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Numerical Approach Based on the Haar Wavelet Collocation Method for Solving a Coupled System with the Caputo–Fabrizio Fractional Derivative

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Abstract: In the present paper, we consider an effective computational method to analyze a coupled dynamical system with Caputo–Fabrizio fractional derivative. The method is based on expanding the approximate solution into a symmetry Haar wavelet basis. The Haar wavelet coefficients are obtained by using the collocation points to solve an algebraic system of equations in mathematical physics. The error analysis of this method is characterized by a good convergence rate. Finally, some numerical examples are presented to prove the accuracy and effectiveness of this method.

Keywords: numerical system; Haar wavelet; collocation method; Caputo–Fabrizio; fractional derivatives; coupled system

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1. Introduction

Recently, various problems in the applied mathematics, physical, biological, and engineering sciences are modeled using fractional calculus. Due to its property of memory effect, this concept has received a great response in the applied sciences. In this regard, many definitions have been given for both the integral and the fractional derivatives, such as the Riemann–Liouville [1], Caputo [2] and Caputo–Fabrizio fractional integrals and derivatives [1–5]. However, the concepts of Riemann–Liouville and Caputo were used to model the phenomena first, which have singularity in their kernels. For this reason, many new definitions of integrals and fractional derivatives have been introduced in the literature. For instance, the Caputo–Fabrizio fractional integral and derivative [1] avoid the singularity problem; this property makes it popular in the scientific community. The main problem facing researchers in solving Caputo–Fabrizio fractional differential equations and systems [6] is the difficulty in finding an analytical solution, which leads them to use numerical methods. In fact, it is known that in many numerical methods, the solutions contain discontinuities at some points, which negatively affects the accuracy and convergence rate. Currently, the Haar wavelet method [7] is a common strategy that aims to improve the convergence rate according to exponential decay. Confirming this, we find that it has been applied to solve many problems, such as ODEs [8] and time-PDEs [9,10], differential equations [11–14], and fractional differential equations (FDEs) [15–21]. Indeed, the Haar wavelet has the advantages of simplicity, orthogonality, and compact support. As a support for the Haar wavelet method, in the present paper, we apply and investigate it to solve the following system [6] for $0 \leq t \leq 1$:

$$\begin{cases} {}^{CF}D^{(\nu)}u(t) = c_1 u(t) + c_2 v(t) + f(t), \\ {}^{CF}D^{(\nu)}v(t) = c_3 u(t) + c_4 v(t) + h(t), \\ u(t=0) = v(t=0) = 0, \end{cases} \quad (1)$$

where $0 < \nu < 1$, f and h are two continuous functions, whereas the operator ${}^{CF}D^{(\nu)}$ is the Caputo–Fabrizio derivative of order ν, and c_i $(i = 1, ..., 4)$ are real constants.

In fact, Ikram Mansouri et al. [6] considered the questions of the existence of a unique solution for this system, where the Adomian Decomposition Method (ADM) is applied to provide an approximate solution to it. However, the Adomian Decomposition Method has a polynomial decay of the convergence rate, and it is expensive to compute its terms and requires a large number of terms to obtain the exact solution. To overcome these defects, the Haar wavelet collocation method is suitable in terms of computation costs and convergence rates. As far as we know, the Haar wavelet approximation method has not been applied to a coupled system with the Caputo–Fabrizio fractional derivative before.

This paper is structured as follows: In the second Section 2, we remember the definition of the Caputo–Fabrizio fractional derivative and the associated fractional integral. In Section 3, we introduce the Haar wavelet family that is associated with our proposed numerical method of the solution. In Section 4, we give illustrative examples to prove the accuracy and effectiveness of our proposed method. Finally, we finish this paper with a concluding section.

2. Basic Knowledge

2.1. Caputo–Fabrizio Fractional Integral and Derivative

We state and recall some definitions and the main properties related to the Caputo–Fabrizio fractional integral and derivative.

Definition 1 ([6])**.** *Assume that the function $f \in H^1(a,b)$ and a constant $\nu \in (0,1)$; then, the Caputo–Fabrizio fractional derivative is defined by*

$${}^{CF}D_a^{(\nu)} f(x) = \frac{M(\nu)}{1-\nu} \int_a^x f'(\tau) \exp\Big(-\nu \frac{t-\tau}{1-\nu}\Big) d\tau, \quad (2)$$

and the associated fractional integral is defined as

$${}^{CF}I_a^{\nu} f(x) = \frac{1}{M(\nu)} \left[(1-\nu)(f(x) - f(a)) + \nu \int_a^x f(\tau) d\tau \right], \quad (3)$$

where M is a normalization function such as

$$M(\nu = 0) = M(\nu = 1) = 1.$$

Lemma 1 ([6])**.** *Assume that*

$$\nu \in (n, n+1), n = [\nu] \geq 0.$$

Let $f \in C^n([a,b])$; then,

1. *If $f(a) = 0$, then*

$${}^{CF}D_a^{(\nu)} \Big({}^{CF}I_a^{\nu} f(x) \Big) = f(x).$$

2. *We have*

$$ {}^{CF}I_a^{\nu}\left({}^{CF}D_a^{(\nu)}f(x)\right) = f(x) + \sum_{i=0}^{n} a_i x^i, a_i \in \mathbb{R}, i = 0, ..., n. $$

3. *If* $\nu \in (0,1)$, *then*

$$ {}^{CF}I_a^{\nu}\left({}^{CF}D_a^{(\nu)}f(x)\right) = f(x) - f(a). $$

In the following theorem and in our study, it is assumed that $M = 1$.

Theorem 1 ([6]). *Let us define* $L_1(t,u,v) = c_1u + c_2v$, $L_2(t,u,v) = c_3u + c_4v$. *Suppose that* L_1 *and* L_2 *are continuous functions; then, the system* (1) *has a unique solution such that*

$$ \mu = \mu_1 + \mu_2 < 1, \tag{4} $$

where $\mu_1 = \max(|c_1|, |c_2|)$, $\mu_2 = \max(|c_3|, |c_4|)$.

