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Sharp Bounds on Toeplitz Determinants for Starlike and Convex Functions Associated with Bilinear Transformations

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Abstract: Starlike and convex functions have gained increased prominence in both academic literature and practical applications over the past decade. Concurrently, logarithmic coefficients play a pivotal role in estimating diverse properties within the realm of analytic functions, whether they are univalent or nonunivalent. In this paper, we rigorously derive bounds for specific Toeplitz determinants involving logarithmic coefficients pertaining to classes of convex and starlike functions concerning symmetric points. Furthermore, we present illustrative examples showcasing the sharpness of these established bounds. Our findings represent a substantial contribution to the advancement of our understanding of logarithmic coefficients and their profound implications across diverse mathematical contexts.

Keywords: univalent functions; starlike and convex functions; symmetric points; logarithmic coefficients; Schwarz functions; Toeplitz determinants

MSC: 30A10; 30H05; 30C35; 30C45; 30C50; 30C55

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1. Introduction, Definitions and Motivation

Within the open unit disk $\mathbb{U}$ in the complex plane, the class $\mathcal{B}$ is defined by functions g expressed as power series $g(z) = \sum_{k=1}^{\infty} a_k z^k$, where $z \in \mathbb{U}$ and the functions are analytic. Let $\mathcal{A}$ be the collection of functions $g \in \mathcal{B}$ satisfying $g(0) = 0$, $g'(0) = 1$. Additionally, $\mathcal{S}$ is defined as a sub-collection of $\mathcal{A}$ consisting of schlicht functions, where a function $g \in \mathcal{S}$ is represented by

$$g(z) = z + \sum_{k=1}^{\infty} a_{k+1} z^{k+1},\ z \in \mathbb{U}. \tag{1}$$

In the domain of geometric function theory, considerable attention has been directed towards subclasses associated with convex and starlike functions. A function g within the class $\mathcal{S}$ is classified as convex if its image $g(\mathbb{U})$ takes on a convex shape, while it is characterized as starlike when the image $g(\mathbb{U})$ exhibits starlikeness with respect to the origin. We denote these sets of functions by $\mathcal{S}^c$ and $\mathcal{S}^*$, respectively.

Undoubtedly, the primary focus of study within univalent functions revolves around the subclasses $\mathcal{S}^c$ and $\mathcal{S}^*$. Throughout this manuscript, it will become evident that both $\mathcal{S}^c$ and $\mathcal{S}^*$ fall within the purview of $\mathcal{S}$, thereby exemplifying natural subsets of $\mathcal{S}$. These subclasses boast a longstanding historical legacy, originating in the early decades of the twentieth century. Starlike functions hold a particular allure, especially considering the inclusion of the Koebe function $K(z) = z/(1-z)^2$ within $\mathcal{S}^*$. This implies that starlike functions have the potential to exhibit growth rates equivalent to those in $\mathcal{S}$. Consequently, the geometric constraints imposed on the image domain of starlike functions contribute to a rich array of properties within $\mathcal{S}^*$, many of which extend to the broader class $\mathcal{S}$. However, some properties either prove false or present open conjectures within $\mathcal{S}$. We begin by presenting the established analytic characterization of starlike functions, as outlined by

Duren [1], in which these functions are defined in terms of having a positive real part. Let $g \in \mathcal{A}$. Then $g \in \mathcal{S}^*$ if, and only if, $\mathrm{Re}(G_1) > 0$, where

$$G_1(z) = \frac{zg'(z)}{g(z)},\ z \in \mathbb{U}. \tag{2}$$

A similar analytic expression applies to convex functions, and a again follows that given by Duren [1]. Let $g \in \mathcal{A}$. Then $g \in \mathcal{S}^c$ if, and only if, $\mathrm{Re}(G_2) > 0$, where

$$G_2(z) = \frac{(zg'(z))'}{g'(z)},\ z \in \mathbb{U}. \tag{3}$$

We note that $\mathcal{S}^c \subset \mathcal{S}^* \subset \mathcal{S}$. In the research by Sakaguchi [2], a category of functions, denoted by $\mathcal{S}_s^*$, was introduced, where a function $g \in \mathcal{S}_s^*$ was defined as starlike concerning symmetric points if, for any $\rho \in (0,1)$ near 1 and for any z_0 located on $|z| = \rho$, the angular velocity of $g(z)$ around $g(-z_0)$ is positive at z_0 as z moves along the circle $|z| = \rho$ in the positive direction. These functions, often referred to as Sakaguchi functions, are noteworthy for their amalgamation of symmetry and starlike properties, establishing them as a unique subclass within the realm of univalent functions. The well-known analytic description of functions that are starlike with respect to symmetric points has an excellent exposition, expressed by $\mathrm{Re}(G_3) > 0$, where

