frac{1}{2} \{(\mu_1 + \mu_2)a + \mu_2(\mu_1 - \mu_2)b + 2n_1 n_2 (\mu_1 - \mu_2) \pi(H)\}. \end{aligned} \quad (24)$$

Here, ∇ and Δ represent the gradient and Laplacian operators, respectively. H denotes the mean curvature vector of N^n , while a and b correspond to the traces of α and β , respectively.

(ii) The equality case in (24) is satisfied if and only if ϕ is a mixed totally geodesic isometric immersion, and $n_1 H_1 = n_2 H_2$, where H_1 and H_2 are the partial mean curvature vectors of H along $N_1^{n_1}$ and $N_2^{n_2}$, respectively. Additionally, $\pi(H) = \frac{1}{n} \sum_{i=1}^n \pi(\sigma(v_i, v_j)) = g(\mathcal{Q}, H)$ holds true.

Proof. By selecting $\{v_1, \dots, v_n\}$ and $\{v_{n+1}, \dots, v_{2m+1}\}$ as an orthonormal tangent and normal frames on N , respectively, and substituting $\chi_1 = \chi_4 = v_i$ and $\chi_2 = \chi_3 = v_j$ into (10) while employing (11), we obtain

$$\begin{aligned}
R(v_i, v_j, v_j, v_i) &= \frac{c-3u^2}{4} \{g(v_i, v_i)g(v_j, v_j) - g(v_i, v_j)g(v_j, v_i)\} \\
&+ \frac{c+u^2}{4} \{g(v_i, \varphi v_i)g(v_j, \varphi v_j) - g(v_i, \varphi v_j)g(v_j, \varphi v_i) \\
&- 2g(v_i, \varphi v_j)g(v_j, \varphi v_i)\} \\
&- \left\{ \frac{c+u^2}{4} + u' \right\} \{g(v_i, v_i)\eta(v_j)\eta(v_j) - g(v_i, v_j)\eta(v_i)\eta(v_i) \\
&+ g(v_j, v_j)\eta(v_i)\eta(v_i)g(v_j, v_i)(v_i)\eta(v_j)\} \\
&+ \mu_1\alpha(v_i, v_j)g(v_j, v_i) - \mu_1\alpha(v_j, v_j)g(v_i, v_i) \\
&+ \mu_2g(v_i, v_j)\alpha(v_j, v_i) - \mu_2g(v_j, v_j)\alpha(v_i, v_i) \\
&+ \mu_2(\mu_1 - \mu_2)g(v_i, v_j)\beta(v_j, v_i) \\
&- \mu_2(\mu_1 - \mu_2)g(v_j, v_j)\beta(v_i, v_i), \\
&- (\mu_1 - \mu_2)g(h(v_j, v_j), \mathcal{Q}^\perp)g(v_i, v_i) \\
&- (\mu_2 - \mu_1)g(h(v_i, v_j), \mathcal{Q}^\perp)g(v_j, v_i) \\
&+ g(\sigma(v_i, v_i), \sigma(v_j, v_j)) - g(\sigma(v_i, v_j), \sigma(v_i, v_j)). \tag{25}
\end{aligned}$$

Through the summation $1 \leq i, j \leq n$ of (25) and the utilization of (15), we arrive at

$$\begin{aligned}
2\tau &= \frac{c-3u^2}{4}n(n-1) + 3\frac{c+u^2}{4}\sum_{i,j=1}^n g^2(Jv_i, v_j) \\
&+ \sum_{i,j=1}^n \{(\mu_1 + \mu_2)(1-n)\alpha(v_j, v_j) + \mu_2(\mu_1 - \mu_2)(1-n)\beta(v_i, v_i) \\
&+ (\mu_2 - \mu_1)(n-1)\pi(\sigma(v_j, v_j))\} \\
&+ \sum_{i,j=1}^n \{g(\sigma(v_i, v_i), \sigma(v_j, v_j)) - g(\sigma(v_i, v_j), \sigma(v_i, v_j))\}.
\end{aligned}$$

The aforementioned expression can be written as follows:

$$\begin{aligned}
2\tau &= \frac{c-3u^2}{4}n(n-1) + 3\frac{c+u^2}{4}(n_1 \cos^2 \theta_1 + n_2 \cos^2 \theta_2) + (\mu_1 + \mu_2)(1-n)a \\
&+ \mu_2(\mu_1 - \mu_2)(1-n)b + (\mu_2 - \mu_1)n(n-1)\pi(H) + n^2\|H\|^2 - \|\sigma\|^2. \tag{26}
\end{aligned}$$

Let us make the assumption that

$$\begin{aligned}
\delta &= 2\tau - \frac{c-3u^2}{4}n(n-1) + 3\frac{c+u^2}{4}(n_1 \cos^2 \theta_1 + n_2 \cos^2 \theta_2) + (\mu_1 + \mu_2)(1-n)a \\
&+ \mu_2(\mu_1 - \mu_2)(1-n)b + (\mu_2 - \mu_1)n(n-1)\pi(H) - \frac{n^2}{2}\|H\|^2. \tag{27}
\end{aligned}$$