2.2. Haar Wavelet Basis

As in [1,7], the Haar wavelets basis on $[0,1)$ consists of the following functions:
The scaling function on $[0,1)$ is $g_1(t) = 1$ for $0 \le t < 1$ and also the wavelets functions

$$ g_i(t) = \begin{cases} 1, \ if \ t \in [\varrho_1, \varrho_2), \\ -1, \ if \ t \in [\varrho_2, \varrho_3), \\ 0, \ otherwise, \end{cases} \tag{5} $$

where $\varrho_1 = \frac{\kappa}{m}$, $\varrho_2 = \frac{\kappa+0.5}{m}$, and $\varrho_3 = \frac{\kappa+1}{m}$, $m = 2^j$, $j = 0,1,...,J$, and J is considered as resolution level for wavelet approximation and $\kappa = 0,...,m-1$ is a translation parameter where $i = m + \kappa + 1$. The value i can take $i = 2M$, $M = 2^J$ as a maximum value.

Any function $f \in L^2([0,1))$ can be expanded as

$$ f(t) = \sum_{i=1}^{+\infty} a_i g_i(t), \tag{6} $$

where

$$ a_i = \int_0^1 f(t) g_i(t) dt. $$

In reality, note that the first $2M$ terms of (6) can be considered, where M is a power of 2 $(M = 2^J)$. That is,

$$ f(t) \equiv f_{2M}(t) = \sum_{i=1}^{2M} a_i g_i(t). \tag{7} $$

The integration β times of $g_1(t)$ together with (5) is as follows:

$$ F_{i,\beta}(t) = \int_0^t \int_0^t ... \int_0^t g_i(t) dt^{\beta}, \tag{8} $$

and we obtain the following formula [1]: $F_{1,\beta}(t) = \frac{t^{\beta}}{\beta!}$ and $\forall i \ge 2$,

$$ F_{i,\beta}(t) = \frac{1}{\beta!} \begin{cases} 0, & if \quad t \in [0, \varrho_1), \\ (t-\varrho_1)^{\beta}, & if \quad t \in [\varrho_1, \varrho_2), \\ (t-\varrho_1)^{\beta} - 2(t-\varrho_2)^{\beta}, & if \quad t \in [\varrho_2, \varrho_3), \\ (t-\varrho_1)^{\beta} - 2(t-\varrho_2)^{\beta} + (t-\varrho_3)^{\beta}, & if \quad t \in [\varrho_3, 1). \end{cases} \tag{9} $$

3. Haar Wavelet Approximation Method

Here, we propose our new numerical method.

3.1. Method of Solution

Consider the system (1) for $0 \le t \le 1$:

$$\begin{cases} {}^{CF}D^{(\nu)}u(t) = c_1 u(t) + c_2 v(t) + f(t), \\ {}^{CF}D^{(\nu)}v(t) = c_3 u(t) + c_4 v(t) + h(t), \\ u(t=0) = v(t=0) = 0. \end{cases} \tag{10}$$

Suppose that

$$\begin{cases} {}^{CF}D^{(\nu)}u(t) \equiv \widetilde{{}^{CF}D^{(\nu)}}u(t) = \sum_{i=1}^{2M} a_i g_i(t), \\ {}^{CF}D^{(\nu)}v(t) \equiv \widetilde{{}^{CF}D^{(\nu)}}v(t) = \sum_{i=1}^{2M} b_i g_i(t), \end{cases} \tag{11}$$

where $a_i, b_i, i = 1, ..., 2M$ are the Haar wavelet coefficients to be determined.

By integrating (11) in the Caputo–Fabrizio sense and taking into account that $u(t=0) = v(t=0) = 0$, we obtain

$$\begin{cases} u(t) \equiv u_{2M}(t) = \sum_{i=1}^{2M} a_i I^{\nu} g_i(t), \\ v(t) \equiv v_{2M}(t) = \sum_{i=1}^{2M} b_i I^{\nu} g_i(t). \end{cases} \tag{12}$$

Using (3), we have

$$I^{(\nu)} g_i(t) = (1-\nu)(g_i(t) - g_i(0)) + \nu F_{i,1}(t), \ \forall i = 1, ..., 2M. \tag{13}$$

Substituting (13) in (12), we find

$$\begin{cases} u(t) \equiv u_{2M}(t) = \sum_{i=1}^{2M} a_i[(1-\nu)(g_i(t) - g_i(0)) + \nu F_{i,1}(t)], \\ v(t) \equiv v_{2M}(t) = \sum_{i=1}^{2M} b_i[(1-\nu)(g_i(t) - g_i(0)) + \nu F_{i,1}(t)]. \end{cases} \tag{14}$$

Using (11) and (14) and (10), we obtain the next system of equations:

$$\begin{cases} \sum_{i=1}^{2M} a_i L(i,t) + \sum_{i=1}^{2M} b_i G(i,t) = -f(t), \\ \sum_{i=1}^{2M} a_i \tilde{L}(i,t) + \sum_{i=1}^{2M} b_i \tilde{G}(i,t) = -h(t), \end{cases} \tag{15}$$

where

$$\begin{cases} L(i,t) = (c_1(1-\nu) - 1)g_i(t) + \nu c_1 F_{i,1}(t) - c_1(1-\nu)g_i(0), \\ G(i,t) = c_2(1-\nu)g_i(t) + \nu c_2 F_{i,1}(t) - c_2(1-\nu)g_i(0), \\ \tilde{L}(i,t) = c_3(1-\nu)g_i(t) + \nu c_3 F_{i,1}(t) - c_3(1-\nu)g_i(0), \\ \tilde{G}(i,t) = (c_4(1-\nu) - 1)g_i(t) + \nu c_4 F_{i,1}(t) - c_4(1-\nu)g_i(0). \end{cases} \tag{16}$$

