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Existence of a Fixed Point and Convergence of Iterates for Self-Mappings of Metric Spaces with Graphs

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Abstract: In the present paper, under certain assumptions, we establish the convergence of iterates for self-mappings of complete metric spaces with graphs which are of a contractive type. The class of mappings considered in the paper contains the so-called cyclical mappings introduced by W. A. Kirk, P. S. Srinivasan and P. Veeramani in 2003 and the class of monotone nonexpansive operators. Our results hold in the case of a symmetric graph.

Keywords: complete metric space; contractive mapping; fixed point; graph

MSC: 47H09; 47H10; 54E50

1. Introduction

Since the seminal result of Banach [1] was reported, the fixed-point theory of nonexpansive maps has been a rapidly growing area of research. See, for example, [2–11] and the references mentioned therein. In particular, convergence of Bregman projections is studied in [2], fixed-point results of Caristi type and Mizoguchi-Takahashi type are obtained in [3], many fixed-point results are nicely collected in [4,5], fixed-point results in b-metric spaces are obtained in [6], the fixed point theory in modular spaces is discussed in [7] and the Rakotch contraction is introduced and studied in [8]. Many generic fixed-point results are collected in [9]. The books [10,11] are devoted to approximate solutions of common fixed-point problems. A great deal of progress has taken place in this area, including studies of feasibility and common fixed-point problems, which find various important applications [10–14]. In particular, the perturbation resilience and superiorization of iterative algorithms are discussed in [12]; inconsistent feasibility problems are considered in [13]; and split inverse problems are analyzed in [14]. Note that the analysis of nonexpansive operators acting on complete metric spaces with graphs is of great interest. See, for example, [15–25] and the references mentioned therein. Many useful examples can be found there. In particular, fixed-point results on a metric spaces with a graph are obtained in [15,16,19,24], Reich-type contractions are studied in [17], extensions of the Kelisky–Rivlin theorem are obtained in [18], contractive mappings are studied in [20], fixed-point results on intuitionistic fuzzy metric spaces with a graph are obtained in [21] and hybrid methods are studied in [22,23,25].

In the present paper, under certain assumptions, we establish the convergence of iterates of self-mappings of complete metric spaces with graphs which are of a contractive type. The class of mappings considered in the paper contains, in particular, the so-called cyclical mappings studied in [26–29]. Our results hold in the case of a symmetric graph.

Assume that (X, ρ) is a complete metric space endowed with the metric ρ. For every point $u \in X$ and every positive number r set

$$B(u, r) = \{v \in X : \rho(u, v) \leq r\}.$$

For each mapping $S : X \to X$, denote by S^0 the identity self-mapping of X and set

$$S^{i+1} = S \circ S^i$$

for each integer $i \geq 0$.

Here, A and B are nonempty and closed subsets of X.

Assume that $\phi : [0, \infty) \to [0, 1]$ is a decreasing function, such that

$$\phi(t) < 1, \ t \in (0, \infty)$$

and $T : A \cup B \to A \cup B$ is a mapping such that

$$T(A) \subset B, \ T(B) \subset A$$

and that for any point $a \in A$ and any point $b \in B$, we have

$$\rho(T(a), T(b)) \leq \phi(\rho(a, b))\rho(a, b).$$

The mapping T is called contractive [9].

In the paper [27], it was shown that the map T has a unique fixed point belonging to the intersection $A \cap B$. In [29], we generalized this result for the case when $T : A \cup B \to X$ is not necessarily a self-mapping of $A \cup B$. Note that if $A \cap B \neq \varnothing$, then the result of [27] is obvious. But this was not assumed in [27]. As a matter of fact, in [27], it was considered a more general case when instead of two sets A, B we have a finite family of m sets where m is any natural number.