Thus, with reference to (26) and (27), we have

$$n^2\|H\|^2 = 2(\delta + \|\sigma\|^2). \tag{28}$$

Accordingly, (28) can be written in the form

$$\left(\sum_{i=1}^n \sigma_{ii}^{n+1}\right)^2 = 2\left\{\delta + \sum_{i=1}^n (\sigma_{ii}^{n+1})^2 + \sum_{i \neq j} (\sigma_{ij}^{n+1})^2 + \sum_{r=n+1}^{2m+1} \sum_{i,j=1}^n (\sigma_{ij}^r)^2\right\} \tag{29}$$

for the orthonormal frame $\{v_1, \dots, v_n\}$. Through the application of algebraic Lemma 1 and relation (29), we determine

$$2\sigma_{11}^{n+1}\sigma_{22}^{n+1} \geq \sum_{i \neq j} (\sigma_{ij}^{n+1})^2 + \sum_{i,j=1}^n \sum_{r=n+2}^{2m+1} (\sigma_{ij}^r)^2 + \delta. \quad (30)$$

If we substitute $w_1 = \sigma_{11}^{n+1}$, $w_2 = \sum_{i=2}^{n_1} \sigma_{ii}^{n+1}$ and $w_3 = \sum_{t=n_1+1}^n \sigma_{tt}^{n+1}$ in the above Equation (29), we find

$$\begin{aligned} \left(\sum_{i=1}^n w_i\right)^2 &= 2\left\{\delta + \sum_{i=1}^n w_i^2 + \sum_{i \neq j \leq n} (\sigma_{ij}^{n+1})^2 + \sum_{r=n+1}^{2m+1} \sum_{i,j=1}^n (\sigma_{ij}^r)^2\right. \\ &\quad \left. - \sum_{2 \leq j \neq k \leq n_1} \sigma_{jj}^{n+1}\sigma_{kk}^{n+1} - \sum_{n_1+1 \leq s \neq t \leq n} \sigma_{ss}^{n+1}\sigma_{tt}^{n+1}\right\}. \end{aligned} \quad (31)$$

Thus, it can be inferred that w_1, w_2, w_3 satisfy Chen's Lemma (for $k = 3$), implying that

$$\left(\sum_{i=1}^3 w_i\right)^2 = 2\left(b + \sum_{i=1}^3 w_i^2\right).$$

Therefore, the inequality $2w_1w_2 \geq b$ holds, and equality is attained if and only if $w_1 + w_2 = w_3$. In the specific case being examined, this implies that

$$\begin{aligned} \sum_{1 \leq j < k \leq n_1} \sigma_{jj}^{n+1}\sigma_{kk}^{n+1} + \sum_{n_1+1 \leq s < t \leq n} \sigma_{ss}^{n+1}\sigma_{tt}^{n+1} \\ \geq \frac{\delta}{2} + \sum_{1 \leq \alpha_3 < \beta_3 \leq n} (\sigma_{\alpha_3\beta_3}^{n+1})^2 + \sum_{r=n+1}^{2m+1} \sum_{\alpha_3\beta_3=1}^n (\sigma_{\alpha_3\beta_3}^r)^2. \end{aligned} \quad (32)$$

The equality sign holds in the above inequality if and only if

$$\sum_{i=1}^{n_1} \sigma_{ii}^{n+1} = \sum_{t=n_1+1}^n \sigma_{tt}^{n+1}. \quad (33)$$

Again, using Gauss equation, we derive

$$n_2 \frac{\Delta_1 f_1}{f_1} + n_1 \frac{\Delta_2 f_2}{f_2} = \tau - \sum_{1 \leq j < k \leq n_1} \kappa(v_j \wedge v_k) - \sum_{n_1+1 \leq s < t \leq n} \kappa(v_s \wedge v_t). \quad (34)$$

Subsequently, by considering (21), we obtain the scalar curvature for the locally conformal almost cosymplectic space form with a quarter-symmetric connection as

$$\begin{aligned} n_2 \frac{\Delta_1 f_1}{f_1} + n_1 \frac{\Delta_2 f_2}{f_2} &= \tau - \frac{c - 3u^2}{8} n_1(n_1 - 1) - 3 \frac{c + u^2}{4} n_1 \cos^2 \theta_1 \\ &\quad - \frac{1}{2} \{(\mu_1 + \mu_2)(1 - n_1)a + \mu_2(\mu_1 - \mu_2)(1 - n_1)b \\ &\quad + (\mu_2 - \mu_1)n_1(n_1 - 1)\pi(H)\} - \sum_{r=n+1}^{2m+1} \sum_{n_1+1 \leq j < k \leq n} (\sigma_{jj}^r \sigma_{kk}^r - (\sigma_{jk}^r)^2) \\ &\quad - \frac{c - 3u^2}{8} n_2(n_2 - 1) - 3 \frac{c + u^2}{4} n_2 \cos^2 \theta_2 \\ &\quad - \frac{1}{2} \{(\mu_1 + \mu_2)(1 - n_2)a + \mu_2(\mu_1 - \mu_2)(1 - n_2)b \\ &\quad + (\mu_2 + \mu_1)n_2(n_2 - 1)\pi(H)\} - \sum_{r=n+1}^{2m+1} \sum_{n_1+1 \leq s < t \leq n} (\sigma_{ss}^r \sigma_{tt}^r - (\sigma_{st}^r)^2). \end{aligned} \quad (35)$$