Define the collocation points $t_\ell = \frac{\ell - 0.5}{2M}$, $\ell = 1, ..., 2M$, and replace them in the system (15); we have the following $4M \times 4M$ linear system of equations:

$$\begin{cases} \sum_{i=1}^{2M} a_i L(i,t_\ell) + \sum_{i=1}^{2M} b_i G(i,t_\ell) = -f(t_\ell), \ \ell = 1, ..., 2M, \\ \sum_{i=1}^{2M} a_i \tilde{L}(i,t_\ell) + \sum_{i=1}^{2M} b_i \tilde{G}(i,t_\ell) = -h(t_\ell), \ \ell = 1, ..., 2M. \end{cases} \tag{17}$$

By solving this system, we obtain the unknown coefficients a_i, b_i, $i = 1, ..., 2M$, and by substituting them into (14), we obtain the numerical solution of the system (1).

3.2. Analysis of Error Estimations

The error of approximation using our proposed method of the solution is studied here.

Lemma 2 ([12])**.** *Let $f \in L^2([0,1))$ be a differentiable function; then, f' is bounded on $(0,1)$ and $\widetilde{f}$ is its Haar wavelet approximation, defined by (7); then,*

$$\left\| f - \widetilde{f} \right\|_{L^2([0,1))} \leq C2^{-J}, \tag{18}$$

where C is a constant.

The Haar wavelet coefficients that are given in (11) can be estimated as follows:

$$a_i = \mathcal{O}\left(\frac{1}{2^{j+1}}\right),\ i = 2^j + \kappa + 1. \tag{19}$$

Proof. We have, first,

$$\begin{aligned} a_i &= \int_0^1 {}^{CF}D^{(\nu)}u(t)g_i(t)dt \\ &= \int_{\varrho_1}^{\varrho_2} {}^{CF}D^{(\nu)}u(t)dt - \int_{\varrho_2}^{\varrho_3} {}^{CF}D^{(\nu)}u(t)dt \\ &= (\varrho_2 - \varrho_1)^{CF}D^{(\nu)}u(\eta_1) - (\varrho_3 - \varrho_2)^{CF}D^{(\nu)}u(\eta_2), \end{aligned}$$

where $\eta_1 \in [\varrho_1, \varrho_2]$ and $\eta_2 \in [\varrho_2, \varrho_3]$.

Since $\varrho_2 - \varrho_1 = \varrho_3 - \varrho_2 = \frac{1}{2m} = \frac{1}{2^{j+1}}$,

$$a_i = \frac{(\eta_1 - \eta_2)}{2^{j+1}} \frac{d^{CF}D^{(\nu)}u}{dt}(\vartheta),\ \vartheta \in [\eta_1, \eta_2],$$

which implies that

$$|a_i| \leq \frac{1}{2^{j+1}} \left\| \frac{d^{CF}D^{(\nu)}u}{dt} \right\|_\infty. \tag{20}$$

On the other hand,

$$\begin{aligned} \frac{d^{CF}D^{(\nu)}u}{dt}(t) &= \frac{1}{1-\nu}\left[u'(t) + \int_0^t u'(\tau)\left(\frac{-\nu}{1-\nu}\right)\exp(-\nu\frac{t-\tau}{1-\nu})d\tau\right] \\ &= \frac{1}{1-\nu}\left[u'(t) - \nu^{CF}D^{(\nu)}u(t)\right]. \end{aligned}$$

This expression leads to

$$\left\| \frac{d^{CF}D^{(\nu)}u}{dt} \right\|_\infty \leq \frac{1}{1-\nu}\|u'\|_\infty + \frac{\nu}{1-\nu}\left\| {}^{CF}D^{(\nu)}u \right\|_\infty. \tag{21}$$

Now, for $\left\| {}^{CF}D^{(\nu)}u \right\|_\infty$, we have

$$\begin{aligned}\left|{}^{CF}D^{(\nu)}u(t)\right| &\leq \frac{1}{1-\nu}\|u'\|_\infty \int_0^t \exp(-\nu\frac{t-\tau}{1-\nu})d\tau \\ &\leq \frac{\|u'\|_\infty}{\nu}\left[1-\exp(-\nu\frac{t}{1-\nu})\right] \\ &\leq \frac{\|u'\|_\infty}{\nu}\left[1-\exp(\frac{-\nu}{1-\nu})\right],\end{aligned}$$

and thus,

$$\left\|{}^{CF}D^{(\nu)}u(t)\right\|_\infty \leq \frac{\|u'\|_\infty}{\nu}\left[1-\exp(\frac{-\nu}{1-\nu})\right], \quad (22)$$

and inserting (22) into (21) yields

$$\left\|\frac{d^{CF}D^{(\nu)}u}{dt}\right\|_\infty \leq \frac{2-\exp(\frac{-\nu}{1-\nu})}{1-\nu}\|u'\|_\infty.$$

Therefore,

$$|a_i| \leq \left(\frac{2-\exp(\frac{-\nu}{1-\nu})}{1-\nu}\|u'\|_\infty\right)\frac{1}{2^{j+1}}.$$

□

Lemma 3. *The function $F_{i,1}$ for $i \geq 2$ verifies the following inequality:*

$$\|F_{i,1}\|_\infty \leq \frac{1}{2^{j+1}},\ i = 2^j + \kappa + 1. \quad (23)$$

Proof. From (9), we have, $i \geq 2$

$$F_{i,1}(t) = \begin{cases} 0, & if \quad t \in [0, \varrho_1), \\ t-\varrho_1, & if \quad t \in [\varrho_1, \varrho_2), \\ -t+2\varrho_2-\varrho_1, & if \quad t \in [\varrho_2, \varrho_3), \\ 2\varrho_2-\varrho_3-\varrho_1, & if \quad t \in [\varrho_3, 1). \end{cases}$$

Note that in the interval $[\varrho_1, \varrho_2)$, the function $F_{i,1}$ is positive and increasing; then, $|F_{i,1}| \leq \varrho_2 - \varrho_1 = \frac{1}{2^{j+1}}$.