More precisely, assume that m is a natural number $A_i \subset X, i = 1, \ldots, m$ are nonempty closed sets, $A_{m+1} = A_1$,

$$T : \cup_{i=1}^m A_i \to \cup_{i=1}^m A_i$$

and for each $i \in \{1, \ldots, m\}$, each $a \in A_i$, each $b \in A_{i+1}$,

$$T(A_i) \subset A_{i+1},$$

$$\rho(T(a), T(b)) \leq \phi(\rho(a, b))\rho(a, b).$$

It was shown in [27] that the map T has a unique fixed point belonging to the intersection $\cap_{i=1}^m A_i$. In this paper, we show that this result follows from our fixed-point result for G-nonexpansive mapping in the space X equipped with a graph G which is obtained in our paper. We can consider the cyclical mapping T as a G-nonexpansive mapping in the space X equipped with a graph G, such that its set of vertices $V(G)$ is $\cup_{i=1}^m A_i$ and

$$E(G) = \cup_{i=1}^m (A_i \times A_{i+1})$$

is its set of edges. It is not difficult to see that for every $(u, v) \in E(G)$,

$$(T(u), T(v)) \in E(G),$$

$$\rho(T(u), T(v)) \leq \phi(\rho(u, v))\rho(u, v),$$

$$(u, T(u)) \in E(G), \ u \in V(G),$$

and if $u, v \in V(G)$, then there exists $j \in \{0, \ldots, m-1\}$, such that $(u, T^j(v)) \in E(G)$. Thus, the analysis of cyclical mappings is reduced to the study of G-nonexpansive mappings in the space X equipped with the graph G under the assumptions stated above. It should be mentioned that there exists a rich literature on cyclical mappings. See, for example, refs. [30–32] and the references mentioned therein. Note that we prove our results under the assumptions which are weaker than the assumptions on T and G presented above. Some of them hold for a general G-contractive mapping. The result, which can be applied to cyclical mapping, is Theorem 3, which can be also applied to the class of monotone

nonexpansive mappings. There is rich literature on the monotone nonexpansive mappings containing numerous examples. See, for example, ref. [33] and the references mentioned therein. Therefore we have two large classes of mappings studied in the literature for which our results can be applied.

2. Strict *G*-Contractions

Assume that (X, ρ) is a complete metric space endowed with a graph G. We denote by $V(G)$ the set of its vertices and by $E(G)$ the set of its edges. Assume that $T : X \to X$ is a mapping such that for every pair of points $x, y \in X$ satisfying $(x, y) \in E(G)$,

$$(T(x), T(y)) \in E(G) \text{ and } \rho(T(x), T(y)) \leq \rho(x, y). \tag{1}$$

It is called a *G*-nonexpansive mapping.

If $\alpha \in (0, 1)$ and for every $(x, y) \in E(G)$, we have

$$\rho(T(x), T(y)) \leq \alpha \rho(x, y),$$

then the map T is called a *G*-strict contraction.

The operator T is called *G*-contractive (or *G*-Rakotch contraction [8]) if there is a decreasing function $\phi : [0, \infty) \to [0, 1]$, such that

$$\phi(t) < 1, \ t \in [0, \infty),$$

and for every $(x, y) \in E(G)$ we have

$$\rho(T(x), T(y)) \leq \phi(\rho(x, y))\rho(x, y).$$

If T is a *G*-strict contraction, then under some conditions, T possesses a unique fixed point [19]. In this section, we prove the following result which shows the convergence of inexact orbits of a *G*-strict contraction under the presence of summable computational errors.

Theorem 1. *Assume that $\alpha \in (0, 1)$, for each $(x, y) \in E(G)$,*

$$\rho(T(x), T(y)) \leq \alpha \rho(x, y), \tag{2}$$

$\{x_n\}_{n=0}^{\infty} \subset X$, *for each integer $n \geq 0$,*

$$(x_n, x_{n+1}) \in E(G) \tag{3}$$

and

$$\sum_{n=0}^{\infty} \rho(x_{n+1}, T(x_n)) < \infty. \tag{4}$$

Then, the sequence $\{x_n\}_{n=0}^{\infty}$ converges and its limit is a fixed point of T if the graph of T is closed.