$$G_3(z) = \frac{2zg'(z)}{g(z) - g(-z)},\ z \in \mathbb{U}. \tag{4}$$

Das and Singh [3] introduced the class of convex functions concerning symmetric points. A similar analytic expression holds for functions that are convex with respect to symmetric points, expressed as $\mathrm{Re}(G_4) > 0$, where

$$G_4(z) = \frac{2(zg'(z))'}{(g(z) - g(-z))'},\ z \in \mathbb{U}. \tag{5}$$

The functions within the class $\mathcal{S}_s^*$ are recognized as close to convex, thus ensuring their univalence. This class encompasses not only starlike functions concerning symmetric points but also extends to convex functions and odd starlike functions with respect to the origin. Addressing the geometric properties of analytic functions, Sakaguchi functions find applications in various facets of complex analysis, including the theory of univalent functions, quasiconformal mappings, and the exploration of conformal maps between diverse regions in the complex plane. The practical utility of conformal mappings and geometric transformations tied to the properties of starlike functions extends to computer graphics and image processing, facilitating the visually pleasing and efficient mapping of shapes, textures, and images.

Consider $\mathcal{H}$ as the collection of functions $\omega \in \mathcal{B}$ satisfying $\omega(0) = 0$ and $|\omega(z)| < 1$ for $z \in \mathbb{U}$. Functions $\omega \in \mathcal{H}$ are said to be Schwarz functions and have the following series form:

$$\omega(z) = \sum_{k=1}^{\infty} b_k z^k. \tag{6}$$

An analytic function g_1 is considered subordinate to g_2 (symbolically $g_1 \prec g_2$), if we have $\omega \in \mathcal{H}$, such that $g_1(z) = g_2(\omega(z))$ for $z \in \mathbb{U}$. Ma and Minda [4] unified various subclasses of starlike and convex functions. They defined

$$\mathcal{S}^*(\phi) = \{g \in \mathcal{S} :\ G_1 \prec \phi\}, \tag{7}$$

and

$$\mathcal{S}^c(\phi) = \{g \in \mathcal{S} :\ G_2 \prec \phi\}, \tag{8}$$

where G_1 and G_2 are given by (2) and (3), respectively. Moreover,

$$\phi(z) = 1 + \sum_{k=1}^{\infty} J_k z^k, \tag{9}$$

is univalent in $\mathbb{U}$ with a positive real part, and all J_ks are real numbers, where $J_1 > 0$. In [5], Ravichandran introduced unified classes $\mathcal{S}_s^*(\phi(z))$ and $\mathcal{S}_s^c(\phi(z))$ of Sakaguchi functions, which are defined as follows:

$$\mathcal{S}_s^*(\phi) = \{g \in \mathcal{S} : \ G_3 \prec \phi\}, \tag{10}$$

and

$$\mathcal{S}_s^c(\phi) = \{g \in \mathcal{S} : \ G_4 \prec \phi\}, \tag{11}$$

where G_3, G_4 and ϕ are defined by (4), (5) and (9), respectively. The classes $\mathcal{S}_s^*(\phi(z))$ and $\mathcal{S}_s^c(\phi(z))$ include various subclasses associated with the exponential function, the cosine hyperbolic function, the lemniscate of Bernoulli, and others. Authors [6] focused on subclasses, such as $\mathcal{S}_s^*(e^z)$ and $\mathcal{S}_s^*(\sqrt{z+1})$, within the realm of Sakaguchi functions, investigating bounded constraints on the initial Taylor–Maclaurin coefficients of functions within these subclasses. Subsequently, Ganesh et al. [7] delved into the class $\mathcal{S}_s^*(e^z)$ and estimated certain coefficient functionals, although most of the results lack sharpness. In a distinct approach introduced in [8], the author established connections between the coefficients of Schwarz functions and the coefficients of corresponding functions in a specified class. In this study, we examine the classes $\mathcal{S}_s^* := \mathcal{S}_s^*(\phi(z))$ and $\mathcal{S}_s^c := \mathcal{S}_s^c(\phi(z))$, where $\phi(z)$ is the bilinear transformation $(1+z)/(1-z)$. Therefore, we can represent these classes as

