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Geometric Features of the Hurwitz–Lerch Zeta Type Function Based on Differential Subordination Method

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Abstract: The interest in special complex functions and their wide-ranging implementations in geometric function theory (GFT) has developed tremendously. Recently, subordination theory has been instrumentally employed for special functions to explore their geometric properties. In this effort, by using a convolutional structure, we combine the geometric series, logarithm, and Hurwitz–Lerch zeta functions to formulate a new special function, namely, the logarithm-Hurwitz–Lerch zeta function (LHL-Z function). This investigation then contributes to the study of the LHL-Z function in terms of the geometric theory of holomorphic functions, based on the differential subordination methodology, to discuss and determine the univalence and convexity conditions of the LHL-Z function. Moreover, there are other subordination and superordination connections that may be visually represented using geometric methods. Functions often exhibit symmetry when subjected to conformal mappings. The investigation of the symmetries of these mappings may provide a clearer understanding of how subordination and superordination with the Hurwitz–Lerch zeta function behave under different transformations.

Keywords: holomorphic function; Hurwitz–Lerch zeta function; univalent function; convolution product; differential subordination

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1. Introduction

Special function theory (SFT) is a significant branch of mathematical sciences as most of those functions are solutions of (fractional or ordinary) differential equations and have great implementations in science and engineering. This term refers to specific mathematical functions that are naturally generalized elementary functions, usually named after early creators. They are formulated as infinite series or integrative representations. Lots of special functions have been posed in a complex domain. Consequently, they are analytical functions. Interesting special functions comprise the Gamma function, Beta function, Hypergeometric function, Hurwitz–Lerch zeta function, Wright function, Meijer G-function, Mittag–Leffler function, and others. Recently, several investigations of generalized versions and implementations of special functions have been conducted. In the 20th century, SFT played a significant role in the development of geometric function theory (GFT), which examines the behavior and geometric features of normalized holomorphic functions on the open unit disk [1]. Specifically, in 1985, De Branges [2] used the class of hypergeometric functions as a mathematical tool for solving the renowned coefficient problem, namely, "Bieberbach conjecture", posed by Bieberbach in 1916, ref. [1], which contributed to the advancement of GFT. Since then, diverse studies on GFT connected to SFT have extensively appeared to develop geometric outcomes, for instance, Sokół et al. [3] and Al-Janaby and

Ahmad [4]. In this context, the famed sort of SFT is the general Hurwitz–Lerch zeta (HL-Z) function formulated on a complex domain. It originally dates back to the 18th century and is included in the analytic class. It is one of the higher transcendental functions of great importance in science that involves number theory and applied statistics [5]. The zeta function was initially introduced by Euler in 1737 on the real domain, and it was stated as follows:

$$\mathscr{E}_{\zeta}(\tau) = \sum_{\mu=1}^{\infty} \frac{1}{\mu^{\tau}}. \tag{1}$$

It is also named the Euler zeta function. Following this, in 1859, Riemann [6] (see also for details, Erdélyi et al. [7] in 1953 and Srivastava and Choi [8] in 2001) imposed extended Euler's formula (1) to the complex domain, called Riemann-zeta function given as follows: for $\mathscr{R}(\tau) > 1$,

$$\mathfrak{R}_{\zeta}(\tau) = \sum_{\mu=1}^{\infty} \frac{1}{\mu^{\tau}} = \sum_{\mu=0}^{\infty} \frac{1}{(\mu+1)^{\tau}}. \tag{2}$$

It continues analytically to $\mathbb{C}$, (except for a simple pole at $\tau = 1$ with residue 1), which is also prominent and named the Euler-Riemann zeta function or ($p-$series). The integral representation of (2) is stated as follows: for $\mathscr{R}(\tau) > 1$,

$$\mathfrak{R}_{\zeta}(\tau) = \frac{1}{\Gamma(\tau)} \int_{0}^{\infty} \frac{\omega^{\tau-1}}{e^{\omega} - 1}\, d\omega, \tag{3}$$

where $\Gamma(\tau) = \int\limits_{0}^{\infty} e^{-\omega} \omega^{\tau-1} d\omega$ indicates the famed Euler's Gamma function proposed by Euler [9]. Riemann zeta function is utilized as a vital tool to prove the remarkable conjecture in number theory, namely, the "prime number distribution theorem," by employing it to yield the merits of prime numbers representing the zeros of this function, which are closely related to the distribution of prime numbers. This eminent conjecture was stated by Gauss and Legendre earlier than 1830 and proved by Hadamard [10] and Poussin in 1896 [11]. Later, in 1882, Hurwitz [12] provided a generalized version of Riemann zeta function as follows:

$$\mathscr{H}_{\zeta}(\tau, \sigma) = \sum_{\mu=0}^{\infty} \frac{1}{(\mu+\sigma)^{\tau}}, \tag{4}$$

where, $\mathscr{R}(\tau) > 1$, and $\sigma \in \mathbb{C} - \mathbb{Z}_0^-$. It continues meromorphically to $\mathbb{C}$, (except for a simple pole at $\tau = 1$ with residue 1) and is called the Hurwitz-zeta function. The integral representation of (4) is stated as follows: for $\mathscr{R}(\tau) > 1$, and $\mathscr{R}(\sigma) > 0$,

$$\mathscr{H}_{\zeta}(\tau, \sigma) = \frac{1}{\Gamma(\tau)} \int_{0}^{\infty} \frac{\omega^{\tau-1}\, e^{-\sigma\omega}}{1 - e^{-\omega}}\, d\omega. \tag{5}$$

Additionally, the specific case of the Hurwitz-zeta function includes the Riemann-zeta function given in (2), that is, for $\sigma = 1$, yields $\mathscr{H}_{\zeta}(\tau, 1) = \mathfrak{R}_{\zeta}(\tau)$. Moreover, in 2002, Yen et al. [13] discussed and attained the integral representation of (5) based on the sum:

$$\mathscr{H}_{\zeta}(\tau, \sigma) = \sum_{\kappa=1}^{\varrho-1} \frac{1}{\Gamma(\tau)} \int_{0}^{\infty} \frac{\omega^{\tau-1}\, e^{-(\sigma+\kappa)\omega}}{1 - e^{-\varrho\omega}}\, d\omega, \tag{6}$$

where $\mathscr{R}(\tau) > 1$, $\mathscr{R}(\sigma) > 0$, and $\varrho \in \mathbb{N}$. In the same year, Nishimoto et al. [14] achieved the specific case of the above representation (6) when $\kappa = 2$. In this framework, a more general special function related to the Hurwitz-zeta function is the Lerch-zeta function or Lipschitz Lerch-zeta function as follows (see [15]):

$$\mathscr{L}_{\zeta}(\rho,\tau,\sigma) = \sum_{\mu=0}^{\infty} \frac{e^{2\mu\pi i\rho}}{(\mu+\sigma)^{\tau}}, \tag{7}$$

where $\sigma \in \mathbb{C} - \mathbb{Z}_0^-$, $\mathscr{R}(\tau) > 1$, when $\rho \in \mathbb{R}\backslash\mathbb{Z}$, and $\mathscr{R}(\tau) > 0$, when $\rho \in \mathbb{Z}$. This function (7) generalizes the Hurwitz-zeta function (4), which occurs when $\rho = 0$, that is, $\mathscr{L}_{\zeta}(\tau,\sigma,0) = \mathscr{H}_{\zeta}(\tau,\sigma)$, and the Riemann-zeta function (2), which occurs when $\rho = 0$, and $\sigma = 1$, that is, $\mathscr{L}_{\zeta}(\tau,1,0) = \mathfrak{R}_{\zeta}(\tau)$. From a historical perspective, this function was proposed by Lipschitz [16] in 1857 for real ρ and $\tau > 0$ but is attributed after Lerch [17] verified in 1887 that, for the imaginary part of $\rho\mathfrak{J}(\rho)$ and $0 < \tau < 1$, it attains a functional equation, namely Lerch's transformation formula, stated as follows:

$$\mathscr{L}_{\zeta}(1-\tau,\sigma,\rho) = (2\pi)^{-\tau}\Gamma(\pi)\Big\{e^{\pi i\tau/2}\, e^{-2\pi i\rho\alpha}\, \mathscr{L}_{\zeta}(\tau,-\sigma,\rho) + e^{-\pi i\tau/2}e^{-2\pi i\alpha(1-\rho)}\, \mathscr{L}_{\zeta}(\tau,\sigma,1-\rho)\Big\}, \tag{8}$$