Proof. For every non-negative integer n, put

$$\Delta_n = \rho(x_{n+1}, T(x_n)). \tag{5}$$

In view of (4) and (5),

$$\sum_{n=0}^{\infty} \Delta_n < \infty. \tag{6}$$

By (2), (3) and (5),

$$\rho(x_{n+1}, x_{n+2}) \leq \rho(x_{n+1}, T(x_n)) + \rho(T(x_n), T(x_{n+1})) + \rho(T(x_{n+1}), x_{n+2})$$

$$\leq \Delta_n + \Delta_{n+1} + \alpha\rho(x_n, x_{n+1}). \tag{7}$$

Assume that $m \geq 0$ is an integer. In view of (7),

$$\rho(x_{m+1}, x_{m+2}) \leq \alpha\rho(x_m, x_{m+1}) + \Delta_m + \Delta_{m+1}, \tag{8}$$

$$\rho(x_{m+2}, x_{m+3}) \leq \alpha\rho(x_{m+1}, x_{m+2}) + \Delta_{m+1} + \Delta_{m+2}$$
$$\leq \alpha^2\rho(x_m, x_{m+1}) + \alpha\Delta_m + \alpha\Delta_{m+1} + \Delta_{m+1} + \Delta_{m+2}. \tag{9}$$

We show that for each integer $k \geq 1$,

$$\rho(x_{m+k}, x_{m+k+1}) \leq \alpha^k\rho(x_m, x_{m+1}) + \sum_{i=0}^{k-1}(\Delta_{i+m} + \Delta_{m+1+i})\alpha^{k-1-i}. \tag{10}$$

It follows from (8), (9) that relation (10) is true for $k = 1, 2$. Assume that $k \geq 1$ is an integer and that relation (10) is true. In view of (7) and (10),

$$\rho(x_{m+k+1}, x_{m+k+2}) \leq \alpha\rho(x_{m+k}, x_{m+k+1}) + \Delta_{m+k} + \Delta_{m+k+1}$$

$$\leq \alpha^{k+1}\rho(x_m, x_{m+1}) + \sum_{i=0}^{k-1}(\Delta_{i+m} + \Delta_{m+1+i})\alpha^{k-i} + \Delta_{m+k} + \Delta_{m+k+1}$$

$$= \alpha^{k+1}\rho(x_m, x_{m+1}) + \sum_{i=0}^{k}(\Delta_{i+m} + \Delta_{m+1+i})\alpha^{k-i}$$

and (10) holds for $k + 1$ too. Thus, we show by induction that (10) holds for each integer $k \geq 1$. By (6) and (10) (with $m = 0$),

$$\sum_{k=0}^{\infty}\rho(x_k, x_{k+1}) \leq \sum_{k=0}^{\infty}\alpha^k\rho(x_0, x_1) + \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(\Delta_i + \Delta_{i+1})\alpha^j < \infty.$$

Therefore, $\{x_k\}_{k=0}^{\infty}$ is a Cauchy sequence and it has a limit. This completes the proof of Theorem 1. □

3. Rakotch G-Contraction

We continue to use the notation, definitions, and assumptions introduced in Section 2. Assume that $\phi : [0, \infty) \rightarrow [0, 1]$ is a decreasing function, such that

$$\phi(t) < 1, \ t \in (0, \infty) \tag{11}$$

and that for each $(x, y) \in E(G)$,

$$\rho(T(x), T(y)) \leq \phi(\rho(x, y))\rho(x, y). \tag{12}$$

The next result demonstrates that exact iterates of a G-contractive mapping are its approximate fixed points.

Theorem 2. *Assume that $x \in X$ and that we have the natural number q and points $y_i \in X$, $i = 0, \ldots, q$, such that*

$$y_0 = x, \ y_q = T(x), \tag{13}$$

$$(y_i, y_{i+1}) \in E(G), \ i = 0, \ldots, q - 1. \tag{14}$$

Then,

$$\lim_{i \to \infty}\rho(T^i(x), T^{i+1}(x)) = 0.$$

Proof. It follows from (1), (13) and (14) that for each integer, $n \geq 0$ and each $i \in \{0, \ldots, q-1\}$,

$$(T^n(y_i), T^n(y_{i+1})) \in E(G), \tag{15}$$

$$T^n(y_0) = T^n(x), \; T^n(y_q) = T^{n+1}(x). \tag{16}$$

In view of (16), it is sufficient to show that for every $i \in \{0, \ldots, q-1\}$, we have