$$\mathcal{S}_s^* = \left\{ g \in \mathcal{S} : \ G_3 \prec \frac{1+z}{1-z}, \ z \in \mathbb{U} \right\}, \tag{12}$$

and

$$\mathcal{S}_s^c = \left\{ g \in \mathcal{S} : \ G_4 \prec \frac{1+z}{1-z}, \ z \in \mathbb{U} \right\}, \tag{13}$$

where G_3 and G_4 are given by (4) and (5), respectively. It is well established that these classes are subclasses of close-to-convex functions with bounded k^{th} Taylor–Maclaurin coefficients, where the bound is set to 1. However, no bounds are known for coefficients of functions g satisfying (10) and (11), excluding cases where $k = 2, 3$. It is worth noting that the functions $g_1(z) = z/(1-z^2)$ and $g_2(z) = -\log(1-z)$ belong to the classes $\mathcal{S}_s^*$ and $\mathcal{S}_s^c$, respectively. Consequently, it becomes evident that these classes are not empty.

The logarithmic coefficients of the function $g \in \mathcal{S}$, denoted as γ_k for $k \in \mathbb{N} := \{1, 2, 3, \cdots\}$, are defined as follows:

$$\frac{1}{2} \log \frac{g(z)}{z} = \sum_{k=1}^{\infty} \gamma_k z^k, \quad z \in \mathbb{U}. \tag{14}$$

The significance of these coefficients becomes evident in various estimates during the analysis of univalent functions (see Milin [9], Chapter 2). The heightened interest in logarithmic coefficients is driven by the fact that, for the class $\mathcal{S}$, exact bounds are established solely for $|\gamma_1|$ (≤ 1) and $|\gamma_2|$ $(\leq 0.5 + e^{-1})$. The logarithmic coefficients γ_k for any $g \in \mathcal{S}$ adhere to the following inequality

$$\sum_{k=1}^{\infty} |\gamma_k|^2 \leq \frac{\pi^2}{6},$$

and the equality is established for the Koebe function. It is noteworthy that the function $k(z)$ does not belong to $\mathcal{S}_s^*$, preventing it from being an extremal function for $\mathcal{S}_s^*$.

If g is given by (14), then differentiating and the equating coefficients yields

$$2\gamma_1 = a_2,\ 4\gamma_2 = 2a_3 - a_2^2 \text{ and } 6\gamma_3 = 3a_4 - 3a_3a_2 + a_2^3. \tag{15}$$

Toeplitz and Hankel matrices share a close relationship. Toeplitz matrices display identical elements along their main diagonals, whereas Hankel matrices exhibit constant values along the converse diagonals. In a seminal work in 2016, Ye and Lim [10] established that matrices of size $m \times m$ across the set of complex values can generally be considered to derive from the product of specific Toeplitz matrices. The importance of Toeplitz determinants and matrices spans different mathematical concepts, presenting a diverse scope of applications [10]. Furthermore, across both the theoretical and applied mathematical realm, Toeplitz determinants take on vital roles, finding applications in fields such as integral equations and analysis, quantum mechanics, signal and image processing, etc. [11].

The research presented by Thomas and Halim [12] introduces the symmetric Toeplitz determinant associated with $g \in \mathcal{S}$ defined by (1). This Toeplitz determinant, denoted as $T_{k,n}(g)$, is expressed as follows:

$$T_{k,n}(g) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{k+n-1} \\ a_{n+1} & a_n & \cdots & a_{k+n-2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{k+n-1} & a_{k+n-2} & \cdots & a_n \end{vmatrix}$$

where $k, n \in \mathbb{N}$. Numerous recent papers have focused on determining upper and lower bounds for Toeplitz determinants associated with functions in $\mathcal{S}$. Ali et al. [13] examined the bounds of $|T_{2,n}(f)|$, $|T_{3,1}(f)|$ and $|T_{3,2}(f)|$ within the classes $\mathcal{S}^*$ and $\mathcal{S}^c$. Cudna et al. [14] investigated sharp upper and lower estimates for $|T_{2,1}(f)|$ and $|T_{3,1}(f)|$ concerning the classes $\mathcal{S}^*(\alpha)$ and $\mathcal{S}^c(\alpha)$, where $0 \leq \alpha < 1$. Obradović and Tuneski [15] delved into the same determinants within the class $\mathcal{S}$ and some of its subclasses. Recently, Sun and Wang [16] derived sharp bounds for second- and third-order Hermitian Toeplitz determinants for the class $\mathcal{S}^c(\alpha)$ of convex functions. Recently, Mandal et al. [17] determined the best possible bounds for second Hankel and Hermitian Toeplitz matrices, involving logarithmic coefficients of inverse functions, which are applied to starlike and convex functions concerning symmetric points. In recent studies, considerable attention has been devoted to exploring interesting properties associated with Teoplitz and Hankel determinants within the realm of analytic functions of certain classes of convex and starlike functions (see, for example, [18–27] and references therein).