In the interval $[\varrho_2, \varrho_3)$, the function $F_{i,1}$ is positive and decreasing; then, $|F_{i,1}| \leq \varrho_2 - \varrho_1 = \frac{1}{2^{j+1}}$.

Otherwise, the function $F_{i,1}$ is null.

Thus, $\|F_{i,1}\|_\infty \leq \frac{1}{2^{j+1}}$. □

Lemma 4. *For $i \geq 2$, we define the function q_i as*

$$q_i(t) = (1-\nu)(g_i(t) - g_i(0)) + \nu F_{i,1}(t), i = 2^j + \kappa + 1, 0 \leq t < 1. \quad (24)$$

Then, we have

1. $\forall i \geq 2$, $|q_i(t)| \leq 2 - \nu$.
2. If $\kappa \neq 0$, then $\int_0^1 |q_i(t)|dt \leq \left(1 - \frac{\nu}{2}\right)\frac{1}{2^j}$.

Proof. 1. $\forall i \geq 2$; we have

$$\begin{aligned} |q_i(t)| &\leq (1-\nu)|g_i(t) - g_i(0)| + \nu \|F_{i,1}(t)\|_\infty \\ &\leq 2(1-\nu) + \nu \frac{1}{2^{j+1}} \\ &\leq 2(1-\nu) + \nu = 2 - \nu. \end{aligned}$$

2. If $\kappa \neq 0$, then $g_i(0) = 0$, and we have

$$\int_0^1 |q_i(t)|dt \leq (1-\nu) \int_0^1 |g_i(t)|dt + \nu \int_0^1 \|F_{i,1}(t)\|_\infty dt.$$

Note that

$$\int_0^1 |g_i(t)|dt = \int_{\varrho_1}^{\varrho_2} dt + \int_{\varrho_2}^{\varrho_3} dt = \frac{1}{2^j}.$$

Hence,

$$\begin{aligned} \int_0^1 |q_i(t)|dt &\leq (1-\nu)\frac{1}{2^j} + \nu \frac{1}{2^{j+1}} \\ &\leq (1 - \frac{\nu}{2})\frac{1}{2^j}. \end{aligned}$$

□

Theorem 2. *Let (u, v) be the exact solution of (1), and let (u_{2M}, v_{2M}) be its approximation formula, which is expressed in (14), so the convergence rate is estimated by*

$$\|u - u_{2M}\|_{L^2([0,1))} = \mathcal{O}\left(\frac{1}{\sqrt{M}}\right), \tag{25}$$

and

$$\|v - v_{2M}\|_{L^2([0,1))} = \mathcal{O}\left(\frac{1}{\sqrt{M}}\right). \tag{26}$$

Proof. From (14), the error square at the Jth-level resolution for the function u can be written as

$$\begin{aligned} E_M^2(u) &= \|u - u_{2M}\|^2_{L^2([0,1))} = \left\| \sum_{i=2M+1}^{+\infty} a_i[(1-\nu)(g_i(t) - g_i(0)) + \nu F_{i,1}(t)] \right\|^2_{L^2([0,1))} \\ &= \sum_{\ell=2M+1}^{+\infty} \sum_{i=2M+1}^{+\infty} a_i a_\ell \int_0^1 q_i(t) q_\ell(t) dt. \end{aligned}$$

By putting $i = 2^j + \kappa + 1$ and $\ell = 2^r + s + 1$, we obtain

$$\begin{aligned}
E_M^2(u) &= \sum_{r=J+1}^{+\infty} \sum_{s=0}^{2^r-1} \sum_{j=J+1}^{+\infty} \sum_{\kappa=0}^{2^j-1} a_{2^j+\kappa+1} a_{2^r+s+1} \int_0^1 q_{2^j+\kappa+1}(t) q_{2^r+s+1}(t) dt \\
&= \sum_{r=J+1}^{+\infty} \sum_{j=J+1}^{+\infty} a_{2^j+1} a_{2^r+1} \int_0^1 q_{2^j+1}(t) q_{2^r+1}(t) dt \\
&\quad + \sum_{r=J+1}^{+\infty} \sum_{s=1}^{2^r-1} \sum_{j=J+1}^{+\infty} a_{2^j+1} a_{2^r+s+1} \int_0^1 q_{2^j+1}(t) q_{2^r+s+1}(t) dt \\
&\quad + \sum_{r=J+1}^{+\infty} \sum_{j=J+1}^{+\infty} \sum_{\kappa=1}^{2^j-1} a_{2^j+\kappa+1} a_{2^r+1} \int_0^1 q_{2^j+\kappa+1}(t) q_{2^r+1}(t) dt \\
&\quad + \sum_{r=J+1}^{+\infty} \sum_{s=1}^{2^r-1} \sum_{j=J+1}^{+\infty} \sum_{\kappa=1}^{2^j-1} a_{2^j+\kappa+1} a_{2^r+s+1} \int_0^1 q_{2^j+\kappa+1}(t) q_{2^r+s+1}(t) dt.
\end{aligned}$$