$$\lim_{n \to \infty} \rho(T^n(y_i), T^n(y_{i+1})) = 0.$$

Assume the contrary. Then, according to (12) and (14), we have $i \in \{0, \ldots, q-1\}$ and $\epsilon > 0$ for which

$$\rho(T^n(y_i), T^n(y_{i+1})) \geq \epsilon, \; n = 0, 1, \ldots. \tag{17}$$

According to (12), (15) and (17), for every non-negative integer n,

$$\rho(T^{n+1}(y_i), T^{n+1}(y_{i+1})) \leq \phi(\rho(T^n(y_i), T^n(y_{i+1})))\rho(T^n(y_i), T^n(y_{i+1}))$$

$$\leq \phi(\epsilon)\rho(T^n(y_i), T^n(y_{i+1})),$$

$$\rho(T^n(y_i), T^n(y_{i+1})) - \rho(T^{n+1}(y_i), T^{n+1}(y_{i+1}))$$

$$\geq (1 - \phi(\epsilon))\rho(T^n(y_i), T^n(y_{i+1})) \geq \epsilon(1 - \phi(\epsilon))$$

and for each natural number n,

$$\rho(y_i, y_{i+1}) \geq \rho(y_i, y_{i+1}) - \rho(T^n(y_i), T^n(y_{i+1}))$$

$$= \sum_{k=0}^{n-1} (\rho(T^k(y_i), T^k(y_{i+1})) - \rho(T^{k+1}(y_i), T^{k+1}(y_{i+1}))$$

$$\geq k\epsilon(1 - \phi(\epsilon)) \to \infty \text{ as } k \to \infty.$$

The contradiction we have reached completes the proof of Theorem 2. $\square$

The next result shows that under certain assumptions, exact iterates of T converge to its fixed point.

Theorem 3. *Assume that $x \in X$ and that there is an integer $q \geq 1$, $y_i \in X$, $i = 0, \ldots, q$, such that*

$$y_0 = x, \; y_q = T(x), \tag{18}$$

$$(y_i, y_{i+1}) \in E(G), \; i = 0, \ldots, q-1. \tag{19}$$

Assume that there exists a natural number m_0, such that the following property holds:
(P) for each pair of natural numbers $i < j$, there exists

$$p \in \{j, \ldots, j + m_0\},$$

such that

$$(T^i(x), T^p(x)) \in E(G).$$

Then, the sequence $\{x_n\}_{n=0}^{\infty}$ converges and its limit is a fixed point of T if the graph of T is closed.

Proof. Using Theorem 2, based on (18) and (19), we determine that

$$\lim_{i \to \infty} \rho(T^i(x), T^{i+1}(x)) = 0. \tag{20}$$

Let $\epsilon > 0$. Choose a positive number

$$\delta < \min\{4^{-1}\epsilon(1 - \phi(\epsilon/2)), \ (2m_0)^{-1}\epsilon\}. \tag{21}$$

According to (20), there exists a natural number n_0 such that for each integer $i \geq n_0$,

$$\rho(T^i(x), T^{i+1}(x)) \leq \delta. \tag{22}$$

Assume that integers j, i satisfy

$$j > i \geq n_0.$$

Property (P) implies that there exists

$$p \in \{j, \ldots, j + m_0\}, \tag{23}$$

such that

$$(T^i(x), T^p(x)) \in E(G). \tag{24}$$

According to (12), (22), and (23),

$$\rho(T^i(x), T^p(x))$$

$$\leq \rho(T^i(x), T^{i+1}(x)) + \rho(T^{i+1}(x), T^{p+1}(x)) + \rho(T^{p+1}(x), T^p(x))$$

$$\leq 2\delta + \phi(\rho(T^i(x), T^p(x)))\rho(T^i(x), T^p(x)),$$

$$\rho(T^i(x), T^p(x))(1 - \phi(\rho(T^i(x), T^p(x)))) \leq 2\delta \tag{25}$$

and if

$$\rho(T^i(x), T^p(x)) > \epsilon/2,$$

then in view of (25),

$$2^{-1}\epsilon(1 - \phi(\epsilon/2)) < 2\delta.$$

This contradicts (21). The contradiction we have reached proves that

$$\rho(T^i(x), T^p(x)) \leq \epsilon/2. \tag{26}$$

Then, implementing (21)–(23) and (26),

$$\rho(T^i(x), T^j(x))$$

$$\leq \rho(T^i(x), T^p(x)) + \rho(T^p(x), T^j(x)) + \rho(T^p(x), T^j(x))$$

$$\leq \epsilon/2 + m_0\delta \leq \epsilon.$$

Thus, $\{T^i(x)\}_{i=0}^{\infty}$ is a Cauchy sequence and this completes the proof of Theorem 3. $\square$