The Toeplitz determinant, characterized by entries corresponding to the logarithmic coefficients of $g \in \mathcal{S}$ in the form (14), is expressed as

$$T_{k,n}(\gamma_g) = \begin{vmatrix} \gamma_n & \gamma_{n+1} & \cdots & \gamma_{k+n-1} \\ \gamma_{n+1} & \gamma_n & \cdots & \gamma_{k+n-2} \\ \vdots & \vdots & \vdots & \vdots \\ \gamma_{k+n-1} & \gamma_{k+n-2} & \cdots & \gamma_n \end{vmatrix}.$$

Consequently, specific expressions are obtained, such as

$$T_{2,1}(\gamma_g) = \gamma_1^2 - \gamma_2^2 \text{ and } T_{2,2}(\gamma_g) = \gamma_2^2 - \gamma_3^2. \tag{16}$$

Inspired by the aforementioned works and recognizing the importance of Toeplitz determinants and logarithmic coefficients, the structure of the rest of this paper is as follows: in Section 2, we establish sharp bounds for the Toeplitz determinants $T_{2,1}(\gamma_g)$ and $T_{2,2}(\gamma_g)$. These determinants are characterized by entries representing the logarithmic coefficients of analytic univalent functions, and we delve into the analysis of the well-known class of Sakaguchi functions within $\mathbb{U}$. To achieve this, we employ the Schwarz function, in accordance with the principle of subordination between analytic functions with respect to symmetric points. Significant advancements have been achieved in this regard, particularly in the context of Toeplitz determinants for univalent functions concerning symmetric points. The findings contribute to a broader understanding of logarithmic coefficients. Finally, Section 3 includes the most relevant concluding remarks from the present work and addresses future research directions.

2. Main Results

To establish our findings, the pivotal role will be played by the subsequent lemma applied to Schwarz functions.

Lemma 1 ([28]). *If $\omega \in \mathcal{H}$ be of the form (6), then*

$$|b_1| \leq 1,$$
$$|b_2| \leq 1 - |b_1|^2,$$
$$|b_3| \leq 1 - |b_1|^2 - \frac{|b_2|^2}{1+|b_1|}.$$

Theorem 1. *Let $g \in \mathcal{S}_s^*$ be given by (1). Then, the inequality*

$$\left|\gamma_1^2 - \gamma_2^2\right| \leq \frac{5}{16}, \tag{17}$$

holds true, and the bound is proven to be sharp.

Proof. Assuming $g \in \mathcal{S}_s^*$, it follows that we have $\omega \in \mathcal{H}$ in the form (6), such that

$$\frac{2zg'(z)}{g(z)-g(-z)} = \frac{1+\omega(z)}{1-\omega(z)}. \tag{18}$$

We begin by observing that, through the equalization of coefficients in (18), we derive

$$a_2 = b_1, \; a_3 = b_1^2 + b_2, \tag{19}$$

and

$$2a_4 = 2b_1^3 + 3b_1b_2 + b_3. \tag{20}$$

Substituting (19) into (15) and simplifying, we obtain

$$16\left(\gamma_1^2 - \gamma_2^2\right) = 4b_1^2 - 4b_2^2 - 4b_1^2b_2 - b_1^4. \tag{21}$$

Next, using Lemma 1 along with the triangle inequality, we obtain

$$16\left|\gamma_1^2 - \gamma_2^2\right| \leq |b_1|^4 + 4|b_1|^2 + 4|b_1|^2\left(1-|b_1|^2\right) + 4\left(1-|b_1|^2\right)^2. \tag{22}$$

Now, setting $\xi = |b_1|$ in (22) and simplifying, we arrive at

$$16\left|\gamma_1^2 - \gamma_2^2\right| \leq \xi^4 + 4 \leq 5 \text{ for } 0 \leq \xi \leq 1.$$

