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Gauss–Bonnet Theorem Related to the Semi-Symmetric Metric Connection in the Heisenberg Group

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Abstract: In this paper, we introduce the notion of the semi-symmetric metric connection in the Heisenberg group. Moreover, by using the method of Riemannian approximations, we define the notions of intrinsic curvature for regular curves, the intrinsic geodesic curvature of regular curves on a surface, and the intrinsic Gaussian curvature of the surface away from characteristic points in the Heisenberg group with the semi-symmetric metric connection. Finally, we derive the expressions of those curvatures and prove the Gauss–Bonnet theorem related to the semi-symmetric metric connection in the Heisenberg group.

Keywords: Heisenberg group; Gauss–Bonnet theorem; semi-symmetric metric connection; sub-Riemannian geometry

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1. Introduction

The Heisenberg group is a non-commutative nilpotent Lie group, which is a special structure of Lie groups. It usually consists of third-order upper triangular matrices whose elements can be taken from some kind of commutative ring, such as the ring of numbers or the ring of integers. The Heisenberg group is a population structure in the space of three-dimensional real numbers, and the product operation is defined as

$$(x_1, y_1, z_1) * (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2 - \frac{1}{2}(x_2 y_1 - x_1 y_2)).$$

The special nature of its structure enables this group to play an important role in mathematics. In 2003, Semmes introduced the notions of the Heisenberg group in analysis and geometry [1]. Subsequently, many researchers began to work in the Heisenberg group. In 2004, Pauls characterized minimal surfaces in terms of a sub-elliptic partial differential equation and proved an existence result for the Plateau problem. Further, he investigated the minimal surface problem in the three-dimensional Heisenberg group [2]. In 2010, Onda calculated the Ricci tensor of the Heisenberg group with the left invariant Lorentz metric g_1 and proved that g_1 satisfies the Ricci soliton equation [3]. In 2013, Yoon and Lee defined translation surfaces in the three-dimensional Heisenberg group H_3 obtained as a product of two planar curves lying in planes, which are not orthogonal, and studied minimal translation surfaces in H_3 [4]. In 2016, Zhao used the tent spaces on the Siegel upper half space to introduce the Hardy–Hausdorff spaces in the Heisenberg group. Finally, the author proved that the predual spaces of Q spaces are the Hardy–Hausdorff spaces in the Heisenberg group [5]. In 2021, Wang proved Gauss–Bonnet theorems associated with two kinds of canonical connections in the Heisenberg group [6]. In the same year, he also proved that the Gauss–Bonnet theorem is associated with two kinds of Schouten–Van Kampen affine connections in the Heisenberg group [7]. All the above studies have achieved good results, and we have found that there are many studies on the sub-Riemannian geometry of curves and surfaces in the Heisenberg group.

On the other hand, the use of semi-symmetric metric connections is very widespread. In [8], Hayden defined the notion of a semi-symmetric metric connection on a Riemannian manifold. Later, Yano investigated a Riemannian manifold endowed with a semi-symmetric metric connection whose curvature tensor vanishes if and only if the Riemannian manifold is conformally flat [9]. In [10], Imai introduced a hypersurface with the semi-symmetric metric connection and obtained the Codazzi–Ricci equations with respect to the semi-symmetric metric connection. In [11], Klepikov and Rodionov classified invariant Ricci solitons on three-dimensional Lie groups with left-invariant Riemannian metrics and semi-symmetric connections. It has been proven that there are invariant Ricci solitons with non-conformal Killing vector fields in this case. According to the relevant studies described above, there is little research on the geometric properties related to semi-symmetric connections in the Heisenberg group. The research on the Gauss–Bonnet theorems related to different connections on between Lie groups can be found at the following references ([12–18]). Under the influence of the above work, this paper attempts to research geometric properties related to the semi-symmetric connection in the Heisenberg group by employing the method of the Riemannian approximation scheme. In this paper, we introduce the sub-Riemannian geometry of curves and surfaces in the Heisenberg group with a semi-symmetric metric connection and we use the Riemannian approximation scheme to compute sub-Riemannian limits of the Gaussian curvature for a Euclidean C_2-smooth surface in the Heisenberg group away from characteristic points and signed geodesic curvature for Euclidean C_2-smooth curves on surfaces. On this basis, we prove the Gauss–Bonnet theorem related to the semi-symmetric metric connection in the Heisenberg group. For future research directions, we want to conduct research related to the different connections of the Gauss–Bonnet theorem on Lie groups.

The paper is organized as follows. In Section 2, we briefly introduce the concept of semi-symmetric metric connection and calculate the corresponding connection components and curvature components in the Heisenberg group. In Section 3, we calculate the sub-Riemannian limit of curvature of curves in the Heisenberg group. In Sections 4 and 5, we compute sub-Riemannian limits of the geodesic curvature of curves on surface and the Riemannian Gaussian curvature of surface in the Heisenberg group with the semi-symmetric metric connection. In Section 6, we prove the Gauss–Bonnet theorem related to the semi-symmetric metric connection in the Heisenberg group. Finally, we summarize the main results and discuss future research directions in Section 7.

2. Riemannian Approximates of $(\mathbb{H}, g_L)$

In this section, we introduce concepts of the Heisenberg group, the semi-symmetric metric connection, and curvature associated with the semi-symmetric metric connection. We also calculate the corresponding expressions.

Firstly, we recall the structure of the Heisenberg group in [6]. Let $\mathbb{H}$ be the Heisenberg group R^3, where the non-commutative group law is given by

$$(x_1, y_1, z_1) * (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2 - \frac{1}{2}(x_2 y_1 - x_1 y_2)),$$

and with the Riemannian metric g given by $g = dx^2 + dy^2 + (dz + \frac{1}{2}(ydx - xdy))^2$, where (x, y, z) are the standard coordinates of R^3.

Let $\widetilde{X_1}$, $\widetilde{X_2}$ and $\widetilde{X_3}$ be the vector fields on $\mathbb{H}$ given by

$$\widetilde{X_1} = \partial x_1 - \frac{x_2}{2}\partial x_3,\ \widetilde{X_2} = \partial x_2 + \frac{x_1}{2}\partial x_3,\ \widetilde{X_3} = \partial x_3, \tag{1}$$

and $span\left\{\widetilde{X_1}, \widetilde{X_2}, \widetilde{X_3}\right\} = T(\mathbb{H})$. One can check the following brackets

$$\left[\widetilde{X_1}, \widetilde{X_2}\right] = \widetilde{X_3},\ \left[\widetilde{X_2}, \widetilde{X_3}\right] = 0,\ \left[\widetilde{X_1}, \widetilde{X_3}\right] = 0. \tag{2}$$

Let $H = span\left\{\widetilde{X_1}, \widetilde{X_2}\right\}$ be the horizontal distribution on $\mathbb{H}$. If we let

$$\omega_1 = dx_1, \omega_2 = dx_2, \omega_3 = dx_3 + \frac{1}{2}(x_2 dx_1 - x_1 dx_2), \tag{3}$$

then $H = ker\omega$. To describe the Riemannian metric on $\mathbb{H}$, let $L > 0$ and define a metric

$$g_L = \omega_1 \otimes \omega_1 + \omega_2 \otimes \omega_2 + L\omega_3 \otimes \omega_3, \tag{4}$$

so that $\widetilde{X_1}, \widetilde{X_2}, \widetilde{X_3}^L := L^{-\frac{1}{2}}\widetilde{X_3}$ are the orthonormal basis on $T(\mathbb{H})$ with respect to g_L. We denote the Riemannian approximants to $\mathbb{H}$ by $(\mathbb{H}, g_L)$.

Next, let $g = g_L$ represent the Riemannian metric on $\mathbb{H}$. If $\nabla_g{}^s = 0$, then ∇^s is called a semi-symmetric metric connection on $\mathbb{H}$. Following [9], a semi-symmetric metric connection ∇^s on $\mathbb{H}$ is given by

$$\nabla^s_{\widetilde{X}}\widetilde{Y} = \nabla^L_{\widetilde{X}}\widetilde{Y} + g(\widetilde{Y}, \widetilde{X_3})\widetilde{X} - g(\widetilde{X}, \widetilde{Y})\widetilde{X_3},$$

for any vector fields $\widetilde{X}$ and $\widetilde{Y}$ on $\mathbb{H}$. Let ∇^L be the Levi-Civita connection on $\mathbb{H}$ with respect to g_L, which is determined by Lemma 2.1 in [7], where $\nabla^L_{\widetilde{X_j}}\widetilde{X_j} = 0, 1 \leq j \leq 3$, $\nabla^L_{\widetilde{X_1}}\widetilde{X_2} = \frac{1}{2}\widetilde{X_3}$, $\nabla^L_{\widetilde{X_2}}\widetilde{X_1} = -\frac{1}{2}\widetilde{X_3}$, $\nabla^L_{\widetilde{X_1}}\widetilde{X_3} = \nabla^L_{\widetilde{X_3}}\widetilde{X_1} = -\frac{L}{2}\widetilde{X_2}$, $\nabla^L_{\widetilde{X_2}}\widetilde{X_3} = \nabla^L_{\widetilde{X_3}}\widetilde{X_2} = -\frac{L}{2}\widetilde{X_1}$. So we have

Lemma 1. *Let $\mathbb{H}$ be the Heisenberg group, then*

$$\begin{aligned}
&\nabla^s_{\widetilde{X_1}}\widetilde{X_1} = -\widetilde{X_3}, \nabla^s_{\widetilde{X_1}}\widetilde{X_2} = \frac{1}{2}\widetilde{X_3}, \nabla^s_{\widetilde{X_1}}\widetilde{X_3} = L\widetilde{X_1} - \frac{L}{2}\widetilde{X_2},\\
&\nabla^s_{\widetilde{X_2}}\widetilde{X_1} = -\frac{1}{2}\widetilde{X_3}, \nabla^s_{\widetilde{X_2}}\widetilde{X_2} = -\widetilde{X_3}, \nabla^s_{\widetilde{X_2}}\widetilde{X_3} = \frac{L}{2}\widetilde{X_1} + L\widetilde{X_2},\\
&\nabla^s_{\widetilde{X_3}}\widetilde{X_1} = -\frac{L}{2}\widetilde{X_2}, \nabla^s_{\widetilde{X_3}}\widetilde{X_2} = \frac{L}{2}\widetilde{X_1}, \nabla^s_{\widetilde{X_3}}\widetilde{X_3} = 0.
\end{aligned} \tag{5}$$

Proof. We will only compute $\nabla^s_{\widetilde{X_1}}\widetilde{X_1}$ as an example. Firstly,

$$\nabla^s_{\widetilde{X_1}}\widetilde{X_1} = \nabla^L_{\widetilde{X_1}}\widetilde{X_1} + g(\widetilde{X_1}, \widetilde{X_3})\widetilde{X_1} - g(\widetilde{X_1}, \widetilde{X_1})\widetilde{X_3},$$

next, we compute

$$\nabla^L_{\widetilde{X_1}}\widetilde{X_1} = 0,$$

$$g(\widetilde{X_1}, \widetilde{X_3})\widetilde{X_1} = 0,$$

$$g(\widetilde{X_1}, \widetilde{X_1})\widetilde{X_3} = \widetilde{X_3},$$

and therefore, we obtain $\nabla^s_{\widetilde{X_1}}\widetilde{X_1} = -\widetilde{X_3}$. Other cases can be calculated using the same method. □

Finally, we finish the curvature of the connection ∇^s by $R^s(\widetilde{X}, \widetilde{Y})\widetilde{Z} = \nabla^s_{\widetilde{X}}\nabla^s_{\widetilde{Y}}\widetilde{Z} - \nabla^s_{\widetilde{Y}}\nabla^s_{\widetilde{X}}\widetilde{Z} - \nabla^s_{[\widetilde{X},\widetilde{Y}]}\widetilde{Z}$, where $\widetilde{X}, \widetilde{Y}, \widetilde{Z} \in \mathbb{H}$, we obtain the following proposition.