Afterward, the Hurwitz Lerch-zeta (HL-Z) function, symbolized by $\mathscr{H}_{\mathscr{L}}(z,\tau,\sigma)$, is provided as follows (see [7] and [18]):

$$\mathscr{H}_{\mathscr{L}}(z,\tau,\sigma) = \sum_{\mu=0}^{\infty} \frac{z^{\mu}}{(\mu+\sigma)^{\tau}}, \tag{9}$$

where $\sigma \in \mathbb{C} - \mathbb{Z}_0^-$, $\tau \in \mathbb{C}$, when $|z| < 1$, and $\mathscr{R}(\tau) > 1$, when $|z| = 1$. It continues meromorphically to $\mathbb{C}$, (except for a simple pole at $\tau = 1$ with residue 1). Notice that the HL-Z function (9) is a generalized version of the Lerch-zeta function (7) $\mathscr{L}_{\zeta}(\tau,\sigma,\rho) = \mathscr{H}_{\mathscr{L}}\big(e^{2\mu\pi i\rho},\tau,\sigma\big)$, the Hurwitz-zeta function (4) $\mathscr{H}_{\zeta}(\tau,\sigma) = \mathscr{H}_{\mathscr{L}}(1,\tau,\sigma)$, and the Riemann-zeta function (2) $\mathfrak{R}_{\zeta}(\tau) = \mathscr{H}_{\mathscr{L}}(1,\tau,1)$. Further, the integral representation of (8) is rendered as follows:

$$\mathscr{H}_{\mathscr{L}}(z,\tau,\sigma) = \frac{1}{\Gamma(\tau)} \int_0^{\infty} \frac{\omega^{\tau-1}\, e^{-\sigma\omega}}{1 - ze^{-\omega}}\, d\omega, \tag{10}$$

where, $\mathscr{R}(\sigma) > 0$, $\mathscr{R}(\tau) > 0$ when $|z| \leqq 1$, $(z \neq 1)$, $\mathscr{R}(\tau) > 1$, when $z = 1$.

Since then, the analysis and investigation of the HL-Z function, its generalized formulas, and its multi-parameters extension have become a catalyst appealing to a lot of scholars. In 1997, Goyal and Laddha [19] adapted the HL-Z function and made it more general:

$$\mathscr{H}^{*}_{\mathscr{L}}(z,\gamma,\tau,\sigma) = \sum_{\mu=0}^{\infty} \frac{(\gamma)_{\mu}}{\mu!}\, \frac{z^{\mu}}{(\mu+\sigma)^{\tau}}, \tag{11}$$

where, $\gamma \in \mathbb{C}$, $\sigma \in \mathbb{C} - \mathbb{Z}_0^-$, $\tau \in \mathbb{C}$, when $|z| < 1$, $\mathscr{R}(\tau - \gamma) > 1$, when $|z| = 1$ and $(t)_{\mu}$ refers to the Pochhammer symbol formulated by [2]:

$$(t)_{\mu} = \frac{\Gamma(t+\mu)}{\Gamma(t)} = f(x) = \begin{cases} 1, & (\mu = 0) \\ t(t+1)\ldots(t+\mu-1), & (\mu \in \mathbb{N}; t \in \mathbb{C}\backslash\{0\}). \end{cases}$$

Furthermore, the integral representation of (11) is described as follows:

$$\mathscr{H}^{*}_{\mathscr{L}}(z,\gamma,\tau,\sigma) = \frac{1}{\Gamma(\tau)} \int_0^{\infty} \frac{\omega^{\tau-1}\, e^{-(\sigma-1)\omega}}{(e^{\omega} - z)^{\tau}}\, d\omega, \tag{12}$$

where, $\mathscr{R}(\sigma) > 0$, $\mathscr{R}(\tau) > 0$ when $|z| \leqq 1$, $(z \neq 1)$, $\mathscr{R}(\tau) > 1$, when $z = 1$. Subsequently, numerous captivating studies involving the HL-Z function have made contributions to complex analysis and other allied areas, such as Srivastava et al. [20], Lin and Srivastava [21], Ferreira and López [22], Murugusundaramoorthy [23], Hadi and Darus [24], Garg et al. [25], Jankov et al. [26], Srivastava et al. [27], Choi and R.K. Parmar [28], Ghanim et al. [29], Ghanim and Al-Janaby [30], Al-Janaby and Ghanim [31], Nisar [32], Nadeem et al. [33], Reynolds and Stauffer [34], and Mehrez and Agarwal [35].

In GFT, a major class is indicated by $\mathscr{H}_{\mathfrak{U}}$ that comprises holomorphic functions θ occurring in $\mathfrak{U} = \{z \in \mathbb{C} : |z| < 1\}$. For $\mu \in \mathbb{N}$, and $\wp \in \mathbb{C}$, state $\mathscr{H}_{\mathfrak{U}}[\wp, \mu] = \{\theta \in \mathscr{H}_{\mathfrak{U}} : \theta(z) = \wp + \wp_\mu z^\mu + \ldots \wp_{\mu+1} z^{\mu+1} + \ldots\}$ and assuming $\mathscr{H}^0_{\mathfrak{U}} = \mathscr{H}_{\mathfrak{U}}[0,1]$ and $\mathscr{H}^1_{\mathfrak{U}} = \mathscr{H}_{\mathfrak{U}}[1,1]$. Closely related to $\mathscr{H}^1_{\mathfrak{U}}$ is the class of holomorphic functions of the formula:

$$\theta(z) = 1 + \sum_{\mu=1}^{\infty} \wp_\mu z^\mu, \tag{13}$$

in $\mathfrak{U}$ with $\mathscr{R}(\theta(z)) > 0$, indicated by $\mathscr{P}$, namely, Caratheodory functions (functions of positive real part). Further, represented by $\mathscr{A}$, a holomorphic class of normalized functions θ described as follows: for $z \in \mathfrak{U}$

