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Inference for Compound Exponential XLindley Model with Applications to Lifetime Data

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Abstract: The creating of novel models essentially stems from the requirement to appropriate describe survival cases. In this study, a novel lifetime model with two parameters is proposed and studied for modeling more types of data used in different study cases, including symmetric, asymmetric, skewed, and complex datasets. The proposed model is obtained by compounding the exponential and XLindley distributions, and it is regarded as a strong competitor for the widely applied symmetrical and non-symmetrical models. Several characteristics and statistical properties are investigated. The unknown parameters of the recommended model for the complete sample are estimated using two estimation methods; notably, maximum likelihood estimation and Bayes techniques based on several loss functions as well as an approximate tool are used to construct the confidence intervals for the unknown parameters of the suggested model. The estimation procedures are compared using a Monte Carlo simulation experiment to demonstrate their effectiveness. In the end, the applicability and flexibility of the recommended model are conducted using two real lifetime datasets. In our illustration, we compare the practicality of the recommended model with several well-known competing distributions.

Keywords: Bayes techniques; confidence interval; lifetime model; maximum likelihood estimation; simulation experiments; XLindley distribution

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1. Introduction

Lifetime models are common statistical procedures which used in fitting and modeling survival events for numerous descriptions of lifetime datasets, particularly engineering and survival sciences. For fitting several kinds of data, many multi-parameter distributions are considered in the statistical literature in the statistical literature. In the last decades, various generated families of lifetime distributions have been introduced to model many datasets. However, a classical distribution is not appropriate to fit such sophisticated data. For this reason, the authors are motivated to obtain a novel extension of the existing distributions using numerous techniques, including adding new parameters by generalizing the distribution or mixing two or more classical distributions. These new statistical models provide greater flexibility in modeling for various applications such as engineering, biomedicine, actuarial science, medicine, insurance, and environmental fields. In this context, Chouia and Zeghdoudi [1] introduced a new extension of Lindley distribution named the XLindley (XL) model. It is one way to describe the lifetime of a process or device, and it can be applied in several areas of study, such as medical science, lifetime, insurance, and hydrology. It can be considered a more efficient model than symmetrical models, notably normal distribution. A random variable (RV) Y_1 is said to have XL distribution if its probability density function (pdf) and survival function (sf) can be expressed, respectively, as follows:

$$g(y_1) = \frac{\theta^2 e^{-\theta y_1}(\theta + y_1 + 2)}{(\theta+1)^2}, \ \ y_1 > 0,\ \theta > 0,$$

and

$$S_1(y_1) = \left(\frac{\theta y_1}{(\theta+1)^2} + 1\right) e^{-\theta y_1}.$$

In the last few decades, several researchers have given special attention to the XL distribution due to its importance in fitting skewed, asymmetric, complex, and lifetime datasets. For example, Fatima et al. [2] provided certain properties of the Poisson Quasi XLindley distribution, and they demonstrated that it is more efficient and works better in analyzing lifetime datasets than other well-known models. Beghriche et al. [3] proposed the inverse XLindley model by applying the inverse method, which is more appropriate in modeling mortality studies. The exponentiated XLindley model was defined by Alomair et al. [4], who established numerous mathematical properties concerning the new distribution. A new flexible generalized XLindley model was considered by Musekwa et al. [5]. Gemeay et al. [6] established the modified XLindley distribution and investigated various tools for estimating the parameters.

In the context of distribution theory, the compound method is one of the most popular choices for fitting several types of datasets, such as skewed and lifetime data. It has been used in numerous domains of studies including economic, biology, actuarial, and environmental (see Abdelghani et al. [7], Meraou et al. [8–11], and Jafari and Tahmasebi [10]). The compound distributions are defined as the minimum or maximum of M independent and identically distributed (i.i.d) RVs. Several authors applied this technique in their works, for example, one may refer to Mahmoudi and Jafari [12] who introduced generalized exponential–power series models by compounding generalized exponential and power series distributions. The inverted Nadarajah–Haghighi power series distributions are considered by Ahsan-ul-Haq et al. [13]. In the same way, the exponential Poisson model was introduced by Cancho et al. [14], and Yousef et al. [15] defined the unit Gompertz power series distribution and estimated the model parameter using the ranked set sampling method. It is worth motioning that the exponential (Exp) model has received considerable attention in the literature. It is efficiency in analyzing engineering, finance, and climatology phenomena. Further, The Exp model can be extensively implemented to fit the failure times of components and systems. Numerous authors applied the Exp model in numerous applications. A RV Y_2 follows the Exp distribution if its pdf and sf can be formulated by

$$h(y_2) = \beta\, e^{-\beta y_2}, \ \ y_2 > 0, \beta > 0,$$

and

$$S_2(y_2) = \ e^{-\beta y_2}.$$

Despite these advancements, when there are different kinds of datasets in survival and lifetime, many existing methods lack flexibility and may not provide the best fit. To overcome this challenge, we defined a novel distribution named the Compound Exponential XLindley (CEXL) model, and it can be used in different areas including lifetime and engineering fields. This proposed model has two parameters and is obtained by compounding the Exp and XL distributions. Let us consider the RVs Y_1 and Y_2 that are i.i.d, and assume that $X = \min(Y_1, Y_2)$. The sf of the random variable of X is

$$S(x) = S_1(x)S_2(x) \tag{1}$$

Additionally, another objective of this study is to explore estimating the CEXL model parameters using two traditional estimation methods, such as maximum likelihood estimation, and Bayesian methods under the square error loss function. For more information about lifetime analysis and Bayesian inference, one may cite the works of Xu et al. [16], Xu et al. [17], Wang et al. [18], Muqrin [19], and Wu and Gui [20]. Also, we construct the confidence intervals for model parameters using the approximate of the MLE method.

The remaining part of the current study is structured as given: The suggested compound model is developed and studied in Section 2. The underlying characteristics of the CEXL distribution such as k-th moment, moment generating function, distribution of order statistics, and certain entropy measures are investigated in Section 3. In Section 4, we provide two estimation procedures for estimating the model parameters. We conduct some numerical simulation experiments in Section 5 to observe the effectiveness of the proposed MLE and Bayes methods. Finally, in Section 6, two lifetime datasets are analyzed for validation purposes. Finally concluding remarking has been obtained for this study In Section 7.

2. Compound Exponential XLindley Model

A continuous RV X is said to follow the proposed CEXL model with parameters θ and β if its cumulative distribution function (cdf) and pdf are expressed, respectively, as follows:

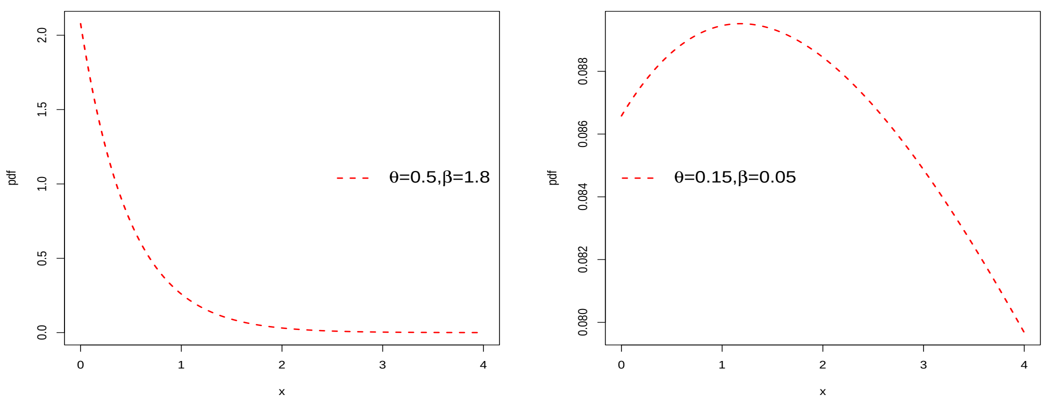
$$F(x) = 1 - \left(\frac{\theta x}{(\theta+1)^2} + 1\right) e^{-(\theta+\beta)x}, \; x > 0, \; \theta, \beta > 0, \tag{2}$$

and

$$f(x) = \frac{1}{(\theta+1)^2}\left(\theta(\theta+\beta)x + (\theta+1)^2(\theta+\beta) - \theta\right) e^{-(\theta+\beta)x}. \tag{3}$$

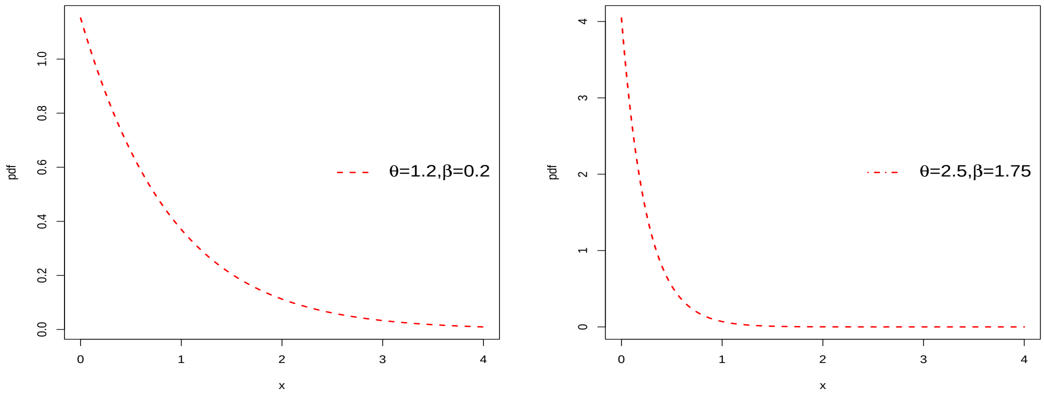
From now on, we assume that $X \sim \text{CEXL}(\theta, \beta)$.

It is evident that the proposed CEXL model contains a sub model as a special case. If θ tends to be 0, the recommended CEXL reduces to an Exp distribution; when β approaches 0, we have an XL distribution. Figure 1 demonstrates the graphs for the pdf of the proposed model given in Equation (3) using several parameters recors. It is highly positively skewed and uni-modal as well, as it is good for modeling skewed datasets.



(a) (0.5,1.8) **(b)** (0.15,0.05)

Figure 1. *Cont.*



(c) (1.2,0.2) **(d)** (2.5,1.75)

Figure 1. Possible pdf shapes of the CEXL model.