Proposition 1. *Let $\mathbb{H}$ be the Heisenberg group, then*

$$\begin{aligned}
&R^s(\widetilde{X_1}, \widetilde{X_2})\widetilde{X_1} = \frac{7L}{4}\widetilde{X_2}, R^s(\widetilde{X_1}, \widetilde{X_2})\widetilde{X_2} = -\frac{7L}{4}\widetilde{X_1}, R^s(\widetilde{X_1}, \widetilde{X_2})\widetilde{X_3} = 0,\\
&R^s(\widetilde{X_1}, \widetilde{X_3})\widetilde{X_1} = -\frac{L}{4}\widetilde{X_3}, R^s(\widetilde{X_1}, \widetilde{X_3})\widetilde{X_2} = -\frac{L}{2}\widetilde{X_3}, R^s(\widetilde{X_1}, \widetilde{X_3})\widetilde{X_3} = \frac{L^2}{2}\widetilde{X_2} + \frac{L^2}{4}\widetilde{X_1},\\
&R^s(\widetilde{X_2}, \widetilde{X_3})\widetilde{X_1} = \frac{L}{2}\widetilde{X_3}, R^s(\widetilde{X_2}, \widetilde{X_3})\widetilde{X_2} = -\frac{L}{4}\widetilde{X_3}, R^s(\widetilde{X_2}, \widetilde{X_3})\widetilde{X_3} = \frac{L^2}{4}\widetilde{X_2} - \frac{L^2}{2}\widetilde{X_1}.
\end{aligned} \tag{6}$$

Proof. We will only compute $R^s(\widetilde{X_1}, \widetilde{X_2})\widetilde{X_1}$ as an example. Firstly, we list the formula based on the curvature associated with the semi-symmetric metric connection

$$R^s(\widetilde{X}, \widetilde{Y})\widetilde{Z} = \nabla^s_{\widetilde{X}} \nabla^s_{\widetilde{Y}} \widetilde{Z} - \nabla^s_{\widetilde{Y}} \nabla^s_{\widetilde{X}} \widetilde{Z} - \nabla^s_{[\widetilde{X}, \widetilde{Y}]} \widetilde{Z}.$$

For example, we compute

$$\nabla^s_{\widetilde{X_1}} \nabla^s_{\widetilde{X_2}} \widetilde{X_1} = -\frac{1}{2}(L\widetilde{X_1} - \frac{L}{2}\widetilde{X_2}),$$

$$\nabla^s_{\widetilde{X_2}} \nabla^s_{\widetilde{X_1}} \widetilde{X_1} = -(\frac{L}{2}\widetilde{X_1} + L\widetilde{X_2}),$$

$$\nabla^s_{[\widetilde{X_1}, \widetilde{X_2}]} \widetilde{X_1} = -\frac{L}{2}\widetilde{X_2},$$

therefore, we obtain $R^s(\widetilde{X_1}, \widetilde{X_2})\widetilde{X_1} = \frac{7L}{4}\widetilde{X_2}$. Other cases can be calculated by using the same method. □

3. The Sub-Riemannian Limit of Curvature of Curves in $(\mathbb{H}, g_L)$

In Section 3, we will compute the sub-Riemannian limit of curvature of curves in $(\mathbb{H}, g_L)$. Our approach is to define sub-Riemannian objects as limits of horizontal objects in $(\mathbb{H}, g_L)$, where a family of metrics g_L is essentially obtained as an anisotropic blow-up of the Riemannian metric g. At the heart of this approach is the fact that the intrinsic geometry does not change with L. Let $\gamma : [a, b] \to (\mathbb{H}, g_L)$ be a regular curve, where $[a, b]$ is an open interval in R.

Definition 1. *Let $\gamma : [a, b] \to (\mathbb{H}, g_L)$ be a Euclidean C^1-smooth curve. We say that γ is regular if $\dot{\gamma} \neq 0$ for every $t \in [a, b]$. Moreover, we say that $\gamma(t)$ is a horizontal point of γ if*

$$\omega(\dot{\gamma}(t)) = \frac{\dot{\gamma}_2(t)}{\gamma_1(t)} - \dot{\gamma}_3(t) = 0,$$

where $\gamma(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t))$.

As is well known, if γ is a curve with arc length parametrization, then the standard definition of curvature for γ in Riemannian geometry is $\widetilde{\kappa}^L_\gamma = \left\| \nabla^s_{\dot{\gamma}} \dot{\gamma} \right\|$. If γ is a curve with an arbitrary parametrization, then we give the definitions as follows:

Definition 2. *Let $\gamma : [a, b] \to (\mathbb{H}, g_L)$ be a Euclidean C^2-smooth regular curve in the Riemannian manifold $(\mathbb{H}, g_L)$. The curvature $\widetilde{\kappa}^L_\gamma$ of γ at $\gamma(t)$ can be defined as*

$$\widetilde{\kappa}^L_\gamma := \sqrt{\frac{\left\| \nabla^s_{\dot{\gamma}} \dot{\gamma} \right\|^2_L}{\|\dot{\gamma}\|^4_L} - \frac{\left\langle \nabla^s_{\dot{\gamma}} \dot{\gamma}, \dot{\gamma} \right\rangle^2_L}{\|\dot{\gamma}\|^6_L}}. \tag{7}$$

Proposition 2. *Let $\gamma : [a, b] \to (\mathbb{H}, g_L)$ be a Euclidean C^2-smooth regular curve in the Riemannian manifold $(\mathbb{H}, g_L)$, then*

$$\begin{aligned}\widetilde{\kappa}_\gamma^L =&\{\{[\ddot{\gamma}_1(t)+L\omega(\dot{\gamma}(t))(\dot{\gamma}_1(t)+\dot{\gamma}_2(t))]^2\\&+[\ddot{\gamma}_2(t)+L\omega(\dot{\gamma}(t))(\dot{\gamma}_2(t)-\dot{\gamma}_1(t))]^2\\&+L[-\dot{\gamma}_1(t)^2-\dot{\gamma}_2(t)^2+\frac{d}{dt}(\omega(\dot{\gamma}(t)))]^2\}\\&\times[\dot{\gamma}_1(t)^2+\dot{\gamma}_2(t)^2+L\omega(\dot{\gamma}(t))^2]^{-2}\\&-\{\dot{\gamma}_1(t)[\ddot{\gamma}_1(t)+L\omega(\dot{\gamma}(t))(\dot{\gamma}_1(t)+\dot{\gamma}_2(t))]\\&+\dot{\gamma}_2(t)[\ddot{\gamma}_2(t)+L\omega(\dot{\gamma}(t))(\dot{\gamma}_2(t)-\dot{\gamma}_1(t))]\\&+L\omega(\dot{\gamma}(t))[-\dot{\gamma}_1(t)^2-\dot{\gamma}_2(t)^2+\frac{d}{dt}(\omega(\dot{\gamma}(t)))]\}^2\\&\times[\dot{\gamma}_1(t)^2+\dot{\gamma}(t)^2+L\omega(\dot{\gamma}(t))^{-3}]\}^{-2}.\end{aligned} \tag{8}$$

In particular, when $\gamma(t)$ is a horizontal point of γ, then

$$\begin{aligned}\widetilde{\kappa}_\gamma^L =&\{[\ddot{\gamma}_1(t)^2+\ddot{\gamma}_2(t)^2+L(-\dot{\gamma}_1(t)^2-\dot{\gamma}_2(t)^2+\frac{d}{dt}(\omega(\dot{\gamma}(t)))]^2\\&\times[\dot{\gamma}_1(t)^2+\dot{\gamma}_2(t)^2]^{-2}-[\dot{\gamma}_1(t)\ddot{\gamma}_2(t)+\dot{\gamma}_2(t)\ddot{\gamma}_2)]^2\times[\dot{\gamma}_1(t)^2+\dot{\gamma}_2(t)^2]^{-3}\}^{-2}.\end{aligned} \tag{9}$$

Proof. By $\gamma(t)=(\gamma_1(t),\gamma_2(t),\gamma_3(t))$, we have

$$\dot{\gamma}(t)=\dot{\gamma}_1(t)\widetilde{X_1}+\dot{\gamma}_2(t)\widetilde{X_2}+\omega(\dot{\gamma}(t))\widetilde{X_3}. \tag{10}$$

By (5), we have

$$\begin{aligned}\nabla^s_{\dot{\gamma}}\widetilde{X_1}&=\dot{\gamma}_1(t)\nabla^s_{\widetilde{X_1}}\widetilde{X_1}+\dot{\gamma}_2(t)\nabla^s_{\widetilde{X_1}}\widetilde{X_2}+\omega(\dot{\gamma}(t))\nabla^s_{\widetilde{X_1}}\widetilde{X_3}\\&=-\dot{\gamma}_1(t)\widetilde{X_3}-\frac{1}{2}\dot{\gamma}_2(t)\widetilde{X_3}-\frac{L}{2}(\omega(\dot{\gamma}(t))\widetilde{X_2},\\\nabla^s_{\dot{\gamma}}\widetilde{X_2}&=\dot{\gamma}_1(t)\nabla^s_{\widetilde{X_1}}\widetilde{X_2}+\dot{\gamma}_2(t)\nabla^s_{\widetilde{X_2}}\widetilde{X_2}+\omega(\dot{\gamma}(t))\nabla^s_{\widetilde{X_3}}\widetilde{X_2}\\&=\frac{1}{2}\dot{\gamma}_1(t)\widetilde{X_3}-\dot{\gamma}_2(t)\widetilde{X_3}+\frac{L}{2}(\omega(\dot{\gamma}(t))\widetilde{X_1},\\\nabla^s_{\dot{\gamma}}\widetilde{X_3}&=\dot{\gamma}_1(t)\nabla^s_{\widetilde{X_1}}\widetilde{X_3}+\dot{\gamma}_2(t)\nabla^s_{\widetilde{X_2}}\widetilde{X_3}+\omega(\dot{\gamma}(t))\nabla^s_{\widetilde{X_3}}\widetilde{X_3}\\&=(L\widetilde{X_1}-\frac{L}{2}\widetilde{X_2})\dot{\gamma}_1(t)+(\frac{L}{2}\widetilde{X_1}+L\widetilde{X_2})\dot{\gamma}_2(t).\end{aligned} \tag{11}$$

By (11), we obtain

$$\begin{aligned}\nabla^s_{\dot{\gamma}}\dot{\gamma}=&\nabla^s_{\dot{\gamma}}\dot{\gamma}_1(t)\widetilde{X_1}+\dot{\gamma}_2(t)\widetilde{X_2}+\omega(\dot{\gamma}(t))\widetilde{X_3}\\=&\ddot{\gamma}_1(t)\widetilde{X_1}+\dot{\gamma}_1(t)\nabla^s_{\dot{\gamma}}\widetilde{X_1}+\ddot{\gamma}_2(t)\widetilde{X_2}+\dot{\gamma}_2(t)\nabla^s_{\dot{\gamma}}\widetilde{X_2}+\frac{d}{dt}(\omega(\dot{\gamma}(t)))\widetilde{X_3}+\omega(\dot{\gamma}(t))\nabla^s_{\dot{\gamma}}\widetilde{X_3}\\=&\ddot{\gamma}_1(t)\widetilde{X_1}+\dot{\gamma}_1(t)(-\widetilde{X_3}\dot{\gamma}_2(t)-\frac{1}{2}\widetilde{X_3}\dot{\gamma}_2(t)-\frac{L}{2}\widetilde{X_2}(\omega(\dot{\gamma}(t)))\\&+\ddot{\gamma}_2(t)\widetilde{X_2}+\dot{\gamma}_2(t)(\frac{1}{2}\widetilde{X_3}\dot{\gamma}_1(t)\widetilde{X_3}\dot{\gamma}_2(t)+\frac{L}{2}\widetilde{X_1}(\omega(\dot{\gamma}(t)))\\&+\frac{d}{dt}(\omega(\dot{\gamma}(t)))\widetilde{X_3}+\omega(\dot{\gamma}(t))[(L\widetilde{X_1}-\frac{L}{2}\widetilde{X_2})\dot{\gamma}_1(t)+(\frac{L}{2}\widetilde{X_1}+L\widetilde{X_2})\dot{\gamma}_2(t)],\\=&[\ddot{\gamma}_1(t)+L\omega(\dot{\gamma}(t))(\dot{\gamma}_1(t)+\dot{\gamma}_2(t))]\widetilde{X_1}\\&+[\ddot{\gamma}_2(t)+L\omega(\dot{\gamma}(t))(\dot{\gamma}_2(t)-\dot{\gamma}_1(t))]\widetilde{X_2}\\&+[-\dot{\gamma}_1(t)^2-\dot{\gamma}_2(t)^2+\frac{d}{dt}(\omega(\dot{\gamma}(t)))]\widetilde{X_3}.\end{aligned} \tag{12}$$