Now, making use of (32) and (35), we obtain

$$\begin{aligned}
 n_2 \frac{\Delta_1 f_1}{f_1} + n_1 \frac{\Delta_2 f_2}{f_2} &\leq \tau - \frac{c - 3f^2}{8} [n(n - 1) - 2n_1 n_2] \\
 &\quad - \frac{c + u^2}{4} 3[n_1 \cos^2 \theta_1 + n_2 \cos^2 \theta_2] \\
 &\quad + \frac{1}{2} \{ (\mu_1 + \mu_2)(2 - n)a + \mu_2(\mu_1 - \mu_2)(2 - n)b \\
 &\quad + (\mu_2 - \mu_1)[n(n - 1) - 2n_1 n_2]\pi(H) \} - \frac{\delta}{2}.
 \end{aligned} \tag{36}$$

By utilizing (27) in the preceding equation, we obtain

$$\begin{aligned}
 \frac{n_2 \Delta_1 f_1}{f_1} + \frac{n_1 \Delta_2 f_2}{f_2} &\leq \frac{n^2}{4} \|H\|^2 + \frac{c - 3u^2}{4} n_1 n_2 \\
 &\quad - \frac{3(c + u^2)}{4} (n_1 \cos^2 \theta_1 + n_2 \cos^2 \theta_2) \\
 &\quad - \frac{1}{2} \{ (\mu_1 + \mu_2)a + \mu_2(\mu_1 - \mu_2)b + 2n_1 n_2(\mu_1 - \mu_2)\pi(H) \},
 \end{aligned} \tag{37}$$

In Equation (24), the equality holds if and only if the expression in Equations (32) and (33) leads to

$$\sum_{r=n+1}^{2m+1} \sum_{i=1}^{n_1} \sigma_{ii}^r = \sum_{r=n+1}^{2m+1} \sum_{t=n_1+1}^n \sigma_{tt}^r = 0, \tag{38}$$

and $n_1 H_1 = n_2 H_2$.

Moreover, from (33), we obtain

$$\sigma_{jt} = 0, \quad \forall 1 \leq j \leq n_1, n + 1 \leq t \leq n, n + 1 \leq r \leq 2m + 1. \tag{39}$$

This demonstrates that ϕ is an immersion that is mixed and totally geodesic. On the other hand, the converse part of (39) is true when considering the immersion of a pointwise bi-slant warped product into a locally almost cosymplectic space form. As a result, we can assert that the proof is fully established. \square

The above theorem readily implies the following corollary.

Corollary 1. *Let $\tilde{N}(c)$ be a $(2m+1)$ -dimensional locally conformal almost cosymplectic manifold, and let $\phi :_{f_2} N_1 \times_{f_1} N_2 \rightarrow \tilde{N}(c)$ denote an isometric immersion of n -dimensional different submanifolds into $\tilde{N}(c)$ equipped with different connections. Then, the following statement holds true:*

$\tilde{N}(c)$	with quarter-symmetric connection	with semi-symmetric connection	with semi-symmetric non-metric connection
Pointwise bi-slant	$ \begin{aligned} &\frac{n_2 \Delta_1 f_1}{f_1} + \frac{n_1 \Delta_2 f_2}{f_2} \leq \\ &\frac{n^2}{4} \ H\ ^2 + \\ &\frac{c - 3u^2}{4} n_1 n_2 - \\ &\frac{3(c + u^2)}{4} (n_1 \cos^2 \theta_1 + \\ &n_2 \cos^2 \theta_2) - \frac{1}{2} \{ (\mu_1 + \\ &\mu_2)a + \mu_2(\mu_1 - \mu_2)b + \\ &2n_1 n_2(\mu_1 - \mu_2)\pi(H) \}, \end{aligned} $	$ \begin{aligned} &\frac{n_2 \Delta_1 f_1}{f_1} + \frac{n_1 \Delta_2 f_2}{f_2} \leq \\ &\frac{n^2}{4} \ H\ ^2 + \\ &\frac{c - 3u^2}{4} n_1 n_2 - \\ &\frac{3(c + u^2)}{4} (n_1 \cos^2 \theta_1 + \\ &n_2 \cos^2 \theta_2) - a \end{aligned} $	$ \begin{aligned} &\frac{n_2 \Delta_1 f_1}{f_1} + \frac{n_1 \Delta_2 f_2}{f_2} \leq \\ &\frac{n^2}{4} \ H\ ^2 + \\ &\frac{c - 3u^2}{4} n_1 n_2 - \\ &\frac{3(c + u^2)}{4} (n_1 \cos^2 \theta_1 + \\ &n_2 \cos^2 \theta_2) - \frac{1}{2} (a + \\ &2n_1 n_2 \pi(H)) \end{aligned} $