The equality case in (17) is realized by the function g_1, which is given as

$$\frac{2zg_1'(z)}{g_1(z)-g_1(-z)} = \frac{1-iz}{1+iz}.$$

It is clear that $g_1 \in \mathcal{S}_s^*$, and for this function, we find

$$\gamma_1 = -\frac{i}{2} \quad \text{and} \quad \gamma_2 = -\frac{1}{4},$$

which demonstrates the sharpness of the bound in (17). □

Theorem 2. *If g is defined in $\mathcal{S}_s^c$ according to (1), then the inequality*

$$\left|\gamma_1^2 - \gamma_2^2\right| \leq \frac{169}{2304}, \tag{23}$$

holds true, and the estimate is sharp.

Proof. Consider $g \in \mathcal{S}_s^c$ expressed as in (1). There exists $\omega \in \mathcal{H}$ represented by (6), such that

$$\frac{2[zg'(z)]'}{[g(z) - g(-z)]'} = \frac{1+\omega(z)}{1-\omega(z)}. \tag{24}$$

By equating coefficients in (24), we derive

$$2a_2 = b_1,\ 3a_3 = b_1^2 + b_2 \tag{25}$$

and

$$8a_4 = 2b_1^3 + 3b_1b_2 + b_3. \tag{26}$$

Using (15), we obtain

$$2304\left(\gamma_1^2 - \gamma_2^2\right) = 144b_1^2 - 64b_2^2 - 80b_1^2b_2 - 25b_1^4. \tag{27}$$

Applying Lemma 1 and the triangle inequality yields

$$2304\left|\gamma_1^2 - \gamma_2^2\right| \leq 25|b_1|^4 + 144|b_1|^2 + 80|b_1|^2\left(1 - |b_1|^2\right) + 64\left(1 - |b_1|^2\right)^2. \tag{28}$$

Setting $\tau = |b_1|$ in (28) and simplifying, we obtain

$$2304\left|\gamma_1^2 - \gamma_2^2\right| \leq 9\tau^4 + 96\tau^2 + 64 \leq 169 \text{ for } 0 \leq \tau \leq 1.$$

To establish the sharpness of the bound in (23), we consider the function g_2, which is defined by

$$\frac{2[zg_2'(z)]'}{[g_2(z) - g_2(-z)]'} = \frac{1+iz}{1-iz}.$$

Clearly, $g_2 \in \mathcal{S}_s^c$ and for this function, we have

$$\gamma_1 = \frac{i}{4} \quad \text{and} \quad \gamma_2 = -\frac{5}{48},$$

demonstrating the sharpness of the bound. □

Theorem 3. *Let $g \in \mathcal{S}_s^*$ be given by (1). Then, the inequality*

$$\left|\gamma_2^2 - \gamma_3^2\right| \leq \frac{37}{144}, \tag{29}$$

holds true, affirming the sharpness of the bound.

Proof. Given that $g \in \mathcal{S}_s^*$, we can derive the following expression from (15):

$$144\left(\gamma_2^2 - \gamma_3^2\right) = 9a_2^4 - 8a_2^6 + 24a_2^4 a_3 - 24a_2^3 a_4 - 36a_2^2 a_3 + 36a_3^2 - 36a_4^2 + 72a_2 a_3 a_4. \quad (30)$$

Further, by substituting the values of a_2, a_3 and a_4 from (19) and (20) into (30) for $g \in \mathcal{S}_s^*$, we obtain

$$144\left(\gamma_2^2 - \gamma_3^2\right) = 28b_1^6 + 9b_1^4 + 60b_1^4 b_2 - 12b_1^3 b_3 + 36b_1^2 b_2 + 27b_1^2 b_2^2 + 36b_2^2 - 9b_3^2 - 18b_1 b_2 b_3. \quad (31)$$