This implies that

$$\begin{aligned}
E_M^2 &\leq \sum_{r=J+1}^{+\infty} \sum_{j=J+1}^{+\infty} |a_{2^j+1}||a_{2^r+1}| \int_0^1 |q_{2^j+1}(t)||q_{2^r+1}(t)| dt \\
&\quad + \sum_{r=J+1}^{+\infty} \sum_{s=1}^{2^r-1} \sum_{j=J+1}^{+\infty} |a_{2^j+1}||a_{2^r+s+1}| \int_0^1 |q_{2^j+1}(t)||q_{2^r+s+1}(t)| dt \\
&\quad + \sum_{r=J+1}^{+\infty} \sum_{j=J+1}^{+\infty} \sum_{\kappa=1}^{2^j-1} |a_{2^j+\kappa+1}||a_{2^r+1}| \int_0^1 |q_{2^j+\kappa+1}(t)||q_{2^r+1}(t)| dt \\
&\quad + \sum_{r=J+1}^{+\infty} \sum_{s=1}^{2^r-1} \sum_{j=J+1}^{+\infty} \sum_{\kappa=1}^{2^j-1} |a_{2^j+\kappa+1}||a_{2^r+s+1}| \int_0^1 |q_{2^j+\kappa+1}(t)||q_{2^r+s+1}(t)| dt.
\end{aligned}$$

By using Lemma (2), Lemma (3), and Lemma (4), we obtain

$$\begin{aligned}
E_M^2(u) &\leq \sum_{r=J+1}^{+\infty} \sum_{j=J+1}^{+\infty} C \frac{1}{2^{j+1}} \frac{1}{2^{r+1}} \int_0^1 (2-\nu)^2 dt \\
&\quad + \sum_{r=J+1}^{+\infty} \sum_{s=1}^{2^r-1} \sum_{j=J+1}^{+\infty} C \frac{1}{2^{j+1}} \frac{1}{2^{r+1}} \int_0^1 (2-\nu)|q_{2^r+s+1}(t)| dt \\
&\quad + \sum_{r=J+1}^{+\infty} \sum_{j=J+1}^{+\infty} \sum_{\kappa=1}^{2^j-1} C \frac{1}{2^{j+1}} \frac{1}{2^{r+1}} \int_0^1 (2-\nu)|q_{2^j+\kappa+1}(t)| dt \\
&\quad + \sum_{r=J+1}^{+\infty} \sum_{s=1}^{2^r-1} \sum_{j=J+1}^{+\infty} \sum_{\kappa=1}^{2^j-1} C \frac{1}{2^{j+1}} \frac{1}{2^{r+1}} \sqrt{\int_0^1 |q_{2^j+\kappa+1}(t)|^2 dt \int_0^1 |q_{2^r+s+1}(t)|^2 dt}.
\end{aligned}$$

Note that

$$\sqrt{\int_0^1 |q_{2^j+\kappa+1}(t)|^2 dt \int_0^1 |q_{2^r+s+1}(t)|^2 dt} =$$

$$\sqrt{\int_0^1 |q_{2^j+\kappa+1}(t)||q_{2^j+\kappa+1}(t)|dt \int_0^1 |q_{2^r+s+1}(t)||q_{2^r+s+1}(t)|dt}$$
$$\leq (2-\nu)\sqrt{\int_0^1 |q_{2^j+\kappa+1}(t)|dt \int_0^1 |q_{2^r+s+1}(t)|dt}$$
$$\leq (2-\nu)\left(1-\frac{\nu}{2}\right)\frac{1}{2^{\frac{j}{2}}}\frac{1}{2^{\frac{r}{2}}}.$$

Then,

$$\begin{aligned}
E_M^2(u) \leq & \sum_{r=J+1}^{+\infty}\sum_{j=J+1}^{+\infty} C\frac{1}{2^{j+1}}\frac{1}{2^{r+1}}(2-\nu)^2 \\
& + \sum_{r=J+1}^{+\infty}\sum_{s=1}^{2^r-1}\sum_{j=J+1}^{+\infty} C\frac{1}{2^{j+1}}\frac{1}{2^{r+1}}(2-\nu)\left(1-\frac{\nu}{2}\right)\frac{1}{2^r} \\
& + \sum_{r=J+1}^{+\infty}\sum_{j=J+1}^{+\infty}\sum_{\kappa=1}^{2^j-1} C\frac{1}{2^{j+1}}\frac{1}{2^{r+1}}(2-\nu)\left(1-\frac{\nu}{2}\right)\frac{1}{2^j} \\
& + \sum_{r=J+1}^{+\infty}\sum_{j=J+1}^{+\infty} C\frac{1}{2^{\frac{j}{2}+1}}\frac{1}{2^{\frac{r}{2}+1}}(2-\nu)\left(1-\frac{\nu}{2}\right) \\
\leq & \frac{C(2-\nu)^2}{4}\left(\frac{1}{2^J}\right)^2 + \frac{C(2-\nu)(1-\frac{\nu}{2})}{4}\left(\frac{1}{2^J}\right)^2 \\
& + \frac{C(2-\nu)(1-\frac{\nu}{2})}{4}\left(\frac{1}{2^J}\right)^2 + \frac{C(2-\nu)(1-\frac{\nu}{2})}{4\left(\sqrt{2}-1\right)^2}\left(\frac{1}{2^{\frac{J}{2}}}\right)^2 \\
\leq & \left[\frac{C(2-\nu)^2}{4} + \frac{C(2-\nu)(1-\frac{\nu}{2})}{2} + \frac{C(2-\nu)(1-\frac{\nu}{2})}{4\left(\sqrt{2}-1\right)^2}\right]\left(\frac{1}{2^{\frac{J}{2}}}\right)^2.
\end{aligned}$$

Therefore,

$$E_M(u) = \mathcal{O}\left(\frac{1}{\sqrt{M}}\right).$$

In a similar way, we prove that $E_M(v) = \mathcal{O}\left(\frac{1}{\sqrt{M}}\right)$.

Now, to verify the convergence analysis, we must prove that the following system converges to the system (1):

$$\begin{cases} \widetilde{{}^{CF}D^{(\nu)}}u(t) = c_1u_{2M}(t) + c_2v_{2M}(t) + f(t) + R_M^1(t),\ 0 \leq t \leq 1, \\ \widetilde{{}^{CF}D^{(\nu)}}v(t) = c_3u_{2M}(t) + c_4v_{2M}(t) + h(t) + R_M^2(t),\ 0 \leq t \leq 1, \end{cases} \tag{27}$$

where R_M^1 and R_M^2 represent the remainders.