	with quarter-symmetric connection	with semi-symmetric connection	with semi-symmetric non-metric connection
Pointwise semi-slant	$\frac{n_2 \Delta_1 f_1}{f_1} + \frac{n_1 \Delta_2 f_2}{f_2} \leq$ $\frac{n^2}{4} \ H\ ^2 +$ $\frac{c-3u^2}{4} n_1 n_2 -$ $\frac{3(c+u^2)}{4} (n_1 n_2 \cos^2 \theta_2) - \frac{1}{2} \{(\mu_1 + \mu_2)a + \mu_2(\mu_1 - \mu_2)b + 2n_1 n_2(\mu_1 - \mu_2)\pi(H)\}$	$\frac{n_2 \Delta_1 f_1}{f_1} + \frac{n_1 \Delta_2 f_2}{f_2} \leq$ $\frac{n^2}{4} \ H\ ^2 +$ $\frac{c-3u^2}{4} n_1 n_2 -$ $\frac{3(c+u^2)}{4} (n_1 n_2 \cos^2 \theta_2) - a$	$\frac{n_2 \Delta_1 f_1}{f_1} + \frac{n_1 \Delta_2 f_2}{f_2} \leq$ $\frac{n^2}{4} \ H\ ^2 +$ $\frac{c-3u^2}{4} n_1 n_2 -$ $\frac{3(c+u^2)}{4} (n_1 n_2 \cos^2 \theta_2) - \frac{1}{2}(a + 2n_1 n_2 \pi(H))$
Pointwise hemi-slant	$\frac{n_2 \Delta_1 f_1}{f_1} + \frac{n_1 \Delta_2 f_2}{f_2} \leq$ $\frac{n^2}{4} \ H\ ^2 +$ $\frac{c-3u^2}{4} n_1 n_2 -$ $\frac{3(c+u^2)}{4} n_2 \cos^2 \theta -$ $\frac{1}{2} \{(\mu_1 + \mu_2)a + \mu_2(\mu_1 - \mu_2)b + 2n_1 n_2(\mu_1 - \mu_2)\pi(H)\}$	$\frac{n_2 \Delta_1 f_1}{f_1} + \frac{n_1 \Delta_2 f_2}{f_2} \leq$ $\frac{n^2}{4} \ H\ ^2 +$ $\frac{c-3u^2}{4} n_1 n_2 -$ $\frac{3(c+u^2)}{4} n_2 \cos^2 \theta - a$	$\frac{n_2 \Delta_1 f_1}{f_1} + \frac{n_1 \Delta_2 f_2}{f_2} \leq$ $\frac{n^2}{4} \ H\ ^2 +$ $\frac{c-3u^2}{4} n_1 n_2 -$ $\frac{3(c+u^2)}{4} n_2 \cos^2 \theta -$ $\frac{1}{2}(a + 2n_1 n_2 \pi(H))$
Pointwise CR	$\frac{n_2 \Delta_1 f_1}{f_1} + \frac{n_1 \Delta_2 f_2}{f_2} \leq$ $\frac{n^2}{4} \ H\ ^2 +$ $\frac{c-3u^2}{4} n_1 n_2 -$ $\frac{3(c+u^2)}{4} n_1 - \frac{1}{2} \{(\mu_1 + \mu_2)a + \mu_2(\mu_1 - \mu_2)b + 2n_1 n_2(\mu_1 - \mu_2)\pi(H)\}$	$\frac{n_2 \Delta_1 f_1}{f_1} + \frac{n_1 \Delta_2 f_2}{f_2} \leq$ $\frac{n^2}{4} \ H\ ^2 +$ $\frac{c-3u^2}{4} n_1 n_2 -$ $\frac{3(c+u^2)}{4} n_1 - a$	$\frac{n_2 \Delta_1 f_1}{f_1} + \frac{n_1 \Delta_2 f_2}{f_2} \leq$ $\frac{n^2}{4} \ H\ ^2 +$ $\frac{c-3u^2}{4} n_1 n_2 -$ $\frac{3(c+u^2)}{4} n_1 - \frac{1}{2}(a + 2n_1 n_2 \pi(H))$

Remark 2. The above result is obtained by using Remark 1 and the definition of semi-slant, hemi-slant, and CR in Theorem 1.

Next, we have the following theorem.