Applying the triangle inequality and Lemma 1 to Equation (31), we obtain the following inequality

$$\begin{aligned} 144\left|\gamma_2^2 - \gamma_3^2\right| \le\ & 28|b_1|^6 + 9|b_1|^4 + 60|b_1|^4|b_2| + 36|b_1|^2|b_2| + 27|b_1|^2|b_2|^2 + 36|b_2|^2 \\ & + \left(12|b_1|^3 + 18|b_1||b_2| + 9\left(1 - |b_1|^2 - \frac{|b_2|^2}{1+|b_1|}\right)\right)\left(1 - |b_1|^2 - \frac{|b_2|^2}{1+|b_1|}\right) \end{aligned} \quad (32)$$

where

$$0 \le |b_1| \le 1 \text{ and } 0 \le |b_2| \le 1 - |b_1|^2.$$

Now, by letting $\alpha = |b_1|$ and $\beta = |b_2|$ in (32), we express the inequality as follows:

$$144\left|\gamma_2^2 - \gamma_3^2\right| \le \Gamma(\alpha, \beta), \quad (33)$$

where Γ is defined by

$$\begin{aligned} \Gamma(\alpha, \beta) = {} & \alpha^4\left(28\alpha^2 + 9\right) + \alpha^2\beta\left(60\alpha^2 + 36\right) + \beta^2\left(27\alpha^2 + 36\right) \\ & + \alpha\left(12\alpha^2 + 18\beta\right)\left(1 - \alpha^2 - \frac{\beta^2}{1+\alpha}\right) + 9\left(1 - \alpha^2 - \frac{\beta^2}{1+\alpha}\right)^2, \end{aligned}$$

Considering Lemma 1, the admissible values for a pair (α, β) are consistent with the compact set defined by (see Figure 1)

$$\Delta = \left\{(\alpha, \beta) : 0 \le \alpha \le 1,\ 0 \le \beta \le 1 - \alpha^2\right\}. \quad (34)$$

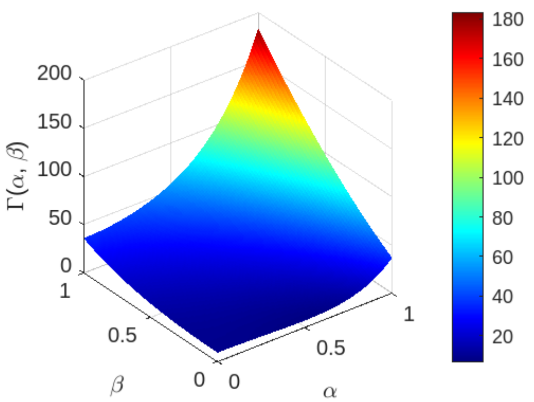

Figure 1. Graph of the region Δ and the function $\Gamma(\alpha, \beta)$ over Δ.

Since Γ is differentiable, the only places where Γ can assume these values are points inside Δ where $\frac{\partial \Gamma}{\partial \alpha} = \frac{\partial \Gamma}{\partial \beta} = 0$ and points on the boundary. Therefore, we show that $\max \Gamma(\alpha, \beta) \le 37$ for $0 \le \alpha \le 1$ and $\beta = 1 - \alpha^2$. All the critical points of Γ within Δ satisfy

$$\frac{\partial\Gamma}{\partial\alpha} = \alpha\beta\left(54\beta + 120\alpha^2\right) + 18x(y - 2\alpha^2 - \beta) + 12\alpha^3(14\alpha^2 + 3) + 2\alpha\beta(60\alpha^2 + 36) - \alpha(12\alpha^2 + 18\beta)y = 0,$$
$$\frac{\partial\Gamma}{\partial\beta} = 2\beta(27\alpha^2 + 36) - 18\alpha x + \alpha^2(60\alpha^2 + 36) + \frac{30\beta x}{\alpha+1} - \frac{2\alpha\beta(12\alpha^2 + 18\beta)}{\alpha+1} = 0,$$

where

$$x = \frac{\beta^2}{\alpha+1} + \alpha^2 - 1 \quad \text{and} \quad y = 2\alpha - \frac{\beta^2}{(\alpha+1)^2},$$

by a numerical computation are the following:

$$(-0.999853, -0.000056),\ (-0.999912, -0.000034),\ (-0.999669, -0.000126),\ (-0.999932, -0.000026),$$
$$(-0.99994, -0.000023),\ (-0.999738, -0.0001),\ (-0.999922, -0.00003),\ (-0.999861, -0.000053),$$
$$(-0.999812, -0.000071),\ (-0.724801, -0.358268),\ (-1.001259, 0.030112),\ (-0.999944, -0.000021),$$
$$(-1.192423, -0.566136),\ (-0.999819, -0.000068),\ (-0.999905, -0.000035),\ (0.87716, -0.565515),$$
$$(-0.999685, -0.000121),\ (-0.999837, -0.000063).$$