$$\theta(z) = z + \sum_{\mu=2}^{\infty} \wp_\mu z^\mu. \tag{14}$$

In this regard, designated by $\mathscr{S}$, the significant majority of investigated class θ of normalized holomorphic functions in $\mathscr{A}$ are univalent. Moreover, the functions in the starlike class $\mathscr{S}^*(t)$ and convex class $\mathscr{CV}(t)$ of order $t \in [0,1)$, which are subclasses of $\mathscr{A}$ consecutively, have the following merits: $\mathscr{R}\left(\frac{z\theta'(z)}{\theta(z)}\right) > t$ exemplify the starlike functions and $\mathscr{R}\left(\frac{z\theta''(z)}{\theta'(z)} + 1\right) > t$ exemplify the convex functions. Accordingly, $\mathscr{S}^*(0) = \mathscr{S}^*$ and $\mathscr{CV}(0) = \mathscr{CV}$, along with the class $\mathscr{S}$ comprises starlike functions and convex functions, achieve chain state as $\mathscr{CV} \subset \mathscr{S}^* \subset \mathscr{S}$, [2]. The convolution operation (Hadamard product), denoted by *, is an eminent mathematical operation attributed to Hadamard. This term presents an interesting new approach to creating convolution operators and special functions. It is formulated as follows: for $\theta_1,\ \theta_2 \in \mathscr{A}$ stated by $\theta_1(z) = z + \sum\limits_{\mu=2}^{\infty} \wp_{\mu,1} z^\mu$ and $\theta_2(z) = z + \sum\limits_{\mu=2}^{\infty} \wp_{\mu,2} z^\mu$, the convolution product of θ_1 and θ_2, written as $\theta_1 * \theta_2$, yields a new holomorphic function stated as [1]

$$(\theta_1 * \theta_2)(z) = z + \sum_{\mu=2}^{\infty} \wp_{\mu,1} \wp_{\mu,2}\, z^\mu. \tag{15}$$

In this regard, the subordinate formula based on holomorphic functions is an extension principle acting on $\mathbb{C}$ (complex domain) of the inequality formula on $\mathbb{R}$ (real line), which was created by Goodman [1] in 1909. It is stated as follows: $\theta_1,\ \theta_2 \in \mathscr{A}$, then θ_1 is subordinate to θ_2, representation $\theta_1 \prec \theta_2$, if there is a function w, holomorphic in $\mathfrak{U}$, with $w(0) = 0$ and $|w(z)| < 1$ such that $\theta_1(z) = \theta_2(w(z))$. Precisely, if θ_2 is univalent, then $\theta_1 \prec \theta_2$, if and only if $\theta_1(0) = \theta_2(0)$ and $\theta_1(\mathfrak{U}) \subset \theta_2(\mathfrak{U})$. Later, Miller and Mocanu [36,37] contributed to developing subordinate discipline. This principle plays a significant role in GFT. In other words, this principle acts as a gist tool of the holomorphic class of functions in which characterizations of functions are inferred from a differential stipulation. Recently, advanced studies have been extensively conducted by famed mathematicians based on subordinate techniques, for instance, Zayed and Bulboacă [38], Attiy et al. [39], Lupaş and Oros [40], Reem and Kassim [41], Abdulnabi et al. [42], Oros and Oros [43], Oros [44], and others. On the other hand, the study of interesting geometric attributes associated with some families of special functions has attracted a lot of investigators and experts, such as Merkes and Scott [45] in 1961, who examined starlike hypergeometric functions. After that, in 1986, Ruscheweyh [46] discussed the order of the starlikeness attribute of hypergeometric functions. Later, Ponnusamy and Vuorinen [47], in 1998, researched the univalence and convexity attributes of confluent hypergeometric functions. In the same year, Ponnusamy and Ronning [48] analyzed starlikeness attributes for convolutions, including hypergeometric series, and Ponnusamy [49] discussed close-to-convexity attributes of Gaussian hypergeometric functions. Recently, in 2023, Layth et al. [50] examined convexity attributes of integro-differential operators indicated by Hurwitz–Lerch zeta-type functions.

Motivated by the aforementioned remarkable contributions, this paper arrives at a new special function, namely, the logarithm-Hurwitz–Lerch zeta function (LHL-Z function), in terms of the convolution construct. Moreover, the univalence and convexity conditions of the LHL-Z function are discussed based on the differential subordination technique. The following terminology and lemmas in the basis of differential subordination theory are required to attain new interesting outcomes:

Definition 1 ([5]). *Let $\mathscr{T} : \mathbb{C}^3 \times \mathfrak{U} \to \mathbb{C}$ and suppose that the function $\hbar(z)$ is univalent in $\mathfrak{U}$. If the function $\mathscr{P}(z)$ is analytic in $\mathfrak{U}$ and satisfies the following second-order differential subordination:*

$$\mathscr{T}\left(\mathscr{P}(z),\ z\mathscr{P}'(z),\ z^2\omega\mathscr{P}''(z);z\right) \prec \hbar(z), \tag{16}$$

then $\mathscr{P}(z)$ is called a solution of the differential subordination (16). Furthermore, a given univalent function $\mathcal{q}(z)$ is called a dominant of the solutions of the differential subordination (16) or, simply, a dominant if $\mathscr{P}(z) \prec \mathcal{q}(z)$ for all $\mathscr{P}(z)$ achieving (16). A dominant $\widetilde{\mathcal{q}}(z)$ that archives $\widetilde{\mathcal{q}}(z) \prec \mathcal{q}(z)$ for all dominants $\mathcal{q}(z)$ of (16) is said to be the best dominant.

Lemma 1 ([5]). *Let $\mathcal{q}$ be a univalent function in $\mathfrak{U}$ and let $\mathscr{O}$ and $\mathfrak{Q}$ be holomorphic in a domain D including $\mathcal{q}(\mathfrak{U})$, with $\mathfrak{Q}(w) \neq 0$, when $w \in \mathcal{q}(\mathfrak{U})$. Setting $\mathcal{q}(z) = z\mathcal{q}'(z)\mathfrak{Q}[\mathcal{q}(z)]$, $\hbar(z) = \mathscr{O}[q(z)] + \mathcal{q}(z)$. Consider that either*

i. $\hbar$ is convex or
ii. $\mathcal{q}$ is starlike.

Furthermore, $\Re\left(\frac{z\mathfrak{h}'(z)}{\mathcal{q}(z)}\right) = \Re\left(\frac{\mathfrak{Q}[\mathcal{q}(z)]}{\mathfrak{Q}[\mathcal{q}(z)]} + \frac{z\mathcal{q}(z)}{\mathcal{q}(z)}\right) > 0$. If $\mathscr{P}$ is holomorphic in $\mathfrak{U}$, with $\mathscr{P}(0) = \mathcal{q}(0)$, $\mathscr{P}(\mathfrak{U}) \subset D$, and $[\mathscr{P}(z)] + z\mathscr{P}'(z)\mathfrak{Q}[\mathscr{P}(z)] \prec \mathscr{O}[\mathcal{q}(z)] + z\mathcal{q}'(z)\mathfrak{Q}[\mathcal{q}(z)] = \hbar(z)$. Then, $\mathscr{P} \prec \mathcal{q}$ and $\mathcal{q}$ is the best dominant.