The following result shows that inexact iterates of a *G*-contractive mapping are its approximate fixed points.

Theorem 4. *Assume that* $\{x_n\}_{n=0}^{\infty} \subset X$, *for each integer* $i \geq 0$,

$$(x_i, x_{i+1}) \in E(G), \tag{27}$$

$$\lim_{i \to \infty} \rho(x_{i+1}, T(x_i)) = 0. \tag{28}$$

Then,

$$\lim_{i \to \infty} \rho(x_i, x_{i+1}) = 0.$$

Proof. Let $\epsilon \in (0,1)$. Based on (12), for each integer $i \geq 0$,

$$\rho(x_{i+1}, x_{i+2}) \leq \rho(x_{i+1}, T(x_i)) + \rho(T(x_i), T(x_{i+1})) + \rho(T(x_{i+1}), x_{i+2})$$

$$\leq \rho(x_{i+1}, T(x_i)) + \rho(T(x_{i+1}), x_{i+2}) + \phi(\rho(x_i, x_{i+1}))\rho(x_i, x_{i+1}). \tag{29}$$

Let

$$\delta \in (0, 4^{-1}\epsilon(1 - \phi(\epsilon/2))). \tag{30}$$

In view of (28), there exists a natural number n_0, such that for each integer $i \geq n_0$,

$$\rho(x_{i+1}, T(x_i)) \leq \delta/8. \tag{31}$$

Assume that $i \geq n_0$ is an integer, such that

$$\rho(x_i, x_{i+1}) > \epsilon/2. \tag{32}$$

It follows from (29)–(32) that

$$\rho(x_{i+1}, x_{i+2}) \leq \delta/4 + \phi(\rho(x_i, x_{i+1}))\rho(x_i, x_{i+1}),$$

$$\rho(x_i, x_{i+1}) - \rho(x_{i+1}, x_{i+2})$$

$$\geq \rho(x_i, x_{i+1})(1 - \phi(\rho(x_i, x_{i+1}))) - \delta/4$$

$$\geq 2^{-1}\epsilon(1 - \phi(\epsilon/2)) - \delta/4 \geq 4^{-1}\epsilon(1 - \phi(\epsilon/2)).$$

Thus, the following property holds:
(a) if an integer $i \geq n_0$ satisfies $\rho(x_i, x_{i+1}) > \epsilon/2$, then

$$\rho(x_i, x_{i+1}) - \rho(x_{i+1}, x_{i+2}) \geq 4^{-1}\epsilon(1 - \phi(\epsilon/2)). \tag{33}$$

Assume that for each integer $i \geq n_0$,

$$\rho(x_i, x_{i+1}) > \epsilon/2.$$

Property (a) implies that for each integer $i \geq n_0$, relation (33) holds and for each natural number Q,

$$\rho(x_{n_0}, x_{n_0+1}) \geq \rho(x_{n_0}, x_{n_0+1}) - \rho(x_{n_0+Q}, x_{n_0+Q+1})$$

$$= \sum_{i=0}^{Q-1} (\rho(x_{n_0+i}, x_{n_0+1+i}) - \rho(x_{n_0+i+1}, x_{n_0+i+2}))$$

$$\geq 4^{-1}Q\epsilon(1 - \phi(\epsilon/2)) \to \infty$$

as $Q \to \infty$. The contradiction we have reached proves that there exists an integer $n_1 \geq n_0$ for which