By subtracting (27) from (1), we obtain

$$\begin{cases} R_M^1(t) = c_1(u(t)-u_{2M}(t)) + c_2(v(t)-v_{2M}(t)) - \left({}^{CF}D^{(\nu)}u(t) - \widetilde{{}^{CF}D^{(\nu)}}u(t)\right),\ 0 \leq t \leq 1, \\ R_M^2(t) = c_3(u(t)-u_{2M}(t)) + c_4(v(t)-v_{2M}(t)) - \left({}^{CF}D^{(\nu)}v(t) - \widetilde{{}^{CF}D^{(\nu)}}v(t)\right),\ 0 \leq t \leq 1. \end{cases} \tag{28}$$

Then,

$$\begin{cases} \|R_M^1\|_{L^2([0,1])} \leq c_1\|u - u_{2M}\|_{L^2([0,1])} + c_2\|v - v_{2M}\|_{L^2([0,1])} + \left\| {}^{CF}D^{(\nu)}u - \widetilde{{}^{CF}D^{(\nu)}u} \right\|_{L^2([0,1])}, \\ \|R_M^2\|_{L^2([0,1])} \leq c_3\|u - u_{2M}\|_{L^2([0,1])} + c_4\|v - v_{2M}\|_{L^2([0,1])} + \left\| {}^{CF}D^{(\nu)}v - \widetilde{{}^{CF}D^{(\nu)}v} \right\|_{L^2([0,1])}, \end{cases}$$

when $M \to +\infty$, we obtain $\|u - u_{2M}\|_{L^2([0,1])} \to 0$, $\|v - v_{2M}\|_{L^2([0,1])} \to 0$, $\left\| {}^{CF}D^{(\nu)}u - \widetilde{{}^{CF}D^{(\nu)}u} \right\|_{L^2([0,1])} \to 0$ and $\left\| {}^{CF}D^{(\nu)}v - \widetilde{{}^{CF}D^{(\nu)}v} \right\|_{L^2([0,1])} \to 0$.

Therefore, $\|R_M^1\|_{L^2([0,1])} \to 0$ and $\|R_M^2\|_{L^2([0,1])} \to 0$. □

4. Numerical Examples

Here, we examine examples of the problem (1) to show the efficiency of our proposed method. All computations are performed using Matlab, and the numerical results are represented in Figures 1–4.

Example 1. *The next coupled system is considered for* $0 \leq t \leq 1$*:*

$$\begin{cases} {}^{CF}D^{(0.25)}u(t) = -\frac{1}{2}u(t) + \frac{1}{2}v(t) + f(t), \\ {}^{CF}D^{(0.25)}v(t) = -\frac{1}{4}u(t) + \frac{1}{4}v(t) + h(t), \\ u(t=0) = v(t=0) = 0, \end{cases}$$

where

$$f(t) = \left(\frac{17}{10}t - \frac{8}{25}\right)\sin(t) + \left(\frac{13}{50} + \frac{9}{10}t\right)\cos(t) + \frac{6}{25}e^{-\frac{t}{3}} - \frac{1}{2}(1+t),$$

and

$$h(t) = \left(\frac{4}{25} + \frac{13}{20}t\right)\sin(t) + \left(\frac{-63}{100} - \frac{19}{20}t\right)\cos(t) - \frac{78}{25}e^{-\frac{t}{3}} - \frac{t}{4} + \frac{15}{4}.$$

The exact solution is given by

$$u(t) = t\sin(t),$$

and

$$v(t) = (1+t)(1-\cos(t)).$$

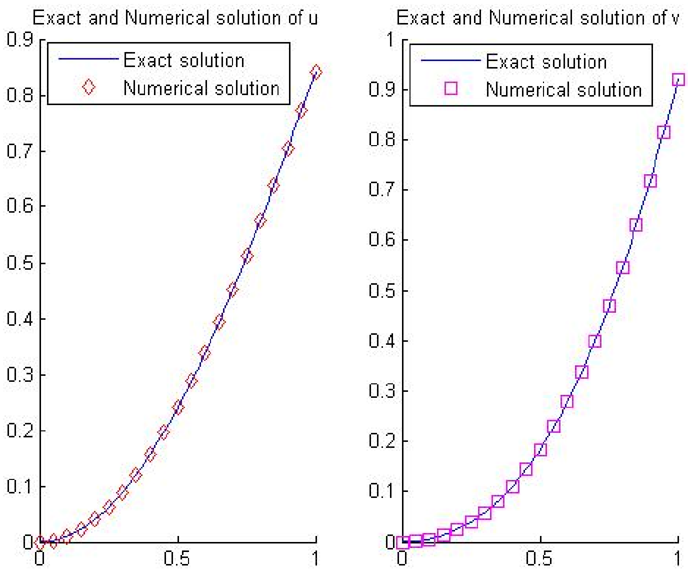

Figure 1. Numerical solution of Example 1 at level J = 3.