By (7) and (10), we have

$$\left\|\nabla^s_{\dot\gamma}\dot\gamma\right\|^2_L = [\ddot\gamma_1(t) + L\omega(\dot\gamma(t))(\dot\gamma_1(t) + L\dot\gamma_2(t))]^2 + [\ddot\gamma_2(t) + L\omega(\dot\gamma(t))(\dot\gamma_2(t) - L\dot\gamma(t))]^2 + [-\dot\gamma_1(t)^2 - \dot\gamma_2(t)^2 + \frac{d}{dt}(\omega(\dot\gamma(t)))]^2, \tag{13}$$

$$\|\dot\gamma\|^4_L = \left[\dot\gamma_1(t)^2 + \dot\gamma_2(t)^2 + L\omega(\dot\gamma(t))^2\right]^2, \tag{14}$$

$$\left\langle\nabla^s_{\dot\gamma}\dot\gamma, \dot\gamma\right\rangle^2_L = \{\dot\gamma_1(t)[\ddot\gamma_1(t) + L\omega(\dot\gamma(t))(\dot\gamma_1(t) + L\dot\gamma_2(t))] + \dot\gamma_2(t)[\ddot\gamma_2(t) + L\omega(\dot\gamma(t))(\dot\gamma_2(t) - L\dot\gamma_1(t))] + L\omega(\dot\gamma(t))[-\dot\gamma_1(t)^2 - \dot\gamma_2(t)^2 + \frac{d}{dt}(\omega(\dot\gamma(t)))]\}^2 \tag{15}$$

and

$$\|\dot\gamma\|^6_L = [\dot\gamma_1(t)^2 + \dot\gamma_2(t)^2 + L\omega(\dot\gamma(t))^2]^3.$$

By Definition 2, we obtain Proposition 2. □

Definition 3. *Let $\gamma : [a,b] \to (\mathbb{H}, g_L)$ be a Euclidean C^2-smooth regular curve in the Riemannian manifold $(\mathbb{H}, g_L)$, the intrinsic curvature $\widetilde{\kappa}^\infty_\gamma$ of γ at $\gamma(t)$ is defined as*

$$\widetilde{\kappa}^\infty_\gamma := \lim_{L\to\infty} \widetilde{\kappa}^L_\gamma,$$

if the limit exists.

We introduce the following notation: for continuous functions $f_1, f_2 : (0, +\infty) \to \mathbb{R}$,

$$f_1(L) \sim f_2(L), \text{ as } L \to +\infty \Leftrightarrow \lim_{L\to\infty} \frac{f_1(L)}{f_2(L)} = 1.$$

Proposition 3. *Let $\gamma : [a,b] \to (\mathbb{H}, g_L)$ be a Euclidean C^2-smooth regular curve in the Riemannian manifold $(\mathbb{H}, g_L)$.*

(1) When $\omega(\dot\gamma(t)) \neq 0$, we have

$$\widetilde{\kappa}^\infty_\gamma = \frac{\sqrt{2(\dot\gamma_1(t)^2 + \dot\gamma_2(t)^2)}}{|\omega(\dot\gamma(t))|}. \tag{16}$$

(2) When $\omega(\dot\gamma(t)) = 0$ and $\frac{d}{dt}(\omega(\dot\gamma(t))) = 0$, we have

$$\widetilde{\kappa}^\infty_\gamma = \frac{|\ddot\gamma_1(t)\dot\gamma_2(t) - \ddot\gamma_2(t)\dot\gamma_1(t)|}{|\dot\gamma_1(t)^2 + \dot\gamma_2(t)^2|}. \tag{17}$$

(3) When $\omega(\dot\gamma(t)) = 0$ and $\frac{d}{dt}(\omega(\dot\gamma(t))) \neq 0$, we have

$$\lim_{L\to\infty} \frac{\widetilde{\kappa}^L_\gamma}{\sqrt{L}} = \frac{|\frac{d}{dt}(\omega(\dot\gamma(t)))|}{|\dot\gamma_1(t)^2 + \dot\gamma_2(t)^2|}. \tag{18}$$

Proof. When $\omega(\dot\gamma(t)) \neq 0$, we have

$$\left\|\nabla^s_{\dot\gamma}\gamma\right\|^2_L \sim 2\omega(\dot\gamma(t))^2(\dot\gamma_1(t)^2 + \dot\gamma_2(t)^2)L^2 \text{ as } L \to +\infty,$$

$$\|\dot\gamma\|^2_L \sim L\omega(\dot\gamma(t))^2,\ \left\langle\nabla^s_{\dot\gamma}\dot\gamma, \dot\gamma\right\rangle^2_L \sim O\left(L^2\right) \text{ as } L \to +\infty,$$

therefore

$$\frac{\left\|\nabla^s_{\dot\gamma}\dot\gamma\right\|^2_L}{\|\dot\gamma\|^4_L} \to \frac{2(\dot\gamma_1(t)^2+\dot\gamma_2(t)^2)}{\omega(\dot\gamma(t))^2}\text{as } L\to+\infty,$$

$$\frac{\left\langle\nabla^s_{\dot\gamma}\dot\gamma,\dot\gamma\right\rangle^2_L}{\|\dot\gamma\|^6_L}\to 0 \text{ as } L\to+\infty,$$

$$\widetilde{\kappa}^{\infty}_{\gamma} = \frac{|\ddot\gamma_1(t)\dot\gamma_2(t)-\ddot\gamma_2(t)\dot\gamma_1(t)|}{|\dot\gamma_1(t)^2+\dot\gamma_2(t)^2|}. \tag{19}$$

When $\omega(\dot\gamma(t))=0$ and $\frac{d}{dt}(\omega(\dot\gamma(t)))=0$, we have

$$\left\|\nabla^s_{\dot\gamma}\gamma\right\|^2_L \sim L\left[\frac{d}{dt}(\omega(\dot\gamma(t)))\right]^2 \text{ as } L\to+\infty,$$

$$\|\dot\gamma\|^2_L=(\dot\gamma_1(t)^2+\dot\gamma_2(t)^2)^2,\ \left\langle\nabla^s_{\dot\gamma}\dot\gamma,\dot\gamma\right\rangle^2_L\sim O(1) \text{ as } L\to+\infty.$$

When $\omega(\dot\gamma(t))=0$ and $\frac{d}{dt}(\omega(\dot\gamma(t)))\neq 0$, we obtain

$$\lim_{L\to\infty}\frac{\widetilde{\kappa}^L_{\gamma}}{\sqrt{L}} = \frac{|\frac{d}{dt}(\omega(\dot\gamma(t)))|}{|\dot\gamma_1(t)^2+\dot\gamma_2(t)^2|}.$$

□

4. The Sub-Riemannian Limit of Geodesic Curvature of Curves on Surfaces in $(\mathbb{H}, g_L)$

In this section, we define the notions of geodesic curvature, intrinsic geodesic curvature, signed geodesic curvature and intrinsic signed geodesic curvature for Euclidean C^2-smooth regular curves in $(\mathbb{H}, g_L)$ and calculate their expressions.

We will determine that a surface $\Sigma^1\subset(\mathbb{H},g_L)$ is regular if Σ^1 is a Euclidean C^2-smooth compact and oriented surface. In particular, we will assume that there exists a Euclidean C^2-smooth function $u:\mathbb{H}\to\mathbb{R}$, such that

$$\Sigma^1=\{(x,y,z)\in\mathbb{G}: u(x,y,z)=0\},$$

and $u_x\partial_x+u_y\partial_y+u_z\partial_z\neq 0$. Let $\nabla^s_H u(X)=\widetilde{X_1}(u)\widetilde{X_1}+\widetilde{X_2}(u)\widetilde{X_2}$. A point $x\in\Sigma^1$ is referred to as characteristic when $\nabla^s_H u(x)=0$. Next, we define the characteristic set by

$$C(\Sigma^1):=\{(x,y,z)\in\Sigma^1|\nabla^s_H u(x,y,z)=0\}.$$

Our computations will be local and will be distanced from feature points of Σ^1. We begin by defining $a:=\widetilde{X_1}u, b:=\widetilde{X_2}u$ and $r:=\widetilde{X_3}^L u$. Let

$$\begin{aligned}&l:=\sqrt{a^2+b^2}, l_L:=\sqrt{a^2+b^2+r^2}, \bar a:=\frac{a}{l},\\&\bar b:=\frac{b}{l}, \bar a_L:=\frac{a}{l_L}, \bar b_L:=\frac{b}{l_L}, \bar r_L:=\frac{r}{l_L}.\end{aligned} \tag{20}$$

In particular, $\bar a^2+\bar b^2=1$. At every non-characteristic point, these functions are well defined.

Let

$$v_L=\bar a_L\widetilde{X_1}+\bar b_L\widetilde{X_2}+\bar r_L\widetilde{X_3}^L, e_1=\bar b\widetilde{X_1}-\bar a\widetilde{X_2}, e_2=\bar r_L\bar a\widetilde{X_1}+\bar r_L\bar b\widetilde{X_2}-\frac{l}{l_L}\widetilde{X_3}^L, \tag{21}$$

where v_L is the Riemannian unit normal vector to Σ^1 and e_1, e_2 form the orthonormal basis of Σ^1. Using $T\Sigma^1$, we define a linear transformation $J_L: T\Sigma^1\to T\Sigma^1$ such that

$$J_L(e_1) := e_2, J_L(e_2) := -e_1. \tag{22}$$

For every $U, V \in T\Sigma^1$, we have $\nabla^{\Sigma^1,s}_U V = \pi \nabla^s_U V$ where $\pi : TH \to T\Sigma^1$ is the projection. So $\nabla^{\Sigma^1,s}$ is the semi-symmetric metric connection on Σ^1 with respect to the metric g_L and

$$\nabla^{\Sigma^1,s}_{\dot\gamma}\dot\gamma = \langle \nabla^s_{\dot\gamma}\dot\gamma, e_1\rangle_L e_1 + \langle \nabla^s_{\dot\gamma}\dot\gamma, e_2\rangle_L e_2, \tag{23}$$

we obtain

$$\begin{aligned}\nabla^{\Sigma^1,s}_{\dot\gamma}\dot\gamma =&\{\bar{b}[\ddot\gamma_1(t) + L\omega(\dot\gamma(t))(\dot\gamma_1(t) + \dot\gamma_2(t))] \\ &- \bar{a}[\ddot\gamma_2(t) + L\omega(\dot\gamma(t))(\dot\gamma_2(t) - \dot\gamma_1(t))]\}e_1 \\ &+ \{\bar{r}_L\bar{a}[\ddot\gamma_1(t) + L\omega(\dot\gamma(t)(\dot\gamma_1(t) + \dot\gamma_2(t)] \\ &+ \bar{r}_L\bar{b}[\ddot\gamma_2(t) + L\omega(\dot\gamma(t))(\dot\gamma_2(t) - \dot\gamma_1(t))] \\ &- \frac{l}{l_L}L^{\frac{1}{2}}[-\dot\gamma_1(t)^2 - \dot\gamma_2(t)^2 + \frac{d}{dt}(\omega(\dot\gamma(t)))]\}e_2.\end{aligned} \tag{24}$$

If $\omega(\dot\gamma(t)) = 0$, then

$$\begin{aligned}\nabla^{\Sigma^1,s}_{\dot\gamma}\dot\gamma =&\{\bar{b}\ddot\gamma_1(t) - \bar{a}\ddot\gamma_2(t)\}e_1 \\ &+ \{\bar{r}_L\bar{a}\ddot\gamma_1(t) + \bar{r}_L\bar{b}\ddot\gamma_2(t) - \frac{l}{l_L}L^{\frac{1}{2}}[-\dot\gamma_1(t)^2 - \dot\gamma_2(t)^2 + \frac{d}{dt}(\omega(\dot\gamma(t)))]\}e_2.\end{aligned} \tag{25}$$

Definition 4. *Let $\Sigma^1 \subset (\mathbb{H}, g_L)$ be a regular surface and $\gamma : [a,b] \to \Sigma$ be a Euclidean C^2-smooth regular curve. We define the geodesic curvature $\widetilde{\kappa}^L_{\gamma,\Sigma^1}$ of γ at $\gamma(t)$, then*

$$\widetilde{\kappa}^L_{\gamma,\Sigma^1} := \sqrt{\frac{\|\nabla^{\Sigma^1,s}_{\dot\gamma}\dot\gamma\|^2_{\Sigma^1,L}}{\|\dot\gamma\|^4_{\Sigma^1,s}} - \frac{\langle \nabla^{\Sigma^1,s}_{\dot\gamma}\dot\gamma, \dot\gamma\rangle^2_{\Sigma^1,L}}{\|\dot\gamma\|^6_{\Sigma^1,L}}}. \tag{26}$$

Definition 5. *Let $\Sigma^1 \subset (\mathbb{H}, g_L)$ be a regular surface and $\gamma : [a,b] \to \Sigma$ be a Euclidean C^2-smooth regular curve. The intrinsic geodesic curvature $\widetilde{\kappa}^\infty_{\gamma,\Sigma^1}$ of γ at $\gamma(t)$ is defined as*

$$\widetilde{\kappa}^\infty_{\gamma,\Sigma^1} := \lim_{L\to+\infty} \widetilde{\kappa}^L_{\gamma,\Sigma^1},$$

if the limit exists.