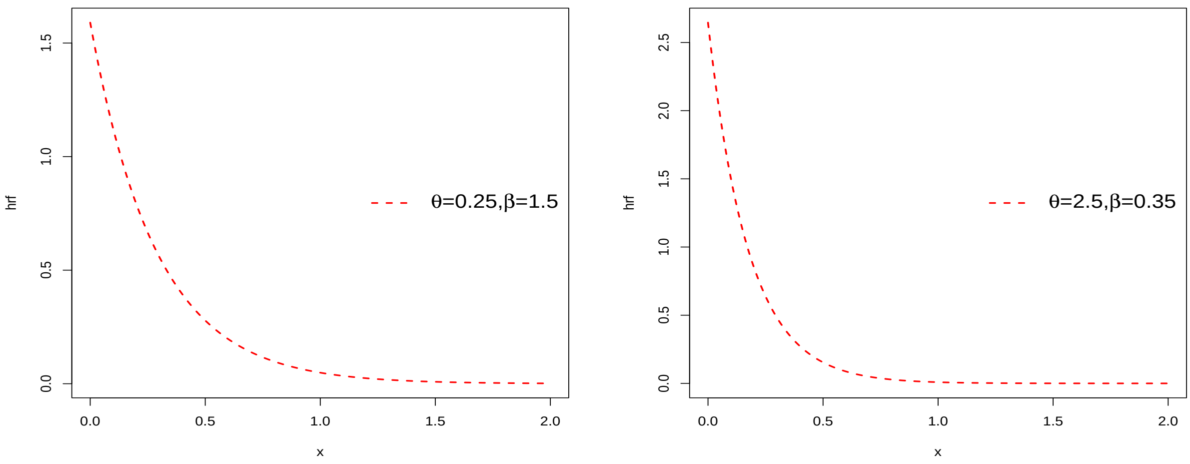
Henceforth, the sf and hazard rate function (hrf) of the RV X are

$$S(x) = \left(\frac{\theta x}{(\theta+1)^2} + 1 \right) e^{-(\theta+\beta)x},$$

and

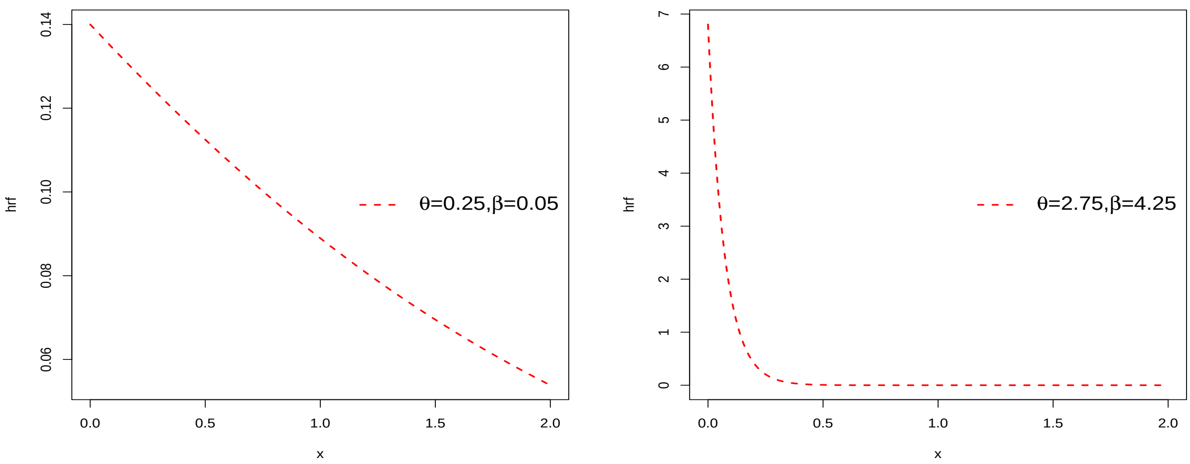
$$h(x) = \frac{1}{(\theta+1)^2} \left(\frac{\theta x}{(\theta+1)^2} + 1 \right)^{-1} \Big(\theta(\theta+\beta)x + (\theta+1)^2(\theta+\beta) - \theta \Big). \tag{4}$$

From the hrf of the CEXL model, $h(0) = \theta + \beta - \dfrac{\theta}{(\theta+1)^2}$ and $h(\infty) = 0$. The graphs for the hrf of the proposed model given in Equation (4) are demonstrated in Figure 2 for different parameter values of θ and β. Clearly, for all parameter values of θ and β, our CEXL distribution has a decreasing hrf, which confirms the flexibility of the recommended model.



(a) (0.25,1.5) **(b)** (2.5,0.35)

Figure 2. *Cont.*



(c) (0.25,0.05) **(d)** (2.75,4.25)

Figure 2. Possible hrf shapes of the CEXL model.

Similarly, the cumulative hazard rate function $H(x)$ and inverse hazard rate function $R(x)$ of the RV X are

$$H(x) = (\theta+\beta)x - \log\left(\frac{\theta x}{(\theta+1)^2} + 1\right),$$

and

$$R(x) = \frac{\frac{1}{(\theta+1)^2}\left(\theta(\theta+\beta)x + (\theta+1)^2(\theta+\beta) - \theta\right) e^{-(\theta+\beta)x}}{1 - \left(\frac{\theta x}{(\theta+1)^2} + 1\right) e^{-(\theta+\beta)x}}.$$

The Odds function of the proposed CEXL model can be defined as the ratio of the cdf and sf. It verifies the non-monotone hrf and can be written as

$$O(x) = \frac{(1+\theta)^2 - (\theta x + (1+\theta)^2)e^{-(\theta+\beta)x}}{(\theta x + (1+\theta)^2)e^{-(\theta+\beta)x}}.$$

3. The Characteristics of the CEXL Model

This section introduces several statistical properties of the proposed CEXL model—notably, the k-th moment, mean, variance, moment generating function, characteristic function, distribution of order statistics, and some entropy measures—because of its importance in distribution theory.

3.1. Moments with Related Measures

Let the RV X have the CEXL model. The proposed expression k-th moment of X is given below:

$$\mu'_k = \frac{\theta\Gamma(k+2)}{(\theta+1)^2(\theta+\beta)^{k+1}} + \frac{\Gamma(k+1)}{(\theta+\beta)^k} - \frac{\theta\Gamma(k+1)}{(\theta+1)^2(\theta+\beta)^{k+1}}, \tag{5}$$

where $\Gamma(n) = (n-1)!$ for $n = 1, 2; \ldots$.

Proof. The expression of k moment of X can be defined as

$$\begin{aligned}
\mu'_k &= \int_0^\infty x^k\, f(x)\, dx \\
&= \frac{1}{(\theta+1)^2}\int_0^\infty x^k \Big[\theta(\theta+\beta)x + (\theta+1)^2(\theta+\beta) - \theta\Big]\, e^{-(\theta+\beta)x}\, dx \\
&= \frac{1}{(\theta+1)^2}\bigg\{\theta(\theta+\beta)\int_0^\infty x^{k+1}e^{-(\theta+\beta)x}\, dx + (\theta+1)^2(\theta+\beta)\int_0^\infty x^k e^{-(\theta+\beta)x}\, dx \\
&- \theta\int_0^\infty x^k e^{-(\theta+\beta)x}\, dx\bigg\} \\
&= \frac{1}{(\theta+1)^2}\bigg\{\theta(\theta+\beta)\frac{\Gamma(k+2)}{(\theta+\beta)^{k+2}} + (\theta+1)^2(\theta+\beta)\frac{\Gamma(k+1)}{(\theta+\beta)^{k+1}} - \theta\frac{\Gamma(k+1)}{(\theta+\beta)^{k+1}}\bigg\} \\
&= \frac{\theta\Gamma(k+2)}{(\theta+1)^2(\theta+\beta)^{k+1}} + \frac{\Gamma(k+1)}{(\theta+\beta)^k} - \frac{\theta\Gamma(k+1)}{(\theta+1)^2(\theta+\beta)^{k+1}}.
\end{aligned}$$

□

Henceforth, the first and second moment of X come out to be

$$\mu'_1 = \frac{1}{\theta+\beta}\left(1 + \frac{\theta}{(\theta+1)^2(\theta+\beta)}\right),$$

and

$$\mu'_2 = \frac{2}{(\theta+\beta)^2}\left(1 + \frac{2\theta}{(\theta+1)^2(\theta+\beta)}\right).$$

The variance and coefficient of variation (CV) of X are

$$V(X) = \frac{1}{(\theta+\beta)^2}\left(2 + \frac{4\theta}{(\theta+1)^2(\theta+\beta)} - \left[1 + \frac{\theta}{(\theta+1)^2(\theta+\beta)}\right]^2\right),$$

and

$$CV = \frac{\left\{2 + \frac{4\theta}{(\theta+1)^2(\theta+\beta)} - \left[1 + \frac{\theta}{(\theta+1)^2(\theta+\beta)}\right]^2\right\}^{1/2}}{1 + \frac{\theta}{(\theta+1)^2(\theta+\beta)}}.$$

At the end, the coefficients for skewness ($\mathcal{S}$) and the kurtosis ($\mathcal{K}$) of the RV X are

$$\mathcal{S} = \frac{\mu_3 - 3\mu_2 + 2\mu_1^3}{(\mu_2 - \mu_1^2)^{3/2}},$$

and

$$\mathcal{K} = \frac{\mu_4 - 4\mu_3 + 6\mu_1^2\mu_2 - 3\mu_1^4}{(\mu_2 - \mu_1^2)^2}.$$

Now, the moment generating function (mgf) and characteristic function (cf) of X are given, respectively, below:

$$M(t) = \int_0^\infty e^{tx}\, f(x)\, dx = \sum_{m=0}^\infty \frac{t^m}{m!}\, \frac{\theta\Gamma(k+2)}{(\theta+1)^2(\theta+\beta)^{k+1}} + \frac{\Gamma(k+1)}{(\theta+\beta)^k} - \frac{\theta\Gamma(k+1)}{(\theta+1)^2(\theta+\beta)^{k+1}},$$

and

$$\phi_i(t) = \int_0^\infty e^{itx}\, f(x)\, dx = \sum_{i=0}^\infty \frac{(it)^m}{m!}\, \frac{\theta\Gamma(k+2)}{(\theta+1)^2(\theta+\beta)^{k+1}} + \frac{\Gamma(k+1)}{(\theta+\beta)^k} - \frac{\theta\Gamma(k+1)}{(\theta+1)^2(\theta+\beta)^{k+1}}.$$

Table 1. Possible statistical properties of the CEXL model for several parameter values.