Example 2. *Let us consider the problem for* $0 \le t \le 1$*:*

$$\begin{cases} {}^{CF}D^{(0.5)}u(t) = \frac{1}{3}u(t) - \frac{1}{2}v(t) + f(t), \\ {}^{CF}D^{(0.5)}v(t) = \frac{1}{8}u(t) + \frac{1}{6}v(t) + h(t), \\ u(t=0) = v(t=0) = 0, \end{cases}$$

where

$$f(t) = \left(-\frac{1}{10}\cos(t) + \frac{13}{15}\sin(t) + \frac{1}{2}\right)e^t - \frac{2}{5}e^{-t},$$

and

$$h(t) = \left(-\frac{31}{30}\cos(t) + \frac{11}{40}\sin(t) + \frac{5}{6}\right)e^t + \frac{1}{5}e^{-t}.$$

The exact solution is defined as

$$u(t) = \sin(t)e^t,$$

and

$$v(t) = (1 - \cos(t))e^t.$$

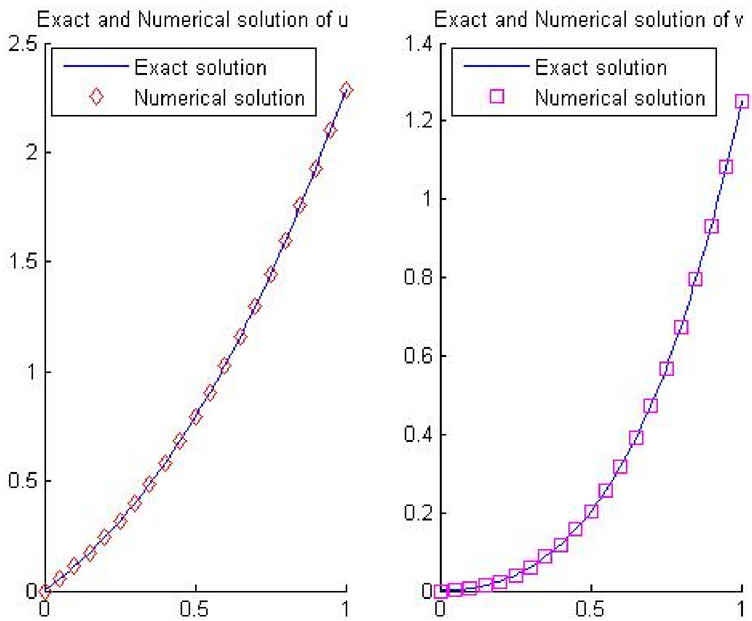

Figure 2. Numerical solution to Example 2 at level J = 3.

Example 3. *Let the next problem for* $0 \le t \le 1$ *be*

$$\begin{cases} {}^{CF}D^{(0.75)}u(t) = -\frac{1}{5}u(t) + \frac{2}{3}v(t) + f(t), \\ {}^{CF}D^{(0.75)}v(t) = \frac{1}{6}u(t) + \frac{1}{12}v(t) + h(t), \\ u(t=0) = v(t=0) = 0, \end{cases}$$

where

$$\begin{aligned} f(t) &= \left(\frac{60}{169} + \frac{93}{65}t\right)\sin(2t) + \left(\frac{48}{13}t - \frac{288}{169}\right)\cos^2(t) \\ &+ \frac{144}{169}e^{-3t} - \frac{24}{13}t + \frac{144}{169} - \frac{2}{3}(t-1)(1-\cos(t)), \end{aligned}$$

and

$$
\begin{aligned}
h(t) &= \left(-\frac{48}{25}+\frac{6}{5}t\right)\sin(t)+\left(-\frac{14}{25}-\frac{2}{5}t\right)\cos(t)-\frac{1}{6}t\sin(2t)-\frac{58}{75}e^{-3t}\\
&- \frac{1}{12}(t-1)(1-\cos(t))+\frac{4}{3}.
\end{aligned}
$$

The exact solution is given by

$$u(t)=t\sin(2t),$$

and

$$v(t)=(t-1)(1-\cos(t)).$$

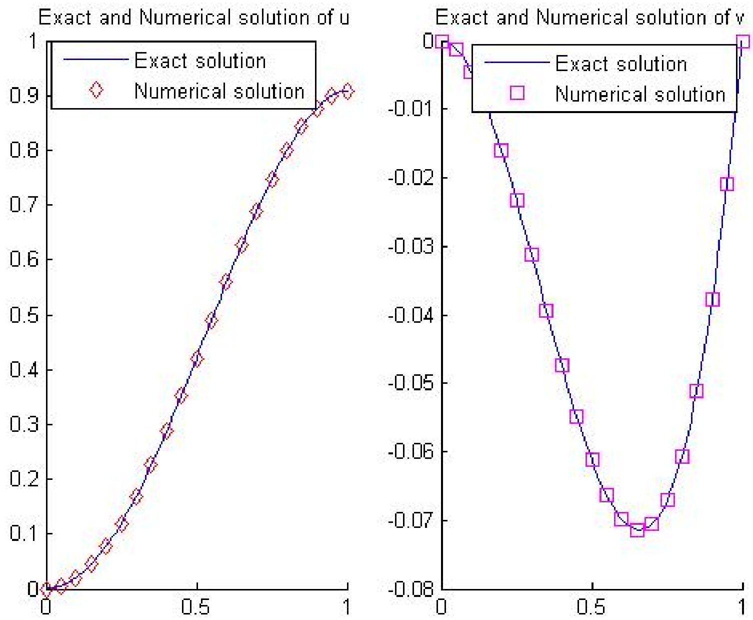

Figure 3. Numerical solution of Example 3 at level J = 3.

Example 4. *We consider the following problem:*

$$
\begin{cases}
{}^{CF}D^{0.9}u(t) = -\frac{1}{3}u(t)+\frac{2}{7}\nu(t)+f(t),\\
{}^{CF}D^{0.9}\nu(t) = \frac{1}{11}u(t)+\frac{5}{13}\nu(t)+h(t),\\
u(t=0)=\nu(t=0)=0,
\end{cases}
$$

where

$$
\begin{aligned}
f(t) &= \frac{1}{1313}\left[\frac{715e^{2t}\cos(t)-585e^{2t}\sin(t)+707\cos(t)+909\sin(t)}{e^t}\right]\\
&\quad -\frac{1422}{1313}e^{-9t}+\frac{1}{3}\cos(t)\sinh(t)-\frac{2}{7}t\sin(t)\cosh(t),\\
h(t) &= \frac{1}{8619845e^t}\Big[-4641455t\sin(t)+5783967\sin(t)\\
&\quad +4693975e^{2t}t\sin(t)+5967585t\cos(t)\Big]\\
&\quad -\frac{1}{8619845e^t}\Big[1468944\cos(t)+3764475e^{2t}\sin(t)\\
&\quad -760500e^{2t}\cos(t)+3840525te^{2t}\cos(t)Big]\\
&\quad +\frac{2229444}{8619845}e^{-9t}-\frac{1}{11}\cos(t)\sinh(t)-\frac{5}{13}t\sin(t)\cosh(t).
\end{aligned}
$$