2. Proposed Logarithm-Hurwitz–Lerch Zeta Function $\mathfrak{T}_{\mathscr{H}}(z,\gamma,\tau,\ell)$

This section investigates a new logarithm-Hurwitz–Lerch zeta function (LHL-Z function) based on the convolution structure. The HL-Z function $\mathscr{H}^*_{\mathscr{L}}(z,\tau,\sigma)$ is considered in (11) as follows:

$$\mathscr{H}^*_{\mathscr{L}}(z,\gamma,\tau,\sigma) = \sum_{\mu=0}^{\infty} \frac{(\gamma)_\mu}{\mu!}\ \frac{z^\mu}{(\mu+\sigma)^\tau}.$$

$(\gamma \in \mathbb{C}, \sigma \in \mathbb{C} - \mathbb{Z}_0^-, \tau \in \mathbb{C}$, when $|z| < 1$, $\mathscr{R}(\tau-\gamma) > 1$, when $|z| = 1)$

In addition, for $\sigma = 1$, the function $\mathscr{H}^*_{\mathscr{L}}(z,\gamma,\tau,\sigma)$ coincides with

$$\mathscr{H}^*_{\mathscr{L}}(z,\gamma,\tau,1) = \sum_{\mu=0}^{\infty} \frac{(\gamma)_\mu}{\mu!}\ \frac{z^\mu}{(\mu+1)^\tau}, \tag{17}$$

Next, utilizing respect geometric series for the modified Koebe function $\mathscr{G}(z)$ and log function $\mathscr{L}(z)$, respectively, are coined as:

$$\mathscr{G}(z) = \frac{z}{(1-z)^{\wp}} = \sum_{\mu=0}^{\infty} \frac{(\wp)_\mu}{\mu!} \text{ and } \mathscr{L}(z) = \frac{-\log(1-z)}{z} = \sum_{\mu=0}^{\infty} \frac{z^\mu}{\mu+1}. \tag{18}$$

Then, from the convolution tool, Equations (17) and (18), it leads to the following new function $\mathfrak{T}_{\mathscr{H}}(z,\gamma,\tau,\ell)$

$$\mathfrak{T}_{\mathscr{H}}(z,\gamma,\tau,\ell) = \mathscr{H}^{*}_{\mathscr{L}}(z,\gamma,\tau,1) * \left(\underbrace{\left[\mathscr{G}(z) * \frac{-\log(1-z)}{z}\right] * \ldots * \left[\mathscr{G}(z) * \frac{-\log(1-z)}{z}\right]}_{\ell-times} \right)$$
$$= \left[\sum_{\mu=0}^{\infty} \frac{(\gamma)_\mu}{\mu!} \frac{z^\mu}{(\mu+1)^\tau}\right] * \left[\sum_{\mu=0}^{\infty} \frac{\mu^\ell\, z^\mu}{(\mu+1)^\ell}\right] = \sum_{\mu=0}^{\infty} \frac{\mu^\ell\, (\gamma)_\mu}{\mu!\, (\mu+1)^{\tau+\ell}} z^\mu.$$

Therefore, LHL-Z function is

$$\mathfrak{T}_{\mathscr{H}}(z,\gamma,\tau,\ell) = \sum_{\mu=0}^{\infty} \frac{\mu^\ell\, (\gamma)_\mu}{(\mu!)^{\ell+1}\, (\mu+1)^{\tau+\ell}} z^\mu. \tag{19}$$

$$(\gamma \in \mathbb{C},\ \sigma \in \mathbb{C} - \mathbb{Z}_0^-,\ \tau \in \mathbb{C},\ \ell \in N \text{ when } |z| < 1)$$

Remark 1. *The following specific cases of LHL-Z function* $\mathfrak{T}_{\mathscr{H}}(z,\gamma,\tau,\ell)$

1. $\mathfrak{T}_{\mathscr{H}}(z\ ,1,0,0) = \sum_{\mu=0}^{\infty} \frac{(1)_\mu}{\mu!} z^\mu = \sum_{\mu=0}^{\infty} z^\mu = \frac{1}{1-z}.$
2. $\mathfrak{T}_{\mathscr{H}}(z\ ,2,0,0) = \sum_{\mu=0}^{\infty} \frac{(2)_\mu}{\mu!} z^\mu = \sum_{\mu=0}^{\infty} \mu z^\mu + \sum_{\mu=0}^{\infty} z^\mu = \frac{1}{(1-z)^2} + \frac{1}{1-z}.$
3. $\mathscr{T}_{\mathscr{H}}(z\ ,3,0,\ell) = \sum_{\mu=0}^{\infty} \frac{\mu^\ell (3)_\mu}{\mu!(\mu+1)^\ell} z^\mu = \sum_{\mu=0}^{\infty} \frac{(\mu+2)}{2(\mu+1)^{\ell-1}} z^\mu.$

3. Geometric Features of $\mathfrak{T}_{\mathscr{H}}(z,\gamma,\tau,\ell)$

This section investigates certain univalence conditions and convexity conditions of the LHL-Z function $\mathfrak{T}_{\mathscr{H}}(z,\gamma,\tau,\ell)$ by utilizing the differential subordination procedure.

Theorem 1. *Let* $\mathscr{Q}$ *be convex in* $\mathfrak{U}$ *with* $\mathscr{Q}(0) = 1$, *if* $\mathfrak{T}_{\mathscr{H}}(z,\gamma,\tau,\ell)$ *stated in (19) achieves the following subordination condition for* $0 < \alpha \leq 1$,

$$(1-\alpha\ )\ \mathfrak{T}_{\mathscr{H}}(z,\gamma,\tau,\ell) + \alpha\ z\mathfrak{T}'_{\mathscr{H}}(z,\gamma,\tau,\ell) \prec\ (1-\alpha\ )\mathscr{Q}(z) + \alpha\ z\mathscr{Q}'(z), \tag{20}$$

then

$$\mathfrak{T}_{\mathscr{H}}(z,\gamma,\tau,\ell) \prec \mathscr{Q}(z),$$

and $\mathscr{Q}$ *is the best dominant.*

Proof. Define the function $\mathscr{P}(z)$ by

$$\mathscr{P}(z) = \mathfrak{T}_{\mathscr{H}}(z,\gamma,\tau,\ell). \tag{21}$$

Noting that $\mathscr{P}$(z) is holomorphic in $\mathfrak{U}$ and $\mathscr{P}(0) = 1$. Differentiating (21), we yield

$$\mathscr{P}'(z) = \mathfrak{T}'_{\mathscr{H}}(z,\gamma,\tau,\ell),\ (z \in \mathfrak{U}). \tag{22}$$

In light of (21) and (22), differential subordination (20) leads to

$$(1-\alpha\ )\ \mathscr{P}(z) + \alpha\ z\mathscr{P}'(z) \prec (1-\alpha\ )\mathscr{Q}(z) + \alpha\ z\mathscr{Q}'(z). \tag{23}$$

Assuming the functions $\mathscr{O}$ and $\mathfrak{Q}$ as follows:

$$\mathfrak{O}(\mathbb{w}) = (1-\alpha)\mathbb{w} \text{and} \mathfrak{Q}(\mathbb{w}) = \alpha. \tag{24}$$

Obviously, the functions $\mathscr{O}$ and $\mathfrak{Q}$ are holomorphic in $\mathfrak{U}$ and $\mathfrak{Q}(\mathbb{w}) = \alpha \neq 0$. From (24) and $\mathbb{w} = \mathscr{q}(z)$, (23) is rewritten as

$$\mathscr{O}[\mathscr{P}(z)] + z\mathscr{P}'(z)\mathfrak{Q}[\mathscr{P}(z)] \prec \mathscr{O}[\mathscr{q}(z)] + z\mathscr{q}'(z)\mathfrak{Q}[\mathscr{q}(z)].$$

Further, let $\mathscr{q}, \hbar : \mathfrak{U} \to \mathbb{C}$ be the functions emergent as follows:

$$\mathscr{q}(z) = z\mathscr{q}'(z)\mathfrak{Q}[\mathscr{q}(z)] = \alpha\, z\mathscr{q}'(z), \tag{25}$$

and

$$\hbar(z) = \mathscr{O}[\mathscr{q}(z)] + \mathscr{q}[\mathscr{q}(z)] = (1-\alpha)\mathscr{q}(z) + \alpha\, z\mathscr{q}'(z). \tag{26}$$