β		Mean	V	CV	$\mathcal{S}$	$\mathcal{K}$
$\theta = 0.2$	0.3	2.5534	5.9164	0.9526	1.8088	4.7294
	0.6	1.4664	2.0630	0.9795	1.9076	5.3646
	0.9	1.0235	1.0240	0.9887	1.9419	5.5624
	1.2	0.7849	0.6073	0.9929	1.9597	5.6942
$\theta = 0.4$	0.3	1.8437	3.0561	0.9482	1.7934	4.6561
	0.6	1.2026	1.3645	0.9713	1.8759	5.1578
	0.9	0.8889	0.7613	0.9816	1.9149	5.4018
	1.2	0.7039	0.4825	0.9869	1.9342	5.5075
$\theta = 0.6$	0.3	1.4001	1.7902	0.9556	1.8165	4.8211
	0.6	0.9956	0.9372	0.9723	1.8785	5.1826
	0.9	0.7705	0.5715	0.9812	1.9146	5.4292
	1.2	0.6276	0.3833	0.9864	1.9344	5.5485
$\theta = 0.8$	0.3	1.1135	1.1599	0.9672	1.8649	5.1215
	0.6	0.8406	0.6769	0.9787	1.9121	5.4423
	0.9	0.6741	0.4406	0.9847	1.9336	5.5846
	1.2	0.5622	0.3090	0.9889	1.9493	5.6663
$\theta = 1$	0.3	0.9178	0.7967	0.9725	1.8779	5.1348
	0.6	0.7235	0.5038	0.9811	1.9113	5.3603
	0.9	0.5963	0.3457	0.9861	1.9309	5.4854
	1.2	0.5069	0.2514	0.9893	1.9445	5.5754
$\theta = 1.2$	0.3	0.7764	0.5780	0.9793	1.9180	5.4884
	0.6	0.6316	0.3867	0.9847	1.9463	5.7368
	0.9	0.5321	0.2764	0.9880	1.9542	5.775
	1.2	0.4596	0.2070	0.9899	1.9554	5.7525

The numerical results of numerous statistical measures, as discussed previously, of the proposed CEXL model using specific parameter values are summarized in Table 1. From these values, it can be deduced that our CEXL distribution is more efficient for explaining more datasets.

3.2. Order Statistics

We draw a random sample of size n $X_1, X_2, \ldots, X_n$ from the CEXL model and $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ represent its order statistics. The pdf of the j-th order statistic $X_{(j)}$ is expressed as follows:

$$d_{(j)}(x) = \frac{n!}{(j-1)!(n-j)!} f(x)[F(x)]^{j-1}[1-F(x)]^{n-j} \tag{6}$$

$$= \frac{n!}{(j-1)!(n-j)!(1+\theta)^2)} \left\{\theta(\theta+\beta)x + (\theta+1)^2(\theta+\beta) - \theta\right\} e^{-(\theta+\beta)x} \tag{7}$$

$$\times \left\{1 - \left(\frac{\theta x}{(\theta+1)^2} + 1\right) e^{-(\theta+\beta)x}\right\}^{j-1} \left\{\left(\frac{\theta x}{(\theta+1)^2} + 1\right) e^{-(\theta+\beta)x}\right\}^{n-j}. \tag{8}$$

The associated cdf of $X_{(j)}$ is

$$\begin{aligned} D_{(j)}(x) &= \sum_{i=j}^{n} F^i(x)[1-F(z)]^{n-i} \\ &= \sum_{i=j}^{n}\sum_{k=0}^{n-i} \binom{n}{i}\binom{n-i}{k}(-1)^i F^{k+i}(x) \\ &= \sum_{i=j}^{n}\sum_{k=0}^{n-i} \binom{n}{i}\binom{n-i}{k}(-1)^i \left\{1 - \left(\frac{\theta x}{(\theta+1)^2} + 1\right) e^{-(\theta+\beta)x}\right\}^{k+i}. \end{aligned}$$

From Equation (6), the probability distribution of maximum $X_{(n)} = \max\{x_1, x_2, \ldots, x_n\}$ and minimum $X_{(1)} = \min\{x_1, x_2, \ldots, x_n\}$ are obtained by setting $j = n$ and $j = 1$, respectively, and they are given as

$$\begin{aligned} d_{(n)}(x) &= \frac{n}{(1+\theta)^2)} \Big\{\theta(\theta+\beta)x + (\theta+1)^2(\theta+\beta) - \theta\Big\} \, e^{-(\theta+\beta)x} \\ &\times \left\{1 - \left(\frac{\theta x}{(\theta+1)^2} + 1\right) e^{-(\theta+\beta)x}\right\}^{n-1}, \end{aligned}$$

and

$$\begin{aligned} d_{(1)}(x) &= \frac{n}{(1+\theta)^2)} \Big\{\theta(\theta+\beta)x + (\theta+1)^2(\theta+\beta) - \theta\Big\} \, e^{-(\theta+\beta)x} \\ &\times \left\{\left(\frac{\theta x}{(\theta+1)^2} + 1\right) e^{-(\theta+\beta)x}\right\}^{n-1}. \end{aligned}$$

3.3. *Information Measure of the CEXL Model*

Here, we discuss several entropy's such as Rényi, Shannon, Havrda and Charvat, Tsallis, Arimoto, and Mathai–Haubold. The proposed entropy measures have a key role in information amounts.

In information theory, Renyi entropy [21] $\varphi_1(\gamma)$ is an important measure, and it is defined as

$$\begin{aligned} \varphi_1(\gamma) &= \frac{1}{1-\gamma} \log\left(\int_0^\infty f^\gamma(x)\,dx\right) \quad \gamma \neq 1, \gamma > 0. \\ &= \frac{1}{1-\gamma} \log\left(\int_0^\infty \left[\frac{1}{(\theta+1)^2}\Big(\theta(\theta+\beta)x + (\theta+1)^2(\theta+\beta) - \theta\Big)\, e^{-(\theta+\beta)x}\right]^\gamma dx\right) \\ &= \frac{1}{1-\gamma} \log\left(\frac{1}{(\theta+1)^{2\gamma}}\, \Psi_{\gamma,\theta,\beta}(x)\right), \end{aligned}$$

with $\Psi_{\gamma,\theta,\beta}(x) = \int_0^\infty \Big[\theta(\theta+\beta)x + (\theta+1)^2(\theta+\beta) - \theta\Big]^\gamma e^{-\gamma(\theta+\beta)x}\,dx.$

Shannon's entropy [22] φ_2 is defined as

$$\begin{aligned} \varphi_2 &= \mathrm{E}(-\log f(x)) \\ &= \mathrm{E}\left(-\log\left[\frac{1}{(\theta+1)^2}\Big(\theta(\theta+\beta)x + (\theta+1)^2(\theta+\beta) - \theta\Big)\, e^{-(\theta+\beta)x}\right]\right) \\ &= 2\log(\theta+1) - \mathrm{E}\Big(\log[\theta(\theta+\beta)x + (\theta+1)^2(\theta+\beta) - \theta]\Big) + \mathrm{E}((\theta+\beta)x) \end{aligned}$$

Further, another uncertainty information measure is Havrda and Charvat entropy [23], $\varphi_3(\gamma)$, and it is expressed as

$$\begin{aligned} \varphi_3(\gamma) &= \frac{1}{2^{1-\gamma}-1}\left[\int_0^\infty (f^\gamma(x)\,dx)^{1/\gamma} - 1\right] \quad \gamma \neq 1, \gamma > 0. \\ &= \frac{1}{2^{1-\gamma}-1}\left[\frac{1}{(\theta+1)^2}\, \Delta_{\gamma,\theta,\beta}(x) - 1\right], \end{aligned}$$

with $\Delta_{\gamma,\theta,\beta}(x) = \int_0^\infty \left(\Big[\theta(\theta+\beta)x + (\theta+1)^2(\theta+\beta) - \theta\Big]^\gamma e^{-\gamma(\theta+\beta)x}\,dx\right)^{\frac{1}{\gamma}}.$

Using the proposed distribution, the Tsallis entropy [24] $\varphi_4(\gamma)$ is defined as

$$\begin{aligned}\varphi_4(\gamma) &= \frac{1}{1-\gamma}\left(1-\int_0^\infty f^\gamma(x)\,dx\right) \quad \gamma \neq 1, \gamma > 0.\\ &= \frac{1}{1-\gamma}\left(1-\frac{1}{(\theta+1)^{2\gamma}}\,\Psi_{\gamma,\theta,\beta}(x)\right).\end{aligned}$$

Next, we consider the Arimoto entropy [25] $\varphi_5(\gamma)$ of the recommended model, which is

$$\begin{aligned}\varphi_5(\gamma) &= \frac{\gamma}{\gamma-1}\left[\int_0^\infty (f^\gamma(x)\,dx)^{1/\gamma}-1\right] \quad \gamma \neq 1, \gamma > 0.\\ &= \frac{\gamma}{\gamma-1}\left[\frac{1}{(\theta+1)^2}\,\Delta_{\gamma,\theta,\beta}(x)-1\right].\end{aligned}$$

Finally, a new extension entropy measure named the Mathai–Haubold entropy [26] $\varphi_6(\gamma)$ is provided in this subsection. It is written as

$$\begin{aligned}\varphi_6(\gamma) &= \frac{1}{\gamma-1}\left(\int_0^\infty f^{2-\gamma}(x)\,dx-1\right) \quad \gamma \neq 1, \gamma > 0.\\ &= \frac{1}{\gamma-1}\left(\int_0^\infty\left[\frac{1}{(\theta+1)^2}\Big(\theta(\theta+\beta)x+(\theta+1)^2(\theta+\beta)-\theta\Big)\,e^{-(\theta+\beta)x}\,dx\right]^{2-\gamma}-1\right)\\ &= \frac{1}{\gamma-1}\left(\frac{1}{(\theta+1)^{2(2-\gamma)}}\,\Phi_{\gamma,\theta,\beta}(x)-1\right),\end{aligned}$$

with $\Phi_{\gamma,\theta,\beta}(x) = \left(\int_0^\infty \Big[\theta(\theta+\beta)x+(\theta+1)^2(\theta+\beta)-\theta\Big]\,e^{-(\theta+\beta)x}\,dx\right)^{2-\gamma}$.

Tables 2 and 3 report certain numerical values of the proposed entropy measures of the CEXL distribution by applying numerous parameter values of θ and β. Also, the 3D curves of these measures are sketched in Figures 3 and 4.

Table 2. Different numerical records of proposed entropy measures at $\gamma = 1.5$.

	θ	$\varphi_1(\gamma)$	φ_2	$\varphi_3(\gamma)$	$\varphi_4(\gamma)$	$\varphi_5(\gamma)$	$\varphi_6(\gamma)$
$\beta = 0.5$	0.25	1.3049	1.4802	1.6362	0.9584	1.0581	−0.5567
	0.5	1.0258	1.1998	1.3699	0.8025	0.8688	−0.4524
	0.75	0.7793	0.9554	1.1018	0.6454	0.6863	−0.3541
	1	0.5706	0.7487	0.8475	0.4965	0.5197	−0.2659
$\beta = 1$	0.25	0.71610	0.8973	1.0275	0.6019	0.6370	−0.3278
	0.5	0.5534	0.7328	0.8252	0.4834	0.5053	−0.2584
	0.75	0.3921	0.5719	0.6079	0.3561	0.3676	−0.1868
	1	0.2452	0.4255	0.3939	0.2307	0.2354	−0.1189
$\beta = 1.5$	0.25	0.3460	0.5286	0.5424	0.3177	0.3268	−0.1657
	0.5	0.2320	0.4132	0.3739	0.2190	0.2232	−0.1127
	0.75	0.1127	0.2937	0.1871	0.1096	0.1106	−0.0556
	1	0.0215	0.2027	0.0366	0.0214	0.0215	−0.0107

Table 3. Different numerical records of proposed entropy measures at $\gamma = 3$.