Proposition 4. *Let $\Sigma^1 \subset (\mathbb{H}, g_L)$ be a regular surface and $\gamma : [a,b] \to \Sigma^1$ be a Euclidean C^2-smooth regular curve.*

(1) When $\omega(\dot\gamma(t)) \neq 0$, we have

$$\widetilde{\kappa}^\infty_{\gamma,\Sigma^1} = \frac{|\bar{b}(\dot\gamma_1(t) + \dot\gamma_2(t)) + \bar{a}(\dot\gamma_2(t) - \dot\gamma_1(t))|}{|\omega(\dot\gamma(t))|}. \tag{27}$$

(2) When $\omega(\dot\gamma(t)) = 0$ and $\frac{d}{dt}(\omega(\dot\gamma(t))) = 0$, we have

$$\widetilde{\kappa}^\infty_{\gamma,\Sigma^1} = 0.$$

(3) When $\omega(\dot\gamma(t)) = 0$ and $\frac{d}{dt}(\omega(\dot\gamma(t))) \neq 0$, we have

$$\lim_{L\to+\infty} \frac{\widetilde{\kappa}^L_{\gamma,\Sigma^1}}{\sqrt{L}} = \frac{|\frac{d}{dt}(\omega(\dot\gamma(t)))|}{[\bar{b}\dot\gamma_1(t) - \bar{a}\dot\gamma_2(t)]^2}. \tag{28}$$

Proof. By (10) and $\dot{\gamma} \in T\Sigma^1$, we have

$$\dot{\gamma}(t) = \dot{\gamma}_1(t)\widetilde{X_1} + \dot{\gamma}_2(t)\widetilde{X_2} + \omega(\dot{\gamma}(t))\widetilde{X_3}.$$

By (23), we have

$$\begin{aligned}\dot{\gamma}(t) &= me_1 + ne_2\\ &= m(\bar{b}\widetilde{X_1} - \bar{a}\widetilde{X_2}) + n(\bar{r}_L\bar{a}\widetilde{X_1} + \bar{r}_L\bar{b}\widetilde{X_2} - \frac{l}{l_L}\widetilde{X_3}^L)\\ &= (m\bar{b} + n\bar{r}_L\bar{a})\widetilde{X_1} + (-m\bar{a} + n\bar{r}_L\bar{b})\widetilde{X_2} - \frac{nl}{l_L}L^{-\frac{1}{2}}\widetilde{X_3}.\end{aligned}$$

Comparing the above equations, we obtain

$$\begin{cases} m\bar{b} + n\bar{r}_L\bar{a} = \dot{\gamma}_1(t),\\ -m\bar{a} + n\bar{r}_L\bar{b} = \dot{\gamma}_2(t),\\ -\frac{nl}{l_L}L^{-\frac{1}{2}} = \omega(\dot{\gamma}(t)),\end{cases}$$

from which

$$\begin{cases} m = \dot{\gamma}_1(t)\bar{b} - \dot{\gamma}_2(t)\bar{a},\\ n = -L^{\frac{1}{2}}\frac{l_L}{l}\omega(\dot{\gamma}(t)).\end{cases}$$

This proves the following:

$$\dot{\gamma} = (\dot{\gamma}_1(t)\bar{b} - \dot{\gamma}_2(t)\bar{a})e_1 - \frac{l_L}{l}L^{\frac{1}{2}}\omega(\dot{\gamma}(t))e_2, \tag{29}$$

by (24), we have

$$\begin{aligned}||\nabla^{\Sigma^1,S}_{\dot{\gamma}}\dot{\gamma}||^2 =&\{\bar{b}[\ddot{\gamma}_1(t) + L\omega(\dot{\gamma}(t))(\dot{\gamma}_1(t) + \dot{\gamma}_2(t))]\\ &- \bar{a}[\ddot{\gamma}_2(t) + L\omega(\dot{\gamma}(t))(\dot{\gamma}_2(t) - \dot{\gamma}_1(t))]\}^2\\ &+\{\bar{r}_L\bar{a}[\ddot{\gamma}_1(t) + L\omega(\dot{\gamma}(t))(\dot{\gamma}_1(t) + \dot{\gamma}_2(t))]\\ &+ \bar{r}_L\bar{b}[\ddot{\gamma}_2(t) + L\omega(\dot{\gamma}(t))(\dot{\gamma}_2(t) - \dot{\gamma}_1(t))]\\ &- \frac{l}{l_L}L^{\frac{1}{2}}[-\dot{\gamma}_1(t)^2 - \dot{\gamma}_2(t)^2 + \frac{d}{dt}(\omega(\dot{\gamma}(t)))]\}^2\\ \sim&[\bar{b}(\dot{\gamma}_1(t) + \dot{\gamma}_2(t)) - \bar{a}(\dot{\gamma}_2(t) - \dot{\gamma}_1(t))]^2\omega(\dot{\gamma}(t))^2L^2\\ &+ [\bar{r}_L\bar{a}(\dot{\gamma}_1(t) + \dot{\gamma}_2(t)) + \bar{r}_L\bar{b}(\dot{\gamma}_2(t) - \dot{\gamma}_1(t))]\omega(\dot{\gamma}(t))^2L^2.\end{aligned} \tag{30}$$

Similarly, when $\omega(\dot{\gamma}(t)) \neq 0$, we have

$$\begin{aligned}\|\dot{\gamma}\|_{\Sigma^1} &= \sqrt{(\dot{\gamma}_1(t)\bar{b} - \dot{\gamma}_2(t)\bar{a})^2 + (\frac{l_L}{l})^2L\omega^2(\dot{\gamma}(t))}\\ &\sim L^{\frac{1}{2}}|\omega(\dot{\gamma}(t))| \text{ as } L \to +\infty,\end{aligned} \tag{31}$$

by (24) and (28), we obtain

$$\begin{aligned}\langle \nabla^{\Sigma^1,s}_{\dot{\gamma}} \dot{\gamma}, \dot{\gamma} \rangle_{\Sigma^1,L} =& (\bar{b}\dot{\gamma}_1(t) - \bar{a}\dot{\gamma}_2(t))\{\bar{b}[\ddot{\gamma}_1(t) + L\omega(\dot{\gamma}(t))(\dot{\gamma}_1(t) + \dot{\gamma}_2(t))] \\ & - \bar{a}[\ddot{\gamma}_2(t) + L\omega(\dot{\gamma}(t))(\dot{\gamma}_2(t) - \dot{\gamma}_1(t))]\} \\ & - \frac{l_L}{l} L^{\frac{1}{2}} \omega(\dot{\gamma}_1(t))\{\bar{r}_L \bar{a}[\ddot{\gamma}_1(t) + L\omega(\dot{\gamma}(t))(\dot{\gamma}_1(t) + \dot{\gamma}_2(t))] \\ & + \bar{r}_L \bar{b}[\ddot{\gamma}_2(t) + L\omega(\dot{\gamma}(t))(\dot{\gamma}_2(t) - \dot{\gamma}_1(t))] \\ & - \frac{l}{l_L} L^{\frac{1}{2}}[-\dot{\gamma}_1(t)^2 - \dot{\gamma}_2(t)^2 + \frac{d}{dt}(\omega(\dot{\gamma}(t)))]\} \\ \sim & M_0 L,\end{aligned} \tag{32}$$

where M_0 does not depend on L. So, we have

$$\begin{aligned}\widetilde{k}^{\infty}_{\gamma,\Sigma^1} &= \lim_{L\to+\infty} \widetilde{k}^{L}_{\gamma,\Sigma^1} \\ &= \frac{|\bar{b}(\dot{\gamma}_1(t) + \dot{\gamma}_2(t)) + \bar{a}(\dot{\gamma}_1(t) - \dot{\gamma}_2(t))|}{|\omega(\dot{\gamma}(t))|}.\end{aligned}$$

When $\omega(\dot{\gamma}(t)) = 0$ and $\frac{d}{dt}(\omega(\dot{\gamma}(t))) = 0$, we have

$$\begin{aligned}\|\nabla^{\Sigma^1,s}_{\dot{\gamma}} \dot{\gamma}\|^2_{\Sigma^1,L} =& (\bar{b}\ddot{\gamma}_1(t) - \ddot{\gamma}_2(t))^2 + [\bar{r}_L \bar{a}\ddot{\gamma}_1(t) + \bar{r}_L \bar{b}\ddot{\gamma}_2(t) - \frac{l}{l_L} L^{\frac{1}{2}}(-\dot{\gamma}_1(t)^2 - \dot{\gamma}_2(t)^2)] \\ \sim & [\bar{b}\ddot{\gamma}_1(t) - \bar{a}\ddot{\gamma}_2(t)]^2 \text{ as } L \to +\infty\end{aligned} \tag{33}$$

and

$$\|\dot{\gamma}\|_{\Sigma^1,L} = |\bar{b}\dot{\gamma}_1(t) - \bar{a}\dot{\gamma}_2(t)|. \tag{34}$$

Let $P = \bar{b}\dot{\gamma}_1(t) - \bar{a}\dot{\gamma}_2(t)$, $Q = \bar{b}\ddot{\gamma}_1(t) - \bar{a}\ddot{\gamma}_2(t)$, then

$$\langle \nabla^{\Sigma^1,s}_{\dot{\gamma}} \dot{\gamma}, \dot{\gamma} \rangle_{\Sigma^1,L} = PQ, \tag{35}$$

we obtain

$$\widetilde{\kappa}^{\infty}_{\gamma,\Sigma^1} = \sqrt{\frac{P^2}{Q^4} - \frac{P^2 Q^2}{Q^6}} = 0.$$

When $\omega(\dot{\gamma}(t)) = 0$ and $\frac{d}{dt}(\omega(\dot{\gamma}(t))) \neq 0$, we have

$$\|\nabla^{\Sigma^1,s}_{\dot{\gamma}} \dot{\gamma}\|^2_{\Sigma^1,L} \sim \frac{l^2}{l_L^2} L[\frac{d}{dt}(\omega(\dot{\gamma}(t)))]^2 \sim L[\frac{d}{dt}(\omega(\dot{\gamma}(t)))]^2,$$

$$\langle \nabla^{\Sigma^1,s}_{\dot{\gamma}} \dot{\gamma}, \dot{\gamma} \rangle_{\Sigma^1,L} = O(1),$$

so, we obtain

$$\lim_{L\to+\infty} \frac{\widetilde{\kappa}^{L}_{\gamma,\Sigma^1}}{\sqrt{L}} = \frac{|\frac{d}{dt}(\omega(\dot{\gamma}(t)))|}{[\bar{b}\dot{\gamma}_1(t) - \bar{a}\dot{\gamma}_2(t)]^2}.$$

Therefore, proposition 4 holds. □

Definition 6. *Let $\Sigma^1 \subset (\mathbb{H}, g_L)$ be a regular surface. Let $\gamma : [a,b] \to \Sigma^1$ be a Euclidean C^2-smooth regular curve. The signed geodesic curvature $\widetilde{\kappa}^{L,s}_{\gamma,\Sigma^1}$ of γ at $\gamma(t)$ is defined as*

$$\widetilde{\kappa}^{L,s}_{\gamma,\Sigma^1} := \frac{\langle \nabla^{\Sigma^1}_{\dot{\gamma}} \dot{\gamma}, J_L(\dot{\gamma}) \rangle_{\Sigma^1,L}}{\|\dot{\gamma}\|^3_{\Sigma^1,L}},$$

where J_L is defined by (22).