Differentiating (25) and performing some calculations, it yields

$$\frac{z\mathscr{q}'(z)}{\mathscr{q}(z)} = \frac{\alpha\, z^2\mathscr{q}''(z) + z\,\mathscr{q}'(z)\,\alpha}{\alpha\, z\mathscr{q}'(z)} = 1 + \frac{z\mathscr{q}''(z)}{\mathscr{q}'(z)}. \tag{27}$$

As $\mathscr{q}$ is convex, it deduces $\mathscr{R}\left(\frac{z\mathscr{q}'(z)}{\mathscr{q}(z)}\right) > 0$. Thus, it leads to $\mathscr{q}$ being starlike. Moreover, from (25) and (26) and performing some computations, it yields

$$\frac{z\hbar'(z)}{\mathscr{q}(z)} = \frac{(1-\alpha)z\,\mathscr{q}'(z) + \alpha\, z^2\mathscr{q}''(z) + \alpha\, z\,\mathscr{q}'(z)}{\alpha\, z\mathscr{q}'(z)} = \frac{(1-\alpha)}{\alpha} + \frac{z\mathscr{q}''(z)}{\mathscr{q}'(z)} + 1.$$

Therefore,

$$\mathscr{R}\left(\frac{z\hbar'(z)}{\mathscr{q}(z)}\right) = \mathscr{R}\left(\frac{1-\alpha}{\alpha}\right) + \mathscr{R}\left(1 + \frac{z\mathscr{q}''(z)}{\mathscr{q}'(z)}\right) > 0.$$

Then, by Lemma1, $\mathscr{P}(z) \prec \mathscr{q}(z)$. Thus, the proof is complete.
The following corollary is acquired by setting $\mathscr{q}(z) = \frac{1+z}{1-z}$ in Theorem 1. □

Corollary 1. *Let $\mathscr{q}(z) = \frac{1+z}{1-z}$ be a function in $\mathfrak{U}$, and $\mathfrak{T}_{\mathscr{H}}(z,\gamma,\tau,\ell)$ be stated in (19), which achieves the following subordination, for $0 < \alpha \leq 1$,*

$$(1-\alpha)\mathfrak{T}_{\mathscr{H}}(z,\gamma,\tau,\ell) + \alpha\, z\mathfrak{T}'_{\mathscr{H}}(z,\gamma,\tau,\ell) \prec \frac{(1-\alpha)(1-z^2)2\alpha z}{(1-z)^2}, \tag{28}$$

then

$$\mathfrak{T}_{\mathscr{H}}(z,\gamma,\tau,\ell) \prec \frac{1+z}{1-z},$$

and $\frac{1+z}{1-z}$ is the best dominant. Therefore, $\mathfrak{T}_{\mathscr{H}}(z,\gamma,\tau,\ell)$ is a Caratheodory function.

Proof. Noting that $\mathscr{q}(z) = \frac{1+z}{1-z}$ is a convex function and $\mathscr{q}(0) = 1$. By Theorem 1, it leads to

$$\mathfrak{T}_{\mathscr{H}}(z,\gamma,\tau,\ell) \prec \frac{1+z}{1-z}.$$

Therefore, $\mathscr{R}(\mathfrak{T}_{\mathscr{H}}(z,\gamma,\tau,\ell)) > 0$. Thus, the proof is complete.
The following corollary is acquired by setting $\mathscr{q}(z) = \frac{1}{1-z}$ in Theorem 1. □

Corollary 2. *Let $\mathscr{q}(z) = \frac{1}{1-z}$ be a function in $\mathfrak{U}$, and $\mathfrak{T}_{\mathscr{H}}(z,\gamma,\tau,\ell)$ be stated in (19), which achieves the following subordination, for $0 < \alpha \leq 1$*

$$(1-\alpha)\mathfrak{T}_{\mathscr{H}}(z,\gamma,\tau,\ell)+\alpha\, z\mathfrak{T}'_{\mathscr{H}}(z,\gamma,\tau,\ell) \prec \frac{(1-\alpha\,)-(1-2\alpha\,)z}{(1-z)^2}, \tag{29}$$

then

$$\mathfrak{T}_{\mathscr{H}}(z,\gamma,\tau,\ell) \prec \frac{1}{1-z},$$

and $\frac{1}{1-z}$ *is the best dominant. Therefore,* $\mathfrak{T}_{\mathscr{H}}(z,\gamma,\tau,\ell)$ *is a Caratheodory function.*

Proof. Observing that $\mathscr{Q}(z)=\frac{1}{1-z}$ is a convex function and $\mathscr{Q}(0)=1$. Using Theorem 1, leads to

$$\mathfrak{T}_{\mathscr{H}}(z,\gamma,\tau,\ell) \prec \frac{1}{1-z} = \mathfrak{T}_{\mathscr{H}}(z,1,0,0).$$

Therefore, $\mathscr{R}(\mathfrak{T}_{\mathscr{H}}(z,\gamma,\tau,\ell))>0$. Thus, the proof is complete. □

Corollary 3. *Let* $\mathscr{Q}(z)=\frac{1+z}{1-z}$ *be a function in* $\mathfrak{U}$*. If* $\frac{2^{\tau+\ell}}{(\gamma)_1}\mathfrak{T}'_{\mathscr{H}}(z,\gamma,\tau,\ell)$*, which achieves the following subordination, for* $0<\alpha\leq 1$*,*

$$(1-\alpha)\left(\frac{2^{\tau+\ell}}{(\gamma)_1}\mathfrak{T}'_{\mathscr{H}}(z,\gamma,\tau,\ell)\right)+\alpha z\left(\frac{2^{\tau+\ell}}{(\gamma)_1}\mathfrak{T}''_{\mathscr{H}}(z,\gamma,\tau,\ell)\right)' \prec \frac{(1-\alpha\,)\,z^2+2\alpha z}{(1+z)^2}, \tag{30}$$

then

$$\frac{2^{\tau+\ell}}{(\gamma)_1}\mathfrak{T}'_{\mathscr{H}}(z,\gamma,\tau,\ell) \prec \frac{1+z}{1-z},$$

and therefore, $\mathfrak{T}_{\mathscr{H}}(z,\gamma,\tau,\ell)$ *is a univalent function in* $\mathfrak{U}$*.*

Proof. Define the function by

$$\mathscr{P}_1(z)=\frac{2^{\tau+\ell}}{(\gamma)_1}\mathfrak{T}'_{\mathscr{H}}(z,\gamma,\tau,\ell).$$

Noticing that $\mathscr{P}_1(z)$ is a holomorphic function and $\mathscr{P}_1(0)=1$. Since $\mathscr{Q}(z)=\frac{1+z}{1-z}$ is a convex function. Theorem 1 produces $\mathscr{P}_1(z)\prec\frac{1+z}{1-z}$. Therefore,

$$\mathscr{R}\left(\frac{2^{\tau+\ell}}{(\gamma)_1}\mathfrak{T}'_{\mathscr{H}}(z,\gamma,\tau,\ell)\right)>0.$$

Thus, $\mathfrak{T}_{\mathscr{H}}(z,\gamma,\tau,\ell)$ is a univalent function. □