$$\rho(x_i, x_{i+1}) \leq \epsilon/2.$$

We show that for each integer $i \geq n_1$,

$$\rho(x_i, x_{i+1}) \leq \epsilon.$$

Assume that an integer $i \geq n_1$ and

$$\rho(x_i, x_{i+1}) \leq \epsilon. \tag{34}$$

In order to complete the proof, it is sufficient to show that

$$\rho(x_{i+1}, x_{i+2}) \leq \epsilon.$$

There are two cases:

$$\rho(x_i, x_{i+1}) \leq \epsilon/2; \tag{35}$$

$$\rho(x_i, x_{i+1}) > \epsilon/2. \tag{36}$$

If (35) holds, then based on (29)–(31) and (35),

$$\rho(x_{i+1}, x_{i+2}) \leq \rho(x_i, x_{i+1}) + \delta/4 \leq \epsilon.$$

If (36) holds, then property (a) implies that

$$\rho(x_{i+1}, x_{i+2}) \leq \rho(x_i, x_{i+1}) \leq \epsilon.$$

This completes the proof of Theorem 4. □

Our final result shows that under certain assumptions, inexact iterates of T converge to its fixed point.

Theorem 5. *Assume that $\{x_i\}_{i=0}^{\infty} \subset X$, for each integer $i \geq 0$,*

$$(x_i, x_{i+1}) \in E(G), \tag{37}$$

$$\lim_{i \to \infty} \rho(x_{i+1}, T(x_i)) = 0 \tag{38}$$

and that there exists a natural number m_0 such that property (P) of Theorem 3 holds. Then, the sequence $\{x_n\}_{n=0}^{\infty}$ converges and its limit is a fixed point of T if the graph of T is closed.

Proof. Using Theorem 4, and (37) and (38), we find that

$$\lim_{i \to \infty} \rho(x_i, x_{i+1}) = 0. \tag{39}$$

Let $\epsilon \in (0, 1)$. Choose a positive number δ, such that

$$\delta < 4^{-1}\epsilon(1 - \phi(\epsilon/2)) \tag{40}$$

and

$$\delta < (4m_0)^{-1}\epsilon; \tag{41}$$

According to (38) and (39), there exists a natural number n_0, such that for each integer $i \geq n_0$,

$$\rho(x_i, x_{i+1}), \ \rho(x_i, T(x_i)) \leq \delta. \tag{42}$$

Assume that integers j, i satisfy

$$j > i \geq n_0.$$

Property (P) implies that there exists

$$p \in \{j, \ldots, j + m_0\}, \tag{43}$$

such that

$$(x_i, x_p) \in E(G). \tag{44}$$

Based on (12) and (44),

$$\rho(T(x_i), T(x_p)) \leq \phi(\rho(x_i, x_p))\rho(x_i, x_p). \tag{45}$$

It follows from (42), (43) and (45) that

$$\rho(x_i, x_p) \leq \rho(x_i, T(x_i)) + \rho(T(x_i), T(x_p)) + \rho(T(x_p), x_{p+1})$$

$$\leq 2\delta + \phi(\rho(x_i, x_p))\rho(x_i, x_p). \tag{46}$$

Assume that

$$\rho(x_i, x_p) > \epsilon/2.$$

In view of (46),

$$2\delta \geq \rho(x_i, x_p)(1 - \phi(\rho(x_i, x_p))) \geq 2^{-1}\epsilon(1 - \phi(\epsilon/2)).$$

This contradicts (40). The contradiction we have reached proves that

$$\rho(x_i, x_p) \leq \epsilon/2,$$

and together with (41)–(43) this implies that

$$\rho(x_i, x_j) \leq \rho(x_i, x_p) + \rho(x_p, x_j) \leq \epsilon/2 + m_0\delta \leq \epsilon.$$

Thus, $\{T^i(x)\}_{i=0}^{\infty}$ is a Cauchy sequence and this completes the proof of Theorem 5. $\square$

4. Conclusions

The main goal of the fixed-point theory is to study the existence of fixed point of nonlinear mappings and convergence of their (inexact) iterates to these fixed points. In this paper, we study a class of *G*-contractive mappings in a complete metric space equipped with a graph under certain assumptions. For this class of mappings, we obtain convergence and existence results. The class contains the class of cyclical nonexpansive mappings and the class of monotone nonexpansive mappings.

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