Definition 7. *Let $\Sigma^1 \subset (\mathbb{H}, g_L)$ be a regular surface. Let $\gamma : [a,b] \to \Sigma^1$ be a Euclidean C^2-smooth regular curve. We define the intrinsic geodesic curvature $\widetilde{\kappa}^{\infty,s}_{\gamma,\Sigma^1}$ of γ at the non-characteristic point $\gamma(t)$ as*

$$\widetilde{\kappa}^{\infty,s}_{\gamma,\Sigma^1} := \lim_{L\to+\infty} \widetilde{\kappa}^{L,s}_{\gamma,\Sigma^1},$$

if the limit exists.

Proposition 5. *Let $\Sigma^1 \subset (\mathbb{H}, g_L)$ be a regular surface. Let $\gamma : [a,b] \to \Sigma^1$ be a Euclidean C^2-smooth regular curve.*

(1) When $\omega(\dot{\gamma}(t)) \neq 0$, we have

$$\widetilde{\kappa}^{\infty,s}_{\gamma,\Sigma^1} = \frac{|\bar{b}(\dot{\gamma}_1(t)+\dot{\gamma}_2(t)) + \bar{a}(\dot{\gamma}_1(t)-\dot{\gamma}_2(t))|}{|\omega(\dot{\gamma}(t))|}.$$

(2) When $\omega(\dot{\gamma}(t)) = 0$ and $\frac{d}{dt}(\omega(\dot{\gamma}(t))) = 0$, we have

$$\widetilde{\kappa}^{\infty,s}_{\gamma,\Sigma^1} = 0.$$

(3) When $\omega(\dot{\gamma}(t)) = 0$ and $\frac{d}{dt}(\omega(\dot{\gamma}(t))) \neq 0$, we have

$$\widetilde{\kappa}^{\infty,s}_{\gamma,\Sigma^1} = \frac{(\bar{a}\dot{\gamma}_2(t) - \bar{b}\dot{\gamma}_1(t))(-\dot{\gamma}_1(t)^2 - \dot{\gamma}_2(t)^2 + \frac{d}{dt}(\omega(\dot{\gamma}(t))))}{|\bar{b}\dot{\gamma}_1(t) - \bar{a}\dot{\gamma}_2(t)|^3}.$$

Proof. By (22) and (29), we obtain

$$\begin{aligned} J_L(\dot{\gamma}) &= (\bar{b}\dot{\gamma}_1(t) - \bar{a}\dot{\gamma}_2(t))J_L(e_1) - \frac{l_L}{l}L^{\frac{1}{2}}\omega(\dot{\gamma}(t))J_L(e_2) \\ &= \frac{l_L}{l}L^{\frac{1}{2}}\omega(\dot{\gamma}(t))e_1 + [(\bar{b}\dot{\gamma}_1(t) - \bar{a}\dot{\gamma}_2(t))]e_2. \end{aligned}$$

Next, we have

$$\begin{aligned} \langle \nabla^{\Sigma^1,s}_{\dot{\gamma}}\dot{\gamma}, J_L(\dot{\gamma})\rangle =& \frac{l_L}{l}L^{\frac{1}{2}}\omega(\dot{\gamma}(t))\{\bar{b}[\ddot{\gamma}_1(t) + L\omega(\dot{\gamma}(t))(\dot{\gamma}_1(t)+\dot{\gamma}_2(t)) \\ &- \bar{a}[\ddot{\gamma}_2(t) + L\omega(\dot{\gamma}(t))(\dot{\gamma}_2(t)-\dot{\gamma}_1(t))]\} \\ &+ \{\{\bar{r}_L\bar{a}[\ddot{\gamma}_1(t) + L\omega(\dot{\gamma}(t))(\dot{\gamma}_1(t)+\dot{\gamma}_2(t))] \\ &+ \bar{r}_L\bar{b}[\ddot{\gamma}_2(t) + L\omega(\dot{\gamma}(t))(\dot{\gamma}_2(t)-\dot{\gamma}_1(t))] \\ &- \frac{l}{l_L}L^{\frac{1}{2}}[-\dot{\gamma}_1(t)^2 - \dot{\gamma}_2(t)^2 + \frac{d}{dt}(\omega(\dot{\gamma}(t)))]\} \\ \sim& \frac{l_L}{l}L^{\frac{3}{2}}\omega(\dot{\gamma}(t))^2[\bar{b}(\dot{\gamma}_1(t)+\dot{\gamma}_2(t)) + \bar{a}(\dot{\gamma}_1(t)-\dot{\gamma}_2(t))] \text{as } L \to +\infty. \end{aligned}$$

So, we obtain

$$\begin{aligned} \widetilde{\kappa}^{L,s}_{\gamma,\Sigma^1} =& \frac{\langle \nabla^{\Sigma^1,s}_{\dot{\gamma}}\dot{\gamma}, J_L(\dot{\gamma})\rangle_{\Sigma^1,L}}{\|\dot{\gamma}\|^3_{\Sigma^1,L}} \\ =& \frac{L^{\frac{3}{2}}\omega(\dot{\gamma}(t))^2[\bar{b}(\dot{\gamma}_1(t)+\dot{\gamma}_2(t)) + \bar{a}(\dot{\gamma}_1(t)-\dot{\gamma}_2(t))]}{L^{\frac{3}{2}}|\omega(\dot{\gamma}(t))|^3}. \end{aligned}$$

Moreover,

$$\widetilde{\kappa}^{\infty,s}_{\gamma,\Sigma^1} = \lim_{L\to+\infty} \widetilde{k}^{L,s}_{\gamma,\Sigma^1} = \frac{|\bar{b}(\dot{\gamma}_1(t)+\dot{\gamma}_2(t)) + \bar{a}(\dot{\gamma}_1(t)-\dot{\gamma}_2(t))|}{|\omega(\dot{\gamma}(t))|}.$$

When $\omega(\dot{\gamma}(t)) = 0$ and $\frac{d}{dt}(\omega(\dot{\gamma}(t))) = 0$, we obtain

$$\begin{aligned}\langle \nabla^{\Sigma^1,s}_{\dot{\gamma}} \dot{\gamma}, J_L(\dot{\gamma}) \rangle_{L,\Sigma^1} =& \bar{b}\dot{\gamma}_1(t) - \bar{a}\dot{\gamma}_2(t)[\bar{r}_L \bar{a} \ddot{\gamma}_1(t) + \bar{r}_L \bar{b} \ddot{\gamma}_2(t) \\ &- \frac{l}{l_L} L^{\frac{1}{2}}(-\dot{\gamma}_1(t)^2 - \dot{\gamma}_2(t)^2)] \\ \sim& M_0 L^{-\frac{1}{2}} \text{ as } L \to +\infty.\end{aligned}$$

So, $\widetilde{\kappa}^{\infty,s}_{\gamma,\Sigma^1} = 0$. When $\omega(\dot{\gamma}(t)) = 0$ and $\frac{d}{dt}(\omega(\dot{\gamma}(t))) \neq 0$, we have

$$\begin{aligned}\langle \nabla^{\Sigma^1,s}_{\dot{\gamma}} \dot{\gamma}, J_L(\dot{\gamma}) \rangle_{L,\Sigma^1} =& \bar{b}\dot{\gamma}_1(t) - \bar{a}\dot{\gamma}_2(t)[\bar{r}_L \bar{a} \ddot{\gamma}_1(t) + \bar{r}_L \bar{b} \ddot{\gamma}_2(t) \\ &- \frac{l}{l_L} L^{\frac{1}{2}}(-\dot{\gamma}_1(t)^2 - \dot{\gamma}_2(t)^2 + \frac{d}{dt}(\omega(\dot{\gamma}(t))))] \\ \sim& (\bar{b}\dot{\gamma}_1(t) - \bar{a}\dot{\gamma}_2(t)(-\frac{l}{l_L})(\frac{d}{dt}(\omega(\dot{\gamma}(t))) - \dot{\gamma}_1(t)^2 - \dot{\gamma}_2(t)^2) \text{ as } L \to +\infty.\end{aligned}$$

We obtain

$$\begin{aligned}\widetilde{\kappa}^{\infty,s}_{\gamma,\Sigma^1} =& \lim_{L \to +\infty} \frac{\widetilde{\kappa}^{L}_{\gamma,\Sigma^1}}{\sqrt{L}} \\ =& \frac{(\bar{a}\dot{\gamma}_2(t) - \bar{b}\dot{\gamma}_1(t))(-\dot{\gamma}_1(t)^2 - \dot{\gamma}_2(t)^2 + \frac{d}{dt}(\omega(\dot{\gamma}(t))))}{|\bar{b}\dot{\gamma}_1(t) - \bar{a}\dot{\gamma}_2(t)|^3}.\end{aligned}$$

□

5. The Sub-Riemannian Limit of the Riemannian Gaussian Curvature of Surfaces in $(\mathbb{H}, g_L)$

In this section, we will compute the sub-Riemannian limit of the Riemannian Gaussian curvature of surfaces in $(\mathbb{H}, g_L)$. To achieve this, we define the second fundamental form II^L of the embedding of Σ^1 into $(\mathbb{H}, g_L)$:

$$II^L = \begin{pmatrix} \langle \nabla^s_{e_1} v_L, e_1 \rangle_L & \langle \nabla^s_{e_1} v_L, e_2 \rangle_L \\ \langle \nabla^s_{e_2} v_L, e_1 \rangle_L & \langle \nabla^s_{e_2} v_L, e_2 \rangle_L \end{pmatrix}.$$

We have the following theorem.

Theorem 1. *For the embedding of Σ^1 into $(\mathbb{H}, g_L)$, the second fundamental form II_L of the embedding of Σ^1 is given by*

$$II^L = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix},$$

where

$$\begin{aligned}h_{11} &= \frac{l}{l_L}(\widetilde{X_1}(\bar{a}) + \widetilde{X_2}(\bar{b})) + \bar{r}_L L^{\frac{1}{2}}, \\ h_{12} &= -\frac{l_L}{l}\langle e_1, \nabla^s_H \bar{r_L} \rangle_L + \bar{r}_L(\bar{b}\widetilde{X_1} - \bar{a}\widetilde{X_2})\frac{l}{l_L} L^{-\frac{1}{2}}, \\ h_{21} &= -\frac{l_L}{l}\langle e_1, \nabla^s_H \bar{r_L} \rangle_L + \frac{1}{2}\bar{r}_L^2, \\ h_{22} &= -\frac{l^2}{l_L^2}\langle e_2, \nabla^s_H (\frac{r}{l}) \rangle_L + \widetilde{X_3}^L(\bar{r}_L) - \bar{r_L}^3.\end{aligned}$$

Proof. Since $\langle e_1, v_L \rangle_L = 0$ and $\langle e_2, v_L \rangle_L = 0$, we have

$$\langle \nabla^s_{e_1} v_L, e_1 \rangle_L = -\langle \nabla^s_{e_1} e_1, v_L \rangle_L, \langle \nabla^s_{e_2} v_L, e_2 \rangle_L = -\langle \nabla^s_{e_2} e_2, v_L \rangle_L.$$

Using the definition of the connection, we have

$$\begin{aligned}
\nabla^s_{e_1} e_1 =& \nabla^s_{\bar{b}\widetilde{X_1} - \bar{a}\widetilde{X_2}} \bar{b}\widetilde{X_1} - \bar{a}\widetilde{X_2} \\
=& \bar{b}(\widetilde{X_1}(\bar{b})\widetilde{X_1} + \bar{b}\nabla^s_{\widetilde{X_1}}\widetilde{X_1} - \widetilde{X_1}(\bar{a})\widetilde{X_2} - \bar{a}\nabla^s_{\widetilde{X_1}}\widetilde{X_2}) \\
& - \bar{a}(\widetilde{X_2}(\bar{b})\widetilde{X_1} + \bar{b}\nabla_{\widetilde{X_2}}\widetilde{X_1} - \widetilde{X_2}](\bar{a})\widetilde{X_2} - \bar{a}\nabla^s_{\widetilde{X_2}}\widetilde{X_2}) \\
=& \bar{b}[\widetilde{X_1}(\bar{b})\widetilde{X_1} + \bar{b}(-\widetilde{X_3}) - \widetilde{X_1}(\bar{a})\widetilde{X_2} - \bar{a}(\frac{1}{2}\widetilde{X_3})] \\
& - \bar{a}[\widetilde{X_2}(\bar{b}\widetilde{X_1} + \bar{b}(-\frac{1}{2}\widetilde{X_3}) - \widetilde{X_2}(\bar{a})\widetilde{X_2} - \widetilde{X_2}(\bar{a})\widetilde{X_2} - \bar{a}(-\widetilde{X_3}) \\
=& (\bar{b}\widetilde{X_1}(\bar{b}) - \bar{a}\widetilde{X_2}(\bar{b}))\widetilde{X_1} - (\bar{b}\widetilde{X_1}(\bar{a}) - \bar{a}\widetilde{X_2}(\bar{a}))\widetilde{X_2} - (\bar{b}^2 + \bar{a}^2).
\end{aligned}$$