Corollary 4. *Let* $\mathscr{Q}(z)=\frac{1+z}{1-z}$ *be a function in* $\mathfrak{U}$*. If* $1+\frac{\mathfrak{T}''_{\mathscr{H}}(z,\gamma,\tau,\ell)}{\mathfrak{T}'_{\mathscr{H}}(z,\gamma,\tau,\ell)}$ *achieves the following subordination, for* $0<\alpha\leq 1$

$$(1-\alpha)\left(1+\frac{\mathfrak{T}''_{\mathscr{H}}(z,\gamma,\tau,\ell)}{\mathfrak{T}'_{\mathscr{H}}(z,\gamma,\tau,\ell)}\right)+\alpha z\left(1+\frac{\mathfrak{T}''_{\mathscr{H}}(z,\gamma,\tau,\ell)}{\mathfrak{T}'_{\mathscr{H}}(z,\gamma,\tau,\ell)}\right)' \prec \frac{(1-\alpha\,)\,z^2+2\alpha z}{(1+z)^2}, \tag{31}$$

then

$$1+\frac{\mathfrak{T}''_{\mathscr{H}}(z,\gamma,\tau,\ell)}{\mathfrak{T}'_{\mathscr{H}}(z,\gamma,\tau,\ell)} \prec \frac{1+z}{1-z},$$

and therefore, $\mathfrak{T}_{\mathscr{H}}(z,\gamma,\tau,\ell)$ *is a convex function in* $\mathfrak{U}$*.*

Proof. Define the function by

$$\mathscr{P}_2(z) = 1 + \frac{\mathfrak{T}''_{\mathscr{H}}(z,\gamma,\tau,\ell)}{\mathfrak{T}'_{\mathscr{H}}(z,\gamma,\tau,\ell)}.$$

Clearly, $\mathscr{P}_2(z)$ is a holomorphic function and $\mathscr{P}(0) = 1$. Since $\mathscr{q}(z) = \frac{1+z}{1-z}$ is a convex function, Theorem 1 attains $\mathscr{P}_2(z) \prec \frac{1+z}{1-z}$. Therefore,

$$\mathscr{R}\left(1 + \frac{\mathfrak{T}''_{\mathscr{H}}(z,\gamma,\tau,\ell)}{\mathfrak{T}'_{\mathscr{H}}(z,\gamma,\tau,\ell)}\right) > 0.$$

Thus, $\mathfrak{T}_{\mathscr{H}}(z,\gamma,\tau,\ell)$ is a convex function. □

Theorem 2. *Let $\mathscr{q}$ be convex in $\mathfrak{U}$ with $\mathscr{q}(0) = 1$. If $\mathfrak{T}_{\mathscr{H}}(z,\gamma,\tau,\ell)$ stated by (19) achieves the following subordination condition, for $0 < \alpha \leq 1$*

$$\alpha\, \mathfrak{T}_{\mathscr{H}}(z,\gamma,\tau,\ell) + z\mathfrak{T}'_{\mathscr{H}}(z,\gamma,\tau,\ell) \prec \alpha\, \mathscr{q}(z) + z\mathscr{q}'(z), \tag{32}$$

then

$$\mathfrak{T}_{\mathscr{H}}(z,\gamma,\tau,\ell) \prec \mathscr{q}(z),$$

and $\mathscr{q}$ is the best dominant.

Proof. From the functions $\mathscr{P}(z)$ and $\mathscr{P}'(z)$ stated in (21) and (22), respectively, (32) is rewritten as

$$\alpha\mathscr{P}(z) + z\mathscr{P}'(z) \prec \alpha\mathscr{q}(z) + z\mathscr{q}'(z). \tag{33}$$

Suppose the functions $\mathscr{O}$ and $\mathfrak{Q}$ to be as follows:

$$\mathfrak{O}(\mathbb{w}) = \alpha\mathbb{w} \text{ and } \mathfrak{Q}(\mathbb{w}) = 1. \tag{34}$$

Evidently, the functions $\mathscr{O}$ and $\mathfrak{Q}$ are holomorphic in $\mathfrak{U}$. From (34) and $\mathbb{w} = \mathscr{q}(z)$, (33) is rewritten as

$$\mathscr{O}[\mathscr{P}(z)] + z\mathscr{P}'(z)\mathfrak{Q}[\mathscr{P}(z)] \prec \mathscr{O}[\mathscr{q}(z)] + z\mathscr{q}'(z)\mathfrak{Q}[\mathscr{q}(z)].$$

Further, let $\mathscr{Q}, \hbar : \mathfrak{U} \to \mathbb{C}$ be the functions given as follows:

$$\mathscr{Q}(z) = z\mathscr{q}'(z)\mathfrak{Q}[\mathscr{q}(z)] = z\mathscr{q}'(z), \tag{35}$$

and

$$\hbar(z) = \mathscr{O}[\mathscr{q}(z)] + \mathscr{q}[\mathscr{q}(z)] = \alpha\mathscr{q}(z) + z\mathscr{q}'(z). \tag{36}$$

Differentiating (35) and performing some calculations, it yields

$$\frac{z\mathscr{Q}'(z)}{\mathscr{Q}(z)} = \frac{z^2\mathscr{q}''(z) + z\mathscr{q}'(z)}{z\mathscr{q}'(z)} = 1 + \frac{z\mathscr{q}''(z)}{\mathscr{q}'(z)}. \tag{37}$$

Since $\mathscr{q}$ is convex, it deduces $\mathscr{R}\left(\frac{z\mathscr{q}'(z)}{\mathscr{q}(z)}\right) > 0$. Thus, it leads to $\mathscr{Q}$ being starlike. Moreover, from (35) and (36) and performing some computations, it yields

$$\frac{z\hbar'(z)}{\mathscr{q}(z)} = \frac{\alpha z\, \mathscr{q}'(z) + z^2\mathscr{q}''(z) + z\mathscr{q}'(z)}{z\mathscr{q}'(z)} = \alpha + \frac{z\mathscr{q}''(z)}{\mathscr{q}'(z)} + 1.$$

Therefore,

$$\mathscr{R}\left(\frac{z\hbar'(z)}{\mathscr{q}(z)}\right) = \mathscr{R}(\alpha) + \mathscr{R}\left(1 + \frac{z\mathscr{q}''(z)}{\mathscr{q}'(z)}\right) > 0.$$

Then, by Lemma 1, $\mathscr{P}(z) \prec \mathscr{Q}(z)$. Thus, the proof is complete.

The following outcome is acquired by setting $\mathscr{P}(z) = \frac{2^{\tau+\ell}}{(\gamma)_1 \mathscr{P}^{\ell}} \mathfrak{T}'_{\mathscr{H}}(z, \gamma, \tau, \ell)$ and $\mathscr{Q}(z) = \frac{1+z}{1-z}$ in Theorem 2. □

Corollary 5. *Let $\mathscr{Q}(z) = \frac{1+z}{1-z}$ be a function in $\mathfrak{U}$. If $\frac{2^{\tau+\ell}}{(\gamma)_1 \mathscr{P}^{\ell}} \mathfrak{T}'_{\mathscr{H}}(z, \gamma, \tau, \ell)$ achieves the following subordination, for $0 < \alpha \leq 1$*

$$\alpha\left(\frac{2^{\tau+\ell}}{(\gamma)_1 \mathscr{P}^{\ell}} \mathfrak{T}'_{\mathscr{H}}(z, \gamma, \tau, \ell)\right) + z\left(\frac{2^{\tau+\ell}}{(\gamma)_1 \mathscr{P}^{\ell}} \mathfrak{T}''_{\mathscr{H}}(z, \gamma, \tau, \ell)\right)' \prec \alpha\left(\frac{1-z}{1+z}\right) - \frac{2z}{1+z^2}, \quad (38)$$

then

$$\frac{2^{\tau+\ell}}{(\gamma)_1} \mathfrak{T}'_{\mathscr{H}}(z, \gamma, \tau, \ell) \prec \frac{1+z}{1-z},$$

and therefore, $\mathfrak{T}_{\mathscr{H}}(z, \gamma, \tau, \ell)$ is a univalent function in $\mathfrak{U}$.