Theorem 2. Let $\tilde{N}(c)$ be a $(2m+1)$ -dimensional locally conformal almost cosymplectic manifold, and let $\phi :_{f_2} N_1 \times_{f_1} N_2 \rightarrow \tilde{N}(c)$ denote an isometric immersion of an n -dimensional pointwise bi-slant doubly warped product submanifold into $\tilde{N}(c)$ with a quarter-symmetric connection. Then, the following statement holds true:

$$\begin{aligned}
 \frac{\Delta_1 f_1}{n_1 f_1} + \frac{\Delta_2 f_2}{n_2 f_2} &\geq \tau - \frac{n^2(n-2)}{2(n-1)} \|H\|^2 - \frac{c-3u^2}{8} (n+1)(n-2) \\
 &\quad - \frac{3(c+u^2)}{8} (n_1 \cos^2 \theta_1 + n_2 \cos^2 \theta_2) \\
 &\quad - \frac{1}{2} \{(\mu_1 + \mu_2)(1-n)a + \mu_2(\mu_1 - \mu_2)(1-n)b \\
 &\quad + (\mu_1 - \mu_2)n(n-1)\pi(H)\}, \tag{40}
 \end{aligned}$$

where $n_i = \dim N_i, i=1,2$, and Δ^i is the Laplacian operator on $N_i, i=1,2$. The equality sign holds in (40) and is guaranteed to hold when the equality sign is present in (53). Furthermore, if $n = 2$, then the equality sign in (40) holds identically.

Proof. Suppose $_{f_2} N_1 \times_{f_1} N_2$ is an isometric immersion of an n -dimensional pointwise bi-slant doubly warped product submanifold into $\tilde{N}(c)$, a manifold with pointwise φ -sectional curvature c and endowed with a quarter symmetric connection. Then, by applying the equation of Gauss, we obtain

$$2\tau = \frac{c-3u^2}{4} \{n(n-1) + 3\frac{c+u^2}{4}(n_1 \cos^2 \theta_1 + n_2 \cos^2 \theta_2) + (\mu_1 + \mu_2)(1-n)a + \mu_2(\mu_1 - \mu_2)(1-n)b + (\mu_2 - \mu_1)n(n-1)\pi(H) + n^2\|H\|^2 - \|\sigma\|^2\}. \quad (41)$$

Now, we consider that

$$\delta = 2\tau - \frac{c-3u^2}{4}(n+1)(n-2) - \frac{n^2(n-2)}{n-1}\|H\|^2 - 3\frac{c+u^2}{4}(n_1 \cos^2 \theta_1 + n_2 \cos^2 \theta_2) - (\mu_1 + \mu_2)(1-n)a - \mu_2(\mu_1 - \mu_2)(1-n)b - (\mu_2 - \mu_1)n(n-1)\pi(H). \quad (42)$$

Then from (41) and (42), it follows that

$$n^2\|H\|^2 = (n-1)\{\|\sigma\|^2 + \delta - \frac{c-3u^2}{2}\}. \quad (43)$$

Given an orthonormal frame $\{v_1, \dots, v_n\}$, the equation can be represented in the following form:

$$\left(\sum_{r=n+1}^{2m+1} \sum_{i=1}^n \sigma_{ii}^r\right)^2 = (n-1)\left\{\delta + \sum_{r=n+1}^{2m+1} \sum_{i=1}^n (\sigma_{ii}^r)^2 + \sum_{r=n+1}^{2m+1} \sum_{i<j} (\sigma_{ij}^r)^2 + \sum_{r=n+2}^{2m+1} \sum_{i,j=1}^n (\sigma_{ij}^r)^2 - \left(\frac{c-3u^2}{2}\right)\right\}, \quad (44)$$

which implies that

$$\begin{aligned} & \left(\sigma_{11}^{n+1} + \sum_{i=2}^{n_1} \sigma_{ii}^{n+1} + \sum_{t=n_1+1}^n \sigma_{tt}^{n+1}\right)^2 = \delta + (\sigma_{11}^{n+1})^2 + \sum_{i=2}^{n_1} (\sigma_{ii}^{n+1})^2 \\ & + \sum_{t=n_1+1}^n (\sigma_{tt}^{n+1})^2 + \sum_{2 \leq j \neq l \leq n_1} \sigma_{jj}^{n+1} \sigma_{ll}^{n+1} \\ & - \sum_{n_1+1 \leq t \neq s \leq n_1} (\sigma_{jj}^{n+1})(\sigma_{ll}^{n+1}) + \sum_{i<j=1}^n (\sigma_{ij}^{n+1})^2 + \sum_{r=n+1}^{2m+1} \sum_{i,j=1}^n (\sigma_{ij}^r)^2 - \left(\frac{c-3u^2}{2}\right). \end{aligned} \quad (45)$$

Let us consider that $b_1 = \sigma_{11}^n + 1$, $b_2 = \sum_{i=2}^{n_1} (\sigma_{ii}^n + 1)^2$ and $b_3 = \sum_{t=n_1+1}^n (\sigma_{tt}^n + 1)^2$. Subsequently, utilizing (1) and (45), we deduce

$$\begin{aligned} & \frac{\delta}{2} - \left(\frac{c-3u^2}{2}\right) + \sum_{i<j=1}^n (\sigma_{ij}^{n+1})^2 + \frac{1}{2} \sum_{r=n+1}^{2m+1} \sum_{i,j=1}^n (\sigma_{ij}^r)^2 \\ & \leq \sum_{2 \leq j \neq l \leq n_1} \sigma_{jj}^{n+1} \sigma_{ll}^{n+1} + \sum_{n_1+1 \leq t \neq s \leq n} \sigma_{tt}^{n+1} \sigma_{ss}^{n+1}. \end{aligned} \quad (46)$$