Since $\bar{a}^2 + \bar{b}^2 = 1$, we have $\bar{a}\widetilde{X_i}\bar{a} + \bar{b}\widetilde{X_i}\bar{b} = 0, i = 1,2,3$. Thus, $\bar{b}\widetilde{X_1}\bar{b} = -\bar{a}\widetilde{X_1}\bar{a}$ and $\bar{b}\widetilde{X_2}\bar{b} = -\bar{a}\widetilde{X_2}\bar{a}$, and we have

$$\nabla^s_{e_1} e_1 = -\bar{a}(\widetilde{X_1}(\bar{a}) + \widetilde{X_2}(\bar{b}))\widetilde{X_1} - \bar{b}(\widetilde{X_1}(\bar{a}) + \widetilde{X_2}(\bar{b}))\widetilde{X_2} - \widetilde{X_3}.$$

Next, we compute the inner product of this with v_L, we obtain

$$\begin{aligned}
h_{11} =& -\langle \nabla^s_{e_1} e_1, v_L \rangle_L \\
=& \bar{a}\bar{a}_L(\widetilde{X_1}(\bar{a}) + \widetilde{X_2}(\bar{b})) + \bar{b}\bar{b}_L(\widetilde{X_1}(\bar{a}) + \widetilde{X_2}(\bar{b})) + \bar{r}_L\sqrt{L} \\
=& \frac{a}{l}\frac{a}{l_L}(\widetilde{X_1}(\bar{a}) + \widetilde{X_2}(\bar{b})) + \frac{b}{l}\frac{b}{l_L}(\widetilde{X_1}(\bar{a}) + \widetilde{X_2}(\bar{b})) + \bar{r}_L\sqrt{L} \\
=& \frac{1}{ll_L}(a^2 + b^2)\widetilde{X_1}(\bar{a}) + \frac{1}{ll_L}(a^2 + b^2)\widetilde{X_2} + (\bar{r}_L\sqrt{L}) \\
=& \frac{l}{l_L}(\widetilde{X_1}(\bar{a}) + \widetilde{X_2}(\bar{b})) + \bar{r}_L\sqrt{L}.
\end{aligned}$$

To compute h_{12} and h_{21}, using the definition of the connection, we have

$$\begin{aligned}
\nabla^s_{e_1} e_2 =& \nabla^s_{\bar{b}\widetilde{X_1} - \bar{a}\widetilde{X_2}} \bar{r}_L\bar{a}\widetilde{X_1} + \bar{r}_L\bar{b}\widetilde{X_2} - \frac{l}{l_L}L^{-\frac{1}{2}}\widetilde{X_3} \\
=& \bar{b}[\widetilde{X_1}(\bar{r}_L\bar{a})\widetilde{X_1} + \bar{r}_L\bar{a}\nabla^s_{X_1}\widetilde{X_1} + \widetilde{X_1}\bar{r}_L\bar{a}\widetilde{X_2} + \bar{r}_L\bar{a}\nabla^s_{\widetilde{X_1}}\widetilde{X_2} \\
& - \widetilde{X_1}(\frac{l}{l_L})L^{-\frac{1}{2}}\widetilde{X_3} - \frac{l}{l_L}L^{-\frac{1}{2}}\nabla^s_{\widetilde{X_1}}\widetilde{X_3}] \\
& - \bar{a}[\widetilde{X_2}(\bar{r}_L\bar{a})\widetilde{X_1} + \bar{r}_L\bar{a}\nabla^s_{\widetilde{X_2}}\widetilde{X_1} + \widetilde{X_2}\bar{r}_L\bar{a}\widetilde{X_2} + \bar{r}_L\bar{a}\nabla^s_{\widetilde{X_2}}\widetilde{X_2} \\
& - \widetilde{X_2}(\frac{l}{l_L})L^{-\frac{1}{2}}\widetilde{X_3} - \frac{l}{l_L}L^{-\frac{1}{2}}\nabla^s_{\widetilde{X_2}}\widetilde{X_3}] \\
=& (\bar{b}\widetilde{X_1}(\bar{r}_L\bar{a}) - \bar{a}\widetilde{X_2}(\bar{r}_L\bar{a}) - \bar{b}(\frac{l}{l_L})L^{\frac{1}{2}} + \frac{1}{2}\bar{a}(\frac{l}{l_L})L^{\frac{1}{2}})\widetilde{X_1} \\
& + (\bar{b}\widetilde{X_1}(\bar{r}_L\bar{b}) - \bar{a}\widetilde{X_2}(\bar{r}_L\bar{b}) + \frac{1}{2}\bar{b}(\frac{l}{l_L})L^{\frac{1}{2}}\widetilde{X_1} + \bar{a}(\frac{l}{l_L})L^{\frac{1}{2}})\widetilde{X_2} \\
& - (\bar{b}\widetilde{X_1}(\frac{l}{l_L})L^{-\frac{1}{2}} - \bar{a}\widetilde{X_2}(\frac{l}{l_L})L^{-\frac{1}{2}})\widetilde{X_3}.
\end{aligned}$$

Then, we calculate the inner product of this with v_L. We use the product rule and the identity $\bar{b}_L\bar{a} = \bar{a}_L\bar{b}$, we gain

$$\begin{aligned}
\langle \nabla^s_{e_1} e_2, v_L \rangle_L =& (\bar{a}_L\bar{b}\bar{a} + \bar{b_L}\bar{b}^2)X_1\bar{r}_L - (\bar{a}_L\bar{a}^2 + \bar{b}_L\bar{a}\bar{b})X_2\bar{r}_L + \bar{a}_L\bar{r}_L\bar{b}\widetilde{X_1}\bar{a} \\
& + \bar{r}_L\bar{b}_L\bar{b}\widetilde{X_1}\bar{b} - \bar{r}_L\bar{a}(\bar{a}_L\widetilde{X_2}\bar{a} + \bar{b}_L\widetilde{X_2}\bar{b}) - \bar{r}_L(\bar{b}\widetilde{X_1}(\frac{l}{l_L}) - \bar{a}\widetilde{X_2}(\frac{l}{l_L}))L^{-\frac{1}{2}}.
\end{aligned}$$

To simplify this, we obtain

$$\begin{aligned}\langle\nabla^s_{e_1}e_2,v_L\rangle_L =&\bar{b}_L\widetilde{X_1}\bar{r}_L-\bar{a}_L\widetilde{X_2}\bar{r}_L-\bar{r}_L(\bar{b}\widetilde{X_1}(\frac{l}{l_L})-\bar{a}\widetilde{X_2}(\frac{l}{l_L}))L^{-\frac{1}{2}}\\ =&\frac{l}{l_L}\langle e_1,\nabla^s_H\bar{r}_L\rangle_L-\bar{r}_L\langle e_1,\nabla^s_H(\frac{l}{l_L})\rangle_L-\bar{r}_L(\bar{b}\widetilde{X_1}(\frac{l}{l_L})-\bar{a}\widetilde{X_2}(\frac{l}{l_L}))L^{-\frac{1}{2}}.\end{aligned}$$

And finally, we employ the identity $(\frac{l}{l_L}-\frac{l_L}{l})\nabla^s_H\bar{r}_L=\bar{r}_L\nabla^s_H(\frac{l}{l_L})$ in the above equation:

$$\langle\nabla^s_{e_1}e_2,v_L\rangle_L=\frac{l_L}{l}\langle e_1,\nabla^s_H\bar{r}_L\rangle_L-\bar{r}_L(\bar{b}\widetilde{X_1}(\frac{l}{l_L})-\bar{a}\widetilde{X_2}(\frac{l}{l_L}))L^{-\frac{1}{2}}.$$

So,

$$\begin{aligned}h_{12}=&-\langle\nabla^s_{e_1}e_2,v_L\rangle_L\\ =&-\frac{l_L}{l}\langle e_1,\nabla^s_H\bar{r}_L\rangle_L+\bar{r}_L(\bar{b}\widetilde{X_1}(\frac{l}{l_L})-\bar{a}\widetilde{X_2}(\frac{l}{l_L}))L^{-\frac{1}{2}},\end{aligned}$$

next

$$\begin{aligned}\nabla^s_{e_2}e_1 =&\nabla^s_{\bar{r}_L\bar{a}\widetilde{X_1}+\bar{r}_L\bar{b}\widetilde{X_2}-\frac{l}{l_L}L^{-\frac{1}{2}}\widetilde{X_3}}(\bar{b}\widetilde{X_1}-\bar{a}\widetilde{X_2})\\ =&\bar{r}_L\bar{a}[\widetilde{X_1}(\bar{b}\widetilde{X_1}+\bar{b}(-\widetilde{X_3})-\widetilde{X_1}(\bar{a})\widetilde{X_2}-\bar{a}(\frac{1}{2})]\\ &+\bar{r}_L\bar{b}[\widetilde{X_2}(\bar{b})\widetilde{X_1}+\bar{b}(-\frac{1}{2}-\widetilde{X_2}(\bar{a})\widetilde{X_2}-\bar{a}(-\widetilde{X_3})]\\ &-\frac{l}{l_L}L^{-\frac{1}{2}}[\widetilde{X_3}(\bar{b})\widetilde{X_1}+\bar{b}(-\frac{L}{2}\widetilde{X_2})-\widetilde{X_3}(\bar{a})\widetilde{X_2}-\bar{a}(\frac{L}{2}\widetilde{X_1})]\\ =&[\bar{r}_L\bar{a}\widetilde{X_1}(\bar{b})+\bar{r}_L\bar{b}\widetilde{X_2}(\bar{b})-(\frac{l}{l_L})L^{-\frac{1}{2}}\widetilde{X_3}(\bar{b})+\frac{1}{2}(\frac{l}{l_L})L^{\frac{1}{2}}\bar{a}]\widetilde{X_1}\\ &-[\bar{r}_L\bar{a}\widetilde{X_1}(\bar{a})+\bar{r}_L\bar{b}\widetilde{X_2}(\bar{a})-(\frac{l}{l_L})L^{-\frac{1}{2}}\widetilde{X_3}(\bar{a})-\frac{1}{2}(\frac{l}{l_L})L^{\frac{1}{2}}\bar{b}]\widetilde{X_2}\\ &-\frac{1}{2}\bar{r}_L\widetilde{X_3}.\end{aligned}$$

Then, we compute the inner product of this with v_L. Using the product rule and the identity $\bar{b}_L\bar{a}=\bar{a}_L\bar{b}$, we obtain

$$\begin{aligned}\langle\nabla^s_{e_2}e_1,v_L\rangle_L =&\bar{a}_L[\bar{r}_L\bar{a}\widetilde{X_1}(\bar{b})+\bar{r}_L\bar{b}\widetilde{X_2}(\bar{b})-(\frac{l}{l_L})L^{-\frac{1}{2}}\widetilde{X_3}(\bar{b})+\frac{1}{2}(\frac{l}{l_L})L^{\frac{1}{2}}\bar{a}]\\ &-\bar{b}_L[\bar{r}_L\bar{a}\widetilde{X_1}(\bar{a})+\bar{r}_L\bar{b}\widetilde{X_2}(\bar{a})-(\frac{l}{l_L})L^{-\frac{1}{2}}\widetilde{X_3}(\bar{a})-\frac{1}{2}(\frac{l}{l_L})L^{\frac{1}{2}}\bar{b}]\\ &-\frac{1}{2}\bar{r}_L^2\\ =&\frac{l}{l_L}\bar{r}_L(\widetilde{X_1}\bar{b}-\widetilde{X_2}\bar{a})-\frac{1}{2}\bar{r}_L^2\\ =&\frac{l_L}{l}\langle e_1,\nabla^s_H\bar{r}_L\rangle_L-\frac{1}{2}\bar{r}_L^2.\end{aligned}$$