	θ	$\varphi_1(\gamma)$	φ_2	$\varphi_3(\gamma)$	$\varphi_4(\gamma)$	$\varphi_5(\gamma)$	$\varphi_6(\gamma)$
$\beta = 0.5$	0.25	1.0572	1.4802	1.1724	0.4396	0.7587	3.6422
	0.5	0.7791	1.1998	1.0526	0.3947	0.6077	1.875
	0.75	0.5288	0.9554	0.8702	0.3263	0.4456	0.9396
	1	0.3163	0.7487	0.6250	0.2344	0.2852	0.4412
$\beta = 1$	0.25	0.4582	0.8973	0.8001	0.3000	0.3948	0.7502
	0.5	0.2971	0.7328	0.5973	0.224	0.2695	0.4057
	0.75	0.1347	0.5719	0.3149	0.1181	0.1289	0.1546
	1	−0.0139	0.4255	−0.0376	−0.0141	−0.0140	−0.0137
$\beta = 1.5$	0.25	0.0848	0.5286	0.2080	0.0780	0.0824	0.0924
	0.5	−0.0282	0.4132	−0.0774	−0.0290	−0.0285	−0.0274
	0.75	−0.1480	0.2937	−0.4593	−0.1722	−0.1555	−0.1281
	1	−0.2621	0.1808	−0.9188	−0.3446	−0.2864	−0.2040

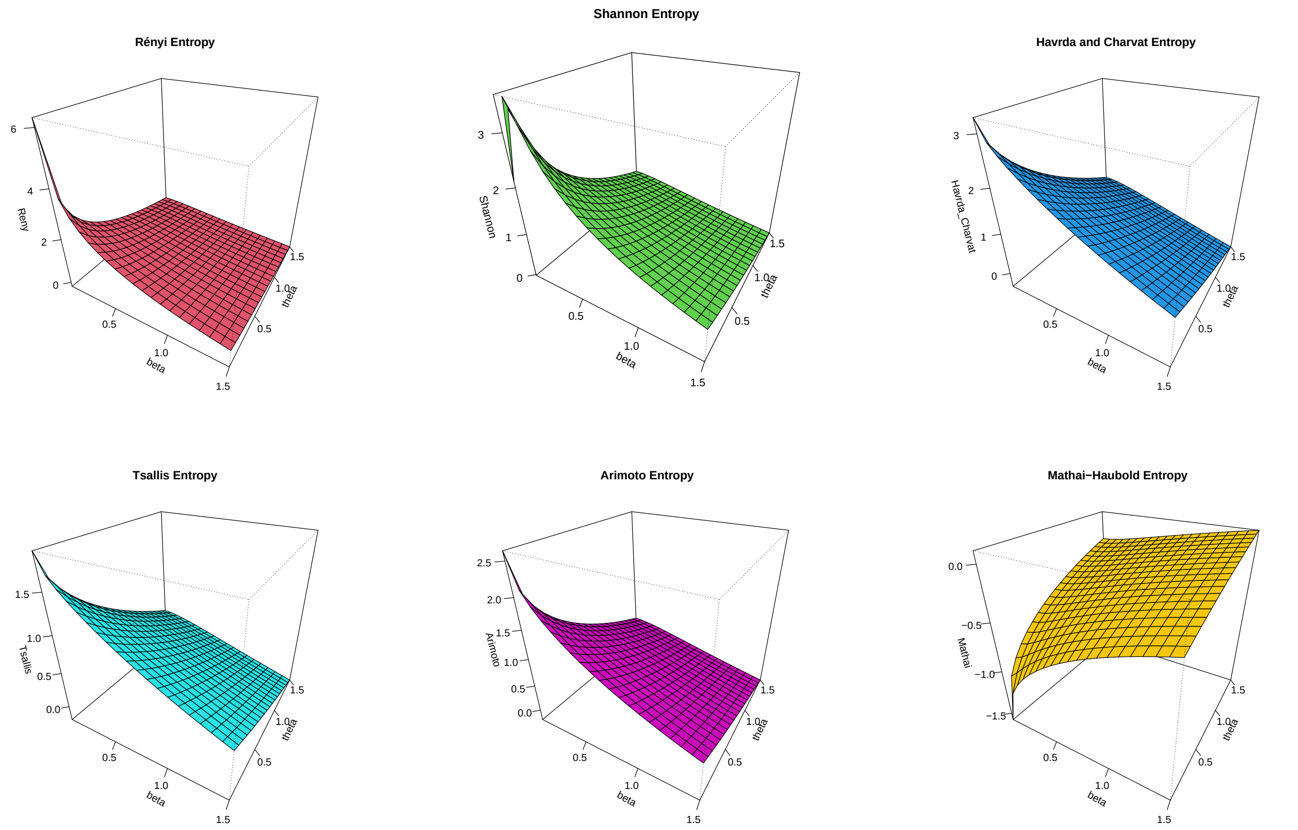


Figure 3. 3D curves of proposed entropy measures at $\gamma = 1.5$.

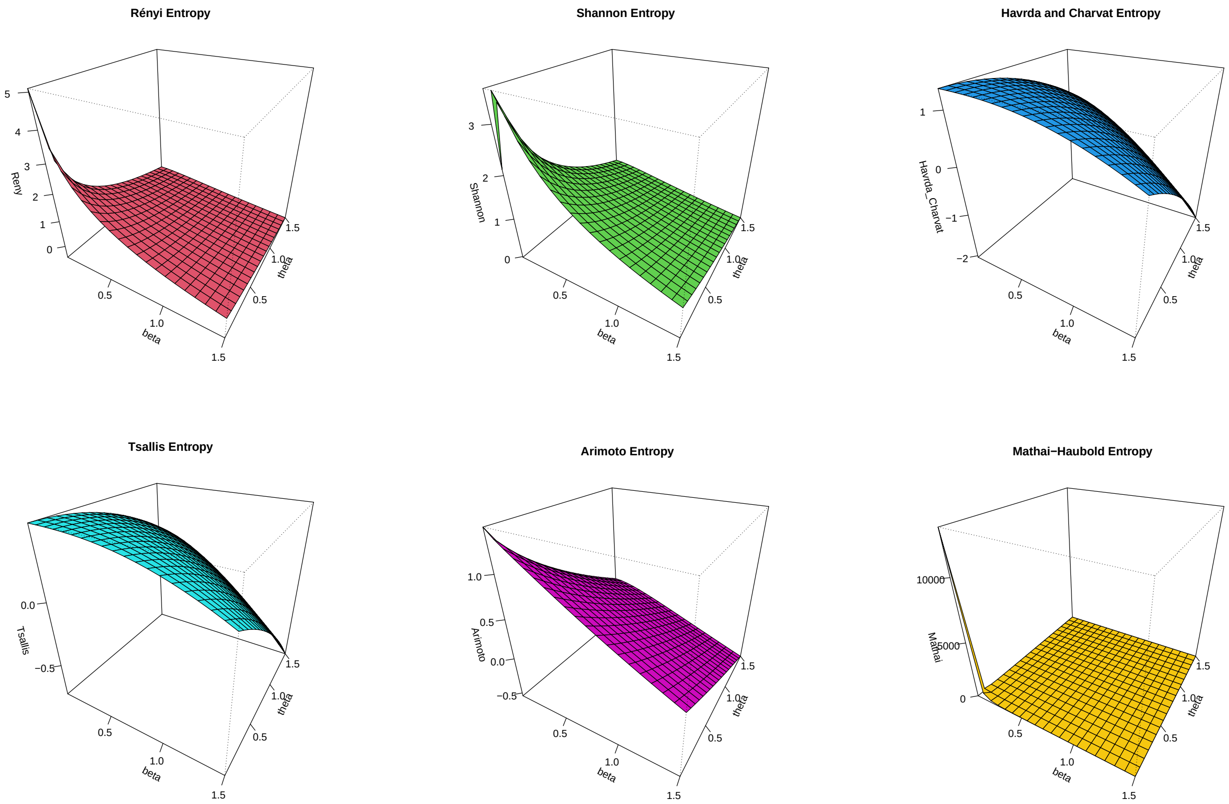


Figure 4. 3D curves of proposed entropy measures at $\gamma = 3$.

4. Statistical Inference of CEXL Model

In this part of the work, we provide statistical inference for complete samples of our CEXL model. In the complete sample, we discuss two estimation processes and also construct the confidence intervals for the model parameters.

4.1. Estimation Based on Maximum Likelihood Method

Let us assume that $\{x_1, \ldots, x_n\}$ is a random sample from the proposed model with parameters θ and β. The corresponding log-likelihood function $\mathcal{LL}(\boldsymbol{\eta})$ is

$$\mathcal{LL}(\boldsymbol{\eta}) = n - 2n\log(\theta+1) - (\theta+\beta)\sum_{i=1}^{n} x_i + \sum_{i=1}^{n}\log\Big[\theta(\theta+1)^2 x_i + (\theta+1)^2(\theta+\beta) - \theta\Big],$$

where $\boldsymbol{\eta} = (\theta, \beta)$. With respect to θ and β, the non linear equations are describes as follows

$$\frac{\partial \mathcal{LL}(\boldsymbol{\eta})}{\partial \theta} = -\frac{2n}{\theta+1} - \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \frac{x_i[2\theta(\theta+1) + (\theta+1)^2] + 2(\theta+1)(\theta+\beta) + (\theta+1)^2 - 1}{\theta(\theta+1)^2 x_i + (\theta+1)^2(\theta+\beta) - \theta}, \tag{9}$$

and

$$\frac{\partial \mathcal{LL}(\boldsymbol{\eta})}{\partial \beta} = -\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \frac{(\theta+1)^2}{\theta(\theta+1)^2 x_i + (\theta+1)^2(\theta+\beta) - \theta}. \tag{10}$$

By solving Equation (10), we can obtain a closed form of the MLE of β, $\hat{\beta}$, which ensures that it exists and it is unique. It can be written as

$$\hat{\beta} = \frac{1}{n(\theta+1)^2}\left(\frac{1}{(\theta+1)^2}\sum_{i=1}^{n} x_i - \theta(\theta+1)\sum_{i=1}^{n} x_i - n\theta\right) - \theta. \tag{11}$$

Now, for θ, it is simple to prove that the MLE of θ, $\hat{\theta}$, can be found as a fixed point solution of the equation

$$f(\theta) = \theta, \tag{12}$$

with

$$f(\theta) = \sum_{i=1}^{n} x_i \left(\frac{2n\theta(\theta+1)^2\, x_i + (\theta+1)^2(\theta+\beta) - \theta}{x_i[2\theta(\theta+1) + (\theta+1)^2] + 2(\theta+1)(\theta+\beta) + (\theta+1)^2 - 1} \right) - \frac{1}{\sum_{i=1}^{n} x_i} - 1.$$

We used the R software to apply the fixed point procedure at j stage to solve Equation (12). From the above equation, it is clear that $\lim\limits_{\theta \longrightarrow 0} f(\theta) = 0$ and $\lim\limits_{\theta \longrightarrow \infty} f(\theta) = c$, where $c = -\frac{1}{\sum_{i=1}^{n} x_i} - 1 < 0$, and since $f'(\theta) < 0$ we can conclude that the function $f(\theta)$ is monotonically decreasing for $0 < \theta < \infty$. Consequently, final estimate of θ exists and it is unique.