Proof. Consider that $\mathscr{P}(z) = \frac{2^{\tau+\ell}}{(\gamma)_1} \mathfrak{T}'_{\mathscr{H}}(z, \gamma, \tau, \ell)$. Noting that $\mathscr{P}(z)$ is a holomorphic function and $\mathscr{P}(0) = 1$. Since $\mathscr{Q}(z) = \frac{1+z}{1-z}$ is a convex function, Theorem 2 produces $\mathscr{P}(z) \prec \frac{1+z}{1-z}$. Therefore,

$$\mathscr{R}\left(\frac{2^{\tau+\ell}}{(\gamma)_1} \mathfrak{T}'_{\mathscr{H}}(z, \gamma, \tau, \ell)\right) > 0.$$

Thus, $\mathfrak{T}_{\mathscr{H}}(z, \gamma, \tau, \ell)$ is a univalent function. □

Corollary 6. *Let $\mathscr{Q}(z) = \frac{1+z}{1-z}$ be a function in $\mathfrak{U}$. If $1 + \frac{\mathfrak{T}''_{\mathscr{H}}(z,\gamma,\tau,\ell)}{\mathfrak{T}'_{\mathscr{H}}(z,\gamma,\tau,\ell)}$ achieves the following subordination, for $0 < \alpha \leq 1$*

$$\alpha\left(1 + \frac{\mathfrak{T}''_{\mathscr{H}}(z, \gamma, \tau, \ell)}{\mathfrak{T}'_{\mathscr{H}}(z, \gamma, \tau, \ell)}\right) + z\left(1 + \frac{\mathfrak{T}''_{\mathscr{H}}(z, \gamma, \tau, \ell)}{\mathfrak{T}'_{\mathscr{H}}(z, \gamma, \tau, \ell)}\right)' \prec \alpha\left(\frac{1-z}{1+z}\right) - \frac{2z}{1+z^2}, \quad (39)$$

then

$$1 + \frac{\mathfrak{T}''_{\mathscr{H}}(z, \gamma, \tau, \ell)}{\mathfrak{T}'_{\mathscr{H}}(z, \gamma, \tau, \ell)} \prec \frac{1+z}{1-z},$$

and therefore, $\mathfrak{T}_{\mathscr{H}}(z, \gamma, \tau, \ell)$ is a convex function in $\mathfrak{U}$.

Proof. Assume that $\mathscr{P}(z) = 1 + \frac{\mathfrak{T}''_{\mathscr{H}}(z,\gamma,\tau,\ell)}{\mathfrak{T}'_{\mathscr{H}}(z,\gamma,\tau,\ell)}$. Clearly, $\mathscr{P}(z)$ is a holomorphic function and $\mathscr{P}(0) = 1$. Since $\mathscr{Q}(z) = \frac{1+z}{1-z}$ is a convex function, Theorem 2 attains $\mathscr{P}(z) \prec \frac{1+z}{1-z}$. Therefore,

$$\mathscr{R}\left(1 + \frac{\mathfrak{T}''_{\mathscr{H}}(z, \gamma, \tau, \ell)}{\mathfrak{T}'_{\mathscr{H}}(z, \gamma, \tau, \ell)}\right) > 0.$$

Thus, $\mathfrak{T}_{\mathscr{H}}(z, \gamma, \tau, \ell)$ is a convex function. □

4. Conclusions

In this research, through the usage of a convolutional structure, a new generation formula for a generalized special function, namely, the logarithm-Hurwitz–Lerch zeta function (LHL-Z function), is formulated in a specific complex domain and discussed along with its geometric features. This function originated in terms of typical geometric series, logarithms, and Hurwitz–Lerch zeta functions. The differential subordination process leads to the conclusion that the LHL-Z function meets the necessary univalence and convexity conditions. In future work, more useful analytical studies can be found by using different holomorphic classes, such as harmonic classes, meromorphic classes, and multivalent classes, based on the suggested generalized LHL-Z function.

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References

1. Goodman, A.W. *Univalent Functions*; Mariner: Tampa, FL, USA, 1983.
2. De Branges, L. A proof of the Bieberbach conjecture. *Acta Math.* **1985**, *154*, 137–152. [CrossRef]
3. Sokół, J.; Ibrahim, R.W.; Ahmad, M.Z.; Al-Janaby, H.F. Inequalities of harmonic univalent functions with connections of hypergeometric functions. *Open Math.* **2015**, *13*, 691–705. [CrossRef]
4. Al-Janaby, H.F.; Ahmad, M.Z. Differential inequalities related to Salagean type integral operator involving extended generalized Mittag-Leffler function. *J. Phys. Conf. Ser.* **2018**, *1132*, 012061. [CrossRef]
5. Srivastava, H.M. A new family of the λ-generalized Hurwitz-Lerch zeta functions with applications. *Appl. Math. Inf. Sci.* **2014**, *8*, 1485–1500. [CrossRef]
6. Riemann, G.F.B. Über die Anzahl der Primzahlen unter einer gegebenen Grösse. *Monatsber. Königl. Preuss. Akad. Wiss. Berlin* **1859**, *2*, 671–680, Reprinted in Das Kontinuum und Andere Monographen (Ed. H. Weyl). New York: Chelsea, 1972.
7. Erdélyi, A.; Magnus, W.; Oberhettinger, F.; Tricomi, F.G. *Tables of Integral Transforms*; McGraw-Hill Book Company: New York, NY, USA, 1954; Volume II.
8. Srivastava, H.M.; Choi, J. *Series Associated with the Zeta and Related Functions*; Kluwer Academic Publishers: Dordrecht, The Netherlands; Boston, MA, USA; London, UK, 2001.
9. Rainville, E.D. *Special Functions*; Macmillan: New York, NY, USA, 1960.
10. Hadamard, J. Sur la distribution des zéros de la fonction $\zeta(s)$ et ses conséquences arithmétiques. *Bull. Société Mathématique Fr.* **1896**, *24*, 199–220. [CrossRef]
11. de la Vallee-Poussin, C.J. *Recherches Analytiques sur la Théorie des Nombres Premiers*; Imprimeur de l'Académie Royale de Belgique: Bruxelles Hayez, Belgium, 1896.
12. Hurwitz, A. Einige Eigenschaften der Dirichletschen Funktionen $F(s) = \sum \left(\frac{D}{n}\right) \frac{1}{n^s}$ die bei der Bestimmung der Klassenanzahlen Binärer quadratischer Formen auftreten. *Z. Für Math. Phys.* **1882**, *27*, 86–101.
13. Yen, C.-E.; Lin, M.-L.; Nishimoto, K. An integral form for a generalized Zeta function. *J. Fract. Calc.* **2002**, *22*, 99–102.
14. Nishimoto, K.; Yen, C.-E.; Lin, M.-L. Some integral forms for a generalized Zeta function. *J. Fract. Calc.* **2002**, *22*, 91–97.
15. Srivastava, H.M. Some generalizations and basic (or q-) extensions of the Bernoulli, Euler and Genocchi polynomials. *Appl. Math. Inform. Sci.* **2011**, *5*, 390–444.
16. Lipschitz, R. Untersuchung einer aus vier Elementen gebildeten Reihe. *J. Reine Angew.* **1857**, *54*, 313–328.
17. Lerch, M. Note sur la fonction $\hbar(w, x, s)\sum_{k=0}^{\infty} \frac{e^{2k\pi ix}}{(w+k)^s}$. *Acta Math.* **1887**, *11*, 19–24. [CrossRef]
18. Whittaker, E.T.; Watson, G.N. *A Course of Modern Analysis: An Introduction to the General Theory of Infinite Processes and of Analytic Functions; With an Account of the Principal Transcendental Functions*, 4th ed.; Cambridge University Press: Cambridge, UK, 1963.
19. Goyal, S.P.; Laddha, R.K. On the generalized Riemann zeta function and the generalized Lambert transform. *Ganita Sandesh* **1997**, *11*, 99–108.
20. Petojević, A. A note about the Pochhammer symbol. *Math. Moravica* **2008**, *12-1*, 37–42. [CrossRef]
21. Lin, S.D.; Srivastava, H.M. Some families of the Hurwitz–Lerch zeta functions and associated fractional derivative and other integral representations. *Appl. Math. Comput.* **2004**, *154*, 725–733. [CrossRef]
22. Ferreira, C.; López, J.L. Asymptotic expansions of the Hurwitz-Lerch zeta function. *J. Math. Anal. Appl.* **2004**, *298*, 210–224. [CrossRef]
23. Murugusundaramoorthy, G. Subordination results and integral means inequalities for k-uniformly starlike functions defined by convolution involving the Hurwitz-Lerch Zeta function. *Stud. Univ. Babeş-Bolyai Math.* **2010**, *LV*, 69–82.
24. Hadi, S.H.; Darus, M. (p, q)-Chebyshev polynomials for the families of biunivalent function associating a new integral operator with (p, q)-Hurwitz zeta function. *Turk. J. Math.* **2022**, *46*, 25. [CrossRef]