Therefore,

$$\begin{aligned}h_{21}=&-\langle\nabla^s_{e_2}e_1,v_L\rangle_L\\ =&-\frac{l_L}{l}\langle e_1,\nabla^s_H\bar{r}_L\rangle_L+\frac{1}{2}\bar{r}_L^2,\end{aligned}$$

because $\langle\nabla^s_{e_2} v_L, e_2\rangle_L = -\langle\nabla^s_{e_2} e_2, v_L\rangle_L$, using the definitions of connection, identities in (5), and grouping terms, we obtain

$$\begin{aligned}\nabla^s_{e_2} e_2 =& \nabla^s_{\bar{r}_L\bar{a}\widetilde{X_1}+\bar{r}_L\bar{b}\widetilde{X_2}-\frac{l}{l_L}L^{-\frac{1}{2}}\widetilde{X_3}}(\bar{r}_L\bar{a}\widetilde{X_1}+\bar{r}_L\bar{b}\widetilde{X_2}-\frac{l}{l_L}L^{-\frac{1}{2}}\widetilde{X_3})\\ =&[\bar{r}_L\bar{a}\widetilde{X_1}\bar{r}_L\bar{a}+\bar{r}_L\bar{b}\widetilde{X_2}\bar{r}_L\bar{a}-(\frac{l}{l_L})L^{-\frac{1}{2}}\widetilde{X_3}\bar{r}_L\bar{a}-\bar{r}_L\bar{a}(\frac{l}{l_L})L^{\frac{1}{2}}-\bar{r}_L\bar{b}(\frac{l}{l_L})L^{\frac{1}{2}}]\widetilde{X_1}\\ &+[\bar{r}_L\bar{a}\widetilde{X_1}\bar{r}_L\bar{b}+\bar{r}_L\bar{b}\widetilde{X_2}\bar{r}_L\bar{a}-(\frac{l}{l_L})L^{-\frac{1}{2}}\widetilde{X_3}\bar{r}_L\bar{b}+\bar{r}_L\bar{a}(\frac{l}{l_L})L^{\frac{1}{2}}-\bar{r}_L\bar{b}(\frac{l}{l_L})L^{\frac{1}{2}}]\widetilde{X_2}\\ &+[\bar{r}_L\bar{a}\widetilde{X_1}(\frac{l}{l_L})L^{-\frac{1}{2}}+\bar{r}_L\bar{b}\widetilde{X_2}(\frac{l}{l_L})L^{-\frac{1}{2}}-(\frac{l}{l_L})L^{-\frac{1}{2}}\widetilde{X_3}(\frac{l}{l_L})L^{-\frac{1}{2}}+\bar{r}_L^2]\widetilde{X_3}.\end{aligned}$$

Taking the inner product with v_L yields

$$\begin{aligned}\langle\nabla^s_{e_2} e_2, v_L\rangle_L =&\bar{a}_L\bar{r}_L\bar{a}^2\widetilde{X_1}\bar{r}_L+\bar{a}_L\bar{r}_L^2\bar{a}\widetilde{X_1}\bar{a}+\bar{a}_L\bar{r}_L\bar{b}\bar{a}\widetilde{X_2}\bar{r}_L+\bar{a}_L\bar{r}_L^2\bar{b}\widetilde{X_2}\bar{a}\\ &-\bar{a}_L\frac{l}{\sqrt{L}l_L}\bar{a}\widetilde{X_3}\bar{r}_L-\bar{a}_L\frac{l}{\sqrt{L}l_L}\bar{r}_L\widetilde{X_3}\bar{a}-\bar{a}_L\bar{r}_L\bar{a}(\frac{l}{l_L})L^{\frac{1}{2}}-\bar{a}_L\bar{r}_L\bar{b}(\frac{l}{l_L})L^{\frac{1}{2}}\\ &+\bar{b}_L\bar{r}_L\bar{a}\bar{b}\widetilde{X_1}\bar{r}_L+\bar{b}_L\bar{r}_L^2\bar{a}\widetilde{X_1}\bar{b}+\bar{b}_L\bar{r}_L^2\bar{b}\widetilde{X_2}\bar{r}_L+\bar{b}_L\bar{r}_L^2\bar{b}\widetilde{X_2}\bar{b}\\ &-\frac{l\bar{b}\bar{b}_L}{l_L\sqrt{L}}\widetilde{X_3}\bar{r}_L-\frac{\bar{b}l\bar{r}_L}{l_L\sqrt{L}}\widetilde{X_3}\bar{b}-\bar{b}_L\bar{r}_L\bar{a}(\frac{l}{l_L})L^{\frac{1}{2}}-\bar{b}_L\bar{r}_L\bar{b}(\frac{l}{l_L})L^{\frac{1}{2}}\\ &+\bar{r}_L[\bar{r}_L\bar{a}\widetilde{X_1}(\frac{l}{l_L})L^{-\frac{1}{2}}+\bar{r}_L\bar{b}\widetilde{X_2}(\frac{l}{l_L})L^{-\frac{1}{2}}-(\frac{l}{l_L}L^{-\frac{1}{2}}\widetilde{X_3}(\frac{l}{l_l})L-\frac{1}{2}+\bar{r}_L^2],\\ \langle\nabla^s_{e_2} e_2, v_L\rangle_L =&\frac{l^2}{l_L^2}\langle e_2, \nabla^s_H(\frac{r}{l})\rangle_L-\widetilde{X_3}^L(\overline{r_L})+\bar{r}_L^3.\end{aligned}$$

We have

$$h_{22} = -\langle\nabla^s_{e_2} e_2, v_L\rangle_L = -\frac{l^2}{l_L^2}\langle e_2, \nabla^s_H(\frac{r}{l})\rangle_L+\widetilde{X_3}^L(\bar{r}_L)-\bar{r}_L^3.$$

□

The Riemannian mean curvature $\mathcal{H}_L$ of Σ^1 is defined by

$$\begin{aligned}\mathcal{H}_L :=& tr\Big(II^L\Big)\\ =&\frac{l}{l_L}(\widetilde{X_1}(\bar{a})+\widetilde{X_2}(\bar{b}))+\bar{r}_L L^{\frac{1}{2}}-\frac{l^2}{l_L^2}\langle e_2, \nabla^s_H(\frac{r}{l})\rangle_L+\widetilde{X_3}^L(\bar{r}_L)-\bar{r}_L^3,\end{aligned}$$

the horizontal mean curvature $\mathcal{H}_\infty$ of $\Sigma^1\in(\mathbb{H}, g_L)$ is given by

$$\mathcal{H}_\infty = \lim_{L\to\infty}\mathcal{H}_L = \widetilde{X_1}(\bar{a})+\widetilde{X_2}(\bar{b}).$$

Let

$$\widetilde{\mathcal{K}}^{\Sigma^1,L}(e_1,e_2) = \langle -R^{\Sigma^1,s}(e_1,e_2)e_1, e_2\rangle_{\Sigma^1,L},\ \widetilde{\mathcal{K}}^L(e_1,e_2) = -\langle R^s(e_1,e_2)e_1,e_2\rangle_L.$$

By the Gauss equation, we obtain

$$\mathcal{K}^{\Sigma^1,L}(e_1,e_2) = \mathcal{K}^L(e_1,e_2)+det(II^L). \tag{36}$$

Proposition 6. *Away from characteristic points, we have*

$$\mathcal{K}^{\Sigma^1,\infty}(e_1,e_2) = -\langle e_1, \nabla^s_H(\frac{\widetilde{X_3}u}{|\nabla^s_H u|})\rangle-\frac{(\widetilde{X_3}u)^2}{l^2}, as\ L\to+\infty.$$

Proof. We compute

$$\begin{aligned}
R^s(e_1,e_2)e_1 =& R^s\left(\bar{b}\widetilde{X_1}-\bar{a}\widetilde{X_2},\bar{r}_L\bar{a}\widetilde{X_1}+\bar{r}_L\bar{b}\widetilde{X_2}-\frac{l}{l_L\sqrt{L}}\widetilde{X_3}\right)\left(\bar{b}\widetilde{X_1}-\bar{a}\widetilde{X_2}\right)\\
=&\bar{r}_L\bar{a}\bar{b}^2R^L\left(\widetilde{X_1},\widetilde{X_1}\right)\widetilde{X_1}+\bar{r}_L\bar{b}^3R^L\left(\widetilde{X_1},\widetilde{X_2}\right)\widetilde{X_1}-\frac{l\bar{b}^2}{l_L\sqrt{L}}R^s\left(\widetilde{X_1},\widetilde{X_3}\right)\widetilde{X_1}\\
&-\bar{r}_L\bar{a}^2\bar{b}R^s\left(\widetilde{X_2},\widetilde{X_1}\right)\widetilde{X_1}-\bar{r}_L\bar{a}\bar{b}^2R^s(\widetilde{X_2},\widetilde{X_2})\widetilde{X_1}+\frac{l\bar{a}\bar{b}}{l_L\sqrt{L}}R^s\left(\widetilde{X_2},\widetilde{X_3}\right)\widetilde{X_1}\\
&-\bar{r}_L\bar{a}^2\bar{b}R^s\left(\widetilde{X_1},\widetilde{X_1}\right)\widetilde{X_2}-\bar{r}_L\bar{a}\bar{b}^2R^s\left(\widetilde{X_1},\widetilde{X_2}\right)\widetilde{X_2}+\frac{l\bar{a}\bar{b}}{l_L\sqrt{L}}R^s\left(\widetilde{X_1},\widetilde{X_3}\right)\widetilde{X_2}\\
&+\bar{r}_L\bar{a}^3R^s\left(\widetilde{X_2},\widetilde{X_1}\right)\widetilde{X_2}+\bar{r}_L\bar{a}^2\bar{b}R^s\left(\widetilde{X_2},\widetilde{X_2}\right)\widetilde{X_2}-\frac{l\bar{a}^2}{l_L\sqrt{L}}R^s\left(\widetilde{X_2},\widetilde{X_3}\right)\widetilde{X_2}\\
=&\frac{7}{4}L\bar{r_L}\bar{a}\widetilde{X_1}+\frac{7}{4}L\bar{r_L}\bar{b}\widetilde{X_2}+\frac{l}{4l_L}L^{\frac{1}{2}}\widetilde{X_3}
\end{aligned}\tag{37}$$

and

$$\begin{aligned}
\mathcal{K}^L(e_1,e_2)&=-\left\langle R^L(e_1,e_2)e_1,e_2\right\rangle_L\\
&=\frac{7}{4}L\bar{r_L}^2+\frac{L}{4}(\frac{l}{l_L})^2.
\end{aligned}\tag{38}$$

By Theorem 1 and $\nabla_H^s(\bar{r}_L)=L^{-\frac{1}{2}}\nabla_H^s\left(\frac{\widetilde{X_3}u}{|\nabla_H^s u|}\right)+O\left(L^{-1}\right)$ as $L\to+\infty$, we get

$$\begin{aligned}
\det\left(II^L\right)&=h_{11}h_{22}-h_{12}h_{21}\\
&=-\langle e_1,\nabla_H^s(\frac{\widetilde{X_3}u}{|\nabla_H^s u|})\rangle-\frac{1}{2}+\widetilde{X_1}(a)+\widetilde{X_2}(b)+O\left(L^{-\frac{1}{2}}\right).
\end{aligned}\tag{39}$$

□

6. A Guass–Bonnet Theorem in $(\mathbb{H},g_L)$

In this section, we will prove the Gauss–Bonnet Theorem in $(\mathbb{H},g_L)$. To prove the Gauss–Bonnet theorem, we need to define the Riemannian length measure and the Rimannian surface measure.