Now, for constructing the confidence intervals (CIs) of the parameters, we use the asymptotic distribution of MLE of $\boldsymbol{\eta}$. Precisely,

$$(\hat{\boldsymbol{\eta}} - \boldsymbol{\eta}) \xrightarrow{D} N_2(\mathbf{0}, \boldsymbol{F}^{-1}(\boldsymbol{\eta})),$$

where $\hat{\boldsymbol{\eta}}$ is the MLE of $\boldsymbol{\eta}$ and $\boldsymbol{F}^{-1}(\boldsymbol{\eta})$ is the inverse of the observed information matrix of $\boldsymbol{\eta}$, which has a size of 2 by 2, and it is presented as

$$\boldsymbol{F}^{-1}(\boldsymbol{\eta}) = \begin{pmatrix} \frac{\partial^2 \mathcal{LL}(\boldsymbol{\eta})}{\partial \theta^2} & \frac{\partial^2 \mathcal{LL}(\boldsymbol{\eta})}{\partial \theta \partial \beta} \\ \frac{\partial^2 \mathcal{LL}(\boldsymbol{\eta})}{\partial \beta \partial \theta} & \frac{\partial^2 \mathcal{LL}(\boldsymbol{\eta})}{\partial \beta^2} \end{pmatrix}$$

Finally, with $\eta_1 = \theta$ and $\eta_2 = \beta$, the lower confidence limit (LCL) and upper confidence limit (UCL) of $(1-\alpha)\%$ CI of Σ_k are

$$\text{LCB} = \hat{\boldsymbol{\eta}}_i - t_{\alpha/2}\sqrt{F^{-1}(\hat{\boldsymbol{\eta}}_i)}, \quad i = 1, 2,$$

and

$$\text{UCB} = \hat{\boldsymbol{\eta}}_i + t_{\alpha/2}\sqrt{F^{-1}(\hat{\boldsymbol{\eta}}_i)}, \quad i = 1, 2,$$

where $t_{\alpha/2}$ is the upper $\alpha/2$ quantile of the standard normal distribution, $N(0,1)$.

4.2. Bayes Procedure

In statistical inference, the Bayesian approach denotes a non-classical method of estimation. It consists of considering it as a random variable that is estimated on the basis of information coming from the sample and taking into account the opinion of the experts, summarized by a law called the a priori law. The choice of the prior distribution is crucial for Bayesian analysis because it directly affects the posterior distributions. Schematically, we can highlight two modes of thinking. The first is subjective and considers that the prior distribution reflects knowledge resulting from professional experiences and reasonable intuitions before observing the data. This information is expressed by a so-called informative law. The second way of thinking is more objective. It is used when there is little information. It is then a question of being able to remain Bayesian in the absence of a priori information. Therefore, we are looking for non-informative prior distributions expressing a priori ignorance but treating the parameters as random.

Let us consider θ and β as random variables following the gamma distribution with parameters $\alpha_1, \beta_1, \alpha_2$, and β_2.

$$\pi_1(\theta) = \frac{\beta_1^{\alpha_1}}{\Gamma(\alpha_1)}\, \theta^{\alpha_1 - 1}\, e^{-\beta_1 \theta}, \quad \theta, \alpha_1, \beta_1 > 0,$$

and

$$\pi_2(\beta) = \frac{\beta_2^{\alpha_2}}{\Gamma(\alpha_2)}\, \beta^{\alpha_2-1}\, e^{-\beta_2\beta}, \quad \beta, \alpha_2, \beta_2 > 0.$$

The joint density of $\boldsymbol{\eta} = (\theta, \beta)$ will be

$$\pi(\boldsymbol{\eta}) = \pi_1(\theta)\pi_2(\beta) \propto \theta^{\alpha_1-1}\, \beta^{\alpha_2-1}\, e^{-\beta_1\theta-\beta_2\beta}.$$

Then, the posterior distribution will be

$$\begin{aligned} \pi^*(\boldsymbol{\eta} \mid x) &= \prod_{i=1}^{n} f(x_i)\, \pi(\boldsymbol{\eta}) \\ &= \theta^{\alpha_1-1}\, \beta^{\alpha_2-1}\, e^{-\beta_1\theta-\beta_2\beta}(\theta+1)^{-2n} \prod_{i=1}^{n}\Big(\theta(\theta+\beta)x_i + (\theta+1)^2(\theta+\beta) - \theta\Big)\, e^{-(\theta+\beta)x_i}. \end{aligned}$$

The Bayes estimator under square error (SE) loss function $\mathcal{D} = (\boldsymbol{\eta} - \hat{\boldsymbol{\eta}})^2$ would result as follows:

$$\hat{\mathcal{D}}_{SE} = \int_{\boldsymbol{\eta}} \mathcal{D}\, \pi^*(\boldsymbol{\eta} \mid x)\, d\boldsymbol{\eta}. \tag{13}$$

The Bayes estimator under linear exponential (LI) loss function $\mathcal{D} = \exp(d(\boldsymbol{\eta} - \hat{\boldsymbol{\eta}})) - (\boldsymbol{\eta} - \hat{\boldsymbol{\eta}})$ would result as follows:

$$\hat{\mathcal{D}}_{LI} = -\frac{1}{d}\log\left(\int_{\boldsymbol{\eta}} e^{-d\mathcal{D}}\, \pi^*(\boldsymbol{\eta} \mid x)\, d\boldsymbol{\eta}\right). \tag{14}$$

Based on the general entropy (GE) loss function $\mathcal{D} = \left(\frac{\hat{\boldsymbol{\eta}}}{\boldsymbol{\eta}}\right)^d - d\log\left(\frac{\hat{\boldsymbol{\eta}}}{\boldsymbol{\eta}}\right) - 1$, the Bayes estimator would result as follows:

$$\hat{\mathcal{D}}_{GE} = \left(\int_{\boldsymbol{\eta}} \mathcal{D}^{-d}\, \pi^*(\boldsymbol{\eta} \mid x)\, d\boldsymbol{\eta}\right)^{-1/d}. \tag{15}$$

The integral in Equations (13)–(15) does not have an explicit form. For this, we applied MCMC technique to achieve an approach for this integral.

5. Simulation Study

Here, several simulation studies are conducted to demonstrate the effectiveness of the recommended estimators (MLE and Bayes estimations) for the recommended CEXL distribution. Recall that all computations are computed using R software.

We use the parameter values (Case 1 = (0.6, 1.2), Case 2 = (0.5, 1.1), and Case 3 = (0.75, 1.3)), and associated the sample sizes $n = 25, 50, 75, 100$ with 1000. For each case and under 1000 replications of the process, we draw a random sample from our CEXL model by applying the following steps of generation:

- We independently generate random samples v_1 and v_2 from the U(0,1) distribution;
- Compute $y_1 = F_1^{-1}(v_1)$, where F_1 denotes the cdf of exponential distribution;
- Compute $y_2 = F_2^{-1}(v_2)$, where F_2 denotes the cdf of XLindley distribution;
- Obtain a random sample from the proposed CEXL model as $X = \min(y_1, y_2)$.

Henceforth, we compute the average estimate (AVEs) with its associated mean square errors (MSEs) of the unknown parameters θ and β using the two procedures listed as MLE and Bayesian under several loss function methods. The results are displayed in Tables 4–6.

Finally, we calculate the 95% simulated CIs for the model parameters with its average lengths (ALs) and coverage probabilities (CPs). Tables 7–9 reported the obtained results.

Table 4. The possible AVE and MSE values of the CEXL model using Case 1.

n		MLE		Bays (SE)		Bays (LI)		Bays (GE)	
		AVE	**MSE**	**AVE**	**MSE**	**AVE**	**MSE**	**AVE**	**MSE**
25	θ	0.6313	0.2477	0.6291	0.0027	0.6319	0.0031	0.6321	0.0033
	β	1.2494	0.2985	1.2447	0.0024	1.2453	0.0027	1.2450	0.0029
50	θ	0.6265	0.2231	0.5906	0.0013	0.5924	0.0016	0.5926	0.0019
	β	1.2290	0.1899	1.1848	0.0023	1.1879	0.0026	1.1865	0.0029
75	θ	0.6165	0.2132	0.6236	0.0010	0.6243	0.0013	0.6243	0.0016
	β	1.2097	0.1557	1.2321	0.0016	1.2329	0.0019	1.2325	0.0021
100	θ	0.5928	0.2019	0.6054	0.0009	0.6095	0.0011	0.6097	0.0014
	β	1.1996	0.1531	1.1952	0.0005	1.1959	0.0008	1.1955	0.0011

Table 5. The possible AVE and MSE values of the CEXL model using Case 2.

n		MLE		Bays (SE)		Bays (LI)		Bays (GE)	
		AVE	**MSE**	**AVE**	**MSE**	**AVE**	**MSE**	**AVE**	**MSE**
25	θ	0.6458	0.2396	0.6095	0.0216	0.6246	0.0219	0.6253	0.0301
	β	0.9337	0.2181	1.0526	0.0219	1.0801	0.0301	1.0706	0.0304
50	θ	0.6056	0.2346	0.5773	0.0114	0.5853	0.0117	0.5864	0.0201
	β	0.9483	0.2029	1.0915	0.0050	1.0986	0.0053	1.0959	0.0056
75	θ	0.5627	0.2013	0.5160	0.0042	0.5218	0.0045	0.5235	0.0048
	β	1.0193	0.1753	1.1338	0.0047	1.1439	0.0050	1.1396	0.5053
100	θ	0.5380	0.1852	0.5294	0.0039	0.5345	0.0042	0.5357	0.0045
	β	1.0673	0.1661	1.1009	0.0040	1.1084	0.0043	1.1055	0.0046

Table 6. The possible AVE and MSE values of the CEXL model using Case 3.