25. Garg, M.; Jain, K.; Kalla, S.L. A further study of general Hurwitz–Lerch zeta function. *Algebras Groups Geom.* **2008**, *25*, 311–319.
26. Jankov, D.; Pogany, T.K.; Saxena, R.K. An extended general Hurwitz-Lerch Zeta function as a Mathieu $(a, \lambda)-$series. *Appl. Math. Lett.* **2011**, *24*, 1473–1476. [CrossRef]
27. Srivastava, H.M.; Jolly, N.; Bansal, M.K.; Jain, R. A new integral transform associated with the λ-extended Hurwitz-Lerch zeta function. *Rev. Real Acad. Cienc. Exactas Físicas Nat. Ser. A Matemáticas* **2019**, *113*, 1679–1692. [CrossRef]
28. Choi, J.; Parmar, R.K. An Extension of the Generalized Hurwitz-Lerch Zeta function of two variables. *Filomat* **2017**, *31*, 91–96. [CrossRef]
29. Ghanim, F.; Batiha, B.; Ali, A.H.; Darus, M. Geometric Properties of a Linear Complex Operator on a Subclass of Meromorphic Functions: An Analysis of Hurwitz-Lerch-Zeta Functions. *Appl. Math. Nonlinear Sci.* **2023**, *8*, 2229–2240. [CrossRef]
30. Ghanim, F.; Al-Janaby, H.F. A certain subclass of univalent meromorphic functions defined by a linear operator associated with the Hurwitz-Lerch zeta function, Rad Hrvatske akademije znanosti i umjetnosti. *Mat. Znan.* **2019**, *23*, 71–83.
31. Al-Janaby, H.F.; Ghanim, F.; Darus, M. On The Third-Order Complex Differential Inequalities of ξ-Generalized-Hurwitz–Lerch Zeta Functions. *Mathematics* **2020**, *8*, 845. [CrossRef]
32. Nisar, K.S. Further extension of the generalized Hurwitz-Lerch Zeta function of two variables. *Mathematics* **2019**, *7*, 48. [CrossRef]
33. Nadeem, R.; Usman, T.; Nisar, K.S.; Baleanu, D. Analytical properties of the Hurwitz–Lerch zeta function. *Adv. Differ. Equ.* **2020**, *2020*, 466. [CrossRef]
34. Reynolds, R.; Stauffer, A. Sum of the Hurwitz-Lerch Zeta Function over Natural Numbers: Derivation and Evaluation. *J. Math.* **2022**, *2022*, 3591775. [CrossRef]
35. Mehrez, K.; Agarwal, P. Integral representations of the multi-parameter Hurwitz–Lerch zeta function and applications. *J. Anal.* **2023**, *31*, 1707–1728. [CrossRef]
36. Miller, S.S.; Mocanu, P.T. Differential subordinations and univalent functions. *Mich. Math. J.* **1981**, *28*, 157–172. [CrossRef]
37. Miller, S.S.; Mocanu, P.T. *Differential Subordinations, Theory and Applications*; A series of Monographs and Textbooks in Pure and Applied Mathematics; CRC Press: New York, NY, USA, 2000; 480p.
38. Zayed, H.; Bulboacă, T. Sandwich results for higher order fractional derivative operators. *Mat. Stud.* **2018**, *49*, 52–66. [CrossRef]
39. Attiy, A.A.; Aouf, M.K.; Ali, E.E.; Yassen, M.F. Differential subordination and superordination results associated with Mittag-Leffler function. *Mathematics* **2021**, *9*, 226. [CrossRef]
40. Lupaş, A.A.; Oros, G.I. Differential Subordination and Superordination Results Using Fractional Integral of Confluent Hypergeometric Function. *Symmetry* **2021**, *13*, 327. [CrossRef]
41. Reem, O.R.; Kassim, A. A Study of a Certain Family of Multivalent Function Associated with Subordination. *Iraqi J. Sci.* **2021**, *62*, 3009–3019.
42. Abdulnabi, F.F.; Al-Janaby, H.F.; Ghanim, F.; Lupaș, A. Some Results on Third-Order Differential Subordination and Differential Superordination for Analytic Functions Using a Fractional Differential Operator. *Mathematics* **2023**, *11*, 18. [CrossRef]
43. Oros, G.I.; Oros, G. Strong differential subordination. *Turk. J. Math.* **2009**, *33*, 249–257. [CrossRef]
44. Oros, G.I. Strong differential superordination. *Acta Univ. Apulensis* **2009**, *19*, 101–106.
45. Merkes, E.P.; Scott, W.T. Starlike hypergeometric functions. *Proc. Am. Math. Soc.* **1961**, *12*, 885–888. [CrossRef]
46. Ruscheweyh, S.; Singh, V. On the order of starlikeness of hypergeometric functions. *J. Math. Anal. App.* **1986**, *113*, 1–11. [CrossRef]
47. Ponnusamy, S.; Vuorinen, M. Univalence and convexity properties for confluent hypergeometric functions. *Complex Var. Theory Appl.* **1998**, *36*, 73–97. [CrossRef]
48. Ponnusamy, S.; Rønning, F. Starlikeness properties for convolutions involving hypergeometric series. *Ann. Univ. Mariae Curie-Sklodowska* **1998**, *52*, 141–155.
49. Ponnusamy, S. Close-to-convexity properties of Gaussian hypergeometric functions. *J. Comput. Appl. Math.* **1998**, *88*, 327–337. [CrossRef]
50. Layth, T.K.; Ahmed, M.A.; Al-Janaby, H.F. Convexity properties for integro-differential operators proposed by Hurwitz-Lerch Zeta Type functions. *Iraqi J. Sci.* **2023**, *64*, 6483–6492.