The equality holds true if and only if

$$\sum_{i=1}^{n_1} \sigma_{ii}^{n+1} = \sum_{t=n_1+1}^n \sigma_{tt}^{n+1}. \quad (47)$$

However, by employing (46) and the definition of scalar curvature, we obtain

$$\begin{aligned} K(v_1 \wedge v_{n_1+1}) &\geq \sum_{r=n+1}^{2m+1} \sum_{j \in P_{1n_1+1}} (\sigma_{1j}^r)^2 \\ &+ \frac{1}{2} \sum_{r=n+1}^{2m+1} \sum_{\substack{i \neq j \\ j \in P_{1n_1+1}}} (\sigma_{ij}^r)^2 + \sum_{r=n+1}^{2n+1} \sum_{j \in P_{1n_1+1}} (\sigma_{n_1+1j}^r)^2 \\ &+ \frac{1}{2} \sum_{r=n+1}^{2m+1} \sum_{i,j \in P_{1n_1+1}} (\sigma_{1j}^r)^2 + \frac{1}{2} \sum_{r=n+1}^{2m+1} \sum_{i,j=1}^{n_1+1} (\sigma_{1j}^r)^2 + \frac{\delta}{2}, \end{aligned}$$

where $P_{1n_1+1} = \{1, \dots, n\} - \{1, n_1 + 1\}$. Thus, it implies that

$$K(v_1 \wedge v_{n_1+1}) = \frac{\delta}{2}. \quad (48)$$

Since, $N = f_2 N_1 \times_{f_1} N_2$ is a C-totally real doubly warped product submanifold, we have $\nabla_{\chi_1} \chi_3 = \nabla_{\chi_3} \chi_1 = (X \ln f_1) \chi_3 + (\chi_3 \ln f_2) \chi_1$ for any unit vector fields χ_1, χ_3 tangent to N_1 and N_2 , respectively.

Then, from (17), (42), and (48):

$$\begin{aligned} \tau &\leq \frac{1}{f_1} \{(\nabla_{v_1} v_1) f_1 - v_1^2 f_1\} + \frac{1}{f_2} \{(\nabla_{v_2} v_2) f_2 - v_2^2 f_2\} \\ &+ \frac{c - 3u^2}{8} (n+1)(n-2) + \frac{n^2(n-2)}{2(n-1)} \|H\|^2 \\ &+ \frac{3(c+u^2)}{8} (n_1 \cos^2 \theta_1 + n_2 \cos^2 \theta_2) \\ &+ \frac{1}{2} \{(\mu_1 + \mu_2)(1-n)a + \mu_2(\mu_1 - \mu_2)(1-n)b + (\mu_2 - \mu_1)n(n-1)\pi(H)\}. \end{aligned} \quad (49)$$

If the equality holds in (49), then by examining the remaining terms in (46) and (48), we deduce the following conditions:

$$\begin{aligned} \sigma_{1j}^r &= 0, \quad \sigma_{jn_1+1}^r = 0, \quad \sigma_{ij}^r = 0, \quad \text{where } i \neq j, \text{ and } r \in \{n+1, \dots, 2m+1\} \\ \sigma_{1j}^r &= \sigma_{jn_1+1}^r = \sigma_{ij}^r = 0, \quad \text{and } \sigma_{11}^r + \sigma_{n_1+1n_1+1}^r = 0. \end{aligned} \quad (50)$$

In a similar fashion, we prolong the relation (49) in the subsequent manner:

$$\begin{aligned} \tau &\leq \frac{1}{f_1} \{(\nabla_{v_\alpha} v_\alpha) f_1 - v_\alpha^2 f_1\} + \frac{1}{f_2} \{(\nabla_{v_\beta} v_\beta) f_2 - v_\beta^2 f_2\} + \frac{c - 3u^2}{8} (n+1)(n-2) \\ &+ \frac{n^2(n-2)}{2(n-1)} \|H\|^2 + 3 \frac{c+u^2}{8} (n_1 \cos^2 \theta_1 + n_2 \cos^2 \theta_2) \\ &+ \frac{1}{2} \{(\mu_1 + \mu_2)(1-n)a + \mu_2(\mu_1 - \mu_2)(1-n)b + (\mu_2 - \mu_1)n(n-1)\pi(H)\} \end{aligned} \quad (51)$$

for any $\alpha = 1, \dots, n_1$ and $\beta = n_1 + 1, \dots, n$.