n		MLE		Bays (SE)		Bays (LI)		Bays (GE)	
		AVE	**MSE**	**AVE**	**MSE**	**AVE**	**MSE**	**AVE**	**MSE**
25	θ	0.6693	0.2162	0.7158	0.0116	0.7308	0.0119	0.7297	0.0121
	β	1.3686	0.3021	1.2315	0.00191	1.2515	0.0194	1.2428	0.0197
50	θ	0.7090	0.2006	0.7195	0.0112	0.7423	0.0115	0.7396	0.0118
	β	1.3527	0.2090	1.2577	0.0059	1.2640	0.0062	1.2610	0.0065
75	θ	0.7061	0.1991	0.7336	0.0034	0.7383	0.0037	0.7378	0.0040
	β	1.3311	0.1567	1.2600	0.0053	1.2685	0.0056	1.2645	0.0059
100	θ	0.7379	0.1762	0.7596	0.0024	0.7703	0.0027	0.7686	0.0030
	β	1.3304	0.1351	1.3125	0.0042	1.3186	0.0045	1.3156	0.0048

Table 7. The possible AL and CP values of the CEXL model using Case 1.

n		MLE		Bays (SE)		Bays (LI)		Bays (GE)	
		AL	**CP**	**AL**	**CP**	**AL**	**CP**	**AL**	**CP**
25	θ	0.5345	0.947	0.5074	0.972	0.5146	0.968	0.5178	0.961
	β	0.6176	0.945	0.5243	0.971	0.5643	0.951	0.5675	0.945
50	θ	0.4873	0.924	0.4137	0.984	0.4357	0.962	0.4487	0.954
	β	0.5474	0.913	0.4782	0.957	0.4964	0.937	0.5079	0.932
75	θ	0.4235	0.924	0.3672	0.972	0.3968	0.943	0.4153	0.938
	β	0.4812	0.912	0.4067	0.984	0.4388	0.963	0.4449	0.954
100	θ	0.3579	0.935	0.2868	0.991	0.3211	0.975	0.2409	0.970
	β	0.3928	0.948	0.3491	0.985	0.3749	0.962	0.3834	0.960

Table 8. The possible AL and CP values of the CEXL model using Case 2.

n		MLE		Bays (SE)		Bays (LI)		Bays (GE)	
		AL	CP	AL	CP	AL	CP	AL	CP
25	θ	0.6658	0.926	0.5837	0.952	0.6017	0.938	0.6155	0.931
	β	0.7853	0.933	0.7129	0.955	0.7369	0.933	0.7451	0.931
50	θ	0.5943	0.938	0.5127	0.964	0.5449	0.948	0.5648	0.943
	β	0.7233	0.947	0.6779	0.974	0.6983	0.962	0.7108	0.959
75	θ	0.4218	0.944	0.4308	0.977	0.4597	0.956	0.4776	0.952
	β	0.6648	0.952	0.5884	0.972	0.6119	0.964	0.6352	0.962
100	θ	0.3764	0.952	0.2977	0.984	0.3234	0.977	0.3581	0.972
	β	0.5981	0.958	0.51237	0.989	0.5436	0.980	0.5648	0.978

Table 9. The possible AL and CP values of the CEXL model using Case 3.

n		MLE		Bays (SE)		Bays (LI)		Bays (GE)	
		AL	CP	AL	CP	AL	CP	AL	CP
25	θ	0.5473	0.936	0.4932	0.962	0.5267	0.951	0.5576	0.948
	β	0.6732	0.936	0.6014	0.963	0.6328	0.955	0.6453	0.950
50	θ	0.4693	0.951	0.4019	0.976	0.4364	0.963	0.4433	0.958
	β	0.6252	0.952	0.5381	0.972	0.5564	0.966	0.5716	0.964
75	θ	0.4157	0.951	0.3447	0.976	0.3659	0.968	0.3902	0.960
	β	0.5791	0.956	0.4623	0.977	0.4917	0.968	0.5113	0.962
100	θ	0.3786	0.964	0.2811	0.988	0.3220	0.976	0.3526	0.971
	β	0.5119	0.959	0.4134	0.986	0.4462	0.977	0.4602	0.975

Concluding Remarks on Simulation

1. For the two proposed estimation techniques, as we grow n, the MSEs diminish in all cases.
2. The MLE and Bayes estimators are consistent and asymptotically unbiased.
3. With considering the MSEs as an optimally criteria, we find that the Bayes estimator based on the SE loss function is the best method of estimation over the MLE.
4. The ALs tend to decrease as n increases in the two suggested estimation methods.
5. For comparing the CIs with considering the AL as an optimally criteria, we find that the CIs constructed based on the Bayes methods are more appropriate than the MLEs.
6. The Bayes estimator usually lies below the nominal level of 95% and is more efficient than the one based on MLE.

6. Real Data Analysis

This section demonstrate the adaptability of our CEXL model using two real datasets for checking the effectiveness and performance among several well-known distributions.

The first dataset consists of the remission times of bladder cancer patients, and it was previously studied by Abouelmagd et al. [27] and Cordeiro et al. [28]. The observation of datasets is written in Table 10.

Table 10. The remission times of bladder cancer patients.

17.36	17.14	17.12	16.62	15.96	14.83	14.77	14.76	14.24	13.80	13.29	13.11	12.63
12.07	12.03	12.02	11.98	11.79	11.64	11.25	10.75	10.66	10.34	10.06	9.74	9.47
9.22	9.02	8.66	8.65	8.53	8.37	8.26	7.93	7.87	7.66	7.63	7.62	7.59
7.39	7.32	7.28	7.26	7.09	6.97	6.94	6.93	6.76	6.54	6.25	5.85	5.71
5.62	5.49	5.41	5.41	5.34	5.32	5.32	5.17	5.09	5.06	4.98	4.87	4.51
4.50	3.02	4.40	4.34	4.33	4.26	4.23	4.18	3.88	3.82	3.70	3.64	3.57
3.52	3.48	3.36	3.36	3.31	3.25	2.87	2.83	2.75	2.69	2.69	2.64	2.62
2.54	2.46	2.26	2.23	2.09	2.07	2.02	2.02	1.76	1.46	1.40	1.35	1.26
1.19	1.05	0.90	0.81	0.51	0.50	0.40	0.20					

The second dataset represents the waiting time (in minutes) of 100 bank customers. The considered data were studied originally by Ghitany et al. [29] and also provided by Bhati et al. [30]. The values of the dataset are reported in Table 11.

Table 11. Waiting time of 100 bank customers (in minutes).

0.8	0.8	1.3	1.5	1.8	1.9	1.9	2.1	2.6	2.7	2.9	3.1	3.2	3.3	3.5
3.6	4	4.	4.2	4.2	4.3	4.3	4.4	4.4	4.6	4.7	4.7	4.8	4.9	4.9
5.0	5.3	5.5	5.7	5.7	6.1	6.2	6.2	6.2	6.3	6.7	6.9	7.1	7.1	7.1
7.1	7.4	7.6	7.7	8	8.2	8.6	8.6	8.6	8.8	8.8	8.9	8.9	9.5	9.6
9.7	9.8	10.7	10.9	11.0	11.0	11.1	11.2	11.2	11.5	11.9	12.4	12.5	12.9	13.0
13.1	13.3	13.6	13.7	13.9	14.1	15.4	15.4	17.3	17.3	18.1	18.2	18.4	18.9	19.0
19.9	20.6	21.3	21.4	21.9	23	27	31.6	33.1	38.5					

The summary statistics for the proposed datasets with the kernel density, TTT, and box plots are displayed, respectively, in Table 12 and Figure 5.

Table 12. Summary statistics for the two considered datasets.

Dataset	Q_1	Median	Mean	Q_3	CV	$\mathcal{S}$	$\mathcal{K}$
I	2.870	5.340	6.408	8.660	3.012	0.738	−0.312
II	0.891	1.717	1.801	2.237	0.851	1.196	1.3517

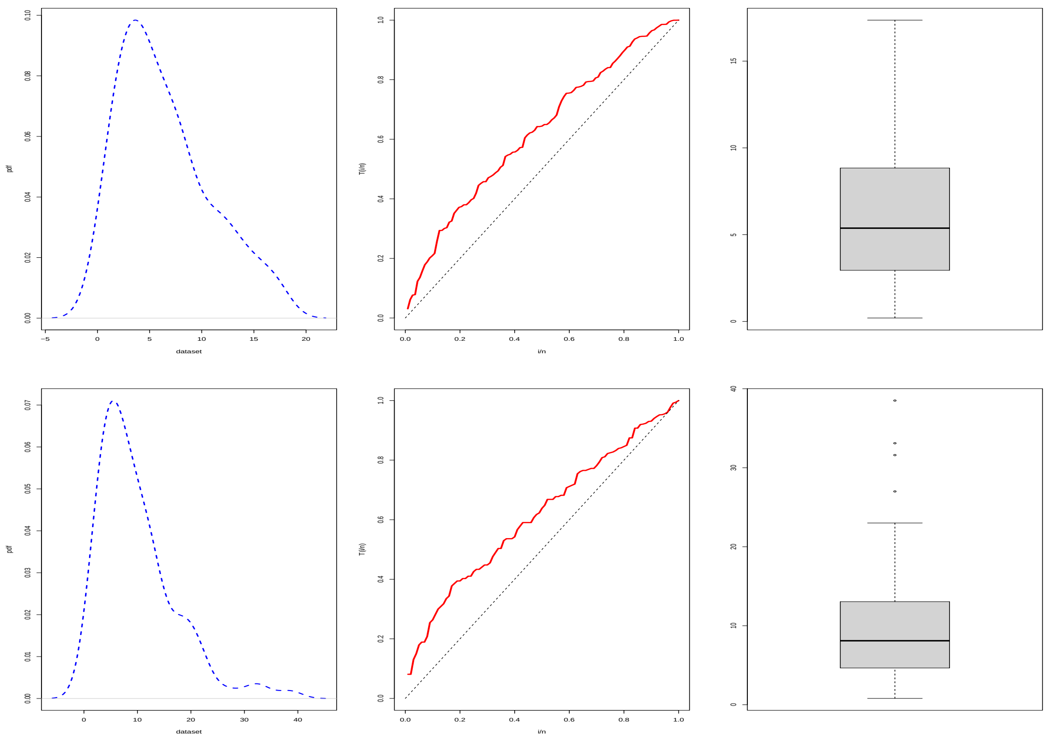


Figure 5. Kernel density, TTT, and box plots of the two proposed datasets.