Thus, there are no solutions within the interior of Δ. Utilizing basic calculus techniques, we can establish that the maximum value of $\Gamma(\alpha, \beta)$ exists on the boundary of Δ. In particular, at the boundary defined by $\alpha = 0$ and $0 \le \beta \le 1$, the function is

$$\Gamma(0, \beta) = 9\beta^4 + 18\beta^2 + 9 \le 36.$$

Likewise, at the boundary where $\beta = 0$ and $0 \le \alpha \le 1$, the function is

$$\Gamma(\alpha, 0) = 28\alpha^6 - 12\alpha^5 + 18\alpha^4 - 18\alpha^2 + 12\alpha^3 + 9 \le 37.$$

Lastly, on the boundary curve defined by $\beta = 1 - \alpha^2$ and $0 \le \alpha \le 1$, the function becomes

$$\Gamma(\alpha, 1 - \alpha^2) = 10\alpha^6 - 27\alpha^4 + 18\alpha^2 + 36$$

with a maximum of 37. The amalgamation of these cases leads to

$$\max \Gamma(\alpha, \beta) = 37$$

and therefore, from (33), we obtain the desired result.

The sharpness of the bound specified in (29) is illustrated through the function g_3, which is defined by

$$\frac{2zg_3'(z)}{g_3(z) - g_3(-z)} = \frac{\sqrt{37} + 6z^2}{\sqrt{37} - 6z^2}.$$

It is evident that $g_3 \in \mathcal{S}_s^*$, and for this function, we observe that

$$\gamma_2 = \frac{\sqrt{37}}{12} \quad \text{and} \quad \gamma_3 = 0,$$

demonstrating the sharpness of the bound in (29). $\square$

Theorem 4. *If g is expressed in $\mathcal{S}_s^c$ according to (1), then the inequality*

$$\left|\gamma_2^2 - \gamma_3^2\right| \le \frac{1}{36}, \tag{35}$$

remains valid, and it has been established that this estimate is sharp.

Proof. In view of (25), (26) and (30), we have

$$2304\left(\gamma_2^2-\gamma_3^2\right)=6b_1^6+2b_1^4b_2+25b_1^4-18b_1^3b_3-9b_1^2b_2^2+80b_1^2b_2-30b_1b_2b_3+64b_2^2-9b_3^2. \quad (36)$$

Therefore, using Lemma 1, we obtain

$$\begin{aligned}2304\left|\gamma_2^2-\gamma_3^2\right| \le\ & 6|b_1|^6+2|b_1|^4|b_2|+25|b_1|^4+18|b_1|^3\left(\frac{|b_2|^2}{1+|b_1|}+|b_1|^2-1\right)+9|b_1|^2|b_2|^2+80|b_1|^2|b_2| \\ & +30|b_1||b_2|\left(\frac{|b_2|^2}{1+|b_1|}+|b_1|^2-1\right)+64|b_2|^2+9\left(\frac{|b_2|^2}{1+|b_1|}+|b_1|^2-1\right)^2.\end{aligned} \quad (37)$$

Consequently, letting $\mu=|b_1|$ and $\nu=|b_2|$ in (37) provides

$$2304\left|\gamma_2^2-\gamma_3^2\right|\le\Lambda(\mu,\nu) \quad (38)$$

where

$$\Lambda(\mu,\nu)=6\mu^6+2\mu^4\nu+25\mu^4-9\mu^2\nu^2+80\mu^2\nu+64\nu^2+\left(18\mu^3+30\mu\nu\right)\left(\frac{\nu^2}{1+\mu}+\mu^2-1\right)-9\left(\frac{\nu^2}{1+\mu}+\mu^2-1\right)^2.$$

In accordance with Lemma 1, the feasible region for the pair (μ,ν) aligns with Δ, which is defined by (34). Given the differentiability of Λ, its values are restricted to points inside Δ, where $\frac{\partial\Lambda}{\partial\mu}=\frac{\partial\Lambda}{\partial\nu}=0$, as well as points on the boundary. Hence, it is necessary to determine the maximum value of $\Lambda(\mu,\nu)$ within Δ (see Figure 2). The critical points of Λ adhere to the following conditions

$$\begin{aligned}\frac{\partial\Lambda}{\partial\mu}&=8\mu\nu(\mu^2+20)+(54\mu^2+30\nu)y-18xy-18\mu\nu^2+x\mu(18\mu^2+30\nu)+\mu^3(36\mu^2+100)=0,\\ \frac{\partial\Lambda}{\partial\nu}&=\nu(128-18\mu^2)+2\mu^2(40+\mu^2)+30\mu y-\frac{36\nu y}{\mu+1}+\frac{2\nu\mu(18\mu^2+30\nu)}{\mu+1}=0,\end{aligned}$$

where

$$x=2\mu-\frac{\nu^2}{(\mu+1)^2}\quad\text{and}\quad y=\frac{\nu^2}{\mu+1}+\mu^2-1.$$