The exact solution is given by

$$u(t) = \cos(t)\sinh(t)$$

and

$$v(t) = t\sin(t)\cosh(t)$$

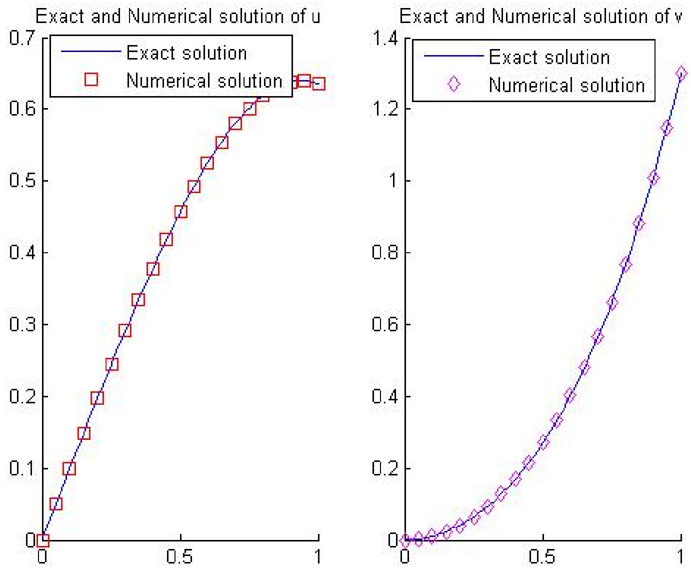

Figure 4. Numerical solution to Example 4 at the level J = 3.

We have computed the maximum absolute errors L_∞ and the rate of convergence R_M, which is defined by

$$R_M = \frac{\log(E_\infty(M/2)/E_\infty(M))}{\log 2},$$

of our proposed method; then, we have obtained the above results, which are presented in Tables 1–4.

Table 1. Comparison of the L_∞ norm with the proposed and ADM methods for Example 1.

J	M	$2M$	$L_\infty(M) = \max\lvert u_i - \tilde{u}_i\rvert$	R_M of Proposed Method	L_∞ of ADM
0	1	2	3.14×10^{-2}		3.02×10^{-1}
1	2	4	1.06×10^{-2}	1.5656	1.32×10^{-1}
2	4	8	3.15×10^{-3}	1.7485	4.28×10^{-2}
3	8	16	8.02×10^{-4}	1.9709	1.52×10^{-2}
4	16	32	2.01×10^{-4}	1.9964	4.19×10^{-3}

Table 2. Comparison of the L_∞ norm with the proposed and ADM methods for Example 2.

J	M	$2M$	$L_\infty(M) = \max\lvert v_i - \tilde{v}_i\rvert$	R_M of Proposed Method	L_∞ of ADM
0	1	2	4.01×10^{-2}		2.50×10^{-1}
1	2	4	1.25×10^{-2}	1.6781	1.15×10^{-1}
2	4	8	3.8×10^{-3}	1.7137	3.92×10^{-2}
3	8	16	9.8×10^{-4}	1.9523	1.27×10^{-2}
4	16	32	2.50×10^{-4}	1.9709	3.50×10^{-3}

Table 3. Comparison of the L_∞ norm with the proposed and ADM methods for Example 3.

J	M	$2M$	$L_\infty(M) = \max\lvert\nu_i - \tilde{\nu}_i\rvert$	R_M of Proposed Method	L_∞ of ADM
0	1	2	3.23×10^{-2}		4.11×10^{-1}
1	2	4	1.15×10^{-2}	1.4854	1.75×10^{-1}
2	4	8	3.55×10^{-3}	1.6915	4.90×10^{-2}
3	8	16	9.0×10^{-4}	1.9781	1.65×10^{-2}
4	16	32	2.4×10^{-4}	1.9069	4.8×10^{-3}

Table 4. Comparison of the L_∞ norm with the proposed and ADM methods for Example 4.

J	M	$2M$	$L_\infty(M) = \max\lvert\nu_i - \tilde{\nu}_i\rvert$	R_M of Proposed Method	L_∞ of ADM
0	1	2	3.80×10^{-2}		3.41×10^{-1}
1	2	4	1.16×10^{-2}	1.7093	1.12×10^{-1}
2	4	8	3.35×10^{-3}	1.7908	3.20×10^{-2}
3	8	16	9.01×10^{-4}	1.8914	1.02×10^{-2}
4	16	32	2.30×10^{-4}	1.9672	3.11×10^{-3}

5. Conclusions

In this work, the Haar wavelet collocation method has been used to solve coupled dynamical systems with the Caputo–Fabrizio fractional derivative. Error analysis shows that our proposed method has an exponential convergence rate. Furthermore, for four examples, the numerical solutions agree very well with the exact solutions. In addition, this method is effective and recommended. As our next work, we will apply the symmetry Haar wavelet collocation method to solve different types of the Caputo–Fabrizio implicit fractional differential equations and coupled systems; see [22,23]. The Haar wavelet collocation method is preferred to investigate the discussed problem. The advantages of this method compared to the conventional collocation method (see [24–26]), like, for example, shooting methods, are as follows:

1. It can decrease computational efforts and is suitable in terms of computation costs and the convergence rate.
2. It is suitable for the analysis of the dynamical system with fractional derivatives.
3. The convergence properties of this class of methods are very helpful.
4. The error analysis of this method is characterized by a good convergence rate.
5. Unlike the collocation method, other methods have several limitations in their applications to unlimited classes of singular problems.

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