The Inverse Weibull (IW), Nadarajah Haghighi (NH), Alpha power transformed exponential (APTE), Zero truncated Poisson exponential (ZTPE), Zero truncated Poisson Lindley (ZTPL), EXP, and XL distributions are used to compare with our recommended CEXL model. The cdfs of the recommended model can be, respectively, expressed as follows:

1. IW:

$$F(x) = e^{-\theta x^{-\beta}}, \quad x; \beta, \theta > 0, \tag{16}$$

2. NH:

$$F(x) = 1 - e^{1-(1+\beta x)^{\theta}}, x > 0\,,\ \theta, \beta > 0.$$

3. APTE:

$$F(x) = \frac{\beta^{1-e^{-\theta x}} - 1}{\beta - 1};\ \ x > 0,\ \ \beta \neq 1\ \beta, \theta > 0.$$

4. ZTPE:

$$F(x) = \frac{e^{\beta(1-e^{-\theta x})} - 1}{e^{\beta} - 1};\ \ x > 0,\ \ \beta, \theta > 0.$$

5. ZTPL:

$$F(x) = \frac{e^{\beta\left(e^{-\theta x}\left(\frac{1+\theta+\theta x}{1+\theta}\right)\right)} - 1}{e^{\beta} - 1};\ \ x > 0,\ \ \beta, \theta > 0.$$

Table 13 summarizes the result of the estimation of the unknown parameters for our CEXL model and other selected distributions using the MLE tool. In order to select more adequate model for modelling the two datasets, we compute some statistic measures, notably, Akaike information criterion ($\mathcal{A}$), Bayesian information criterion ($\mathcal{B}$), and Kolmogorov–Smirnov (KS) with its associated p-values. Also, Table 13 displays these results. The values of $\mathcal{A}$, $\mathcal{B}$, and KS for our proposed CEXL model are smaller in comparison to the existing well-known distributions, which implies that our CEXL model is best fitting model for analyzing the two datasets than the other fitted distributions. Figures 6–9, respectively, represent the estimated pdf, cdf, and sf for the two datasets using our and the fitting models. These figures also highlight that the CEXL model performed better than the competing models.

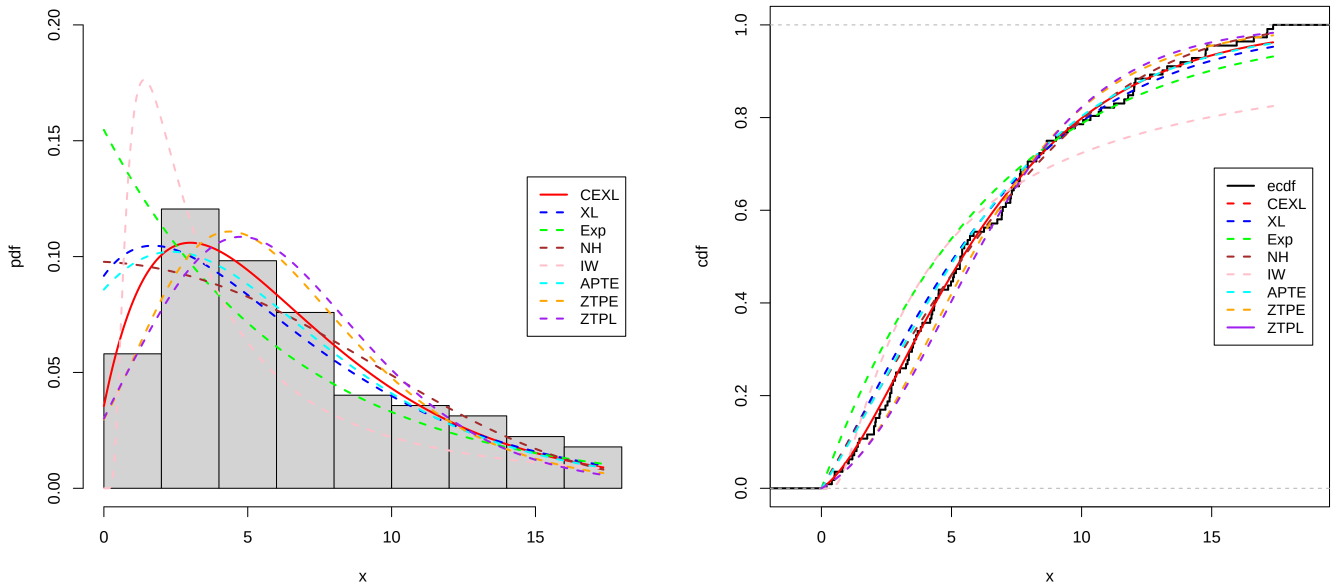


Figure 6. Estimation plots of pdf and cdf of the fitting distributions using the first dataset.

Table 13. Estimated, comparison criterion, and goodness-of-fit statistics for the two datasets.

Data	Distribution	$\hat{\theta}$	$\hat{\beta}$	KS	p-Value	$\mathcal{A}$	$\mathcal{B}$
	CEXL	0.9964	0.7110	0.0416	0.9901	624.656	630.093
	IW	2.8333	0.9419	0.1752	0.0020	693.262	698.699
	NH	15.222	0.0064	0.0788	0.4900	627.229	631.666
I	APTE	0.2273	5.5335	0.0774	0.5121	626.765	629.012
	ZTPE	0.2929	3.5944	0.0627	0.7701	626.090	628.296
	ZTPL	0.3875	2.1794	0.0779	0.5048	626.928	629.134
	Exp		0.1546	0.1522	0.0110	644.061	646.780
	XL		0.2527	0.0885	0.3438	628.414	632.133
	CEXL	0.5538	0.3426	0.0568	0.9026	637.851	643.061
	IW	6.5345	1.1631	0.1166	0.1316	672.762	677.972
	NH	3.4053	0.0206	0.1066	0.2056	650.897	656.107
II	APTE	0.1453	4.6153	0.1062	0.2092	647.508	652.718
	ZTPE	0.1994	3.7903	0.0804	0.5376	643.232	648.442
	ZTPL	0.2533	1.8766	0.0865	0.4427	644.887	650.097
	Exp		0.1012	0.1730	0.0050	660.041	6662.646
	XL		0.1744	0.0905	0.3851	643.523	646.128

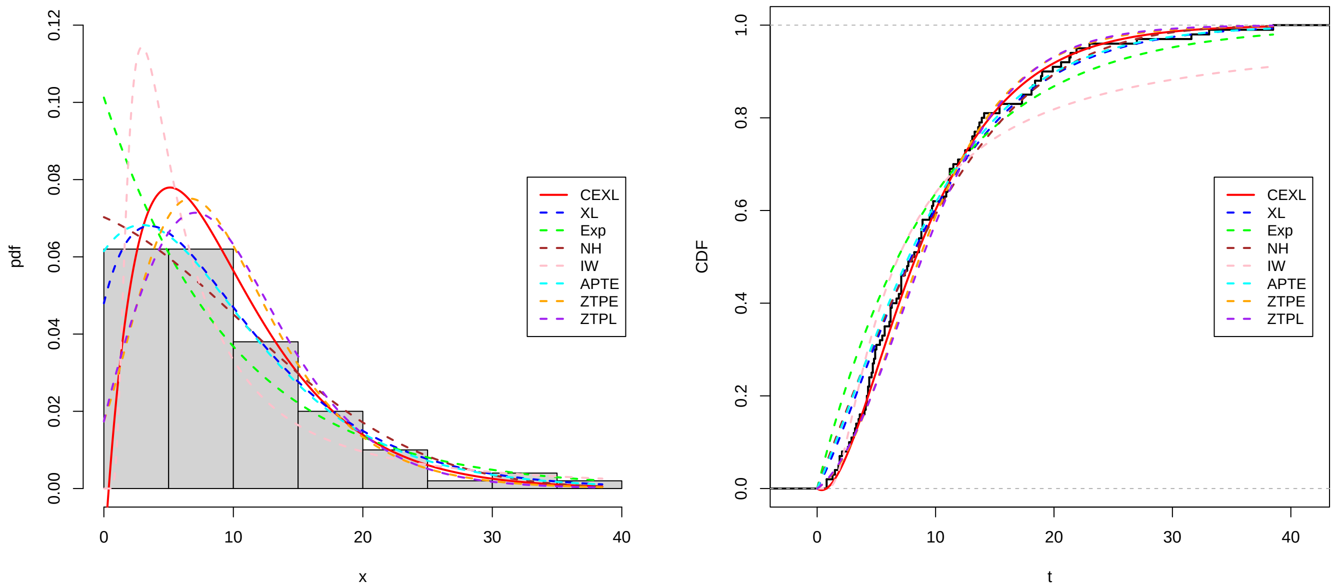


Figure 7. Estimation plots of pdf and cdf of the fitting distributions using the second dataset.

Next, we consider the two proposed datasets employing the Bayesian estimation under all suggested loss functions. The obtained results are presented in Table 14.

Table 14. Bayesian estimation under several loss functions for parameters of CEXL model using the two suggested datasets.

Dataset	Par	SE	LI	GE
I	θ	0.9918	0.9943	0.9934
	β	0.7414	0.7436	0.7433
II	θ	0.5675	0.5736	0.5746
	β	0.3351	0.3372	0.3391

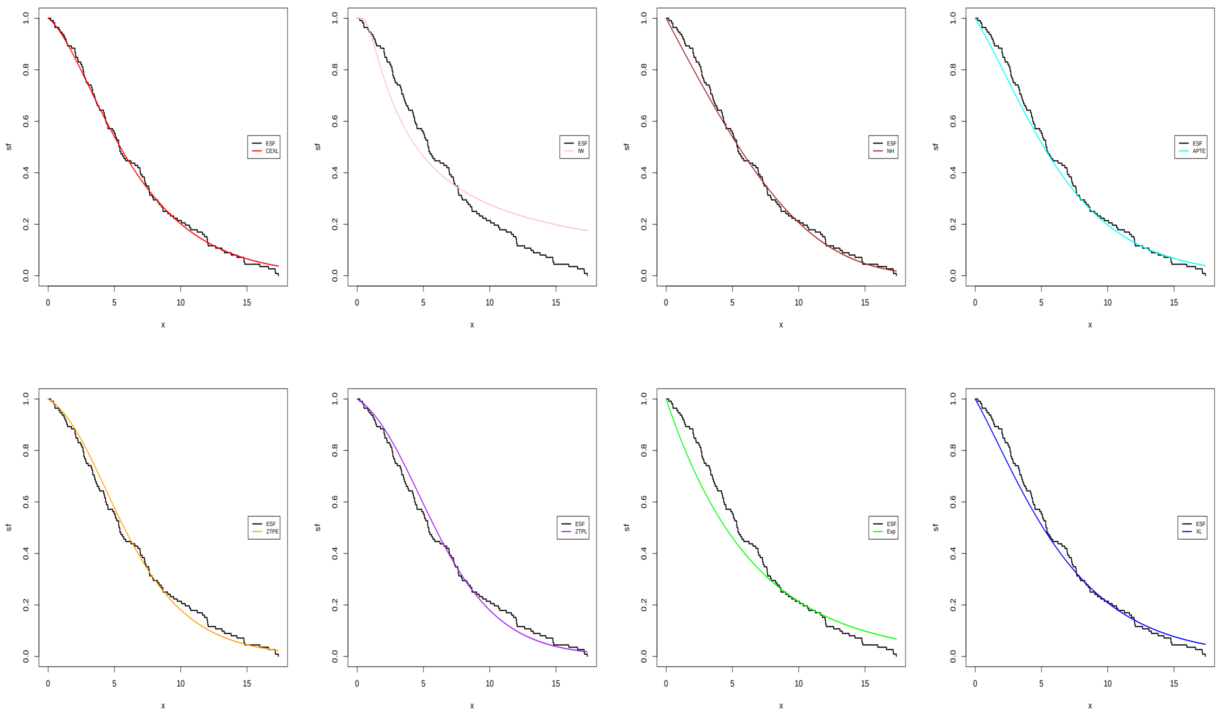


Figure 8. Plots of the esf and fitted sfs for various fitting models using the first dataset.