We consider the case of a regular curve $\gamma:[a,b]\to(\mathbb{H},g_L)$. We define the Riemannian length measure by

$$ds_L=\|\dot{\gamma}\|_L dt.$$

Lemma 2. *Let $\gamma:[a,b]\to(\mathbb{H},g_L)$ be a Euclidean C^2-smooth and regular curve. Let*

$$ds:=|\omega(\dot{\gamma}(t))|dt,$$

$$d\bar{s}:=\frac{1}{2}\frac{1}{|\omega(\dot{\gamma}(t))|}\left(-\dot{\gamma}_1(t)^2+\dot{\gamma}_2(t)^2\right)dt.$$

Then,

$$\lim_{L\to\infty}\frac{1}{\sqrt{L}}\int_\gamma ds_L=\int_a^b ds.$$

When $\omega(\dot{\gamma}(t))\neq 0$, we have

$$\frac{1}{\sqrt{L}}ds_L=ds+d\bar{s}L^{-1}+O\left(L^{-2}\right)\ as\ L\to+\infty.$$

When $\omega(\dot{\gamma}(t)) = 0$, we have

$$\frac{1}{\sqrt{L}}ds_L = \frac{1}{\sqrt{L}}\sqrt{-\dot{\gamma}_1(t)^2 + \dot{\gamma}_2(t)^2}dt.$$

Proof. Since

$$|\dot{\gamma}(t)|_L = \frac{1}{\sqrt{L}}ds_L = \frac{1}{\sqrt{L}}\sqrt{-\dot{\gamma}_1(t)^2 + \dot{\gamma}_2(t)^2 + L\omega(\dot{\gamma})^2},$$

we have

$$\lim_{L\to\infty}\frac{1}{\sqrt{L}}\int_\gamma |\dot{\gamma}(t)|_L dt \tag{40}$$

$$= \int_a^b \lim_{L\to\infty}\frac{1}{\sqrt{L}}|\dot{\gamma}(t)|_L dt$$

$$= \int_a^b \lim_{L\to\infty}|\dot{\gamma}(t)|_L = \frac{1}{\sqrt{L}}ds_L = \frac{1}{\sqrt{L}}\sqrt{-\dot{\gamma}_1(t)^2 + \dot{\gamma}_2(t)^2 + L\omega(\dot{\gamma})^2}dt \tag{41}$$

$$= \int_a^b |\omega(\dot{\gamma}(t))|dt$$

$$= \int_a^b ds.$$

When $\omega(\dot{\gamma}(t)) \neq 0$, we have

$$\frac{1}{\sqrt{L}}ds_L = \sqrt{L^{-1}}|\dot{\gamma}(t)|_L = \frac{1}{\sqrt{L}}ds_L = \frac{1}{\sqrt{L}}\sqrt{-\dot{\gamma}_1(t)^2 + \dot{\gamma}_2(t)^2 + L\omega(\dot{\gamma})^2}dt.$$

Using the Taylor expansion, we can prove

$$\frac{1}{\sqrt{L}}ds_L = ds + d\bar{s}L^{-1} + O\Big(L^{-2}\Big) \text{ as } L \to +\infty.$$

From the definition of ds_L and $\omega(\dot{\gamma}(t)) = 0$, we get

$$\frac{1}{\sqrt{L}}ds_L = \frac{1}{\sqrt{L}}\sqrt{-\dot{\gamma}_1(t)^2 + \dot{\gamma}_2(t)^2}dt.$$

□

Proposition 7. *Let $\Sigma^1 \subset (\mathbb{H}, g_L)$ be a Euclidean C^2-smooth surface, $\Sigma^1 = \{u = 0\}$ and $d\sigma_{\Sigma^1,L}$ denote the surface measure on Σ^1 with respect to the Riemannian metric g_L. Let*

$$d\sigma_{\Sigma^1} := \Big(\bar{a}\omega_2 - \bar{b}\omega_1\Big) \wedge \omega,\ d\bar{\sigma}_{\Sigma^1} := \frac{\widetilde{X_3}u}{l}\omega_1 \wedge \omega_2 - \frac{\Big(\widetilde{X_3}u\Big)^2}{2l^2}\Big(\bar{a}\omega_2 - \bar{b}\omega_1\Big) \wedge \omega.$$

Then

$$\frac{1}{\sqrt{L}}d\sigma_{\Sigma^1,L} = d\sigma_{\Sigma^1} + d\bar{\sigma}_{\Sigma^1}L^{-1} + O\Big(L^{-2}\Big),\ \text{as } L \to +\infty. \tag{42}$$

If $\Sigma^1 = f(D)$ with $f = f(u_1, u_2) = (f_1, f_2, f_3) : D \subset \mathbb{R}^2 \to (\mathbb{H}, g_L)$, then

$$\lim_{L\to\infty}\frac{1}{\sqrt{L}}\int_\Sigma d\sigma_{\Sigma^1,L}$$

$$= \int_D \{[\frac{f_1[((f_1)_{u_2}(f_3)_{u_1} - (f_1)_{u_1}(f_3)_{u_2})]}{2} + (f_2)_{u_1}(f_3)_{u_2}]^2$$

$$- [\frac{f_2[((f_1)_{u_2}(f_3)_{u_1} - (f_1)_{u_1}(f_3)_{u_2})]}{2}]^2 + (f_1)_{u_1}(f_2)_{u_2} - (f_1)_{u_2}(f_2)_{u_1}]^2\}^{\frac{1}{2}}du_1du_2.$$

Proof. It is well known that

$$g_L(\widetilde{X}_1,\cdot)=\omega_1,\ g_L(\widetilde{X}_2,\cdot)=\omega_2,\ g_L(\widetilde{X}_3,\cdot)=L\omega.$$

We define $\widetilde{e}_1^*:=g_L(e_1,\cdot)$, $\widetilde{e}_2^*:=g_L(e_2,\cdot)$, then

$$\widetilde{e}_1^*=\bar{b}\omega_1-\bar{a}\omega_2,\ \widetilde{e}_2^*=\bar{r}_L\bar{a}\omega_1+\bar{r}_L\bar{b}\omega_2-\frac{l}{l_L}L^{\frac{1}{2}}\omega.$$

Therefore,

$$\frac{1}{\sqrt{L}}d\sigma_{\Sigma^1,L}=\frac{1}{\sqrt{L}}\widetilde{e}_1^*\wedge\widetilde{e}_2^*=\frac{l}{l_L}\left(\bar{a}\omega_2-\bar{b}\omega_1\right)\wedge\omega+\frac{1}{\sqrt{L}}\bar{r}_L\omega_1\wedge\omega_2.$$

Recalling

$$\bar{r}_L=\frac{\left(\widetilde{X}_3u\right)L^{-\frac{1}{2}}}{\sqrt{a^2+b^2+L^{-1}\left(\widetilde{X}_3u\right)^2}}$$

and the Taylor expansion

$$\frac{1}{l_L}=\frac{1}{l}-\frac{1}{2l^3}\left(\widetilde{X}_3u\right)^2L^{-1}+O\left(L^{-2}\right)\text{ as }L\to+\infty,$$

we have

$$\begin{aligned}f_{u_1}=&(f_1)_{u_1}\partial x_1+(f_2)_{u_1}\partial x_2+(f_3)_{u_1}\partial x_3\\=&(f_1)_{u_1}\widetilde{X}_1+(f_2)_{u_1}\widetilde{X}_2+\sqrt{L}[\frac{(f_2)_{u_1}}{2}f_2-\frac{(f_1)_{u_2}}{2}f_1+(f_3)_{u_1}]\widetilde{X_3}^L.\end{aligned}$$

Similarly,

$$f_{u_2}=(f_2)_{u_1}\widetilde{X}_1+(f_2)_{u_2}\widetilde{X}_2+\sqrt{L}[\frac{(f_2)_{u_2}}{2}f_2-\frac{(f_1)_{u_2}}{2}f_1+(f_3)_{u_2}]\widetilde{X_3}^L.$$

Let

$$\bar{v}_L=\begin{vmatrix}-\widetilde{X}_1 & \widetilde{X}_2 & \widetilde{X_3}^L\\(f_1)_{u_1} & (f_2)_{u_1} & W\\(f_1)_{u_2} & (f_2)_{u_2} & V\end{vmatrix},\qquad(43)$$

where

$$W=\sqrt{L}[\frac{(f_2)_{u_1}}{2}f_2-\frac{(f_1)_{u_2}}{2}f_1+(f_3)_{u_1}]$$

$$V=\sqrt{L}[\frac{(f_2)_{u_2}}{2}f_2-\frac{(f_1)_{u_2}}{2}f_1+(f_3)_{u_2}].$$

We know that $d\sigma_{\Sigma^1,L}=\sqrt{\det(g_{ij})}du_1du_2$, $g_{ij}=g_L\left(f_{u_i},f_{u_j}\right)$ and

$$\det(g_{ij})=\|\bar{v}_L\|_L^2=-\langle\bar{v}_L,\bar{v}_L\rangle.\qquad(44)$$

So, by using the Lebesgue Dominated Convergence, we obtain Proposition 7. □

Theorem 2. *Let $\Sigma^1\subset(\mathbb{H},g_L)$ be a regular surface with a large but finite number of boundary components $(\partial\Sigma^1)_i$, $i\in\{1,\cdots,n\}$, given by Euclidean C^2-smooth regular and closed curves $\gamma_i:[0,2\pi]\to(\partial\Sigma^1)_i$. Suppose that the characteristic set $C(\Sigma^1)$ satisfies $\mathcal{H}^1(C(\Sigma^1))=0$ where $\mathcal{H}^1(C(\Sigma^1))$ denotes the Euclidean 1-dimensional Hausdorff measure of $C(\Sigma^1)$ and that $\left\|\nabla_H^s u\right\|_H^{-1}$*

is locally summable with respect to the Euclidean 2*-dimensional Hausdorff measure near the characteristic set* $C(\Sigma^1)$*, then*

$$\int_{\Sigma^1} \widetilde{\mathcal{K}}^{\Sigma^1,\infty} d\sigma_{\Sigma^1} + \sum_{i=1}^{n} \int_{\gamma_i} \widetilde{\kappa}^{\infty,s}_{\gamma_i,\Sigma^1} ds = 0.$$

Proof. Based on similar discussions in [12–18], we assume that all points satisfy $\omega(\dot{\gamma}_i(t)) \neq 0$ and $\frac{d}{dt}(\omega(\dot{\gamma}_i(t))) \neq 0$ on the curve γ_i. Since our proof of Proposition 6 is based on the approximation argument relying on the Lebesgue Dominated Convergence Theorem, the finite sets are negligible. So

$$\widetilde{\kappa}^{L,s}_{\gamma_i,\Sigma^1} = \widetilde{\kappa}^{\infty,s}_{\gamma_i,\Sigma^1} + O\Big(L^{-\frac{1}{2}}\Big). \tag{45}$$

Using the Gauss–Bonnet theorem, we obtain

$$\int_{\Sigma^1} \widetilde{\mathcal{K}}^{\Sigma^1,L} \frac{1}{\sqrt{L}} d\sigma_{\Sigma^1,L} + \sum_{i=1}^{n} \int_{\gamma_i} \widetilde{\kappa}^{L,s}_{\gamma_i,\Sigma^1} \frac{1}{\sqrt{L}} ds_L = 2\pi \frac{\chi(\Sigma^1)}{\sqrt{L}}. \tag{46}$$

Let L reach infinity, and then, using the dominated convergence theorem, we obtain

$$\int_{\Sigma^1} \widetilde{\mathcal{K}}^{\Sigma^1,\infty} d\sigma_{\Sigma^1} + \sum_{i=1}^{n} \int_{\gamma_i} \widetilde{\kappa}^{\infty,s}_{\gamma_i,\Sigma^1} ds = 0.$$

□

7. Conclusions

This paper discusses the interesting question of the Gauss–Bonnet theorem in the Heisenberg group in relation to the semi-symmetric metric connection from the Riemannian approximation scheme. The primary result of this paper is Theorem 2, which is Gauss–Bonnet type theorem related to the semi-symmetric metric connection in the Heisenberg group. To prove Theorem 2, we determine the sub-Riemannian limit of the curvature of curves, sub-Riemannian limits of the geodesic curvature of curves on surfaces, and the Riemannian Gaussian curvature of surfaces in the Heisenberg group with the semi-symmetric metric connection.

In future work, we plan to study Gauss–Bonnet theorems in the Heisenberg group with the semi-symmetric non-metric connection and other three-dimensional Riemannian Lie groups which were classified in [19]. The Gauss–Bonnet theorem connects the intrinsic differential geometry of a surface with its topology and has many applications. Therefore, it will be interesting to extend the Gauss–Bonnet theorem to other different Lie groups. We believe that the results to be obtained will have some geometric applications.

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