When we add up α ranging from 1 to n_1 and β ranging from $n_1 + 1$ to n_2 , the result is

$$\begin{aligned}
 n_1 \cdot n_2 \cdot \tau \leq & \frac{n_2 \cdot \Delta_1 f_1}{f_1} + \frac{n_1 \cdot \Delta_2 f_2}{f_2} + \frac{c - 3u^2}{8} n_1 \cdot n_2 (n + 1)(n - 2) \\
 & + n_1 \cdot n_2 \frac{n^2(n - 2)}{2(n - 1)} \|H\|^2 + 3n_1 \cdot n_2 \frac{c + u^2}{8} (n_1 \cos^2 \theta_1 + n_2 \cos^2 \theta_2) \\
 & + \frac{1}{2} n_1 \cdot n_2 \{(\mu_1 + \mu_2)(1 - n)a + \mu_2(\mu_1 - \mu_2)(1 - n)b \\
 & + (\mu_2 - \mu_1)n(n - 1)\pi(H)\}.
 \end{aligned} \tag{52}$$

Similarly, the equality sign holds in (52) identically. Thus, the equality sign in (49) holds for each $\alpha \in \{1, \dots, n_1\}$ and $\beta \in \{n_1 + 1, \dots, n\}$. Then, we obtain

$$\begin{cases} \sigma_{\alpha j}^r = 0, & \sigma_{ij}^r = 0, & \sigma_{ij}^r = 0, \text{ where } i \neq j, \text{ and } r \in \{n + 1, \dots, 2m + 1\} \\ \sigma_{\alpha j}^r = \sigma_{ij}^r = \sigma_{ij}^r = 0, & \text{and} \\ \sigma_{\alpha\alpha}^r + \sigma_{\beta\beta}^r = 0, & i, j \in P_{1n_1+1}, r = n + 2, \dots, 2m + 1. \end{cases} \tag{53}$$

Moreover, if $n = 2$, then $n_1 = n_2 = 1$. Thus, from (17), we obtain

$$\tau = \Delta_1 f_1 + \Delta_2 f_2.$$

Therefore, the theorem is conclusively proven by the observed equality in (40). \square

The above theorem readily implies the following corollary.

Corollary 2. Let $\tilde{N}(c)$ be a $(2m + 1)$ -dimensional locally conformal almost cosymplectic manifold, and let $\phi :_{f_2} N_1 \times_{f_1} N_2 \rightarrow \tilde{N}(c)$ denote an isometric immersion of n -dimensional different submanifolds N into $\tilde{N}(c)$ with different connections. Then, the following statement holds true:

$\tilde{N}(c)$ N	with quarter-symmetric connection	with semi-symmetric connection	with semi-symmetric non-metric connection
Pointwise bi-slant	$ \begin{aligned} & \frac{\Delta_1 f_1}{n_1 f_1} + \frac{\Delta_2 f_2}{n_2 f_2} \geq \\ & \tau - \frac{n^2(n-2)}{2(n-1)} \ H\ ^2 - \\ & \frac{c-3u^2}{8} (n + 1)(n - 2) - \\ & \frac{3(c+u^2)}{8} (n_1 \cos^2 \theta_1 + \\ & n_2 \cos^2 \theta_2) - \frac{1}{2} \{(\mu_1 + \\ & \mu_2)(1 - n)a + \mu_2(\mu_1 - \\ & \mu_2)(1 - n)b + (\mu_1 - \\ & \mu_2)n(n - 1)\pi(H)\} \end{aligned} $	$ \begin{aligned} & \frac{\Delta_1 f_1}{n_1 f_1} + \frac{\Delta_2 f_2}{n_2 f_2} \geq \\ & \tau - \frac{n^2(n-2)}{2(n-1)} \ H\ ^2 - \\ & \frac{c-3u^2}{8} (n + 1)(n - 2) - \\ & \frac{3(c+u^2)}{8} (n_1 \cos^2 \theta_1 + \\ & n_2 \cos^2 \theta_2) + (n - 1)a \end{aligned} $	$ \begin{aligned} & \frac{\Delta_1 f_1}{n_1 f_1} + \frac{\Delta_2 f_2}{n_2 f_2} \geq \\ & \tau - \frac{n^2(n-2)}{2(n-1)} \ H\ ^2 - \\ & \frac{c-3u^2}{8} (n + 1)(n - 2) - \\ & \frac{3(c+u^2)}{8} (n_1 \cos^2 \theta_1 + \\ & n_2 \cos^2 \theta_2) - \frac{1}{2} [\mu_1(1 - \\ & n)a + n(n - 1)\pi(H)] \end{aligned} $
Pointwise semi-slant	$ \begin{aligned} & \frac{\Delta_1 f_1}{n_1 f_1} + \frac{\Delta_2 f_2}{n_2 f_2} \geq \\ & \tau - \frac{n^2(n-2)}{2(n-1)} \ H\ ^2 - \\ & \frac{c-3u^2}{8} (n + 1)(n - \\ & 2) - \frac{3(c+u^2)}{8} (n_1 + \\ & n_2 \cos^2 \theta_2) - \frac{1}{2} \{(\mu_1 + \\ & \mu_2)(1 - n)a + \mu_2(\mu_1 - \\ & \mu_2)(1 - n)b + (\mu_1 - \\ & \mu_2)n(n - 1)\pi(H)\} \end{aligned} $	$ \begin{aligned} & \frac{\Delta_1 f_1}{n_1 f_1} + \frac{\Delta_2 f_2}{n_2 f_2} \geq \\ & \tau - \frac{n^2(n-2)}{2(n-1)} \ H\ ^2 - \\ & \frac{c-3u^2}{8} (n + 1)(n - \\ & 2) - \frac{3(c+u^2)}{8} (n_1 + \\ & n_2 \cos^2 \theta_2) + (n - 1)a \end{aligned} $	$ \begin{aligned} & \frac{\Delta_1 f_1}{n_1 f_1} + \frac{\Delta_2 f_2}{n_2 f_2} \geq \\ & \tau - \frac{n^2(n-2)}{2(n-1)} \ H\ ^2 - \\ & \frac{c-3u^2}{8} (n + 1)(n - \\ & 2) - \frac{3(c+u^2)}{8} (n_1 + \\ & n_2 \cos^2 \theta_2) - \frac{1}{2} [\mu_1(1 - \\ & n)a + n(n - 1)\pi(H)] \end{aligned} $