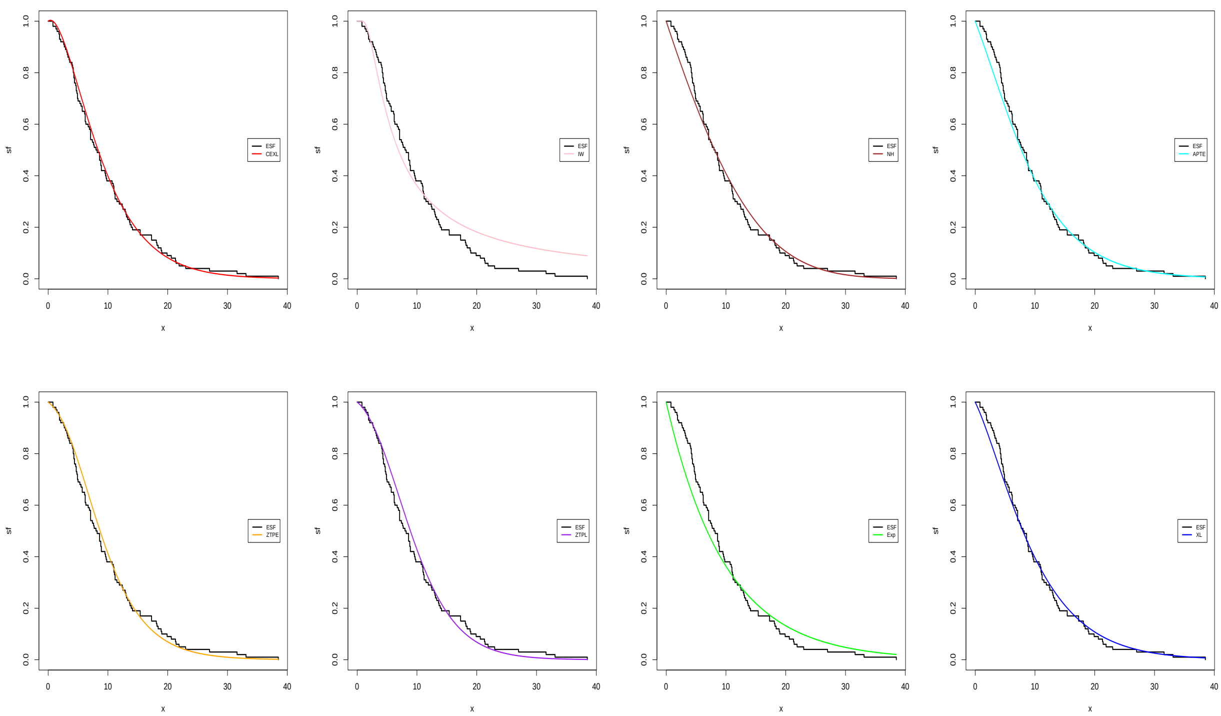


Figure 9. Plots of the esf and fitted sfs for various fitting models using the second dataset.

7. Conclusions

This study introduces a new lifetime model with two parameters obtained by compounding the exponential and XLindley distributions. Numerous distributional and statistical properties are established. Moreover, the estimation of model parameters is considered by applying two estimation techniques, and for simulation analysis, we perform several experiments for examining the potential of the proposed estimation techniques. It is demonstrated that Bayes under the square error loss function has great efficiency in estimating the unknown parameters among the MLE, LI, and GE methods. Finally, for validation purposes, two real lifetime datasets are applied, and it is shown that our CEXL distribution is the best fitting model compared among other famous competing distributions. For future researches, we may apply several censored samples for estimating the unknown parameters of the CEXL distribution. Also, it is better that to applied this new model environmental and engineering fields.

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References

1. Chouia, S.; Zeghdoudi, H. The xlindley distribution: Properties and application. *J. Stat. Appl.* **2021**, *20*, 318–327. [CrossRef]
2. Fatimah, M.A.; Muhammad, A.H.; Muhammad, N.S.; Hussain, E.H.; Ehab, M.A.; Hassan, M.A.; Manahil, S.M.; Etaf, A.; Yusuf, M. Discrete Poisson Quasi-XLindley distribution with mathematical properties, regression model, and data analysis. *J. Radiat. Res. Appl. Sci.* **2024**, *17*, 100874. [CrossRef]
3. Beghriche, A.; Yusra, A.T.; Bakr, E.M.; Zeghdoudi, H.; Gemeay, A.M.; Moyazzem, H.; Hassan, A.M. The Inverse XLindley Distribution: Properties and Application. *IEEE Access* **2023**, *11*, 47272–47281. [CrossRef]
4. Alomair, A.M.; Mukhtar, A.; Saadia, T.; Muhammad, A.; Junaid, T. An exponentiated XLindley distribution with properties, inference and applications. *Heliyon* **2024**, *10*, e25472. [CrossRef]
5. Musekwa, R.; Makubate, B. A flexible generalized XLindley distribution with application to engineering. *Sci. Afr.* **2024**, *24*, e02192. [CrossRef]
6. Gemeay, A.M.; Beghriche, A.; Sapkota, L.P.; Zeghdoudi, H.; Makumi, N.; Bakr, M.E.; Balogun, O.S. Modified XLindley distribution: Properties, estimation, and applications. *AIP Adv.* **2023**, *13*, 095021. [CrossRef]
7. Abdelghani, R.J.; Meraou, M.A.; Raqab, M.Z. Bivariate compound distribution based on Poisson maxima of gamma variates and related applications. *Int. J. Appl. Math.* **2021**, *34*, 957. [CrossRef]
8. Meraou, M.A.; Raqab, M.Z.; Kundu, D.; Alqallaf, F. Inference for compound truncated Poisson log-normal model with application to maximum precipitation data. *Commun. Stat. Simul. Comput.* **2024**, *1*, 1–22. [CrossRef]
9. Meraou, M.A.; Raqab, M.Z. Statistical properties and different estimation procedures of poisson Lindley distribution. *J. Stat. Theory Appl.* **2021**, *20*, 33–45. [CrossRef]
10. Meraou, M.A.; Al-Kandari, N.; Raqab, M.Z.; Kundu, D. Analysis of skewed data by using compound Poisson exponential distribution with applications to insurance claims. *J. Stat. Comput. Simul.* **2021**, *92*, 928–956. [CrossRef]
11. Meraou, M.A.; Al-Kandari, N.; Raqab, M.Z. Univariate and bivariate compound models based on random sum of variates with application to the insurance losses data. *J. Stat. Theory Pract.* **2022**, *16*, 56. [CrossRef]
12. Mahmoudi, E.; Jafari, A.A. Generalized exponential-power series distributions. *Comput. Stat. Data Anal.* **2012**, *56*, 4047–4066. [CrossRef]
13. Ahsan-ul-Haq, M.; Shahzad, M.K.; Tariq, S. The Inverted Nadarajah–Haghighi Power Series Distributions. *Int. J. Appl. Comput* **2024**, *10*, 11. [CrossRef]
14. Cancho, V.G.; Louzada-Neto, F.; Barriga, G.D. The Poisson-exponential lifetime distribution. *Comput. Stat. Data Anal.* **2011**, *55*, 677–686. [CrossRef]
15. Yousef, M.; Hassan, A.; Almetwally, E. Statistical Inference for the Unit Gompertz Power Series Distribution Using Ranked Set Sampling with Applications. *Assiut Univ. J. Multidiscip. Sci. Res.* **2024**, *53*, 154–189. [CrossRef]
16. Xu, A.; Wang, B.; Zhu, D.; Pang, J.; Lian, X. Bayesian reliability assessment of permanent magnet brake under small sample size. *IEEE Trans. Reliab.* **2024**, 1–11. [CrossRef]
17. Xu, A.; Zhou, S.; Tang, Y. A unified model for system reliability evaluation under dynamic operating conditions. *IEEE Trans. Reliab.* **2021**, *70*, 65–72. [CrossRef]
18. Wang, W.; Cui, Z.; Chen, R.; Wang, Y.; Zhao, X. Regression analysis of clustered panel count data with additive mean models. *Stat. Pap.* **2023**, *2023*, 11–28. [CrossRef]

19. Muqrin A.A. Bayesian and non-Bayesian inference for the compound Poisson log-normal model with application in finance. *Alex. Eng. J.* **2024**, *90*, 24–43. [CrossRef]
20. Wu, M.; Gui, W. Estimation and Prediction for Nadarajah- Haghighi Distribution under Progressive Type-II Censoring. *Symmetry* **2021**, *13*, 999. [CrossRef]
21. Rényi, A. On measures of entropy and information. In *Proceedings of the 4th Berkeley Symposium on Mathematical Statistics and Probability*; University of California Press: Berkeley, CA, USA, 1960; pp. 547–561.
22. Shannon, C.E. A mathematical theory of communication. *Bell Syst. Tech. J.* **1948**, *27*, 379–423. [CrossRef]
23. Havrda, J.; Charvat, F.S. Quantification method of classification processes: Concept of structural-entropy. *Kybernetika* **1967**, *3*, 30–35.
24. Tsallis, C. Possible generalization of Boltzmann-Gibbs statistics. *J. Stat. Phys.* **1988**, *52*, 479–487. [CrossRef]
25. Arimoto, S. Information-theoretical considerations on estimation problems. *Inf. Control* **1971**, *19*, 181–194. [CrossRef]
26. Mathai, A.M.; Haubold, H.G. On generalized distributions and pathways. *Phys. Lett.* **2008**, *372*, 2109–2113. [CrossRef]
27. Abouelmagd, T.H.M.; Al-mualim, S.; Afify, A.Z.; Munir, A.; Al-Mofleh, H. The odd Lindley Burr XII distribution with applications. *Pak. J. Stat.* **2018**, *34*, 15–32.
28. Cordeiro, G.M.; Afify, A.Z.; Yousof, H.M.; Pescim, R.R.; Aryal, G.R. The exponentiated Weibull-H family of distributions: Theory and applications. *Med. J. Math.* **2017**, *14*, 155. [CrossRef]
29. Ghitany, M.; Al-Mutairi, D.; Balakrishnan, N.; Al-Enezi, I. Power Lindley Distribution and Associated Inference. *Comput. Stat. Data Anal.* **2013**, *64*, 20–33. [CrossRef]
30. Bhati, D.; Malik, M.A.; Vaman, H.J. Lindley-Exponential Distribution: Properties and Applications. *Metron* **2015**, *73*, 335–357. [CrossRef]