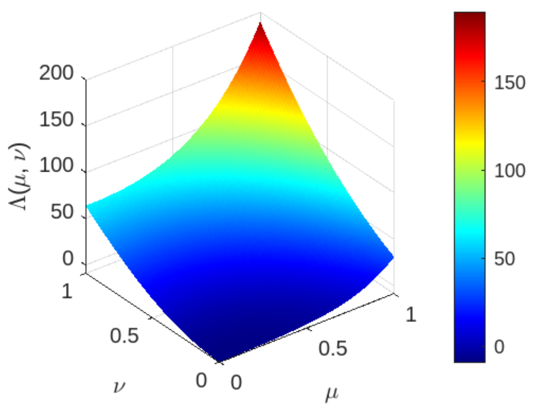

Figure 2. Graph of $\Lambda(\mu,\nu)$ within the region $0\le\mu\le1$ and $0\le\nu\le\mu^2$.

The system of two equations described above does not possess a solution within the interior of Δ. Consequently, the function $\Lambda(\mu,\nu)$ cannot attain a maximum within the interior of Δ. Due to the continuity of Λ on Δ, its maximum value is realized on the boundary of Δ. As a result, we obtain

$$\begin{aligned}\Lambda(\mu,0) &= 6\mu^6+18\mu^5+16\mu^4-18\mu^3+18\mu^2-9 \le 31, \quad \text{for} \quad \mu\in[0,1],\\ \Lambda(0,\nu) &= -9\nu^4+82\nu^2-9\le 64, \quad \text{for} \quad \nu\in[0,1],\\ \Lambda(\mu,1-\mu^2) &= -26\mu^6+89\mu^4-96\mu^2+64\le 64, \quad \text{for} \quad \mu\in[0,1].\end{aligned}$$

Combining the considerations from the above-discussed cases yields

$$\max\Lambda(\mu,\nu)=64.$$

Consequently, from (38), the desired result is obtained.

To demonstrate the sharpness of the bound in (35), we examine the function g_4, which is defined as

$$\frac{2\left[zg_4'(z)\right]'}{[g_4(z)-g_4(-z)]'}=\frac{1+z^2}{1-z^2}.$$

Evidently, $g_4\in\mathcal{S}_s^c$, and for this function, we find

$$\gamma_2=\frac{1}{6} \quad \text{and} \quad \gamma_3=0,$$

indicating the sharpness of the bound. □

3. Concluding Remarks and Observations

The primary focus of exploring coefficient problems in various categories of analytic functions, whether multivalent or univalent, revolves around expressing the coefficients of functions within a particular class using the coefficients of related functions that exhibit a positive real part. This approach allows coefficient functionals to be analyzed by applying established inequalities for the class $\mathcal{S}$. The investigation in this paper has extensively delved into Toeplitz determinants featuring logarithmic coefficients for symmetric points in convex and starlike functions' associated bilinear transformation, resulting in the establishment of sharp bounds.

The importance of logarithmic coefficients enhances the appeal of the proposed problem, making it a topic worthy of consideration and interest. Nevertheless, there has been insufficient work conducted to establish sharp bounds for Toeplitz determinants with logarithmic coefficients for convex, starlike, and their associated subclasses. So, it becomes possible to investigate the bounds of $T_{2,3}(\gamma_g)$ and $T_{3,2}(\gamma_g)$ for a given class and its associate subclasses, defined as

$$T_{2,3}(\gamma_g)=\begin{vmatrix}\gamma_3 & \gamma_4\\ \gamma_4 & \gamma_3\end{vmatrix} \quad \text{and} \quad T_{3,2}(\gamma_g)=\begin{vmatrix}\gamma_2 & \gamma_3 & \gamma_4\\ \gamma_3 & \gamma_2 & \gamma_3\\ \gamma_4 & \gamma_3 & \gamma_2\end{vmatrix}$$

where γ_2 and γ_3 are given by (15) and $8\gamma_4=4a_5-4a_4a_2+4a_3a_2^2-2a_3^2-a_2^4$.

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