	with quarter-symmetric connection	with semi-symmetric connection	with semi-symmetric non-metric connection
Pointwise hemi-slant	$\frac{\Delta_1 f_1}{n_1 f_1} + \frac{\Delta_2 f_2}{n_2 f_2} \geq \tau - \frac{n^2(n-2)}{2(n-1)} \ H\ ^2 - \frac{c-3u^2}{8}(n+1)(n-2) - \frac{3(c+u^2)}{8} n_2 \cos^2 \theta - \frac{1}{2} \{(\mu_1 + \mu_2)(1-n)a + \mu_2(\mu_1 - \mu_2)(1-n)b + (\mu_1 - \mu_2)n(n-1)\pi(H)\}$	$\frac{\Delta_1 f_1}{n_1 f_1} + \frac{\Delta_2 f_2}{n_2 f_2} \geq \tau - \frac{n^2(n-2)}{2(n-1)} \ H\ ^2 - \frac{c-3u^2}{8}(n+1)(n-2) - \frac{3(c+u^2)}{8} n_2 \cos^2 \theta + (n-1)a$	$\frac{\Delta_1 f_1}{n_1 f_1} + \frac{\Delta_2 f_2}{n_2 f_2} \geq \tau - \frac{n^2(n-2)}{2(n-1)} \ H\ ^2 - \frac{c-3u^2}{8}(n+1)(n-2) - \frac{3(c+u^2)}{8} n_2 \cos^2 \theta - \frac{1}{2} [\mu_1(1-n)a + n(n-1)\pi(H)]$
Pointwise CR	$\frac{\Delta_1 f_1}{n_1 f_1} + \frac{\Delta_2 f_2}{n_2 f_2} \geq \tau - \frac{n^2(n-2)}{2(n-1)} \ H\ ^2 - \frac{c-3u^2}{8}(n+1)(n-2) - \frac{3(c+u^2)}{8} n_1 - \frac{1}{2} \{(\mu_1 + \mu_2)(1-n)a + \mu_2(\mu_1 - \mu_2)(1-n)b + (\mu_1 - \mu_2)n(n-1)\pi(H)\}$	$\frac{\Delta_1 f_1}{n_1 f_1} + \frac{\Delta_2 f_2}{n_2 f_2} \geq \tau - \frac{n^2(n-2)}{2(n-1)} \ H\ ^2 - \frac{c-3u^2}{8}(n+1)(n-2) - \frac{3(c+u^2)}{8} n_1 + (n-1)a$	$\frac{\Delta_1 f_1}{n_1 f_1} + \frac{\Delta_2 f_2}{n_2 f_2} \geq \tau - \frac{n^2(n-2)}{2(n-1)} \ H\ ^2 - \frac{c-3u^2}{8}(n+1)(n-2) - \frac{3(c+u^2)}{8} n_1 - \frac{1}{2} [\mu_1(1-n)a + n(n-1)\pi(H)]$

Remark 3. The above result is obtained by using Remark 1 and the definition of semi-slant, hemi-slant, CR in Theorem 2.

4. Conclusions

In this paper, we have established geometric inequalities that provide valuable insights into the relationship between the squared mean curvature and the warping functions of a doubly warped product pointwise bi-slant submanifold. These findings have been achieved within the framework of locally conformal almost cosymplectic manifolds, which are equipped with a quarter-symmetric metric connection. Furthermore, the paper also investigates the cases of equality in these inequalities, shedding light on the specific conditions under which these geometric relationships hold true. This analysis enhances our understanding of the intricate interplay between curvature and warping functions in the context of bi-slant submanifolds. The findings presented here significantly contribute to the existing body of knowledge in the field of locally conformal almost cosymplectic manifolds. Further exploration and utilization of these results are encouraged to advance our understanding of this area of mathematics.

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